NO-SABOTAGE UNDER CONDITIONAL MEAN RISK SHARING OF DEPENDENT-BY-MIXTURE INSURANCE LOSSES

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REPRINT | 2025 / 23



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May 27, 2024

Abstract

Conditional mean risk sharing defines an allocation rule to distribute total losses among participants in an insurance pool. Under this risk-sharing scheme, the no-sabotage condition holds when conditional expectations of individual losses given their sum are comonotonic. This property has been widely studied considering independent risks, often assuming that they possess log-concave densities or letting the pool size tend to infinity. This paper further considers the no-sabotage condition for dependent risks which do not necessarily obey log-concave distributions. Sufficient conditions are derived in order to fulfill the no-sabotage requirement. Several examples are proposed to illustrate the applicability of the results. As an illustration, the results are applied to the case where losses follow a multivariate Pareto distribution. In this particular case, a closed-form expression is obtained using divided differences.

Keywords: Conditional mean risk sharing; no-sabotage condition; conditional independence; Efron's monotonicity

1 Introduction

The conditional mean risk-sharing rule has been proposed by Denuit and Dhaene (2012) building upon the work by Landsberger and Meilijson (1994). It is a risk allocation mechanism assigning to each participant within an insurance pool the conditional expectation of the loss brought to the pool, given the total losses experienced by the entire pool. It has been axiomatized by Jiao et al. (2022). The reader is referred e.g. to Blier-Wong et al. (2022), Clemente et al. (2023, 2024), Denuit and Robert (2023), Ghossoub et al. (2024) and Levantesi and Piscopo (2022) for contributions to the study of the conditional mean risk-sharing rule, as well as to Feng (2023) for a general presentation of the topic.

Formally, let us consider a pool formed by n economic agents with respective insurance losses denoted as X_i, $i = 1, \ldots, n$, with $0 < \mathrm{E}[X_i] < \infty$ for every i. According to the conditional mean risk-sharing rule, the amount allocated to participant i is

$$m_i(s) = \mathrm{E}[X_i \mid S = s], \text{ where } S = \sum_{i=1}^{n} X_i.$$

The conditional expectations $m_i(\cdot)$ are thus used by participants to allocate the total loss S among them.

In the context of risk sharing, a desirable property is the so-called "no-sabotage condition", also referred to as the "comonotonicity property". Under the conditional mean risk-sharing rule, there is no sabotage opportunity when all conditional expectations $m_i(\cdot)$ are non-decreasing. To understand why this property plays an important role, assume that the pool comprises a participant i such that $m_i(\cdot)$ decreases over a range of values of S. When S falls in that range, this participant may then be tempted to exaggerate the loss X_i reported to the pool since this would decrease his or her contribution to total losses S. This behavior is clearly undesirable. Non-decreasingness of every $m_i(\cdot)$ thus appears to be a reasonable and useful requirement in the context of risk sharing, since it aligns interests for all participants. From a methodological point of view, the absence of sabotage is also intimately related to Pareto optimality as seen e.g. in Denuit and Dhaene (2012) and Dhaene et al. (2023). In the remainder of the paper, this property is referred to as the no-sabotage condition.

The study of the stochastic monotonicity of the random variable X_i in the sum S is often referred to as "Efron's monotonicity" after Efron (1965). It has attracted a lot of attention in the literature. Some authors, including Denuit and Robert (2021) or Denuit et al. (2024) study monotonicity for conditional expectations while some others, including Saumard and Wellner (2018) or Oudghiri (2021), study the monotonicity in terms of a suitable stochastic order relation. The latter property is generally stronger. For instance, monotonicity in the usual stochastic order or in the likelihood ratio order implies the monotonicity of the expected values and, therefore, that the no-sabotage condition holds true.

Since Efron (1965), it is well known that there is no sabotage opportunity when X_1, X_2, …, X_n are independent random variables with log-concave densities. Both independence and log-concavity are assumptions that might be too strong in some applications to insurance. Even if independence reasonably holds in many situations, there are cases where correlation arises, for instance when all losses are influenced by a common environment. Also, log-concavity might be regarded as a too strong assumption because log-concave densities are

associated with light-tailed asymptotic behavior, as discussed in Asmussen and Lehtomaa (2017).

The dependence structure among individual losses influences the monotonicity of $m_i(\cdot)$. It is thus interesting to investigate Efron's monotonicity considering dependent random variables. So far, this problem has been mainly studied by considering two absolutely continuous dependent risks, providing conditions on their joint probability density function. Given two random variables X_1 and X_2 with joint probability density function $f(\cdot,\cdot)$ and sum $S = X_1 + X_2$, Saumard and Wellner (2018) studied the stochastic monotonicity of X_i in S in terms of the usual stochastic order by providing conditions on the second derivatives of $-\ln f$. In a similar setting, Pellerey and Navarro (2022) provided sufficient conditions to ensure the monotonicity in terms of the likelihood ratio order. Ortega-Jiménez et al. (2022) extended some of these results and provided conditions involving the marginals and the density of the survival copula.

Intuitively, we expect that sabotage opportunities should disappear under positive dependence (i.e. when all risks X_i are likely to be large or to be small together). Whatever the marginals, there is indeed an extreme case of positive dependence where the no-sabotage condition always holds true: comonotonicity. Recall that the random vector $(X_1, \dots X_n)$ is said to be comonotonic when there exists a random variable Z and non-decreasing functions g_i such that $X_i = g_i(Z)$ for $i = 1, \dots n$. Comonotonic risks X_i can always be represented as non-decreasing functions of their sums so that every $m_i(\cdot)$ is non-decreasing for any choice of marginal distributions. Often, the no-sabotage condition can be restored under weaker positive dependence structures. For example, if the random vector $(X_1, \dots X_n)$ is exchangeable, which is the case when the components are independent and identically distributed given a common factor Z, then every $m_i(\cdot)$ is just the arithmetic mean of $X_1, \dots X_n$, that is, S/n. Also, the conditions provided in Pellerey and Navarro (2022) are related with total positivity (TP2) of the joint density and all the copula-related examples provided by Saumard and Wellner (2018) consider parameter values implying positive dependence.

Efron's monotonicity for dependent variables has been established by Pellerey and Navarro (2022) under suitable conditions on the joint density of X_i and $S - X_i = \sum_{j \neq i} X_j$. These conditions apply in particular to random vectors following a multivariate Normal distributions or having a Schur-constant joint survival function. In this paper, we aim to address the problem by considering common mixture models. This construction assumes that all risks are influenced by a common environment and can be considered as being independent when this environment is controlled. Risks are then typically decomposed into $X_i = g_i(Z, Y_i)$ for $i = 1, \dots n$, where the random variables $Z, Y_1, \dots, Y_n$ are independent. Compared to comonotonicity recalled previously, there are sources of randomness Y_i specific to each risk while a common factor Z impacts on all individual losses. Many standard models used by actuaries belong to the family of common mixture models. See e.g. Wang (1998). Notice that this model agrees with the dependence structures considered so far in the literature because dependence by mixture is a positive dependence notion under certain assumptions on the marginals. See e.g. Chapter 7 in Denuit et al. (2005).

The remainder of the paper is organized as follows. Section 2 offers motivating examples in the bivariate case. The effect of heavy-tailed losses on the monotonicity of $m_i(\cdot)$ is considered there, departing from both independence and log-concavity. The numerical illustrations clearly show how the dependence structure can impact on the no-sabotage condition. Section

3 studies the no-sabotage requirement when considering common mixture models. Sufficient conditions are provided, based on the monotonicity of the conditional expectations given the common environmental parameter. Alternative conditions for no-sabotage are obtained by considering the size-biased representation formula with common mixture models. Section 4 studies the no-sabotage condition when individual losses follow a multivariate Pareto distribution of Mardia's type II. A closed-form expression is derived for $m_i(\cdot)$ using divided differences. Finally, Section 5 discusses the results as well as some open problems related to risk reduction.

2 Motivating examples

Broadly speaking, positive dependence refers to the intuitive concept of random variables being likely to be large or to be small together. This concept can be formalized in many different ways, including the notion of total positivity. See e.g. Denuit et al. (2005) for a general overview of the topic in actuarial science. Recall that a positive bivariate function $p(\cdot,\cdot)$ is said to be totally positive of order 2, abbreviated as TP2, when

$$p\left(x_1,y_1\right)p\left(x_2,y_2\right)\geq p\left(x_1,y_2\right)p\left(x_2,y_1\right) \text{ for all } x_1\leq x_2 \text{ and } y_1\leq y_2.$$

If $p(\cdot,\cdot)$ possesses a second partial derivative then it is TP2 if, and only if, $\frac{\partial^2}{\partial x\partial y}\ln p(x,y)\geq 0$ for all x and y. Also, recall that, given two random variables Y and Z with respective probability density functions or probability mass functions f and g, Y is said to be smaller than Z in the likelihood ratio order, henceforth denoted by $Y\preceq_{\mathrm{LR}} Z$, if $x\mapsto\frac{f(x)}{g(x)}$ decreases over the union of the supports of Y and Z. In particular, if $Y\preceq_{\mathrm{LR}} Z$ then $\mathrm{E}[Y]\leq\mathrm{E}[Z]$.

