

Groupoids and skeletal categories form a pretorsion theory in $\mathbf{Cat}$

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Abstract

We describe a pretorsion theory in the category $\mathbf{Cat}$ of small categories: the torsion objects are the groupoids, while the torsion-free objects are the skeletal categories, *i.e.*, those categories in which every isomorphism is an automorphism. We infer these results from two unexpected properties of coequalizers in $\mathbf{Cat}$ that identify pairs of objects: they are faithful and reflect isomorphisms.

1 Introduction

A *pretorsion theory* in a category $\mathcal{C}$ consists of two classes of objects, respectively called the *torsion* and *torsion-free* objects, together with axioms which mirror those for a *torsion theory* in an abelian category. The objects which are both *torsion* and *torsion-free* are called *trivial*. When $\mathcal{C}$ has a zero object which is taken to be the only trivial object, we recover the classical notion. Hence, in direct generalization of its abelian origins, the current setting is applicable in an arbitrary category, even in the absence of a zero object.

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The original example of a torsion theory is, of course, that in the category of abelian groups, with the usual notions of torsion group and torsion-free group. Tens of papers have been devoted to torsion theories in various non-abelian contexts, including [9, 11, 31, 21, 19, 10, 30, 12, 22]. For a first easy example of a pretorsion theory, consider the category **PreOrd** of preordered sets. Choosing the equivalence relations as torsion objects and the partial orders as torsion-free objects, one obtains a pretorsion theory on **PreOrd** (see [14]) with the discrete objects as trivial ones. This fact has been generalized to the category **PreOrd**($\mathcal{C}$) of preordered objects in a Barr-exact category $\mathcal{C}$ (see [16, 5, 6]). Other examples of pretorsion theories have been studied in [15, 7, 17, 33, 20].

A preordered set may be seen as a small category with at most one arrow between any two objects. On the other hand, given a small category $\mathcal{C}$, one gets a preorder on its set of objects by declaring $A \leq B$ when there is at least one morphism from A to B . In [33], a small category is called “torsion” or “torsion free” when it is the case for the corresponding preordered set of its objects; it is proved that this yields a pretorsion theory on **Cat**.

In the present paper, we adopt a totally different approach, based on the structure of the category of arrows, not just on the existence of arrows. In our pretorsion theory on **Cat**, a small category $\mathcal{C}$ is

torsion when $\mathcal{C}$ is a groupoid, *i.e.*, when every morphism in $\mathcal{C}$ is an isomorphism;

torsion-free when $\mathcal{C}$ is a skeletal category, *i.e.*, when every isomorphism in $\mathcal{C}$ is an automorphism (see [25]).

Of course, when the category $\mathcal{C}$ is just a preordered set, we recapture the situation studied in [14]. This was also the case for the pretorsion theory studied in [33].

The results in the present paper rely heavily on a careful study of some particular coequalizers in **Cat**: the coequalizers of pairs of functors defined on a discrete category. Such coequalizers have properties that are atypical for a quotient functor, the most striking ones being their faithfulness and the reflection of isomorphisms. Establishing these properties is quite cumbersome and reduces to so-called “word problems” on formal chains of arrows in the quotient graph used to construct the coequalizer. The work of John Isbell on *Dominions* (1968, see [23]) and the *Diplomarbeit* of Reinhard Börger (1977, see [8]) provided us with the necessary tools for handling these problems.

We conclude the paper with showing the existence of the so-called $\mathcal{Z}$ -kernels and $\mathcal{Z}$ -cokernels in $\mathbf{Cat}$, that is, kernels and cokernels defined relatively to the ideal of trivial morphisms: those morphisms factoring through a trivial object. Thanks to the results in [7] it then follows that there is a *stable category* naturally associated with this pretorsion theory, that satisfies an interesting universal property.

2 Pretorsion theories

Let us recall that an ideal $\mathcal{Z}$ in a category $\mathcal{C}$ is a class of arrows such that for every arrow $f \in \mathcal{Z}$, one has $fu \in \mathcal{Z}$ and $vf \in \mathcal{Z}$, for all arrows u, v composable with f [13]. When $\mathcal{C}$ has a zero object, the zero morphisms constitute an ideal.

