Introduction

Algebraic theories and algebraic categories were introduced by Lawvere in his pioneering Ph.D. thesis as invariant formulations of Birkhoff's universal algebra (see [30]). The innovative and revelatory approach consists of describing the syntax and the semantics in the following way. An *algebraic theory* is a concrete mathematical object – the concept – namely a set of variables together with formal symbols and equalities between these terms. Stated otherwise, an algebraic theory is a small category with finite products. An algebra or model of the theory is a set-theoretical interpretation – a possible meaning – or, more categorically, a finite product-preserving functor from the theory into the category of sets. We call the category of models of an algebraic theory an *algebraic category*. This description allows us to investigate "simultaneously" all those structures like groups, rings, modules, lattices, ... characterised by the existence of one or several operations defined everywhere and subjected to axioms expressed by equalities. The peculiar feature of the subject is the interplay between properties of the theory and properties of the models: how we can recollect aspects of the syntax knowing about the semantics and conversely. We mention that Lawvere's investigation concerned the one-sorted version; whereas, at the same period, many-sorted algebraic theories were studied by Bénabou (see [8]).

By generalising the theory we do generalise the models. This concept is the fascinating aspect of the subject and the reference point of our project. Replacing finite products by finite limits provides the notion of a locally finitely presentable category of models. A theory, in this case, is a small category $\mathcal{C}$ with finite limits; a model of the theory is a set-valued functor $F : \mathcal{C} \longrightarrow \mathbf{set}$ which preserves finite limits. If we consider the case of absence of limits, other generalisations are possible. A theory is a mere small category $\mathcal{C}$; a model of the theory is a flat functor $F : \mathcal{C} \longrightarrow \mathbf{set}$, namely a (small) filtered colimit of representable functors. This leads to the notion of accessible category of models. On the other hand, a model of the theory $\mathcal{C}$ could be just a set-valued functor $F : \mathcal{C} \longrightarrow \mathbf{set}$. In this case, we deal with presheaf categories.

Accessible categories admit an intrinsic axiomatic characterisation in terms of generators: they are those categories with a "good" set of generators. And locally presentable categories are those accessible categories which are cocomplete (or, equivalently, complete).

In this manuscript, we are interested in the study of categories of models. We pursue our task by means of considering models of different theories. We approach the investigation from two different – but related – points of view, as the structure of the thesis itself will denote.

The first idea is "unification": the theories are small categories with a specified class $\mathbb{D}$ of limits, the models of the theory are set-valued functors preserving the class of limits $\mathbb{D}$ under consideration. It is an attempt of describing "at the same time" similar situations, as the ones previously mentioned, where $\mathbb{D}$ can be the class of finite products, or the class of finite limits, or something else. This wants to be a generalisation of the notion of models of a theory, which encloses in one all the known cases of models, from presheaves to locally finitely presentable categories. The analysis leads to the definition of $\mathbb{D}$-theories and, consequently, of categories of $\mathbb{D}$-models. The theory of locally $\mathbb{D}$-presentable categories has been recently developed by Adámek, Borceux, Lack and Rosický in [1]. Starting from a doctrine – that is, an (essentially small) full subcategory of **cat**, the category of all small categories – $\mathbb{D}$, the notion of $\mathbb{D}$-limit – the limit of a functor with domain in $\mathbb{D}$ – the notion of $\mathbb{D}$-complete category – a category with all $\mathbb{D}$-limits – and the notion of $\mathbb{D}$-continuous functor – a functor, between $\mathbb{D}$-complete categories, which preserves all $\mathbb{D}$-limits – have been introduced. By generalising further these issues, one is led to define $\mathbb{D}$-filtered colimits: they are – as you might expect – those colimits commuting in **set** with $\mathbb{D}$-limits. Then, all notions of $\mathbb{D}$-presentable object, of $\mathbb{D}$-accessibility and of locally $\mathbb{D}$-presentability take shape quite naturally. We mention at once that this kind of generalisation is not possible for any class $\mathbb{D}$ of limits. In fact, a further assumption is required on the doctrine $\mathbb{D}$: the condition is called *soundness*. Roughly speaking, soundness is the hypothesis needed in order to achieve the analogous – to the locally finitely presentable case – implication that for a diagram $S : \mathcal{I} \longrightarrow \mathcal{C}$, for a small category $\mathcal{C}$, its category of cocones is connected implies that the category $\mathcal{C}$ is filtered. The definition might appear quite technical (and it is so) but it is the one needed to establish the expected correspondence between $\mathbb{D}$-flat functors and $\mathbb{D}$-continuous functors, precisely as in the case of finite limits. More explicitly, under the assumption on the doctrine $\mathbb{D}$ to be sound, a $\mathbb{D}$-flat functor $F : \mathcal{A} \longrightarrow \mathcal{B}$, defined on a $\mathbb{D}$-complete category $\mathcal{A}$, is, moreover, $\mathbb{D}$-continuous. We support the general theory of locally $\mathbb{D}$-presentable categories with examples. For our purposes, the illuminating examples are mainly three: the doctrine FIN of finite limits, the doctrine FINPR of finite products and the empty doctrine. The corresponding categories are, therefore, locally finitely presentable categories, (finitary multisorted) varieties and presheaf categories, respectively. We achieve a duality theorem, in terms of a contravariant biequivalence:

$$\mathbb{D}\text{-}\mathbf{Th}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{LP}$$

where $\mathbb{D}$-**Th** is the 2-category of essentially small Cauchy complete and $\mathbb{D}$-complete categories, $\mathbb{D}$-continuous functors and natural transformations, whereas $\mathbb{D}$-**LP** is the 2-category of locally $\mathbb{D}$-presentable categories, $\mathbb{D}$-accessible right adjoint functors and natural transformations (see [18]). This is the theorem which provides a unified proof of Gabriel-Ulmer duality for locally finitely presentable categories (see [23]), Adámek-Lawvere-Rosický duality for varieties (see [2]) and Morita duality for presheaf categories (see, for instance, [10]). Moreover, following the suggestion of the locally finitely presentable case, we investigate further these issues to finally propose a new characterisation of locally $\mathbb{D}$-presentable categories in terms of strong generators. Explicitly, we prove that a cocomplete category is locally $\mathbb{D}$-presentable if and only if it has a strong generator consisting of $\mathbb{D}$-presentable objects (see [17]). We mention that some work in this area has been undertaken independently by Diers ([20]).

The second point of view is "specialisation". We go back to a "case by case" analysis, in order to study *localizations* (namely, fully faithful right adjoint functors whose left adjoint preserves finite limits) of algebraic categories and localizations of presheaf categories. These are still categories of models of the corresponding theory.

Motivations to the study of localizations can be historically traced in homological algebra, for instance. The goal, which a wide class of mathematicians pursue, is to do homological algebra in a sheaf theoretical context. The fundamental, basic remark is that categories of presheaves of abelian groups or modules are abelian categories, meaning, roughly speaking, that techniques of homological algebra as "diagram lemmas" can be applied to categories of presheaves and performed quite easily. From where, the need of carrying this concept to categories of sheaves has arisen. This is the fruitful idea behind the formal definition of localization: homological algebra is inherited via localizations. More precisely, a localization of an abelian category is again abelian. The crucial reference point is that categories of sheaves are precisely the localizations of presheaf categories (see [25, 36]). The next, natural setting where to develop homological algebra and, therefore, where to study localizations are semi-abelian categories and, in particular, semi-abelian algebraic categories: the category of groups, the category of rings, the category Lie algebras, ... are all examples of such. As references for this more recent research developments, we mention to [12, 24]. Finally, to provide some context for localization, we mention the following example: given a ring A and a prime ideal I, by localising the ring A over the ideal I, one obtains the category of modules over the local ring A_I as a localization of the category of modules over the ring A.