The following result provides the actuary with a sufficient condition for $\{X_i|S=s\}$ to be increasing in s with respect to the likelihood ratio order.

Property 2.1. *Let (X_1,X_2) be an absolutely continuous random vector with joint probability density function $f(\cdot,\cdot)$ possessing second partial derivatives with respect to its arguments. If*

$$\frac{\partial^2}{\partial x\partial y}\ln f(x,y)\geq\frac{\partial^2}{\partial y^2}\ln f(x,y) \text{ for all } x \text{ and } y \tag{2.1}$$

then $X_1\mid S=s_1\preceq_{\mathrm{LR}} X_1\mid S=s_2$ for all $s_1\leq s_2$.

Proof. According to Proposition 2.1 in Pellerey and Navarro (2022), X_1 increases in S with respect to the likelihood ratio order if, and only if, $f(x,s-x)$ is TP2 in (x,s). For simplicity of notation, let us denote as $\partial_i^k f$ the kth derivative of f with respect to the ith argument. If $\frac{\partial^2}{\partial x\partial y}\ln f(x,y)\geq\frac{\partial^2}{\partial y^2}\ln f(x,y)$ then

$$\frac{f(x,y)\;\partial_1\partial_2 f(x,y)-\partial_1 f(x,y)\;\partial_2 f(x,y)}{f(x,y)^2}-\frac{\partial_2^2 f(x,y)\;f(x,y)-\left(\partial_2 f(x,y)\right)^2}{f(x,y)^2}\geq 0. \tag{2.2}$$

Now, $f(x, s-x)$ is TP2 in (s, x) if, and only if, $\frac{\partial^2}{\partial x \partial s} \ln f(x, s-x) \geq 0$. Since

$$\begin{aligned}\frac{\partial^2}{\partial x \partial s} \ln f(x, s-x) &= \frac{\big(\partial_1 \partial_2 f(x, s-x) - \partial_2^2 f(x, s-x)\big) f(x, s-x)}{f(x, s-x)^2} \\ &\quad - \frac{\big(\partial_1 f(x, s-x) - \partial_2 f(x, s-x)\big) \partial_2 f(x, s-x)}{f(x, s-x)^2} \\ &= \frac{f(x, s-x)\ \partial_1 \partial_2 f(x, s-x) - \partial_1 f(x, s-x) \partial_2 f(x, s-x)}{f(x, s-x)^2} \\ &\quad - \frac{\partial_2^2 f(x, s-x) \partial_2 f(x, s-x) - \left(\partial_2 f(x, s-x)\right)^2}{f(x, s-x)^2},\end{aligned}$$

the conclusion follows from (2.2) by considering $y = s - x$. □

Property 2.1 can be seen as an extension of the result obtained by Lehmann (1966) to the case where X_1 and X_2 are dependent. Indeed, if X_1 and X_2 are independent with respective marginal probability density functions f_1 and f_2 then the sufficient condition (2.2) reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} \ln f_2(y) \leq 0 \Leftrightarrow f_2 \text{ log-concave.}$$

If the joint probability density function of (X_1, X_2) is TP2 then the sufficient condition (2.2) in Property 2.1 may be satisfied even if f_1 is not log-concave. Precisely, (2.2) allows for moderate departures from log-concavity as the second partial derivative $\frac{\partial^2}{\partial y^2} \ln f(x, y)$ may be negative as long as it does not exceed the second mixed partial derivative $\frac{\partial^2}{\partial x \partial y} \ln f(x, y)$. See also Property 2.4 in Pellerey and Navarro (2022).

Intuitively, Property 2.1 suggests that as far as the TP2 behavior of the joint density prevails over the logconvexity of the conditional density, Efron's monotonicity holds in the likelihood ratio order. The condition in Property 2.1 and the intuition are analogous to Corollary 6 of Saumard and Wellner (2018) dealing with a weaker form of Efron's monotonicity, where the usual stochastic order replaces the likelihood ratio order. Also, Ortega-Jiménez et al. (2022) provided alternative sufficient conditions ensuring that $f(x, s-x)$ is TP2 in (x, s) which characterizing Efron's monotonicity in the likelihood ratio order. These conditions are given in terms of the density of the survival copula and require one of the components to be Exponentially distributed.

Let $f(\cdot|y)$ denote the conditional probability density function of X_1 given $X_2 = y$. Condition (2.1) can then be rewritten as

$$\frac{\partial^2}{\partial x \partial y} \ln f\left(x, y\right) - \frac{\partial^2}{\partial y^2} \ln f\left(x|y\right) \geq \frac{\mathrm{d}^2}{\mathrm{d}y^2} \ln f_2\left(y\right)$$

or

$$\frac{\partial}{\partial y}\left(\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \ln f\left(x|y\right)\right) \geq \frac{\mathrm{d}^2}{\mathrm{d}y^2} \ln f_2\left(y\right).$$

Let us now use Property 2.1 to derive an example which illustrates how the dependence structure can impact on the monotonicity of $m_i(\cdot)$.

Example 2.2. *Recall that the Pareto Type I distribution, henceforth denoted as $P(I)(\theta, \alpha)$ with $\alpha > 1$ and $\theta > 0$ corresponds to the survival function*

$$\overline{F}(x) = \theta^{\alpha-1} x^{-(\alpha-1)}, \;\; x > \theta.$$

Consider $X_1 \sim P(I)(\theta_1, \alpha_1)$ and $X_2 \sim P(I)(\theta_2, \alpha_2)$ with $\alpha_1 > \alpha_2 + 1$ and $\theta_2 \geq \theta_1 > 0$ under two different dependence structures:

(1) Assume that X_1 and X_2 are independent. Then, we know from Proposition 3.3 in Denuit et al. (2024) that there is an interval where $m_1(\cdot)$ is decreasing when $\alpha_1 > \alpha_2 + 1$.

(2) Assume that X_1 and X_2 are dependent with joint survival function

$$\overline{F}(x, y) = \left(\left(\frac{x}{\theta_1} \right)^{\alpha_1/\alpha_2} + \frac{y}{\theta_2} - 1 \right)^{-\alpha_2}.$$

The corresponding joint probability density function is given by

$$f(x, y) = \frac{\alpha_1(\alpha_2 + 1)}{\theta_1^{\alpha_1} \theta_2^{\alpha_2}} x^{\frac{\alpha_1 - \alpha_2}{\alpha_2}} \left(\left(\frac{x}{\theta_1} \right)^{\alpha_1/\alpha_2} + \frac{y}{\theta_2} - 1 \right)^{-(\alpha_2+2)}.$$

The partial derivatives appearing in Property 2.1 are as follows:

$$\begin{aligned} \frac{\partial^2}{\partial y^2} \log f(x, y) &= \frac{\alpha_2 + 2}{\theta_2^2} \left(\left(\frac{x}{\theta_1} \right)^{\alpha_1/\alpha_2} + \frac{y}{\theta_2} - 1 \right)^{-2}, \\ \frac{\partial^2}{\partial x \partial y} \log f(x, y) &= \left(\frac{x}{\theta_1} \right)^{\frac{\alpha_1 - \alpha_2}{\alpha_2}} \frac{\alpha_1(\alpha_2 + 2)}{\alpha_2 \theta_2 \theta_1} \left(\left(\frac{x}{\theta_1} \right)^{\alpha_1/\alpha_2} + \frac{y}{\theta_2} - 1 \right)^{-2}. \end{aligned}$$

Since $x \geq \theta_1$, $\alpha_1 > \alpha_2 + 1$ and $\theta_2 \geq \theta_1$, it is straightforward that (2.1) is valid. We deduce from Property 2.1 that $\{X_1 \mid S = s\}$ increases in s in the likelihood ratio order. Therefore, $m_1(\cdot)$ increases in s and the no-sabotage requirement holds. Let us mention that the conditions of Property 2.1 are met for all $\alpha_1 \geq \alpha_2$ and $\theta_2 \geq \theta_1$.

The dependence structure in item (2) of Example 2.2 corresponds to the Clayton survival copula. As the dependence parameter and the tail index of the second variable are assumed to be equal (to α_2), independence cannot be recovered as a particular case of the bivariate distribution in item (2). Example 2.2 serves to illustrate that even in the cases where independent losses lead to sabotage scenarios, there may exist a positive dependence structure weaker than comonotonicity under which the no-sabotage requirement is met for the same set of marginal distributions.

Coming back to independence, consider again two random variables X_1 and X_2 possessing regularly varying densities with tail indices whose difference is smaller than 1. Property 3.6. in Denuit et al. (2024) ensures that $m_1(s) \to \infty$ as $s \to \infty$. In addition, Properties 4.7 and 4.9. in Denuit et al. (2024) provide the asymptotic behavior of $m_1(\cdot)$. The next example shows that even with equal tail indices, $m_1(\cdot)$ may decrease over a range of central values despite being asymptotically increasing in s.

