

ON SOME WEAKLY COERCIVE QUASILINEAR PROBLEMS WITH FORCING

By

ANDRZEJ SZULKIN AND MICHEL WILLEM

Abstract. We consider the forced problem $-\Delta_p u - V(x)|u|^{p-2}u = f(x)$, where Δ_p is the p -Laplacian ($1 < p < \infty$) in a domain $\Omega \subset \mathbb{R}^N$, $V \geq 0$ and $Q_V(u) := \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} V|u|^p dx$ satisfies the condition (A) below. We show that this problem has a solution for all f in a suitable space of distributions. Then we apply this result to some classes of functions V which in particular include the Hardy potential (1.5) and the potential $V(x) = \lambda_{1,p}(\Omega)$, where $\lambda_{1,p}(\Omega)$ is the Poincaré constant on an infinite strip.

1 Introduction

Our purpose is to solve the forced problem

$$-\Delta_p u - V(x)|u|^{p-2}u = f(x)$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $f \in \mathcal{D}'(\Omega)$, the space of distributions on the domain Ω .

Our assumptions are the following:

- (A) Let $1 < p < \infty$, $V \in L^r_{loc}(\Omega)$ with r as in (1.2), Ω is a domain in $\mathbb{R}^N$, $V \geq 0$ and for all test functions $u \in \mathcal{D}(\Omega) \setminus \{0\}$,

$$(1.1) \quad Q_V(u) := \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} V|u|^p dx > 0.$$

There exists $1 < q \leq p$ such that $p - 1 < q$,

$$(1.2) \quad r = 1 \ (N < q), \quad 1 < r < +\infty \ (N = q), \quad 1/r + (p - 1)/q^* = 1 \ (N > q)$$

and there exists $W \in \mathcal{C}(\Omega)$, $W > 0$, such that for all $u \in \mathcal{D}(\Omega)$,

$$(1.3) \quad \left(\int_{\Omega} (|\nabla u|^q + |u|^q) W dx \right)^{p/q} \leq Q_V(u).$$

Let us recall that $q^* := Nq/(N - q)$.

Our first example of V is the **quadratic Hardy potential** ($N \geq 3, p = 2$):

$$(1.4) \quad V(x) := \left(\frac{N-2}{2} \right)^2 |x|^{-2}.$$

The corresponding forced problem is solved in [7] using the Brezis–Vazquez remainder term for the quadratic Hardy inequality ([4] and [14]).

A second example is the **Hardy potential** ($1 < p < N$):

$$(1.5) \quad V(x) := \left(\frac{N-p}{p} \right)^p |x|^{-p}.$$

The corresponding forced problem is partially solved in [3] using the Abdellaoui–Colorado–Peral remainder term for the Hardy inequality [1].

A third example is a potential $V \in L_{\text{loc}}^\infty(\Omega)$ such that (1.3) is satisfied with $p = q$ (see [13]). Assumption (1.3) corresponds to a remainder term in a weighted Sobolev space and it implies “weak” coercivity of Q_V . However, there exist many different kinds of remainder terms; see, e.g., the references in [1] and [14].

If $p = 2$, the natural energy space is the completion H of $\mathcal{D}(\Omega)$ with respect to the norm $[Q_V(u)]^{1/2}$, and it suffices to have $V \in L_{\text{loc}}^1(\Omega)$. Then H is a Hilbert space with an obvious inner product. The following result is immediate:

Theorem 1.1. *For each $f \in H^*$ (the dual space of H) the problem*

$$-\Delta u - V(x)u = f(x)$$

has a unique solution $u \in H$.

This is an extension of Lemma 1.1' in [7] though the argument is exactly the same as there. As we shall see at the beginning of Section 6, more can be said about u and H^* if V is the Hardy potential (1.4) in a bounded domain.

When $p \neq 2$, one can expect no uniqueness as in Theorem 1.1; see [6], pp. 11–12 and [9], Section 4. Hence the functional $[Q_V(u)]^{1/p}$ is no longer convex, so it cannot serve as a norm, and the second conjugate functional Q_V^{**} was used in [13] to define the energy space.

Our goal is to generalize all the above results by using only the Hahn–Banach theorem to define an energy space and to obtain a priori bounds.

We also consider the case of constant potential

$$V \equiv \lambda_{1,p}(\Omega) := \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in \mathcal{D}(\Omega), \int_{\Omega} |u|^p dx = 1 \right\}$$

on the cylindrical domain $\Omega = \omega \times \mathbb{R}^M$ where $\omega \subset \mathbb{R}^{N-M}$ is bounded. Then we have

$$(1.6) \quad \lambda_{1,p}(\Omega) = \lambda_{1,p}(\omega) > 0;$$

see Section 5.

The paper is organized as follows. In Section 2 we prove an almost everywhere convergence result for the gradients. In Section 3 we solve a sequence of approximate problems. In Section 4 we state and prove our main result. Section 5 is devoted to the proof of (1.6) and to remainder terms for the Poincaré inequality. In Section 6 we apply the main result to the potentials mentioned above.

2 Almost everywhere convergence of the gradients

Let us recall a classical result (see [12]).