Given an ideal $\mathcal{Z}$ in a category $\mathcal{C}$, an arrow k is the $\mathcal{Z}$ -kernel of an arrow f when $fk \in \mathcal{Z}$ and, if $fm \in \mathcal{Z}$ for some arrow m , then m factors uniquely through k . The uniqueness condition forces k to be a monomorphism. When $\mathcal{Z}$ is the ideal of zero morphisms, we recapture the usual notion of kernel. There is of course a dual notion of $\mathcal{Z}$ -cokernel. A pair of composable morphisms

$$K \xrightarrow{k} A \xrightarrow{q} Q$$

is a short $\mathcal{Z}$ -exact sequence when k is the $\mathcal{Z}$ -kernel of q and q is the $\mathcal{Z}$ -cokernel of k .

The following definition was introduced in [14] and then thoroughly investigated in [15]:

Definition 2.1 *A pretorsion theory in a category $\mathcal{C}$ consists of a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects, both of them closed under isomorphisms, whose elements are the torsion and the torsion-free objects of the pretorsion theory, respectively. The objects in $\mathcal{T} \cap \mathcal{F}$ are called trivial, and the ideal $\mathcal{Z}$ of trivial morphisms is that of those arrows factoring through a trivial object.*

These data must satisfy the following two axioms:

PT1 *every arrow $f: A \rightarrow B$ with $A \in \mathcal{T}$ and $B \in \mathcal{F}$ is trivial;*

PT2 *for every object $A \in \mathcal{C}$, there exists a short $\mathcal{Z}$ -exact sequence*

$$K \xrightarrow{k} A \xrightarrow{q} Q$$

with $K \in \mathcal{T}$ and $Q \in \mathcal{F}$.

3 Some coequalizers in Cat revisited

Let us first exhibit a result borrowed from J. Isbell (see [23]) and R. Börger (see [8]).

Definition 3.1 (Def. 4.3 in [8]) *Let $\mathcal{B}$ be a small category. Let M be the free monoid on the set of arrows of $\mathcal{B}$. An element of M – that is, an arbitrary finite sequence of arrows in $\mathcal{B}$ – is reduced when*

- *the sequence does not contain any pair of consecutive arrows which turn out to be composable in $\mathcal{B}$;*
- *the sequence does not contain any identity morphism of $\mathcal{B}$.*

Let us clarify that, when speaking of a *pair of consecutive composable arrows* in a sequence, we always refer to a sequence $(f_1, \dots, f_n)$ and an index $i < n$ such that the codomain of f_i is equal to the domain of f_{i+1} . To avoid any confusion with the (standard) direction in which we write arrows, for clarity we shall sometimes write the composite $f_{i+1}f_i$ in the reverse order, as $f_i \triangleright f_{i+1}$, so that the shorter sequence in which the pair (f_i, f_{i+1}) is replaced by the composite of the two arrows, may then be written as

$$(f_1, \dots, f_{i-1}, f_i \triangleright f_{i+1}, f_{i+2}, \dots, f_n).$$

The unit element of M is of course the empty sequence, which we denote by $()$. By a congruence on the monoid M is meant an equivalence relation on M such that $x \sim y$ implies $xz \sim yz$ and $zx \sim zy$ for all z .

Proposition 3.2 (Statement 1.1 in [23], Satz 4.4 in [8])

Under the conditions of Definition 3.1, consider the smallest congruence S on M such that

- $((u, v), u \triangleright v) \in S$ *for every pair (u, v) of composable morphisms in $\mathcal{B}$;*
- $(\text{id}_B, ()) \in S$ *for every object $B \in \mathcal{B}$.*

Then every element of M is S -equivalent to a unique reduced element.

Proof In 1968, after referring to the work of Mersch (see [27]), Isbell states as item 1.1 of his paper [23] a result which is essentially our Proposition 3.2; however, the proof is only sketched: it makes explicit the representation in

terms of reduced chains, leaving off any details of a proof. Isbell simply says that these are analogous to those for free groups (see [1] and [32]).

In *Satz 4.4* of his 1977 *Diplomarbeit* [8], Börger presents a long and carefully written proof, with all technical details. Since this work has never been published, we took the liberty of making a scan of it available, together with a detailed sketch of his proof: see the appendix to this paper. $\square$

Let us now switch to coequalizers in $\mathbf{Cat}$. Their construction has been described by many authors, with varying levels of detail: see for example Section 5.1 in [4], Section 4 in [3], Section 11 in [29], and so on. All of them use the notion of *congruence* on a category $\mathcal{G}$, that is: an equivalence relation on each hom-set $\mathcal{G}(X, Y)$ such that $f \sim g$ implies $fh \sim gh$ and $kf \sim kg$ for all arrows h, k composable with f .