Example 2.3. *Recall that the Pareto Type II distribution, henceforth denoted as* $P(II)(\theta, \alpha, \vartheta)$*, with* $\alpha > 1$*,* $\theta > 0$*,* $\vartheta \geq 0$ *has survival function*

$$\overline{F}(x) = \theta^{\alpha-1} \left(x + \theta - \vartheta\right)^{-(\alpha-1)}, \quad x > \vartheta.$$

Consider independent risks $X_i \sim P(II)(\theta_i, \alpha, 0)$ *with* $\alpha > 2$*. As* X_1 *and* X_2 *have regularly varying densities with the same index* α*, we know from Property 3.6 in Denuit et al. (2024) that* $m_1(s) \to \infty$ *as* $s \to \infty$*. Although, considering* θ_2 *sufficiently greater that* θ_1*, there are cases where* $m_1(\cdot)$ *is not monotonic. For instance, Figure 1 illustrates such a scenario considering* $\alpha = 3$*,* $\theta_1 = 1$*,* $\theta_2 = 10$*. We can see there that* $m_1(\cdot)$ *decreases in the interval* $(17.2469, 58.4238)$*.*

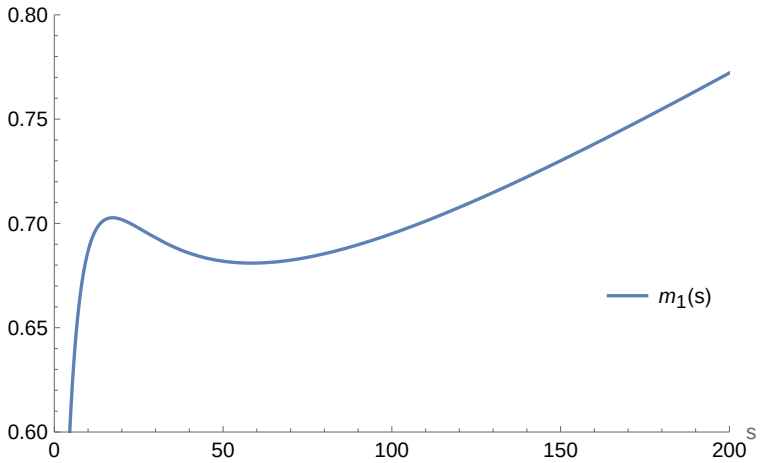


Figure 1: Conditional expectation $m_1(\cdot)$, with $X_i \sim P(II)(\theta_i, \alpha, 0)$ for $i = 1, 2$ considering $\alpha = 3$, $\theta_1 = 1$, $\theta_2 = 10$.

Example 2.3 shows that there may be sabotage opportunities even under the simplest case of heavy-tailed random variables that we could picture, that is, two independent Pareto random variables with the same index. In this paper, we aim to identify the sufficient conditions under which the no-sabotage condition is met for dependent losses. Many of these cases allow for marginals which may lead to a sabotage scenario under independence. In Section 4, a multivariate extension of the Pareto distribution is considered, the multivariate Pareto distribution of Mardia's type II with heavy-tailed marginals (Pareto type II distributions) and equal tail indices, as in Example 2.3, but with a dependence structure such that the no-sabotage condition holds true.

3 No-sabotage condition in common mixture models

3.1 From conditional to unconditional no-sabotage

In this section, we consider a very general setting where individual losses X_i are impacted by a non-negative random variable Λ. No assumption is made about the dependence structure within the random vector $(\Lambda, \boldsymbol{X})$ where $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ gather individual losses

$X_1, X_2, \ldots, X_n$. The common shock Λ can be additive or multiplicative, or may correspond to a random parameter entering the distribution of individual losses $X_1, X_2, \ldots, X_n$.

The next result derives sufficient conditions to meet the no-sabotage requirement in this general setting.

Proposition 3.1. *Consider the random vector* $(\Lambda, \boldsymbol{X})$ *with non-negative components and define* $S = \sum_{i=1}^n X_i$. *If*

(a) $(s, \lambda) \mapsto \mathrm{E}[X_i \mid S = s, \Lambda = \lambda]$ *is non-decreasing,*

(b) $\Lambda \mid S = s_1 \preceq_{\mathrm{ST}} \Lambda \mid S = s_2$ *for all* $s_1 \leq s_2$, *that is,*

$$\mathrm{P}[\Lambda > \lambda \mid S = s_1] \leq \mathrm{P}[\Lambda > \lambda \mid S = s_2] \text{ for all } \lambda,$$

then $m_i(\cdot)$ *is non-decreasing.*

Proof. We can rewrite

$$m_i(s) = \mathrm{E}\big[\mathrm{E}[X_i \mid S = s, \Lambda]\big|S = s\big] = \int_0^\infty \mathrm{E}[X_i \mid S = s, \Lambda = \lambda]\mathrm{d}F_{\Lambda|S=s}(\lambda).$$

For $s_2 \geq s_1$, we have

$$\begin{aligned} m_i(s_2) &= \int_0^\infty \mathrm{E}[X_i \mid S = s_2, \Lambda = \lambda]\mathrm{d}F_{\Lambda|S=s_2}(\lambda) \\ &\geq \int_0^\infty \mathrm{E}[X_i \mid S = s_1, \Lambda = \lambda]\mathrm{d}F_{\Lambda|S=s_2}(\lambda). \end{aligned}$$

Now, the function $g_s(\cdot)$ defined as $g_s(\lambda) = \mathrm{E}[X_i|S = s, \Lambda = \lambda]$ is non-decreasing in λ so that

$$\begin{aligned} m_i(s_2) &\geq \int_0^\infty g_{s_1}(\lambda)\mathrm{d}F_{\Lambda|S=s_2}(\lambda) \\ &\geq \int_0^\infty g_{s_1}(\lambda)\mathrm{d}F_{\Lambda|S=s_1}(\lambda) \\ &= m_i(s_1), \end{aligned}$$

which ends the proof. □

Proposition 3.1 can be interpreted as follows. Condition (a) ensures that X_i gets larger on average when Λ increases. It can be seen as a conditional version of the no-sabotage condition. Condition (b) means that Λ is stochastically increasing in the sum. Both conditions result in a positive relationship between Λ and $X_1, X_2, \ldots, X_n$ ensuring that the conditional version of the no-sabotage condition stated under condition (a) also holds unconditionally, when Λ is not known anymore.

Often, $X_1, X_2, \ldots, X_n$ are independent given Λ so that condition (a) in Proposition 3.1 corresponds to no-sabotage in the independent case. Several theoretical results can be applied to check this. For instance, no-sabotage holds when losses are independent and identically distributed, possess log-concave densities or when the size of the pool is large. Condition (b) in Proposition 3.1 then ensures that the unconditional version of no-sabotage still holds provided Λ stochastically increases in S.

Remark 3.2. The result stated under Proposition 3.1 is still valid when the random variable Λ is replaced with a random vector $\boldsymbol{\Lambda}$. Precisely, if $(s, \boldsymbol{\lambda}) \mapsto \mathrm{E}[X_i \mid S = s, \boldsymbol{\Lambda} = \boldsymbol{\lambda}]$ is non-decreasing and $\boldsymbol{\Lambda}$ increases in S with respect to the multivariate version of $\preceq_{\mathrm{ST}}$ then $m_i(\cdot)$ is non-decreasing.

The result stated under Proposition 3.1 holds true when $\preceq_{\mathrm{ST}}$ is replaced with any stronger stochastic order relation in condition (b). This is the case with $\preceq_{\mathrm{LR}}$, for instance, with the added benefit that the increasingness condition can be reversed, as shown next.

Proposition 3.3. *Consider a random vector $(\Lambda, \boldsymbol{X})$ with non-negative components and define $S = \sum_{i=1}^{n} X_i$. Assume that $X_1, X_2, \ldots, X_n$ are either absolutely continuous, valued in $\{0, 1, 2, \ldots\}$ or zero-augmented absolutely continuous random variables. If*

(a) $(s, \lambda) \mapsto \mathrm{E}[X_i \mid S = s, \Lambda = \lambda]$ is non-decreasing in both s and λ when the other variable is fixed.

(b) $S \mid \Lambda = \lambda_1 \preceq_{\mathrm{LR}} S \mid \Lambda = \lambda_2$ for all $\lambda_1 \leq \lambda_2$.

then $m_i(\cdot)$ is non-decreasing.