This thesis sits inside a wider project, which is still in our minds. In the original classical spirit of Category Theory, our purpose is to produce a new notion which generalises localizations and allows us to treat them in the more general setting of the theory of locally $\mathbb{D}$-presentable categories. More explicitly, for a limit doctrine $\mathbb{D}$, we wish to define a proper notion of $\mathbb{D}$-localization, which should make “good sense” at least while specialising $\mathbb{D}$ in the three main examples we always keep in mind while dealing with $\mathbb{D}$-theories, namely, the doctrine FIN, the doctrine FINPR and the empty doctrine. A $\mathbb{D}$-*localization* $i : \mathcal{B} \longrightarrow \mathcal{A}$ should be a fully faithful functor i together with a left adjoint $L : \mathcal{A} \longrightarrow \mathcal{B}$ which preserves $\mathbb{D}$-limits (where $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{D}$-complete categories). Clearly, in the case of the doctrine $\mathbb{D}$ of finite limits, this definition returns the classical one of localization. The aim of some advanced investigation is to classify $\mathbb{D}$-localizations of categories of set-valued $\mathbb{D}$-continuous functors, that is, categories of the form $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, for some small $\mathbb{D}$-complete category $\mathcal{A}$, which are precisely all locally $\mathbb{D}$-presentable categories. In the case of the doctrine of finite limits, this project has already been undertaken by Day and Street (see [19]): they characterise localizations of locally presentable categories. Moreover, we mention that some work has been done, in a slightly different optic, when specifying $\mathbb{D}$ as the empty doctrine: in [39] (Theorem 17.3.3) they classify localizations of presheaf categories in terms of dense functors. Unfortunately, when we tried to adapt these kinds of results to the case of the doctrine FINPR, we did not succeed in describing the localizations of the categories of models. The nature of the problem is partially grounded on the condition on the left adjoint functor L to preserves finite products. These troubles in finding a general way of investigating $\mathbb{D}$-localizations, for a limit doctrine $\mathbb{D}$, determined the beginning of a deep investigation on the character of those functors $k : \mathcal{C} \longrightarrow \mathcal{E}$ defined on a category $\mathcal{C}$ with finite coproducts. It led us to the discovery of the characterisation theorems for localizations described in Chapter 3, for the presheaf case, and in Chapter 4, for the algebraic case.

Of course, localizations of presheaf categories and of algebraic categories are well-known and well-studied mathematical objects: to mention some references, consider Giraud's theorem for the former case ([25, 34]) and the theorems by Vitale for the latter one ([41, 42]). The innovation in this work consists of a different approach to the matter of localizations. We wish to characterise all functors $k : \mathcal{C} \longrightarrow \mathcal{E}$ from a small category $\mathcal{C}$ – with or without finite products, depending on the case treated – into a category $\mathcal{E}$ which satisfies certain properties, depending on the corresponding case, which realise the category $\mathcal{E}$ as a localization of the category $[\mathcal{C}^{op}, \mathbf{set}]$ of set-valued functors, or of the category $\mathbf{Finprod}[\mathcal{C}^{op}, \mathbf{set}]$ of set-valued finite product-preserving functors, which is precisely $\mathbf{Mod}(\mathcal{C})$. In other words, provided some category $\mathcal{E}$, we establish theorems which describe all its realisations as a localization of some appropriate category of set-valued functors. The suggestion comes from a Gabriel-Popescu theorem, which characterises Grothendieck categories (namely, cocomplete abelian categories with a generator and filtered colimits commuting with finite limits) as localizations of the additive functor category $\mathbf{Add}[\mathcal{C}^{op}, \mathbf{Ab}]$ of additive presheaves defined on a small pre-additive category $\mathcal{C}$, and its generalisation in the recent paper [32]. Following these lines, we achieve a classification of localizations, in both the presheaf and the algebraic context and, moreover, a classification of *geometric morphisms* (namely, functors together with a finite limit-preserving left adjoint). We are able, in fact, to state independent conditions on the functor $k : \mathcal{C} \longrightarrow \mathcal{E}$, which provide fully faithfulness of the embedding into the functor category on one side, and left exactness of the left adjoint functor on the other. It is precisely the nature of these properties which allows us to distinguish the classification of geometric morphisms from that of localizations. It will be redundant to list, in this introduction, the various conditions we came across for the functor k, but it is still worthwhile to mention that some of these might have – for those readers familiar with the subject – the same flavour of the properties characterising the *Lemme de Comparaison*, as in [28], whereas some others remind of the notion of *filtering functor*, as in [34]. In fact, we introduce the new notion of *special filtering* functor, which encapsulates the deep nature of the exact completion of the full subcategory of the category of models consisting of the free ones.