Given a pair (F, G) of functors between small categories

$$\mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \xrightarrow{G} \end{array} \mathcal{B} \xrightarrow{Q} \mathcal{Q}$$

their coequalizer $\mathcal{Q}$ can thus be obtained in the following way.

- As far as objects are concerned, $\mathbf{Ob}(\mathcal{Q})$ is the quotient of $\mathbf{Ob}(\mathcal{B})$ by the equivalence relation generated by $F(A) \sim G(A)$, for every object $A \in \mathcal{A}$.
- First one constructs a graph $\mathcal{G}_0$ on this quotient set of objects, by putting every arrow $f: A \rightarrow B$ in $\mathcal{B}$ as an arrow from $[A]$ to $[B]$ in $\mathcal{G}_0$.
- Next one considers the category $\mathcal{G}$ having the same objects as $\mathcal{G}_0$; its arrows are the non-empty finite chains of “composable” morphisms of $\mathcal{G}_0$ (in the sense that the $\mathcal{G}_0$ -codomain of a $\mathcal{G}_0$ -morphism is the $\mathcal{G}_0$ -domain of the next one), together with an empty chain from each object of $\mathcal{G}_0$ to itself; the composition is just concatenation.
- The coequalizer $\mathcal{Q}$ is the quotient of $\mathcal{G}$ by the congruence generated by:
 - Q1** when two consecutive morphisms in a $\mathcal{G}_0$ -chain are composable in $\mathcal{B}$, the chain is equivalent to the one obtained when replacing the corresponding pair by its composite in $\mathcal{B}$;
 - Q2** when an identity morphism appears in a $\mathcal{G}_0$ -chain, the chain is equivalent to the one obtained when dropping that identity;

Q3 when a morphism of the form $F(f)$ appears in a $\mathcal{G}_0$ -chain, the chain is equivalent to the one obtained when replacing $F(f)$ by $G(f)$.

The following result is then an immediate consequence of the Isbell-Börger result presented above.

Proposition 3.3 *With the notation above, let us assume that the category $\mathcal{A}$ is discrete. Given an arrow in the coequalizer $\mathcal{Q}$, there then exists a unique reduced arrow in $\mathcal{G}$ representing it.*

Proof A non-identity arrow in $\mathcal{Q}$ is represented by a triple (A, ξ, B) where, with the notation of Definition 3.1, $\xi \in M$ is a sequence of consecutive arrows in $\mathcal{G}_0$, the first one having A as domain in $\mathcal{B}$, and the last one having B as codomain in $\mathcal{B}$. An identity arrow in $\mathcal{Q}$ is represented by a triple $(B, (), B)$, where $()$ is the empty sequence and B is an object of $\mathcal{B}$.

Since $\mathcal{A}$ is discrete, condition Q2 above indicates at once that condition Q3 can be omitted, because the only possible values for f in Q3 are identities. Thus the congruence defining $\mathcal{Q}$ from $\mathcal{G}$ is constructed using only conditions Q1 and Q2, just as the congruence on M in Proposition 3.2. Therefore, the result follows at once from Proposition 3.2. $\square$

Let us now infer various interesting consequences from this last proposition.

Proposition 3.4 *The coequalizer of two functors defined on a discrete category is faithful.*

Proof With the notation above, let $f, g: A \rightrightarrows B$ be two non-identity morphisms in $\mathcal{B}$ such that $Q(f) = Q(g)$. Then (f) and (g) are reduced morphisms in $\mathcal{G}$ representing the same arrow of $\mathcal{Q}$, thus they are equal by Proposition 3.3.

Next, if $f = \text{id}_B$ and $g: B \rightarrow B$, with $Q(f) = Q(g)$, g must be an identity, otherwise we would again have two reduced morphisms $()$ and (g) of $\mathcal{G}$ representing the same morphism of $\mathcal{Q}$. $\square$

Lemma 3.5 *Consider the coequalizer $\mathcal{Q}$ of two functors $F, G: \mathcal{A} \rightrightarrows \mathcal{B}$ defined on a discrete category $\mathcal{A}$. With the notation above, a morphism in $\mathcal{Q}$ is an isomorphism if, and only if, its reduced form in $\mathcal{G}$ is empty or composed of isomorphisms in $\mathcal{B}$.*