Proof. First, suppose that all random variables involved are absolutely continuous. Let $(s, \lambda) \mapsto f(s, \lambda)$ denote the joint probability density function of (S, Λ), $s \mapsto f(s|\lambda)$ denote the conditional probability density function of S given $\Lambda = \lambda$, and $\lambda \mapsto f(\lambda|s)$ denote the conditional probability density function of Λ given $S = s$. Condition (b) means that

$$\frac{f(s_1|\lambda_2)}{f(s_1|\lambda_1)} \leq \frac{f(s_2|\lambda_2)}{f(s_2|\lambda_1)} \text{ for } \lambda_1 \leq \lambda_2 \text{ and } s_1 \leq s_2$$

$$\Leftrightarrow \frac{f(s_1, \lambda_2)}{f(s_1, \lambda_1)} \leq \frac{f(s_2, \lambda_2)}{f(s_2, \lambda_1)} \text{ for } \lambda_1 \leq \lambda_2 \text{ and } s_1 \leq s_2$$

$$\Leftrightarrow \frac{f(\lambda_2|s_1)}{f(\lambda_1|s_1)} \leq \frac{f(\lambda_2|s_2)}{f(\lambda_1|s_2)} \text{ for } \lambda_1 \leq \lambda_2 \text{ and } s_1 \leq s_2$$

which is equivalent to $\Lambda \mid S = s_1 \preceq_{\mathrm{LR}} \Lambda \mid S = s_2$ for all $s_1 \leq s_2$ and is stronger than condition (b) in Proposition 3.1. This ends the proof of the absolutely continuous case. The proof is similar when $X_1, \ldots, X_n$ are valued in $\{0, 1, 2, \ldots\}$. In this case, it suffices to replace all densities with corresponding probability mass functions. If $X_1, \ldots, X_n$ are zero-augmented random variables with positive probability masses at the origin and probability density functions over $(0, \infty)$, the proof follows as for the continuous case by Proposition 2.2 (iii) in Denuit (2019) since we know that $m_i(0) = 0$. □

Notice that condition (b) in Proposition 3.3 is equivalent to require that the joint probability density function of (S, Λ) to be TP2.

Remark 3.4 (Conditional mean independence)**.** Let us now consider the particular case where $(s, \lambda) \mapsto \mathrm{E}[X_i \mid S = s, \Lambda = \lambda]$ appearing in condition (a) of Propositions 3.1-3.3 is constant with respect to λ. This is related to the concept of conditional mean independence.

Given a triplet (U_1, U_2, W) of random variables, recall that conditional mean independence of U_1 given W conditioning on U_2 holds if

$$\mathrm{E}[U_1|U_2, W] = \mathrm{E}[U_1|U_2]. \tag{3.1}$$

Here, and throughout this paper, all equalities between random variables are assumed to hold almost surely (that is, with probability 1). Concept (3.1) is used in econometrics to decide whether additional features need to be included in a regression model. If $(s, \lambda) \mapsto \mathrm{E}[X_i \mid S = s, \Lambda = \lambda]$ is constant with respect to λ then X_i is conditionally mean independent of Λ conditioning on S. This is useful in particular when results derived by Furman et al. (2018) to characterize the case where $m_i(s)$ is proportional to s apply conditionally to Λ.

3.2 Common mixture models

As defined in Wang (1998), the common mixture model consists in conditionally independent risks. Intuitively speaking, this dependence results from the existence of an external mechanism impacting on all losses $X_1, X_2, \ldots, X_n$. The external mechanism is generally viewed as a common environment described by some positive random variable Λ. Formally, individual losses $X_1, X_2, \ldots, X_n$ are independent given the environmental parameter Λ, but become dependent when Λ is unknown. Typically, Λ represents the magnitude of a natural catastrophe or any hidden feature influencing all risks, simultaneously. The interested reader is referred e.g. to Wang (1998) or to Chapter 7 in Denuit et al. (2005) for a detailed discussion on the common mixture construction.

Henceforth, we denote the conditional distribution function of X_i given $\Lambda = \lambda$ as

$$F_i(x|\lambda) = \mathrm{P}[X_i \leq x|\Lambda = \lambda], \;\; x \geq 0,$$

and the conditional mean of X_i given $\Lambda = \lambda$ as

$$\mu_i(\lambda) = \mathrm{E}[X_i|\Lambda = \lambda].$$

Unconditionally, $F_i(x) = \mathrm{P}[X_i \leq x]$ and $\mu_i = \mathrm{E}[X_i]$. We assume throughout the paper that $0 < \mu_i < \infty$ holds true for every i. Furthermore, the size-biased transform of $F_i(\cdot|\lambda)$ is denoted as $\widetilde{F}_i(\cdot|\lambda)$ and given by

$$\widetilde{F}_i(x|\lambda) = \frac{\mathrm{E}\big[X_i \mathrm{I}[X_i \leq x]\big|\Lambda = \lambda\big]}{\mu_i(\lambda)}, \;\; x \geq 0.$$

We are now ready to derive an alternative to the conditions in Proposition 3.3 based on the representation formula derived by Denuit and Robert (2022).

Proposition 3.5. *Consider a random vector $(\Lambda, \boldsymbol{X})$ with non-negative components and define $S = \sum_{i=1}^n X_i$. Assume that $X_1, X_2, \ldots, X_n$ are either absolutely continuous, valued in $\{0, 1, 2, \ldots\}$ or zero-augmented absolutely continuous random variables. Assume that the joint distribution function $F_{\boldsymbol{X}}$ of $\boldsymbol{X}$ can be written as*

$$F_{\boldsymbol{X}}(x_1, \ldots, x_n) = \int_0^\infty \left(\prod_{i=1}^n F_i(x_i|\lambda) \right) \mathrm{d}F_\Lambda(\lambda), \;\; \boldsymbol{x} \in \mathbb{R}^n. \tag{3.2}$$

Define the random vector $\mathbf{Y}^{[i]}$ *with joint distribution function given by*

$$F_{\mathbf{Y}^{[i]}}(t_1,\ldots,t_n)=\int_0^\infty\left(\prod_{j\neq i}F_j(t_j|\lambda)\right)\widetilde{F}_i(t_i|\lambda)\mathrm{d}F_{\Lambda_i^*}(\lambda),\ \ \boldsymbol{t}\in\mathbb{R}^n, \tag{3.3}$$

where

$$\mathrm{d}F_{\Lambda_i^*}(\lambda)=\frac{\mu_i(\lambda)}{\mu_i}\mathrm{d}F_\Lambda(\lambda),\ \ \lambda\geq 0. \tag{3.4}$$

Then, $m_i(\cdot)$ *is non-decreasing if, and only if,* $S\preceq_{\mathrm{LR}}\sum_{j=1}^n Y_j^{[i]}$.

Proof. Let us give the proof in the absolutely continuous case. The other cases easy follow. According to Propositions 3.1 and 4.2 in Denuit and Robert (2022),

$$m_i(s)=\mathrm{E}\left[X_i\right]\frac{f_{Y_1^{[i]}+\ldots+Y_n^{[i]}}(s)}{f_{X_1+\ldots+X_n}(s)},\ \ s\geq 0. \tag{3.5}$$

The no-sabotage condition is then equivalent to $X_1+\ldots+X_n\preceq_{\mathrm{LR}}Y_1^{[i]}+\ldots+Y_n^{[i]}$, as announced. □

Notice that when $\mu_i(\lambda)=\lambda\mu_i$ and $\mathrm{E}[\Lambda]=1$, $\Lambda_i^\star$ defined in (3.4) is the size-biased transform of Λ.

4 Application to Type II Mardia Pareto distribution

4.1 No-sabotage condition in multiplicative background risk model

In the multiplicative background risk model, also known as the scale mixture model, risks $X_1,\ldots,X_n$ can be written as $X_i=ZW_i$ for $i=1,\ldots,n$, where $W_1,\ldots,W_n,Z$ are independent non-negative random variables. The common factor Z impacting on every X_i plays the role of the environmental parameter Λ. It can again be interpreted as the background or systemic risk while $W_1,\ldots,W_n$ are the stand-alone risks.

Multiplicative background risk models with W_i obeying the Exponential distribution have attracted a lot of attention. See, for instance, Furman et al. (2018) or Sarabia et al. (2018). A relevant example is the multivariate Type II Mardia Pareto distribution, discussed in Section 4.3, where Z follows a Gamma distribution. See also Furman et al. (2021). For further discussion on this construction, we refer the reader to Asimit et al. (2016) and references therein.

Analogously to the additive mixture model, we can study sufficient conditions so that Proposition 3.3 holds for scale mixture models. Assume that $X_i=ZW_i$, $i=1,\ldots,n$, for independent non-negative random variables $W_1,\ldots,W_n,Z$. Then, $S=Z\sum_{i=1}^n W_i$. Denote $m_{W_i}(s)=\mathrm{E}[W_i\mid\sum_{j=1}^n W_j=s]$ and assume that $m_{W_i}(\cdot)$ admits a derivative $m'_{W_i}(\cdot)$. Condition (a) in Proposition 3.3 requires

$$m_i(s\mid\lambda)=\mathrm{E}[X_i\mid S=s,Z=\lambda]=\lambda m_{W_i}\left(\frac{s}{\lambda}\right)$$

to be non-decreasing in (s, λ). For any fixed λ, $m_i(s \mid \lambda)$ is non decreasing in s if $m'_{W_i} \geq 0$. Condition (a) in Proposition 3.3 is valid provided

$$\frac{\partial}{\partial \lambda} m_i(s \mid \lambda) = m_{W_i}(s) - \frac{s}{\lambda} m'_{W_i}\left(\frac{s}{\lambda}\right) \geq 0 \Leftrightarrow m'_{W_i}(t) \leq \frac{m_{W_i}(t)}{t} \text{ for all t.}$$

Again, the conditions imposed on the derivative of the conditional expectation $m_{W_i}(\cdot)$ are restrictive. Alternative conditions thus appear to be useful.