Lemma 2.1. *For every $1 < p < 2$ there exists $c > 0$ such that for all $x, y \in \mathbb{R}^N$, $(x, y) \neq (0, 0)$,*

$$c|x - y|^2 / (|x| + |y|)^{2-p} \leq (|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y).$$

For every $p \geq 2$ there exists $c > 0$ such that for all $x, y \in \mathbb{R}^N$,

$$c|x - y|^p \leq (|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y).$$

Corollary 2.2. *Let $(x_n) \subset \mathbb{R}^N$. If $(|x_m|^{p-2}x_m - |x_n|^{p-2}x_n) \cdot (x_m - x_n) \rightarrow 0$ as $m, n \rightarrow \infty$, then $x_m - x_n \rightarrow 0$.*

Proof. For $p \geq 2$ the conclusion is immediate from the above lemma. If $1 < p < 2$, the conclusion will follow if we can show that the sequence (x_n) must be bounded. Fix m such that $(|x_m|^{p-2}x_m - |x_n|^{p-2}x_n) \cdot (x_m - x_n) \leq 1$ for all $n \geq m$. Then $c|x_m - x_n|^2 / (|x_m| + |x_n|)^{2-p} \leq 1$ which implies boundedness of (x_n) . $\square$

We define the truncation T by

$$Ts := s, \quad |s| \leq 1; \quad Ts := s/|s|, \quad |s| > 1.$$

The following theorem is a variant of a result which may be found, e.g., in [2], [5] and [11]. Here we provide a very simple argument, similar to that in [5].

Theorem 2.3. *Let $1 < q \leq p$ and $(u_n) \subset W_{loc}^{1,p}(\Omega)$ be such that for all $\omega \subset\subset \Omega$*

- (a) $\sup_n \|u_n\|_{W^{1,q}(\omega)} < +\infty$,
- (b) $\lim_{m,n \rightarrow \infty} \int_{\omega} (|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n) \cdot \nabla T(u_m - u_n) dx = 0$.

Then there exists a subsequence (n_k) and $u \in W_{loc}^{1,q}(\Omega)$ such that

$$(2.1) \quad u_{n_k} \rightarrow u \quad \text{and} \quad \nabla u_{n_k} \rightarrow \nabla u \quad \text{almost everywhere on } \Omega.$$

Proof. It suffices to prove that for all $\omega \subset\subset \Omega$ there exist a subsequence (n_k) and $u \in W^{1,q}(\omega)$ satisfying (2.1) a.e. on ω and to use a diagonal argument.

Let $\omega \subset\subset \Omega$. By assumption (a), extracting if necessary a subsequence, we can assume that for some $u \in W^{1,q}(\omega)$,

$$u_n \rightarrow u \text{ in } L^q(\omega), \quad u_n \rightarrow u \text{ a.e. on } \omega.$$

Let us define

$$\begin{aligned} E_{m,n} &:= \{x \in \omega : |u_m(x) - u_n(x)| < 1\}, \\ e_{m,n} &:= (|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n) \cdot \nabla (u_m - u_n). \end{aligned}$$

Then $e_{m,n} \geq 0$ and by assumption (b),

$$\lim_{m,n \rightarrow \infty} \int_{\omega} e_{m,n} \chi_{E_{m,n}} dx = 0.$$

Since $\chi_{E_{m,n}} \rightarrow 1$ a.e. on ω as $m, n \rightarrow \infty$, it follows, extracting if necessary a subfamily, that $e_{m,n} \rightarrow 0$ a.e. on ω as $m, n \rightarrow \infty$. By Corollary 2.2, $|\nabla u_m - \nabla u_n| \rightarrow 0$ a.e. on ω as $m, n \rightarrow \infty$. Hence $\nabla u_n \rightarrow v$ a.e. on ω for some v . Since by assumption (a),

$$\sup_n \|\nabla u_n\|_{L^q(\omega, \mathbb{R}^N)} < \infty,$$

it follows from Proposition 5.4.7 in [16] that

$$\nabla u_n \rightharpoonup v \text{ in } L^q(\omega, \mathbb{R}^N).$$

We conclude that $v = \nabla u$. □

3 Approximate problems

In this section we assume that (1.1) is satisfied, $V \in L_{loc}^1(\Omega)$ and $f \in W^{-1,p'}(\Omega)$. Let $0 < \varepsilon < 1$. We shall apply Ekeland's variational principle to the functional

$$\varphi(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{1-\varepsilon}{p} \int_{\Omega} V|u|^p dx + \frac{\varepsilon}{p} \int_{\Omega} |u|^p dx - \langle f, u \rangle \equiv Q_{V_\varepsilon}(u) - \langle f, u \rangle$$

defined on $W_0^{1,p}(\Omega)$. By (1.1), for every $u \in W_0^{1,p}(\Omega)$, $V|u|^p \in L^1(\Omega)$ and

$$\int_{\Omega} V|u|^p dx \leq \int_{\Omega} |\nabla u|^p dx.$$

Moreover, the functional

$$u \mapsto \int_{\Omega} V|u|^p dx$$

is continuous and Gâteaux-differentiable on $W_0^{1,p}(\Omega)$ (see [16, Theorem 5.4.1]). It is clear that, on $W_0^{1,p}(\Omega)$,

$$(3.1) \quad -\|f\|_{W^{-1,p'}(\Omega)} \|u\|_{W^{1,p}(\Omega)} + \frac{\varepsilon}{p} \|u\|_{W^{1,p}(\Omega)}^p \leq \varphi(u).$$