Proof By 3.3, consider the reduced form of a non-identity isomorphism

$$(f_1, \dots, f_n), \quad f_i: A_i \longrightarrow B_i, \quad Q(B_i) = Q(A_{i+1}), \quad B_i \neq A_{i+1},$$

in $\mathcal{Q}$ and the reduced form of its inverse

$$(g_1, \dots, g_m), \quad g_i: C_i \longrightarrow D_i, \quad Q(D_i) = Q(C_{i+1}), \quad D_i \neq C_{i+1};$$

thus, in particular, $Q(A_1) = Q(C_m)$ and $Q(C_1) = Q(B_n)$. The sequence

$$(f_1, \dots, f_n, g_1, \dots, g_m)$$

must therefore be equivalent to the empty sequence. If $B_n \neq C_1$, this last sequence is reduced and equivalent to the empty sequence, which is impossible by the uniqueness condition in Proposition 3.3. Thus $B_n = C_1$ and we can shorten the sequence to

$$(f_1, \dots, f_{n-1}, f_n \triangleright g_1, g_2, \dots, g_m).$$

But the domain of $f_n \triangleright g_1$ is $A_n \neq B_{n-1}$, and the codomain of $f_n \triangleright g_1$ is $D_1 \neq C_2$. So, if $f_n \triangleright g_1$ is not an identity, the shortened sequence is reduced, which contradicts again the fact that it is equivalent to the empty sequence. Thus $f_n \triangleright g_1$ is an identity in $\mathcal{A}$ and, in particular, $A_n = D_1$. Next, looking at the other composite

$$(g_1, \dots, g_m, f_1, \dots, f_n)$$

which must also be an identity morphism in $\mathcal{Q}$, we conclude that g_1 is a monomorphism in $\mathcal{Q}$ and f_n is an epimorphism in $\mathcal{Q}$. But by Proposition 3.4 the functor Q is faithful, thus in particular it reflects monomorphisms and epimorphisms. Therefore g_1 is a monomorphism in $\mathcal{B}$ and f_n is an epimorphism in $\mathcal{B}$. And since $f_n \triangleright g_1$ is an identity in $\mathcal{B}$, g_1 is both a retraction and a monomorphism while f_n is both a section and an epimorphism; they are thus inverse isomorphisms.

So, one can further shorten the situation and obtain the sequence

$$(f_1, \dots, f_{n-1}, g_2, \dots, g_m).$$

One repeats inductively the same process as above, up to the moment when we have used all the components of one of the two original reduced sequences. In this way we end up, let us say, with

$$(f_1, \dots, f_k)$$

(the case of any remaining g_i s is, of course, analogous). But if not empty, this sequence is reduced, since the sequence $(f_1, \dots, f_n)$ was reduced, and it must therefore be equivalent to the empty sequence. This is impossible, again by the uniqueness condition in Proposition 3.3. Thus, we in fact ended up with an empty sequence.

This concludes the proof of one of the two stated implications; the other one is obvious. $\square$

Proposition 3.6 *The coequalizer of two functors defined on a discrete category reflects isomorphisms.*

Proof With the same notation as above, let f be a morphism in $\mathcal{B}$ such that $Q(f)$ is an isomorphism. If f is an identity, there is nothing to prove. Otherwise, the morphism (f) of $\mathcal{G}$ is in reduced form and, by assumption, an isomorphism in $\mathcal{Q}$. By Lemma 3.5, f is an isomorphism in $\mathcal{B}$. $\square$

4 The pretorsion theory in Cat

The following result extends, to small categories, the pretorsion theory studied in [14] in the case of preordered sets, *i.e.*, of those categories having at most one arrow between any two objects.

Theorem 4.1 *One gets a pretorsion theory in the category Cat of small categories when choosing*

- *the groupoids as torsion objects;*
- *the skeletal categories as torsion-free objects.*

Proof The trivial objects are thus those categories in which all arrows are automorphisms. We shall write $\mathbf{Iso}(\mathcal{C})$ for the groupoid of isomorphisms of a small category $\mathcal{C}$ and $\mathbf{Aut}(\mathcal{C})$ for the groupoid of its automorphisms. $\mathbf{Aut}(\mathcal{C})$ is thus a trivial category, in the sense of Definition 2.1. So, a functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ is trivial when it factors through $\mathbf{Aut}(\mathcal{B})$. In particular every functor $F: \mathcal{G} \longrightarrow \mathcal{C}$ from a groupoid $\mathcal{G}$ to a torsion-free category $\mathcal{C}$ is trivial. This takes care of axiom PT1.