The classifying theorems contained in Chapter 3 and Chapter 4 might propose a new way of facing the issue of $\mathbb{D}$-localizations. Provided a category $\mathcal{E}$, we should hope to reach some results describing all its realisations as a $\mathbb{D}$-localization of some appropriate category of $\mathbb{D}$-continuous set-valued functors. The desirable achievement would be to establish some theorems giving necessary and sufficient conditions on a $\mathbb{D}$-continuous functor $k : \mathcal{C} \longrightarrow \mathcal{E}$, from a small $\mathbb{D}$-complete category $\mathcal{C}$ into a cocomplete, exact category $\mathcal{E}$ with $\mathbb{D}$-filtered colimits commuting with $\mathbb{D}$-limits, to realise $\mathcal{E}$ as a $\mathbb{D}$-localization of the category of $\mathbb{D}$-continuous set-valued functors defined on $\mathcal{C}$. This remains part of our outgoing project, still open to fruitful investigations.

In Chapter 1, we illustrate the theory of locally $\mathbb{D}$-presentable categories. We recall the basic definitions the reader will find useful to approach this thesis: starting from the notion of a doctrine $\mathbb{D}$, we present the whole series of $\mathbb{D}$-limit, $\mathbb{D}$-filtered colimit, $\mathbb{D}$-presentable object, $\mathbb{D}$-accessibility and locally $\mathbb{D}$-presentability definitions. The fundamental results concerning this material are listed and provided with (sketches of the) proofs. Classical examples constantly support this new-born theory.

In Chapter 2, we build our contravariant biequivalence between the 2-category $\mathbb{D}$-**Th**, of essentially small $\mathbb{D}$-complete and Cauchy complete categories, $\mathbb{D}$-continuous functors and natural transformations, and the 2-category $\mathbb{D}$-**LP**, of locally $\mathbb{D}$-presentable categories, $\mathbb{D}$-accessible right adjoint functors and natural transformations. This Duality theorem provides the desired, expected generalisation of Gabriel-Ulmer duality for locally finitely presentable categories, Adámek-Lawvere-Rosický duality for varieties and Morita duality for presheaf categories. Moreover, we state our characterisation of locally $\mathbb{D}$-presentable categories in terms of strong generators. As applications of the Duality theorem, we compare different doctrines and, consequently, different categories of models, providing a useful scheme to relate locally finitely presentable categories, (finitary multisorted) varieties and presheaves categories.

In Chapter 3, we briefly mention the fundamental definitions of geometric morphism and localization. Moreover, the notion of filtering functor is recalled. Given a small category $\mathcal{A}$ and a cocomplete, exact and extensive category $\mathcal{B}$, we establish the crucial equivalence between the category $\mathbf{GeoMor}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]]$, of geometric morphisms between $\mathcal{B}$ and $[\mathcal{A}^{op}, \mathbf{set}]$ and geometric transformations between those, and the category $\mathbf{Filt}[\mathcal{A}, \mathcal{B}]$, of filtering functors from $\mathcal{A}$ to $\mathcal{B}$ and all natural transformations. We present some revelatory proprieties on functors $k : \mathcal{C} \longrightarrow \mathcal{E}$, which will enlighten our characterisation of localizations. Combining these conditions together, we state our theorem classifying localizations of presheaf categories. As a corollary, we present a classification of geometric morphisms into presheaf categories. Finally, an alternative proof of our classifying theorem based on sheaf theoretical techniques is offered.

In Chapter 4, we outline the basics for algebraic theories and algebraic categories, by listing the crucial results. We specify the coproduct completion and the exact completion in the case of an algebraic theory. This leads to the notion of *special filtering* functor and, further, to an equivalence between the category $\mathbf{GeoMor}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})]$, of geometric morphisms between $\mathcal{E}$ and $\mathbf{Mod}(\mathcal{C})$ and geometric transformations between those, and the category $\mathbf{SFilt}[\mathcal{C}, \mathcal{E}]$, of special filtering functors which preserve finite coproducts from $\mathcal{C}$ to $\mathcal{E}$ and all natural transformations, provided $\mathcal{C}$ is an algebraic theory and $\mathcal{E}$ is a cocomplete and exact category in which filtered colimits commute with finite limits. Thanks to this equivalence, we are able to establish our theorem describing all realisations of a given category as localization of some (one sorted) algebraic category. As in the presheaf case, a direct corollary of this result is a classifying theorem for geometric morphisms into algebraic categories.

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