Let us establish the following technical lemma that will be useful in the remainder of this section.

Lemma 4.1. *A probability density function is such that $f\left(\frac{x}{y}\right)$ is TP2 in x and y if, and only if, $g(x) = f\left(\mathrm{e}^x\right)$ is log-concave.*

Proof. Clearly, $f\left(\frac{x}{y}\right) = f(\exp(\ln x - \ln y))$. Hence, as the logarithm is a strictly increasing function, considering $g(x) = f\left(e^x\right)$, it means that $f\left(\frac{x}{y}\right)$ is TP2 in (x, y) if and only if $g\left(z - w\right)$ is TP2 in z, w. That is, if g is log-concave. □

Let us now proceed similarly as in the proof of Theorem 1.C.9. in Shaked and Shanthikumar (2007) to derive a result that will be needed after.

Property 4.2. *Let W_1, W_2, and Z be absolutely continuous positive independent random variables such that $W_1 \preceq_{\mathrm{LR}} W_2$. If the probability density function $f_Z(\cdot)$ of Z verifies the assumption of Lemma 4.1 then $W_1 Z \preceq_{\mathrm{LR}} W_2 Z$.*

Proof. For $i = 1, 2$, the probability density function of the product $W_i Z$ is given by

$$f_{W_i Z}(w) = \int_0^\infty f_Z\left(\frac{w}{t}\right) f_{W_i}(t) \mathrm{d}t, \;\; w > 0.$$

The assumption $W_1 \preceq_{\mathrm{LR}} W_2$ means that $f_{W_i}(t)$ is TP2 in t and $i \in \{1, 2\}$. By assumption, $f_Z(\frac{w}{t})$ is TP2 in t and w, is TP_2. Therefore, the basic composition formula in Karlin (1968) ensures that $f_{W_i Z}(w)$ is TP2 in $i \in \{1, 2\}$ and w. This ends the proof since this is equivalent to $W_1 Z \preceq_{\mathrm{LR}} W_2 Z$. □

Now, we can obtain alternative sufficient conditions based on Proposition 3.5.

Property 4.3. *Let us consider a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ such that $X_i = Z W_i$, $i = 1, \ldots, n$, for independent non-negative random variables $W_1, \ldots, W_n, Z$. Let Z_i^* be distributed according to $\mathrm{d}F_{Z_i^*}(\lambda) = \frac{\mathrm{E}[X_i(\lambda)]}{\mathrm{E}[X_i(Z)]} \mathrm{d}F_Z(\lambda)$. If $\sum_{j=1}^n W_j$ and $\sum_{j=1, j\neq i}^n W_j$ have log-concave densities and if the probability density functions of Z_i^* and $\sum_{j=1}^n W_j$ verify the assumption of Lemma 4.1 then $m_i(\cdot)$ is non-decreasing.*

Proof. Under the scale mixture model, the characterization of Proposition 3.5 can be rewritten as $S^{[i]} \succeq_{\mathrm{LR}} S$, where

$$S = Z\left(\sum_{j=1}^n W_j\right) \text{ and } S^{[i]} = Z_i^*\left(\widetilde{W_i} + \sum_{j\neq i} W_j\right)$$

$\widetilde{W_i}$ is a size-biased version of W_i, with distribution function

$$\mathrm{P}[\widetilde{W_i} \leq t] = \frac{\mathrm{E}\big[W_i \mathrm{I}[W_i \leq t]\big]}{\mathrm{E}[W_i]}, \ \ t \geq 0,$$

all the random variables being independent. Now, $S^{[i]} \preceq_{\mathrm{LR}} S$ is verified if the following inequalities hold:

(I1) $Z_i^* \left(\widetilde{W_i} + \sum_{j \neq i} W_j\right) \succeq_{\mathrm{LR}} Z_i^* \left(\sum_{j=1}^n W_j\right)$,

(I2) $Z_i^* \left(\sum_{j=1}^n W_j\right) \succeq_{\mathrm{LR}} Z \left(\sum_{j=1}^n W_j\right)$.

Since $\sum_{j \neq i} W_j$ has log-concave density, we deduce from Theorem 1.C.9. in Shaked and Shanthikumar (2007) that

$$\widetilde{W_i} \succeq_{\mathrm{LR}} W_i \Rightarrow \widetilde{W_i} + \sum_{j \neq i} W_j \succeq_{\mathrm{LR}} \sum_{j=1}^n W_j.$$

Since the probability density function of Z_i^* verifies the assumption of Lemma 4.1, (I1) is true by Proposition 4.2. As $\mu_i(\lambda) = \lambda \mathrm{E}\,[W_i]$ increases in λ, we have $Z_i^* \succeq_{\mathrm{LR}} Z$. Since the probability density function of $\sum_{j=1}^n W_j$ also verifies the assumption of Lemma 4.1, (I2) holds by Proposition 4.2 and therefore $S^{[i]} \succeq_{\mathrm{LR}} S$, which ends the proof. □

4.2 Definition of Type II Mardia Pareto distribution and common mixture representation

Let $\alpha > 0$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i > 0$ for all $i = 1, \ldots, n$. Considers a random vector $\boldsymbol{X}$ with joint survival function is given by

$$\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) = \left(1 + \sum_{i=1}^n \frac{x_i}{\sigma_i}\right)^{-\alpha}, \quad x_i \geq 0, \quad i = 1, 2, \ldots, n.$$

This is henceforth denoted as $\boldsymbol{X} = (X_1, \ldots, X_n) \sim MP^{(n)}(\boldsymbol{\sigma}, \alpha)$. See Arnold (1983) for a detailed presentation.

Recall that if $X_i = W_i/\Lambda$ for $i = 1, \ldots, n$ where the random variables $W_1, \ldots, W_n, \Lambda$ are independent and such that $W_i \sim$ Exponential (σ_i), $i = 1, \ldots, n$, and $\Lambda \sim$ Gamma$(\alpha, 1)$ then $\boldsymbol{X} \sim MP^{(n)}(\boldsymbol{\sigma}, \alpha)$.

4.3 No-sabotage under Type II Mardia Pareto distribution

If $\boldsymbol{X} \sim MP^{(n)}(\boldsymbol{\sigma}, \alpha)$ then $X_i \sim P(II)(\sigma_i, \alpha, 0)$ for every $i = 1, \ldots, n$. As we saw in Example 2.3, there may exist a sabotage scenario with these marginals in the independent case. This section establishes that this is not possible when losses follow the multivariate Pareto distribution of Mardia's type II.

Given $\boldsymbol{X} \sim MP^{(n)}(\boldsymbol{\sigma}, \alpha)$, let us check whether the hypotheses of Property 4.3 are fulfilled with $Z = \frac{1}{\Lambda}$. Let us note that

$$\mathrm{E}\left[W_i/\Lambda\right] = \int_0^\infty E\left[W_i/\lambda\right] \mathrm{d}F_\Lambda(\lambda) = \int_0^\infty \frac{\sigma_i}{\lambda} \frac{\lambda^{\alpha-1}\mathrm{e}^{-\lambda}}{\Gamma(\alpha)} \mathrm{d}\lambda = \frac{\sigma_i}{\alpha} \int_0^\infty \frac{\lambda^{\alpha-2}\mathrm{e}^{-\lambda}}{\Gamma(\alpha-1)} \mathrm{d}\lambda = \frac{\sigma_i}{\alpha}.$$

Hence, as

$$\mathrm{d}F_{\Lambda^*}(\lambda) = \frac{\mathrm{E}\left[W_i/\lambda\right]}{\mathrm{E}\left[W_i/\Lambda\right]} \mathrm{d}F_\Lambda(\lambda) = \frac{\lambda^{\alpha-2}\mathrm{e}^{-\lambda}}{\Gamma(\alpha-1)} \mathrm{d}\lambda,$$

we can write $Z_i^* = 1/\Lambda^*$ where $\Lambda^* \sim \mathrm{Gamma}(\alpha - 1, 1)$.

To study whether the probability density functions of the random variables verify the assumption of Lemma 4.1, let us first give the following technical Lemma which will be needed after.