Consider now an arbitrary small category $\mathcal{C}$; it contains the groupoid $\text{Iso}(\mathcal{C})$. We form the following coequalizer in $\mathbf{Cat}$:

$$\coprod_{\sigma \text{ iso}} \mathbf{1} \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} \mathcal{C} \xrightarrow{Q} \mathcal{Q}$$

where the left-hand category is a copower of the terminal category $\mathbf{1}$, indexed by the set of all isomorphisms of $\mathcal{C}$. This is of course a discrete category. The functors d_0 and d_1 are those which, on the component indexed by an isomorphism σ , map the unique object of $\mathbf{1}$ respectively to the domain and codomain of σ . On one hand, all the isomorphisms of $\mathcal{C}$ are thus mapped by Q to automorphisms in $\mathcal{Q}$. On the other hand, by Lemma 3.5, an isomorphism in $\mathcal{Q}$ is an identity or a composite of images of isomorphisms in $\mathcal{C}$. It is therefore a composite of automorphisms in $\mathcal{Q}$ and, thus, an automorphism in $\mathcal{Q}$. Consequently, $\mathcal{Q}$ is skeletal.

It remains to be proved that we have obtained a short $\mathcal{Z}$ -exact sequence (i, Q) :

$$\begin{array}{ccccc} \mathcal{X} & & & & \\ \vdots \downarrow & \searrow G & & & \\ \text{Iso}(\mathcal{C}) & \xrightarrow{i} & \mathcal{C} & \xrightarrow{Q} & \mathcal{Q} \\ & & \searrow H & & \vdots \downarrow \\ & & & & \mathcal{Y} \end{array}$$

Consider first a functor $G: \mathcal{X} \rightarrow \mathcal{C}$ such that QG is trivial. This means that, for every arrow $x \in \mathcal{X}$, $G(x)$ is mapped by Q to an automorphism and, thus, an isomorphism. But by Proposition 3.6, every $G(x)$ is then an isomorphism in $\mathcal{C}$ and, thus, G factors through $\text{Iso}(\mathcal{C})$. That factorization is unique since i is an inclusion functor. On the other hand, if $H: \mathcal{C} \rightarrow \mathcal{Y}$ is such that Hi is trivial, then every isomorphism of $\mathcal{C}$ is mapped by H to an automorphism, thus its domain and its codomain are identified by H . This proves that $Hd_0 = Hd_1$, and we get the expected unique factorization through the coequalizer Q of (d_0, d_1) . $\square$

To the best of our knowledge, the following consequence of Theorem 4.1 has not yet been stated in the literature.

Corollary 4.2 *The full subcategory of skeletal categories is reflexive in the category of small categories.*

Proof In a category $\mathcal{C}$ provided with a pretorsion theory, the full subcategory of torsion objects is coreflexive and that of torsion free objects is reflexive (see [15]). $\square$

Remark 4.3 *Comparison with the notion of skeleton.*

Applying a strong (potentially class-based) version of the axiom of choice one easily sees that every category $\mathcal{C}$ is equivalent to a skeletal full subcategory $\mathcal{S}$ of $\mathcal{C}$ (see [28]), called its *skeleton* (see [25]). It is important to note that, in the notation of the proof of Theorem 4.1, such a skeleton $\mathcal{S}$ (uniquely determined only up to equivalence) is by no means the $\mathcal{Z}$ -cokernel $\mathcal{Q}$ and, thus, the skeletal reflection of $\mathcal{C}$, as given (without the use of any choice principle) by Corollary 4.2. For example, consider the category $\mathcal{C}$ with two objects, the identities on these, and a unique isomorphism (f, f^{-1}) between the two objects. Both candidates for a skeleton of $\mathcal{C}$ are isomorphic to the terminal category $\mathbf{1}$. However, in the $\mathcal{Z}$ -cokernel $\mathcal{Q}$ as in Theorem 4.1 with its single object and its identity morphism, there are also all the powers f^n , $(f^{-1})^n$. It is thus the monoid $(\mathbb{Z}, +)$, viewed as a one-object category. $\square$

Remark 4.4 *About a possible “internalization” of Theorem 4.1.*

The construction of the pretorsion theory on preordered sets (see [14]) may easily be carried over to the case of preordered objects in a Barr-exact category (see [16]), because it refers only to finite limits and coequalizers of kernel pairs (see Barr’s metatheorem [2]). The case of the pretorsion theory in Theorem 4.1 is strikingly different, because the arguments that we have developed – and in particular Börger’s result in the Appendix – are highly set theoretical and cannot be carried out as such in a quite arbitrary category.