Hence φ is bounded below and by Ekeland's variational principle ([15, Theorem 2.4]) there exists a sequence $(u_n) \subset W_0^{1,p}(\Omega)$ such that

$$(3.2) \quad \varphi(u_n) \rightarrow c := \inf_{W_0^{1,p}(\Omega)} \varphi \quad \text{and} \quad \varphi'(u_n) \rightarrow 0 \text{ in } W^{-1,p'}(\Omega).$$

We deduce from (3.1) that

$$(3.3) \quad \sup_n \|u_n\|_{W^{1,p}(\Omega)} < +\infty.$$

Going if necessary to a subsequence, we can assume the existence of $u \in W_0^{1,p}(\Omega)$ such that

$$(3.4) \quad u_n \rightarrow u \quad \text{a.e. on } \Omega.$$

Lemma 3.1. *Let $\zeta \in \mathcal{D}(\Omega)$. Then*

$$\lim_{m,n \rightarrow \infty} \int_{\Omega} [|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n] \cdot \zeta \nabla T(u_m - u_n) dx = 0.$$

Proof. Because of (3.2), we have that

$$(3.5) \quad -\Delta_p u_n - (1 - \varepsilon)V|u_n|^{p-2} u_n + \varepsilon|u_n|^{p-2} u_n = f + g_n,$$

where $g_n \rightarrow 0$ in $W^{-1,p'}(\Omega)$. Testing (3.5) with $\zeta T(u_n - u_m)$, we see that it suffices to prove that

$$(3.6) \quad \lim_{m,n \rightarrow \infty} \int_{\Omega} |\nabla u_m|^{p-1} |\nabla \zeta| |T(u_m - u_n)| dx = 0,$$

$$(3.7) \quad \lim_{m,n \rightarrow \infty} \int_{\Omega} |u_m|^{p-1} |\zeta| |T(u_m - u_n)| dx = 0,$$

$$(3.8) \quad \lim_{m,n \rightarrow \infty} \int_{\Omega} V|u_m|^{p-1} |\zeta| |T(u_m - u_n)| dx = 0.$$

By (3.4) and the fact that $|T(u_n - u_m)| \leq 1$, $\lim_{m,n \rightarrow \infty} \int_{\text{spt } \zeta} |T(u_n - u_m)| dx = 0$. Hence

$$\lim_{m,n \rightarrow \infty} \int_{\text{spt } \zeta} |T(u_m - u_n)|^p dx = 0, \quad \lim_{m,n \rightarrow \infty} \int_{\text{spt } \zeta} V|T(u_m - u_n)|^p dx = 0$$

and (3.6), (3.7) follow from (3.3) and Hölder's inequality. Since

$$\begin{aligned} \int_{\text{spt } \zeta} V|u_m|^{p-1}|T(u_m - u_n)|dx \\ \leq \left(\int_{\text{spt } \zeta} V|u_m|^p dx \right)^{(p-1)/p} \left(\int_{\text{spt } \zeta} V|T(u_m - u_n)|^p dx \right)^{1/p}, \end{aligned}$$

using (1.1) and (3.3) also (3.8) follows. $\square$

Theorem 3.2. *There exists $u \in W_0^{1,p}(\Omega)$ such that $\varphi(u) = \inf_{W_0^{1,p}(\Omega)} \varphi$ and $\varphi'(u) = 0$.*

Proof. Assumption (b) of Theorem 2.3 (with $q = p$) follows from Lemma 3.1. Extracting a subsequence, we can assume that

$$(3.9) \quad \nabla u_n \rightarrow \nabla u \quad \text{a.e. on } \Omega.$$

By (3.5) we have that, for every $\zeta \in \mathcal{D}(\Omega)$,

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \zeta dx - (1 - \varepsilon) \int_{\Omega} V|u_n|^{p-2} u_n \zeta dx + \varepsilon \int_{\Omega} |u_n|^{p-2} u_n \zeta dx = \langle f + g_n, \zeta \rangle.$$

Using (3.3), (3.4), (3.9) and Proposition 5.4.7 in [16], we obtain

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta dx - (1 - \varepsilon) \int_{\Omega} V|u|^{p-2} u \zeta dx + \varepsilon \int_{\Omega} |u|^{p-2} u \zeta dx = \langle f, \zeta \rangle,$$

so that $\varphi'(u) = 0$. As in [10], the homogeneity of $\mathcal{Q}_{V_\varepsilon}$ implies

$$\begin{aligned} \inf_{W_0^{1,p}(\Omega)} \varphi &= \lim_{n \rightarrow \infty} \varphi(u_n) \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{p} \langle \varphi'(u_n), u_n \rangle + \left(\frac{1}{p} - 1 \right) \langle f, u_n \rangle \right] \\ &= \left(\frac{1}{p} - 1 \right) \langle f, u \rangle = \varphi(u) - \frac{1}{p} \langle \varphi'(u), u \rangle = \varphi(u). \end{aligned} \quad \square$$

4 Main result

In this section we assume that (A) is satisfied and we define, on $\mathcal{D}'(\Omega)$,

$$\|f\| := \sup\{ \langle f, u \rangle : u \in \mathcal{D}(\Omega), \mathcal{Q}_V(u) = 1 \}$$

so that

$$(4.1) \quad \langle f, u \rangle \leq \|f\| [\mathcal{Q}_V(u)]^{1/p}.$$

On the spaces

$$X := \{f \in \mathcal{D}'(\Omega) : \|f\| < \infty\} \quad \text{and} \quad Y := W_0^{1,q}(\Omega, Wdx)$$

we respectively use the norm defined above and the natural norm. Note that the space X has been introduced by Takáč and Tintarev in [13].