Lemma 4.4. *Let $Y \sim \mathrm{Exponential}(\lambda)$ and let X be an independent positive random variable with density f_X. Let us consider $g_X(x) = f_X\left(\mathrm{e}^x\right)$. And let f_S be the density function of $S = X + Y$ with $g_S(x) = f_S\left(\mathrm{e}^x\right)$. If $\partial_y^2 \ln g_X(y) < -\lambda \mathrm{e}^y$, then $\partial_y^2 \ln g_S(y) < -\lambda \mathrm{e}^y$.*

Proof. Starting from

$$f_S(s) = \int_0^s f_X(x) \lambda \mathrm{e}^{-\lambda(s-x)} \mathrm{d}x = \lambda \mathrm{e}^{-\lambda s} \int_0^s f_X(x) \mathrm{e}^{\lambda x} \mathrm{d}x$$

we see that

$$g_s(s) = f_S(\mathrm{e}^s) = \lambda \mathrm{e}^{-\lambda \mathrm{e}^s} \int_{-\infty}^s f_X(\mathrm{e}^y) \mathrm{e}^{\lambda \mathrm{e}^y} \mathrm{e}^y \mathrm{d}y.$$

So, we have

$$\partial_y^2 \ln g_s(s) = -\lambda \mathrm{e}^s + \partial_y^2 \ln \left(\int_{-\infty}^s f_X(\mathrm{e}^y) \mathrm{e}^{\lambda \mathrm{e}^y} \mathrm{e}^y \mathrm{d}y \right).$$

Now, let us note that, by the assumptions, $\partial_y^2 \ln f_X(\mathrm{e}^y) + \lambda \mathrm{e}^y \leq 0$ and therefore, $h_1(y) = f_X(\mathrm{e}^y) \mathrm{e}^{\lambda \mathrm{e}^y} \mathrm{e}^y$ is log concave. By Lemma 3 in Bagnoli and Bergstrom (2006), it follows that $h_2(s) = \int_{-\infty}^s f_X(\mathrm{e}^y) \mathrm{e}^{\lambda \mathrm{e}^y} \mathrm{e}^y \mathrm{d}y$ is also log-concave and we can finally conclude that $\partial_y^2 \ln g_s(s) \leq -\lambda \mathrm{e}^s$. □

As a consequence of this result, the probability density function of the sum of n Exponentially distributed random variables verifies the assumption of Lemma 4.1. The proof is given in Appendix A. A direct consequence of this statement is that, by Property 4.3, the no-sabotage condition holds when losses obey the multiplicative background risk model with idiosyncratic Exponentially distributed risk factors provided the background risk possesses a probability density function satisfying the assumption of Lemma 4.1. This is the case in particular for the Type II Mardia Pareto distribution, as proved in Appendix A. Hence,

$$\boldsymbol{X} \sim MP^{(n)}(\boldsymbol{\sigma}, \alpha) \Rightarrow m_i(\cdot) \text{ non-decreasing for every } i = 1, 2, \ldots, n,$$

and the no-sabotage requirement holds.

4.4 Analytical expression for allocations using divided differences

Now that the no-sabotage condition is known to be verified, it is interesting to give a closed-form expression for the contributions. This is possible using divided differences and results from Chiragiev and Landsman (2007).

Proposition 4.5. *Let* $\boldsymbol{X} \sim MP^{(n)}(\boldsymbol{\sigma}, \alpha)$ *with* $\alpha > 0$, $\sigma_i > 0$ *for all* $i = 1, \ldots, n$ *and* $\sigma_i \neq \sigma_j$ *for all* $i \neq j$. *Then,*

$$m_k(s) = \frac{\frac{\alpha\, \sigma_k^{n-2}}{\prod_{\substack{j\neq k \\ 1\leq j\leq n}} (\sigma_j - \sigma_k)} s \left(1 + \frac{s}{\sigma_k}\right)^{-(\alpha+1)} + \sum_{\substack{i\neq k \\ 1\leq i\leq n}} \frac{\sigma_k \sigma_i^{n-1} \left(\left(1+\frac{s}{\sigma_i}\right)^{-\alpha} - \left(1+\frac{s}{\sigma_k}\right)^{-\alpha}\right)}{(\sigma_i - \sigma_k) \prod_{\substack{j\neq i \\ 1\leq j\leq n}} (\sigma_j - \sigma_i)}}{(-1)^{n+1} \sum_{i=1}^{n} \frac{\alpha \sigma_i^{n-2} \left(1+\frac{s}{\sigma_i}\right)^{-(\alpha+1)}}{\prod_{j\neq i} (\sigma_i - \sigma_j)}}. \tag{4.1}$$

The proof of Proposition 4.5 can be found in Appendix B. In particular, considering risk pooling with two participants, we have

$$(X_1, X_2) \sim MP^{(2)}\big((\sigma_1, \sigma_2), \alpha\big) \Rightarrow m_1(s) = \frac{\alpha\, s \left(1 + \frac{s}{\sigma_1}\right)^{-(\alpha+1)} + \frac{\sigma_1 \sigma_2}{(\sigma_1 - \sigma_2)} \left(\left(1 + \frac{s}{\sigma_2}\right)^{-\alpha} - \left(1 + \frac{s}{\sigma_1}\right)^{-\alpha}\right)}{\alpha \left(\left(1 + \frac{s}{\sigma_1}\right)^{-(\alpha+1)} - \left(1 + \frac{s}{\sigma_2}\right)^{-(\alpha+1)}\right)}.$$

The following example illustrates the behavior of $m_i(\cdot)$ if risks follow a Type II Mardia Pareto distribution.

Example 4.6. *Let us consider a pool formed by ten random losses* $X_1, \ldots, X_{10}$ *such that* $\boldsymbol{X} \sim MP^{(10)}(\boldsymbol{\sigma}, \alpha)$ *with* $\alpha = 5$, $\boldsymbol{\sigma} = (1, 2, \ldots, 9, 10)$. *Figure 2 displays the graph of* $m_i(\cdot)$ *for* $i = 1, 3, 6, 10$. *We can observe that they are monotonic, in accordance with the results of Section 4.3.*

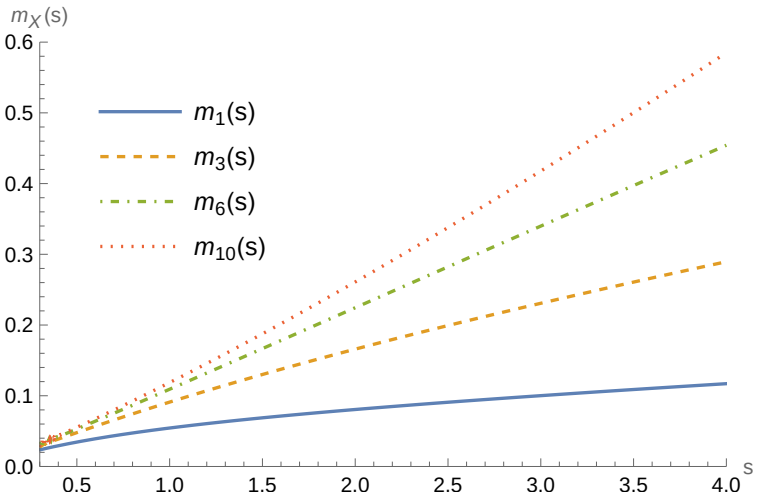


Figure 2: Contributions $m_i(\cdot)$ when $\boldsymbol{X} \sim MP^{(10)}(\boldsymbol{\sigma}, \alpha)$ with $\alpha = 5$, $\boldsymbol{\sigma} = (1, 2, \ldots, 9, 10)$.

5 Conclusions

This paper identified sufficient conditions which ensure that the no-sabotage requirement is satisfied under the conditional mean risk-sharing rule. In the literature, this problem has been mainly studied in the bivariate case. This paper considered larger pools described by the common mixture model. This model often induces a positive dependence structure among the losses, which goes in line with the intuition suggesting that no-sabotage occurs more likely with positively dependent risks.

Considering losses subject to the common mixture model, we obtained sufficient conditions to ensure the monotonicity of the conditional expectations under two different approaches. Firstly, conditions are provided based on the monotonicity of Λ in S with respect to the usual stochastic order $\preceq_{\mathrm{ST}}$, or of S in Λ with respect to the likelihood ratio order $\preceq_{\mathrm{LR}}$, together with the monotonicity of the conditional expectations both in the sum and the environmental parameter Λ. A second approach is motivated by the representation formula of the conditional mean risk-sharing rule for common mixture models derived in Denuit and Robert (2022). The no-sabotage condition is characterized by the comparison with respect to the likelihood ratio order $\preceq_{\mathrm{LR}}$ of two sums, where the dominating one comes naturally from the representation formula. This approach appears to be useful for additive and scale mixture models. Sufficient conditions are provided in these cases on the basis of properties of the environmental (or background) and the stand-alone risks. Finally, the multivariate Type II Mardia Pareto distribution has been studied in detail. This model verifies the no-sabotage condition and closed-form expressions are provided for the contributions.

A promising field of applications of the analytical results derived in this paper is cyber risk insurance. Faure and Nieuwesteeg (2018) and Eling et al. (2021) carefully discussed cyber risk management, including retention, pooling and transfer. In particular, cyber risk pooling can be implemented through the establishment of a mutual insurer, like the new cyber mutual insurer MIRIS (acronym for Mutual Insurance and Reinsurance for Information Systems) based in Brussels. Heavy-tailedness in cyber risk severities is well documented in literature. See for instance Maillart and Sornette (2010) or Eling and Wirfs (2019). In addition, as discussed in Eling (2020), risk management in the field of cyber risk is highly influenced by systemic risk and the inclusion of models which consider heavy tails and tail dependence among cyber risk is necessary. Modeling this systemic risk through a common mixture model appears to be a natural idea. For instance, Peters et al. (2023) study cyber event severity distributions, aggregate cyber-related losses by business sector and explore the copula dependence structure for pairs of sectors. These authors found that the survival Joe copula provided the best fit in most scenarios. Joe's copula is Archimedean and, therefore, can be written as a common mixture model. This suggests the suitability of the results here derived for cyber insurance applications.

To end with, let us discuss an open question related to risk reduction by conditional mean risk sharing. Risk reduction is another open question. Recall that the random variable Y is said to be smaller than the random variable Z in the convex order, denoted as $Y \preceq_{\mathrm{CX}} Z$, when $\mathrm{E}[g(Y)] \leq \mathrm{E}[g(Z)]$ holds for every convex function g such that the expectations exist. Jensen inequality then shows that $m_i(S) \preceq_{\mathrm{CX}} X_i$ which guarantees that joining the pool is the preferred option for all risk-averse agents when total losses are allocated according to the conditional mean risk-sharing rule, since this means paying the preferred $m_i(S)$ rather

than X_i. This is always true, whatever the dependence structure.

A more delicate problem is whether enlarging the pool is regarded as beneficial by all risk-averse agents. Denoting as $S_n = \sum_{i=1}^n X_i$ and $m_{i,n}(S_n) = \mathrm{E}[X_i|S_n]$ for $i = 1, 2, \ldots, n$ where the subscript "n" has been added because we vary the size of the pool, this amounts to determine whether $m_{i,n+1}(S_{n+1}) \preceq_{\mathrm{CX}} m_{i,n}(S_n)$ holds true. It is known from Denuit and Robert (2023) that this is the case when losses X_i are independent. For comonotonic losses, X_i and $m_{i,n}(S_n)$ are identically distributed so that there is indifference between joining the pool or not. A natural question is whether risk reduction still holds for conditionally independent losses X_i. We leave this problem to a subsequent work.

Acknowledgements

Michel Denuit and Patricia Ortega-Jimenez gratefully acknowledge funding from the FWO and F.R.S.-FNRS under the Excellence of Science (EOS) programme, project ASTeRISK (40007517).

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Appendix

A Assumption in Lemma 4.1 for some distribution families

Property A.1. *Let $W_1, \ldots, W_n$ be independent random variables such that $W_i \sim \text{Exponential}(1/\lambda_i)$. Then, the probability density function of $\sum_{i=1}^n W_i$ verifies the assumption of Lemma 4.1.*

Proof. Let us consider the variables sorted in terms of the parameters verifying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let us denote $S_i = \sum_{j=1}^i W_j$, by f_j the density of W_j and g_j the function $g_j(x) = f_j(\mathrm{e}^x)$. Now, proceed by induction:

- For $i = 2$. We have that $\partial_y^2 \ln f_1(\mathrm{e}^y) = -\lambda_1 \mathrm{e}^y$, then, as $\lambda_1 \geq \lambda_2$, by Lemma 4.4, $\partial_y^2 \ln g_2(y) \leq -\lambda_2 \mathrm{e}^y$.
- Assume that the result holds for i and let us establish its validity for $i+1$. Let us assume that $\partial_y^2 \ln g_i(y) \leq -\lambda_i \mathrm{e}^y$. Then, by Lemma 4.4, we have that $\partial_y^2 \ln g_{i+1}(y) \leq -\lambda_i \mathrm{e}^y$. Since $\lambda_i \geq \lambda_{i+1}$, we have that $\partial_y^2 \ln g_{i+1}(y) \leq -\lambda_{i+1} \mathrm{e}^y$.

Hence, $\partial_y^2 \ln g_n(y) \leq -\lambda_n \mathrm{e}^y < 0$ holds true. This means that g_n is log-concave. The result then follows from the equivalence stated under Lemma 4.1. □

Property A.2. *Let $\gamma \sim \text{Gamma}(\alpha, 1)$. Then the probability density function of $X = \frac{1}{\gamma}$ satisfies the assumption of Lemma 4.1.*

Proof. If $\gamma \sim \text{Gamma}(\alpha, 1)$, then $f_\gamma(x) = \frac{x^{\alpha-1}\mathrm{e}^{-x}}{\Gamma(\alpha-1)}$. Note that $\mathrm{P}\left[\frac{1}{X} < x\right] = \mathrm{P}\left[\frac{1}{x} < X\right]$ and, therefore, if $X \sim f_X$, then $f_{1/X}(x) = \frac{1}{x^2} f_X(1/x)$. Hence:

$$\partial_t \partial_w \log f_{1/\gamma}\left(\frac{w}{t}\right) = \partial_t \partial_w \log\left(\left(\frac{t}{w}\right)^2 f_\gamma\left(\frac{t}{w}\right)\right) = \partial_t \partial_w \log\left(\frac{t}{w}\right)^2 + \partial_w \partial_t \log f_\gamma\left(\frac{t}{w}\right)$$

Now, let us note that $\partial_t \partial_w \log\left(\frac{t}{w}\right)^2 = 0$ and

$$\begin{aligned}
\partial_w \partial_t \log f_\gamma\left(\frac{t}{w}\right) &= \partial_w \left(\frac{1}{w} \frac{\frac{(\alpha-1)\left(\frac{t}{w}\right)^{\alpha-2}\mathrm{e}^{-\frac{t}{w}} - \left(\frac{t}{w}\right)^{\alpha-1}\mathrm{e}^{-\frac{t}{w}}}{\Gamma(\alpha-1)}}{\frac{\left(\frac{t}{w}\right)^{\alpha-1} e^{-\frac{t}{w}}}{\Gamma(\alpha-1)}}\right) \\
&= \partial_w \left(\frac{1}{w}\left((\alpha-1)\left(\frac{w}{t}\right) - 1\right)\right) \\
&= \partial_w \left((\alpha-1)\left(\frac{1}{t}\right) - \frac{1}{w}\right) \\
&= \frac{1}{w^2} \geq 0.
\end{aligned}$$

This ends the proof. □

B Proof of Proposition 4.5

In this proof we use the notations of Chiragiev and Landsman (2007). We can write $m_k(s) = \frac{1}{f_S(s)} \int_0^s x f_{X_k,S}(x,s)\mathrm{d}x$. From expression (4.6) in (Chiragiev and Landsman, 2007) we have

$$f_S(s) = \sum_{i=1}^{n} \frac{\alpha \sigma_i^{n-2} \left(1 + (s/\sigma_i)\right)^{-(\alpha+1)}}{\prod_{j \neq i} (\sigma_i - \sigma_j)}.$$

Let us use the computations appearing in the proof of Theorem 1 of Chiragiev and Landsman (2007) for the mixture representation of the density functions. Recall that the density g of a random variable following the Gamma(α, β) distribution is given by $g(x) = \frac{x^{\alpha-1}\mathrm{e}^{-\beta x}\beta^\alpha}{\Gamma(\alpha)}$. In the following expressions, we denote $\lambda_i = \lambda/\sigma_i$ and $\pi(\lambda)$ the density of $\Lambda \sim$ Gamma$(\alpha, 1)$ for $\alpha > 2$:

$$\begin{aligned}
\int_0^s x f_{X_k,S}(x,s)\mathrm{d}x &= \int_0^\infty \int_0^s x f_{(X_k,S)|\lambda}(x,s)\mathrm{d}x\pi(\lambda)\mathrm{d}\lambda \\
&= \int_0^\infty \int_0^s x f_{X_k|\lambda}(x) f_{S-X_k|\lambda}(s-x)\mathrm{d}x\pi(\lambda)\mathrm{d}\lambda \\
&= \int_0^\infty (-1)^{n-1} \left(\prod_{j=1}^{n} \lambda_j\right) h_k(s, \lambda_1; \lambda_2; ...; \lambda_{k-1}; \lambda_{k+1}; ...; \lambda_n)\pi(\lambda)\mathrm{d}\lambda,
\end{aligned} \tag{B.1}$$

where $h_k(s, \lambda_1; \lambda_2; ...; \lambda_{k-1}; \lambda_{k+1}; ...; \lambda_n)$ is the nth-order divided difference in λ of the function h_k given by

$$h_k(s, \lambda_i) = \frac{s\mathrm{e}^{-\lambda_k s}}{\lambda_k - \lambda_i} + \frac{\mathrm{e}^{-\lambda_k s}}{(\lambda_k - \lambda_i)^2} - \frac{\mathrm{e}^{-\lambda_i s}}{(\lambda_k - \lambda_i)^2}.$$

From Remark 1 in Chiragiev and Landsman (2007), this divided difference can be expressed as

$$\begin{aligned}
h_k(s, \lambda_1; \lambda_2; ...; \lambda_{k-1}; \lambda_{k+1}; ...; \lambda_n) &= \sum_{\substack{i \neq k \\ 1 \leq i \leq n}} \frac{h_k(s, \lambda_i)}{\prod_{\substack{j \neq k,i \\ 1 \leq j \leq n}} (\lambda_i - \lambda_j)} \\
&= \sum_{\substack{i \neq k \\ 1 \leq i \leq n}} \frac{h_k(s, \lambda/\sigma_i)}{\lambda^{n-2} \prod_{\substack{j \neq k,i \\ 1 \leq j \leq n}} \left(\frac{1}{\sigma_i} - \frac{1}{\sigma_j}\right)}.
\end{aligned}$$