Lemma 4.1. *Let $f \in Y^*$. Then $f \in X$ and*

$$\|f\| \leq \|f\|_{Y^*}.$$

Proof. Let $u \in \mathcal{D}(\Omega)$. By assumption (1.3) we have

$$\langle f, u \rangle \leq \|f\|_{Y^*} \|u\|_Y \leq \|f\|_{Y^*} [\mathcal{Q}_V(u)]^{1/p}.$$

□

Lemma 4.2. (a) *Let $u \in W_0^{1,p}(\Omega)$. Then*

$$\|u\|_Y \leq \|u\|_{X^*} \leq [\mathcal{Q}_V(u)]^{1/p} \leq \|u\|_{W^{1,p}(\Omega)}.$$

(b) *Let $f \in X$. Then $f \in W^{-1,p'}(\Omega)$ and*

$$\|f\|_{W^{-1,p'}(\Omega)} \leq \|f\|.$$

Proof. (a) Let $u \in \mathcal{D}(\Omega)$. Using the Hahn–Banach theorem and the preceding lemma, we obtain

$$\|u\|_Y = \sup_{\substack{f \in Y^* \\ \|f\|_{Y^*} \leq 1}} \langle f, u \rangle \leq \sup_{\substack{f \in X \\ \|f\| \leq 1}} \langle f, u \rangle = \|u\|_{X^*}.$$

It follows from (4.1) that

$$\sup_{\substack{f \in X \\ \|f\| \leq 1}} \langle f, u \rangle \leq [\mathcal{Q}_V(u)]^{1/p}.$$

Since $V \geq 0$, it is clear that

$$[\mathcal{Q}_V(u)]^{1/p} \leq \|u\|_{W^{1,p}(\Omega)}.$$

Now it is easy to conclude by density of $\mathcal{D}(\Omega)$.

(b) If $f \in X$ and $u \in \mathcal{D}(\Omega)$, then

$$\langle f, u \rangle \leq \|f\| [\mathcal{Q}_V(u)]^{1/p} \leq \|f\| \|u\|_{W^{1,p}(\Omega)}.$$

□

Let $f \in X$ and let $(\varepsilon_n) \subset]0, 1[$ be such that $\varepsilon_n \downarrow 0$. Then $f \in W^{-1,p'}(\Omega)$, so by Theorem 3.2, for every n there exists $u_n \in W_0^{1,p}(\Omega)$ such that

$$(4.2) \quad -\Delta_p u_n - (1 - \varepsilon_n)V|u_n|^{p-2}u_n + \varepsilon_n|u_n|^{p-2}u_n = f$$

and u_n minimizes the functional

$$\varphi_n(v) := \frac{1}{p} \int_{\Omega} |\nabla v|^p dx - \frac{1 - \varepsilon_n}{p} \int_{\Omega} V|v|^p dx + \frac{\varepsilon_n}{p} \int_{\Omega} |v|^p dx - \langle f, v \rangle$$

on $W_0^{1,p}(\Omega)$. In fact below we shall not use the minimizing property of u_n but only the fact that (4.2) holds.

Lemma 4.3. *Let $f \in X$. Then*

$$\sup_n \|u_n\|_Y \leq \sup_n \mathcal{Q}_V(u_n) < \infty.$$

Proof. Lemma 4.2 and equation (4.2) imply that

$$\begin{aligned} \|u_n\|_{X^*}^p &\leq \mathcal{Q}_V(u_n) \leq \mathcal{Q}_V(u_n) + \varepsilon_n \int_{\Omega} (V+1)|u_n|^p dx \\ &= \langle f, u_n \rangle \leq \|f\|_X \|u_n\|_{X^*}. \end{aligned}$$

Since $p > 1$, we obtain the conclusion. $\square$

Going if necessary to a subsequence, we can assume the existence of $u \in Y$ such that

$$(4.3) \quad u_n \rightarrow u \quad \text{a.e. on } \Omega.$$

Lemma 4.4. *Let $\zeta \in \mathcal{D}(\Omega)$. Then*

$$\lim_{m,n \rightarrow \infty} \int_{\Omega} [|\nabla u_m|^{p-2}u_m - |\nabla u_n|^{p-2}u_n] \cdot \zeta \nabla T(u_m - u_n) dx = 0.$$

Proof. Because of (4.2), as in the proof of Lemma 3.1 it suffices to show that

$$\begin{aligned} \lim_{m,n \rightarrow \infty} \int_{\Omega} |\nabla u_m|^{p-1} |\nabla \zeta| |T(u_m - u_n)| dx &= 0, \\ \lim_{m,n \rightarrow \infty} \int_{\Omega} |u_m|^{p-1} |\zeta| |T(u_m - u_n)| dx &= 0, \\ \lim_{m,n \rightarrow \infty} \int_{\Omega} V|u_m|^{p-1} |\zeta| |T(u_m - u_n)| dx &= 0. \end{aligned}$$

We assume that $N > q$. The other cases are similar but simpler. By Lemma 4.3, (u_n) is bounded in $W_{\text{loc}}^{1,q}(\Omega)$, so by the Sobolev theorem, (u_n) is bounded in $L_{\text{loc}}^{q^*}(\Omega)$. Since by (4.3),

$$\lim_{m,n \rightarrow \infty} \int_{\text{spt } \zeta} |T(u_m - u_n)|^{(\frac{q}{p-1})'} dx = 0 \quad \text{and} \quad \lim_{m,n \rightarrow \infty} \int_{\text{spt } \zeta} V^r |T(u_m - u_n)|^r dx = 0,$$

it is easy to conclude using (1.2) and Hölder's inequality. Note in particular that

$$\begin{aligned} & \int_{\text{spt } \zeta} V |u_m|^{p-1} |T(u_m - u_n)| dx \\ & \leq \left(\int_{\text{spt } \zeta} |u_m|^{q^*} dx \right)^{(p-1)/q^*} \left(\int_{\text{spt } \zeta} V^r |T(u_m - u_n)|^r dx \right)^{1/r}. \quad \square \end{aligned}$$

Theorem 4.5. Assume (A) is satisfied and $f \in \mathcal{D}'(\Omega)$ is such that

$$(4.4) \quad \sup\{\langle f, u \rangle : u \in \mathcal{D}(\Omega), \mathcal{Q}_V(u) = 1\} < +\infty.$$

Then there exists $u \in W_0^{1,q}(\Omega, Wdx)$ such that, in $\mathcal{D}'(\Omega)$,

$$(4.5) \quad -\Delta_p u - V(x)|u|^{p-2}u = f(x).$$

Proof. Let $\zeta \in \mathcal{D}(\Omega)$. By (4.2) we have

$$(4.6) \quad \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \zeta dx - (1 - \varepsilon_n) \int_{\Omega} V |u_n|^{p-2} u_n \zeta dx + \varepsilon_n \int_{\Omega} |u_n|^{p-2} u_n \zeta dx = \langle f, \zeta \rangle.$$

Let us recall that (u_n) is bounded in $W_{\text{loc}}^{1,q}(\Omega)$ and $u_n \rightarrow u$ a.e. on Ω . Assumption (b) of Theorem 2.3 follows from Lemma 4.4. Extracting a subsequence, we can assume that

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e. on } \Omega.$$

We assume that $N > q$ and we choose ω such that

$$\text{spt } \zeta \subset \omega \subset \subset \Omega.$$

Using Proposition 5.4.7 in [16], we obtain

$$|\nabla u_n|^{p-2} \nabla u_n \rightarrow |\nabla u|^{p-2} \nabla u \quad \text{in } L^{\frac{q}{p-1}}(\omega) \quad \text{and} \quad |u_n|^{p-2} u_n \rightarrow |u|^{p-2} u \quad \text{in } L^{\frac{q^*}{p-1}}(\omega).$$

It follows then from (4.6) that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta dx - \int_{\Omega} V |u|^{p-2} u \zeta dx = \langle f, \zeta \rangle. \quad \square$$

The following variant of Theorem 4.5 will be needed in one of the applications in Section 6; see Theorem 6.1.

Theorem 4.6. Theorem 4.5 remains valid if we replace (1.2) (case $N > q$) in (A) by the conditions

$$(4.7) \quad \begin{aligned} & V \in L^r_{\text{loc}}(\Omega) \quad \text{where } 1/r + (p-1)(p-q)/q = 1 \\ & \text{and} \quad \int_{\Omega} V^{q/p} |u|^q dx \leq C \int_{\Omega} |\nabla u|^q dx \quad \text{for some } C > 0. \end{aligned}$$

Proof. The argument is similar, except that we must show that the third limit in the proof of Lemma 4.4 is zero also when (4.7) holds and that we can pass to the limit in the second integral in (4.6). Using Lemma 4.3, Hölder's inequality, (4.7) and the fact that $r = q/[(q - p + 1)p] \geq 1$, we obtain

$$\begin{aligned} & \lim_{m,n \rightarrow \infty} \int_{\text{spt } \zeta} V |u_m|^{p-1} |\zeta| |T(u_m - u_n)| dx \\ & \leq \lim_{m,n \rightarrow \infty} \left(\int_{\text{spt } \zeta} V^{q/p} |u_m|^q dx \right)^{(p-1)/q} \left(\int_{\text{spt } \zeta} V^r |T(u_m - u_n)|^{pr} dx \right)^{1/(pr)} = 0. \end{aligned}$$

Let $E \subset \text{spt } \zeta$. Similarly as above, we have

$$\int_E V |u_n|^{p-1} dx \leq \left(\int_E V^{q/p} |u_n|^q dx \right)^{(p-1)/q} \left(\int_E V^r dx \right)^{1/(pr)} \leq D \left(\int_E V^r dx \right)^{1/(pr)}.$$

Since the integrand on the right-hand side is in $L^1(\text{spt } \zeta)$, it follows that $V |u_n|^{p-1}$ are uniformly integrable and we can pass to the limit in the second integral in (4.6) according to the Vitali theorem; see, e.g., [16, Theorem 3.1.9]. $\square$

Note that in the case $q = p$ this result is stronger than Theorem 4.5 because $V \in L^1_{\text{loc}}(\Omega)$ is allowed for any p .