Hence, expression (B.1) can be rewritten as

$$\sum_{\substack{i \neq k \\ 1 \leq i \leq n}} \int_0^\infty \frac{(-1)^{n-1}\lambda^n \left(\prod_{j=1}^n \frac{1}{\sigma_j}\right)}{\lambda^{n-2} \prod_{\substack{j \neq k,i \\ 1 \leq j \leq n}} \left(\frac{1}{\sigma_i} - \frac{1}{\sigma_j}\right)} \left(\frac{s\mathrm{e}^{-\frac{\lambda}{\sigma_k}s}}{\lambda\left(\frac{1}{\sigma_k} - \frac{1}{\sigma_i}\right)} + \frac{\mathrm{e}^{-\frac{\lambda}{\sigma_k}s}}{\lambda^2\left(\frac{1}{\sigma_k} - \frac{1}{\sigma_i}\right)^2} - \frac{\mathrm{e}^{-\frac{\lambda}{\sigma_i}s}}{\lambda^2\left(\frac{1}{\sigma_k} - \frac{1}{\sigma_i}\right)^2} \right) \pi(\lambda)\mathrm{d}\lambda$$

$$= \sum_{\substack{i \neq k \\ 1 \leq i \leq n}} \frac{(-1)^n \left(\prod_{j=1}^n \frac{1}{\sigma_j}\right)}{\prod_{\substack{j \neq i \\ 1 \leq j \leq n}} \left(\frac{1}{\sigma_i} - \frac{1}{\sigma_j}\right)} \int_0^\infty \left(\lambda s\mathrm{e}^{-\frac{\lambda}{\sigma_k}s} + \frac{\mathrm{e}^{-\frac{\lambda}{\sigma_k}s}}{\left(\frac{1}{\sigma_k} - \frac{1}{\sigma_i}\right)} - \frac{\mathrm{e}^{-\frac{\lambda}{\sigma_i}s}}{\left(\frac{1}{\sigma_k} - \frac{1}{\sigma_i}\right)} \right) \lambda^{\alpha-1} \frac{e^{-\lambda}}{\Gamma(\alpha)}\mathrm{d}\lambda. \tag{B.2}$$

Now, we can split $\int_0^s x f_{X_k,S}(x,s)\mathrm{d}x = (I)+(II)+(III)$, with

$$
\begin{aligned}
(I) &= \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{(-1)^n\left(\prod_{j=1}^n \frac{1}{\sigma_j}\right)}{\prod_{\substack{j\neq i\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_i}-\frac{1}{\sigma_j}\right)} \int_0^\infty s\mathrm{e}^{-\frac{\lambda}{\sigma_k}s}\lambda^\alpha \frac{\mathrm{e}^{-\lambda}}{\Gamma(\alpha)}\mathrm{d}\lambda \\
&= \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{(-1)^n\left(\prod_{j=1}^n \frac{1}{\sigma_j}\right)}{\prod_{\substack{j\neq i\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_i}-\frac{1}{\sigma_j}\right)} \left(1+\frac{s}{\sigma_k}\right)^{-(\alpha+1)} s\ \alpha \underbrace{\int_0^\infty \frac{\lambda^\alpha e^{-(1+\frac{s}{\sigma_k})\lambda}\left(1+\frac{s}{\sigma_k}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\mathrm{d}\lambda}_{=1,\text{ since is the density of Gamma}(\alpha+1,1+\frac{s}{\sigma_k}).} \\
&= (-1)^n\left(\prod_{j=1}^n \frac{1}{\sigma_j}\right)\left(1+\frac{s}{\sigma_k}\right)^{-(\alpha+1)} s\ \alpha \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{1}{\prod_{\substack{j\neq i\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_i}-\frac{1}{\sigma_j}\right)} \\
&= (-1)^n\left(\prod_{j=1}^n \frac{1}{\sigma_j}\right)\left(1+\frac{s}{\sigma_k}\right)^{-(\alpha+1)} s\ \alpha \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{1}{\left(\prod_{j=1}^n \frac{1}{\sigma_j}\right)\frac{1}{\sigma_i^{n-2}}\prod_{\substack{j\neq i\\ 1\leq j\leq n}}(\sigma_j-\sigma_i)} \\
&= (-1)^n\left(1+\frac{s}{\sigma_k}\right)^{-(\alpha+1)} s\ \alpha \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{\sigma_i^{n-2}}{\prod_{\substack{j\neq i\\ 1\leq j\leq n}}(\sigma_j-\sigma_i)} \qquad \text{(B.3)} \\
(II) &= \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{(-1)^n\left(\prod_{j\neq k} \frac{1}{\sigma_j}\right)}{\prod_{\substack{j\neq i\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_i}-\frac{1}{\sigma_j}\right)} \int_0^\infty \frac{\mathrm{e}^{-\frac{\lambda}{\sigma_k}s}}{\left(\frac{1}{\sigma_k}-\frac{1}{\sigma_i}\right)}\lambda^{\alpha-1}\frac{\mathrm{e}^{-\lambda}}{\Gamma(\alpha)}\mathrm{d}\lambda \\
&= (-1)^n\left(\prod_{j=1}^n \frac{1}{\sigma_j}\right)\left(1+\frac{s}{\sigma_k}\right)^{-\alpha} \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{1}{\left(\frac{1}{\sigma_k}-\frac{1}{\sigma_i}\right)\prod_{\substack{j\neq i\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_i}-\frac{1}{\sigma_j}\right)} \\
&= (-1)^n\left(1+\frac{s}{\sigma_k}\right)^{-\alpha} \sigma_k \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{\sigma_i^{n-1}}{(\sigma_i-\sigma_k)\prod_{\substack{j\neq i\\ 1\leq j\leq n}}(\sigma_j-\sigma_i)} \qquad \text{(B.4)} \\
(III) &= -\sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{(-1)^n\left(\prod_{j=1}^n \frac{1}{\sigma_j}\right)}{\prod_{\substack{j\neq i\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_i}-\frac{1}{\sigma_j}\right)} \int_0^\infty \frac{\mathrm{e}^{-\frac{\lambda}{\sigma_i}s}}{\lambda\left(\frac{1}{\sigma_k}-\frac{1}{\sigma_i}\right)}\lambda^{\alpha-1}\frac{\mathrm{e}^{-\lambda}}{\Gamma(\alpha)}\mathrm{d}\lambda \\
&= (-1)^{n+1}\left(\prod_{j=1}^n \frac{1}{\sigma_j}\right) \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{\left(1+\frac{s}{\sigma_i}\right)^{-\alpha}}{\left(\frac{1}{\sigma_k}-\frac{1}{\sigma_i}\right)\prod_{\substack{j\neq i\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_i}-\frac{1}{\sigma_j}\right)} \\
&= (-1)^{n+1}\sigma_k \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{\sigma_i^{n-1}\left(1+\frac{s}{\sigma_i}\right)^{-\alpha}}{(\sigma_i-\sigma_k)\prod_{\substack{j\neq i\\ 1\leq j\leq n}}(\sigma_j-\sigma_i)}. \qquad \text{(B.5)}
\end{aligned}
$$

By properties (2.2) and (2.3) of the divided differences in Chiragiev and Landsman (2007),

$\sum_{i=1}^{n} \frac{1}{\prod_{j\neq i}\left(\frac{1}{\sigma_i}-\frac{1}{\sigma_j}\right)} = 0$ and therefore $\sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{1}{\prod_{\substack{j\neq i\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_i}-\frac{1}{\sigma_j}\right)} = \frac{-1}{\prod_{\substack{j\neq k\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_k}-\frac{1}{\sigma_j}\right)}$. Hence

$$\begin{aligned}(I) &= (-1)^{n+1}\left(\prod_{j=1}^{n}\frac{1}{\sigma_j}\right)\frac{\alpha}{\prod_{\substack{j\neq k\\ 1\leq j\leq n}}\left(\frac{1}{\sigma_k}-\frac{1}{\sigma_j}\right)}\left(1+\frac{s}{\sigma_k}\right)^{-(\alpha+1)} s \\ &= (-1)^{n+1}\frac{\alpha\ \sigma_k^{n-2}}{\prod_{\substack{j\neq k\\ 1\leq j\leq n}}(\sigma_j-\sigma_k)} s\left(1+\frac{s}{\sigma_k}\right)^{-(\alpha+1)}. \qquad \text{(B.6)}\end{aligned}$$

Therefore, we can write $\int_0^s x f_{X_k,S}(x,s)\mathrm{d}x$ as

$$(-1)^{n+1}\left(\frac{\alpha\ \sigma_k^{n-2}}{\prod_{\substack{j\neq k\\ 1\leq j\leq n}}(\sigma_j-\sigma_k)} s\left(1+\frac{s}{\sigma_k}\right)^{-(\alpha+1)} + \sigma_k \sum_{\substack{i\neq k\\ 1\leq i\leq n}} \frac{\sigma_i^{n-1}\left(\left(1+\frac{s}{\sigma_i}\right)^{-\alpha}-\left(1+\frac{s}{\sigma_k}\right)^{-\alpha}\right)}{(\sigma_i-\sigma_k)\prod_{\substack{j\neq i\\ 1\leq j\leq n}}(\sigma_j-\sigma_i)}\right)$$

and (4.1) follows. This ends the proof.