Remark 4.7. Let $V_+ \in L^1_{\text{loc}}(\Omega)$ and $V_+ \geq 0$. It is possible to treat the more general case

$$-\Delta_p u + V_+(x) |u|^{p-2} u - V(x) |u|^{p-2} u = f(x)$$

by replacing $W_0^{1,p}(\Omega)$ by $W_0^{1,p}(\Omega) \cap L^p_{V_+}(\Omega)$ with the norm

$$\left(\int_{\Omega} |\nabla u|^p dx \right)^{1/p} + \left(\int_{\Omega} V_+ |u|^p dx \right)^{1/p}.$$

5 Poincaré inequality with remainder term

Let $\Omega := \omega \times \mathbb{R}^M$, where ω is a domain in $\mathbb{R}^{N-M}$ and $N > M > p$. For $x \in \Omega$ we shall write $x = (y, z)$, where $y \in \omega$ and $z \in \mathbb{R}^M$. Recall from the introduction that

$$\lambda_{1,p}(\Omega) := \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in \mathcal{D}(\Omega), \int_{\Omega} |u|^p dx = 1 \right\}.$$

It is well known that $\lambda_{1,p}(\omega) = \lambda_{1,p}(\Omega)$ if $p = 2$; see, e.g., [8], Lemma 3. We shall show that this is also true for general $p \in (1, \infty)$.

Lemma 5.1. $\lambda_{1,p}(\Omega) = \lambda_{1,p}(\omega)$.

Proof. First we show that $\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(\omega)$. Let $u \in \mathcal{D}(\Omega)$, $\|u\|_{L^p(\Omega)} = 1$. Then

$$\begin{aligned} \int_{\Omega} |\nabla u|^p dx &\geq \int_{\Omega} |\nabla_y u|^p dx = \int_{\mathbb{R}^M} dz \int_{\omega} |\nabla_y u|^p dy \\ &\geq \int_{\mathbb{R}^M} dz \lambda_{1,p}(\omega) \int_{\omega} |u|^p dy = \lambda_{1,p}(\omega) \int_{\Omega} |u|^p dx = \lambda_{1,p}(\omega). \end{aligned}$$

Taking the infimum on the left-hand side we obtain the conclusion.

To show the reverse inequality, let $u(x) = v(y)w(z)$, where $v \in \mathcal{D}(\omega) \setminus \{0\}$ and $w \in \mathcal{D}(\mathbb{R}^M) \setminus \{0\}$. For each $\varepsilon > 0$ there exists $C_{\varepsilon} > 0$ such that

$$|\nabla u|^p \leq (|w| |\nabla_y v| + |v| |\nabla_z w|)^p \leq (1 + \varepsilon) |w|^p |\nabla_y v|^p + C_{\varepsilon} |v|^p |\nabla_z w|^p.$$

Hence

$$\lambda_{1,p}(\Omega) \leq \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx} \leq \frac{(1 + \varepsilon) \int_{\omega} |\nabla_y v|^p dy}{\int_{\omega} |v|^p dy} + \frac{C_{\varepsilon} \int_{\mathbb{R}^M} |\nabla_z w|^p dz}{\int_{\mathbb{R}^M} |w|^p dz}.$$

Taking the infimum with respect to v and w , we obtain

$$\lambda_{1,p}(\Omega) \leq (1 + \varepsilon) \lambda_{1,p}(\omega).$$

Since ε has been chosen arbitrarily, it follows that $\lambda_{1,p}(\Omega) \leq \lambda_{1,p}(\omega)$. □

Now we state the main result of this section.

Theorem 5.2. *For each $u \in \mathcal{D}(\Omega)$ the following holds:*

(a) *If $p \geq 2$, then*

$$\int_{\Omega} (|\nabla u|^p - \lambda_{1,p}(\Omega) |u|^p) dx \geq \left(\frac{M-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|z|^p} dx.$$

(b) *If $1 < p < 2$, then*

$$\int_{\Omega} (|\nabla u|^p - \lambda_{1,p}(\Omega) |u|^p) dx \geq 2^{(p-2)/2} \left(\frac{M-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|z|^p} dx.$$

Proof. (a) Let $p \geq 2$. Then $(a+b)^{p/2} \geq a^{p/2} + b^{p/2}$ for all $a, b \geq 0$, hence

$$|\nabla u|^p \geq |\nabla_y u|^p + |\nabla_z u|^p.$$

Using this, Lemma 5.1 and Hardy's inequality in $\mathbb{R}^M$, we obtain

$$\begin{aligned} &\int_{\Omega} (|\nabla u|^p - \lambda_{1,p}(\Omega) |u|^p) dx \\ &\geq \int_{\mathbb{R}^M} dz \int_{\omega} (|\nabla_y u|^p - \lambda_{1,p}(\Omega) |u|^p) dy + \int_{\omega} dy \int_{\mathbb{R}^M} |\nabla_z u|^p dz \\ &\geq \int_{\omega} dy \int_{\mathbb{R}^M} |\nabla_z u|^p dz \geq \int_{\omega} dy \left(\frac{M-p}{p} \right)^p \int_{\mathbb{R}^M} \frac{|u|^p}{|z|^p} dz \\ &= \left(\frac{M-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|z|^p} dx. \end{aligned}$$

(b) Let $1 < p < 2$. It is easy to see that $(a + b)^{p/2} \geq 2^{(p-2)/2}(a^{p/2} + b^{p/2})$ for such p and all $a, b \geq 0$. Hence

$$|\nabla u|^p \geq 2^{(p-2)/2}(|\nabla_y u|^p + |\nabla_z u|^p)$$

and we can proceed as above. $\square$

Here we have not excluded the case $\lambda_{1,p}(\omega) = 0$ but the result is only interesting if $\lambda_{1,p}(\omega)$ is positive. A sufficient condition for this is that ω has finite measure.

6 Applications

In this section we work out some applications of Theorem 4.5 for the potentials V mentioned in the introduction. Let Ω be a domain in $\mathbb{R}^N$. If Ω is bounded, $0 \in \Omega$ and V is the quadratic Hardy potential (1.4), then more can be said about the solution u and the space H^* in Theorem 1.1. Given $1 \leq q < 2$, we have

$$\int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} V(x)u^2 dx \geq C(q, \Omega) \|u\|_{W^{1,q}(\Omega)}^2$$

for all $u \in \mathcal{D}(\Omega)$ ([14], Theorem 2.2). We see that $V \in L^r(\Omega)$ if $N/(N-1) < q < 2$ (r is as in (1.2)) and (1.3) holds with constant W . So $H^* = X$ and we also have $Y = W_0^{1,q}(\Omega)$ where X, Y are as in Section 4. Hence by Lemma 4.2 and Theorem 4.5, $H^* \subset W^{-1,2}(\Omega)$ and the solution u is in $W_0^{1,q}(\Omega)$ for any $1 \leq q < 2$.

Let $x = (y, z) \in \Omega \subset \mathbb{R}^k \times \mathbb{R}^{N-k}$, where $N \geq k > p > 1$, and consider the Hardy potential

$$V(x) := \left(\frac{k-p}{p} \right)^p |y|^{-p}.$$

Theorem 6.1. *Let Ω be a bounded domain containing the origin and let V be the Hardy potential above. Then for each f satisfying (4.4) and each $1 \leq q < p$ there exists a solution $u \in W_0^{1,q}(\Omega)$ to (4.5).*

Recall from Lemma 4.2 that if f satisfies (4.4), then $f \in W^{-1,p'}(\Omega)$.

Proof. According to Lemma 2.1 in [3] (see also [1], Theorem 1.1), for each $1 < q < p$ there exists a constant $C(q, \Omega)$ such that

$$\mathcal{Q}_V(u) \geq C(q, \Omega) \left(\int_{\Omega} |\nabla u|^q dx \right)^{p/q}, \quad u \in \mathcal{D}(\Omega).$$

So the Poincaré inequality implies that (1.3) holds for some constant W . Since also (4.7) holds if q is sufficiently close to p , we obtain the conclusion using Theorem 4.6 ($(k(p-1)/(k-1) < q < p$ is needed in order to have $V \in L'_{\text{loc}}(\Omega)$ with r as in (4.7)).

This result extends Theorem 3.1 in [3] where it was assumed that $f \in L^\gamma(\Omega)$ for some $\gamma > (p^*)'$.

In our next theorem we essentially recover the main result (Theorem 4.3) of [13].

Theorem 6.2. *Let Ω be a domain and let $V \in L^\infty_{\text{loc}}(\Omega)$, $V \geq 0$. Suppose that*

$$(6.1) \quad \mathcal{Q}_V(u) \geq \int_{\Omega} \tilde{W}|u|^p dx \quad \text{for all } u \in \mathcal{D}(\Omega) \text{ and some } \tilde{W} \in \mathcal{C}(\Omega), \tilde{W} > 0.$$

Then for each f satisfying (4.4) there exists a solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ to (4.5).

Proof. According to Proposition 3.1 in [13] (see also (2.6) there), (6.1) implies that $\mathcal{Q}_V$ satisfies (1.3) with $q = p$. Hence our Theorem 4.5 applies.

Theorem 6.3. *Let $\Omega = \omega \times \mathbb{R}^M$, where $\omega \subset \mathbb{R}^{N-M}$ is a domain such that $\lambda_{1,p}(\omega) > 0$, $N > M > p$, and let $V(x) = \lambda_{1,p}(\Omega)$. Then for each f satisfying (4.4) there exists a solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ to (4.5).*

Proof. Let $x = (y, z) \in \omega \times \mathbb{R}^M$. It follows from Theorem 5.2 that

$$\mathcal{Q}_V(u) = \int_{\Omega} |\nabla u|^p dx - \lambda_{1,p}(\Omega) \int_{\Omega} |u|^p dx \geq \int_{\Omega} \tilde{W}|u|^p dx \quad \text{for all } u \in \mathcal{D}(\Omega),$$

where $\tilde{W}(x) = C_p/(1 + |z|^p)$ and C_p is the constant on the right-hand side of respectively (a) and (b) of Theorem 5.2. So the conclusion follows from Theorem 6.2.

Below we give an example showing that the solution we obtain need not be in $W^{1,p}(\Omega)$.

Example. Let $\Omega = \omega \times \mathbb{R}^M$, where $\omega \subset \mathbb{R}^{N-M}$ is such that $\lambda_1 := \lambda_{1,2}(\omega) > 0$ and put

$$\mathcal{Q}(u) := \int_{\Omega} (|\nabla u|^2 - \lambda_1 u^2) dx.$$

By the definition of λ_1 , for each n we can find $u_n \in \mathcal{D}(\Omega)$ such that

$$\left(1 - \frac{1}{n}\right) \int_{\Omega} |\nabla u_n|^2 dx \leq \lambda_1 \int_{\Omega} u_n^2 dx.$$

Hence

$$\mathcal{Q}(u_n) \leq \frac{1}{n} \int_{\Omega} |\nabla u_n|^2 dx.$$

By normalization, we can assume

$$\mathcal{Q}(u_n) = \frac{1}{n^2}.$$

Translating along $\mathbb{R}^M$, we may assume $\text{spt } u_n \cap \text{spt } u_m = \emptyset$ if $n \neq m$. We define

$$f_n := -\Delta u_n - \lambda_1 u_n \quad \text{and} \quad f := \sum_{n=1}^{\infty} f_n \quad \text{in } \mathcal{D}'(\Omega).$$

Then $f \in X$ because

$$\|f\| = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} < \infty,$$

and $u = \sum_{n=1}^{\infty} u_n$ is a weak solution for the equation

$$-\Delta u - \lambda_1 u = f(x).$$

Moreover, u is the unique solution in H as follows from Theorem 1.1. But

$$\int_{\Omega} |\nabla u|^2 dx = \sum_{n=1}^{\infty} \int_{\Omega} |\nabla u_n|^2 dx \geq \sum_{n=1}^{\infty} \frac{1}{n} = +\infty,$$

so $u \notin W^{1,2}(\Omega)$.

Acknowledgements. We would like to thank the referee for useful suggestions.

REFERENCES

- [1] B. Abdellaoui, E. Colorado and I. Peral, *Some improved Caffarelli–Kohn–Nirenberg inequalities*, Calc. Var. Partial Differential Equations **23** (2005), 327–345.
- [2] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. **19** (1992), 581–597.
- [3] B. Brandolini, F. Chiacchio and C. Trombetti, *Some remarks on nonlinear elliptic problems involving Hardy potentials*, Houston J. Math. **33** (2007), 617–630.
- [4] H. Brezis and J. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), 443–469.
- [5] S. de Valeriola and M. Willem, *On some quasilinear critical problems*, Adv. Nonlinear Stud. **9** (2009), 825–836.
- [6] M. del Pino, M. Elgueta and R. Manásevich, *A homotopic deformation along p of a Leray–Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t, u) = 0$, $u(0) = u(T) = 0$, $p > 1$* , J. Differential Equations **80** (1989), 1–13.
- [7] L. Dupaigne, *A nonlinear elliptic PDE with the inverse square potential*, J. Anal. Math. **86** (2002), 359–398.
- [8] M. J. Esteban, *Nonlinear elliptic problems in strip-like domains: symmetry of positive vortex rings*, Nonlinear Anal. **7** (1983), 365–379.
- [9] J. Fleckinger-Pellé, J. Hernández, P. Takáč and F. de Thélin, *Uniqueness and positivity for solutions of equations with the p -Laplacian*, in *Reaction Diffusion Systems*, New York, 1998, pp. 141–155.
- [10] J. P. García Azorero and I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Differential Equations **144** (1998), 441–476.

- [11] J. M. Rakotoson, *Quasilinear elliptic problems with measures as data*, Differential Integral Equations **4** (1991), 449–457.
- [12] J. Simon, *Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$* , in *Journées d'Analyse Non Linéaire*, Springer, Berlin, 1978, pp. 205–227.
- [13] P. Takáč and K. Tintarev, *Generalized minimizer solutions for equations with the p -Laplacian and a potential term*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), 201–221.
- [14] J. Vazquez and E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential*, J. Funct. Anal. **173** (2000), 103–153.
- [15] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, MA, 1996.
- [16] M. Willem, *Functional Analysis. Fundamentals and Applications*, Birkhäuser/Springer, New York, 2013.

Andrzej Szulkin

DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
106 91 STOCKHOLM, SWEDEN
email: andrzej@math.su.se

Michel Willem

INSTITUTE DE RECHERCHE EN MATH. ET PHYS.
UNIVERSITÉ CATHOLIQUE DE LOUVAIN
1348 LOUVAIN-LA-NEUVE, BELGIUM
email: michel.willem@uclouvain.be

(Received September 23, 2017 and in revised form May 4, 2018)