In most classical algebraic categories (such as the categories of groups, Lie algebras, rings, modules, etc.) and, more generally, in every regular n -permutable category for $n \geq 2$ (see [26]), in particular in any regular

Mal'tsev category, the internal categories are always groupoids. Thus internalizing Theorem 4.1 to the context of n -permutable categories, is equivalent to proving that the category of internal skeletal groupoids – *i.e.* those whose domain and codomain morphisms are equal – is reflexive in the category of internal groupoids. This is indeed the case whenever the base category is regular with coequalizers of equivalence relation, as will be proved in a subsequent paper. $\square$

5 The existence of $\mathcal{Z}$ -kernels and $\mathcal{Z}$ -cokernels

Let us now prove the existence of all $\mathcal{Z}$ -kernels and $\mathcal{Z}$ -cokernels in $\mathbf{Cat}$, with respect to the ideal $\mathcal{Z}$ of trivial morphisms as in Theorem 4.1.

Proposition 5.1 *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor in $\mathbf{Cat}$. Its $\mathcal{Z}$ -kernel is given by the functor k in the following pullback*

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathbf{Aut}(\mathcal{B}) \\ \downarrow k & & \downarrow \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

Proof Given $G: \mathcal{X} \longrightarrow \mathcal{A}$, the functor FG is trivial precisely when it factors through $\mathbf{Aut}(\mathcal{B})$, thus through the pullback. $\square$

The case of $\mathcal{Z}$ -cokernels is more involved. Inspired by considerations in [21], we prove first:

Lemma 5.2 *The $\mathcal{Z}$ -cokernel of the identity functor on a small category $\mathcal{A}$ exists and can be constructed in the following way:*

- consider first the category of fractions inverting all the arrows of $\mathcal{A}$

$$p: \mathcal{A} \longrightarrow \mathcal{G} = \mathcal{A}[\mathbf{Ar}(\mathcal{A})^{-1}];$$

- consider next the canonical short $\mathcal{Z}$ -exact sequence of $\mathcal{G}$:

$$K \xrightarrow{k} G \xrightarrow{q} Q.$$

Then k is isomorphic to the identity on $\mathcal{G}$ while the composite

$$\mathcal{A} \xrightarrow{p} \mathcal{G} \xrightarrow{q} \mathcal{Q}$$

is the $\mathcal{Z}$ -cokernel of F .

Proof The construction of the $\mathcal{Z}$ -exact sequence in Theorem 4.1 shows at once that the $\mathcal{Z}$ -kernel part of the sequence is the identity on $\mathcal{G}$. But by construction of $\mathcal{Q}$, all the (iso)morphisms of $\mathcal{G}$ are mapped to automorphisms, so that qp must be trivial.

Next, given a functor $G: \mathcal{A} \rightarrow \mathcal{X}$ such that $G = G \circ \text{id}_{\mathcal{A}}$ is trivial, G inverts all the arrows of $\mathcal{A}$ and thus factors uniquely as a functor H through the groupoid $\mathcal{G}$ of fractions.

$$\begin{array}{ccccccc} \mathcal{A} & \xlongequal{\quad} & \mathcal{A} & \xrightarrow{p} & \mathcal{G} & \xrightarrow{q} & \mathcal{Q} \\ & & & \searrow G & \searrow H & & \downarrow L \\ & & & & & & \mathcal{X} \end{array}$$

But $H = H \circ \text{id}_{\mathcal{G}}$ is trivial because G and thus H map all morphisms of $\mathcal{A}$, and thus also their inverses in $\mathcal{G}$, to automorphisms in $\mathcal{X}$. Therefore H factors uniquely as a functor L through the $\mathcal{Z}$ -cokernel q of $\text{id}_{\mathcal{G}}$. The uniqueness is obvious since both p and q are epimorphisms. $\square$

Proposition 5.3 *The $\mathcal{Z}$ -cokernel of an arbitrary functor $F: \mathcal{A} \rightarrow \mathcal{B}$ in $\mathbf{Cat}$ exists and is given by the pushout of the $\mathcal{Z}$ -cokernel of $\text{id}_{\mathcal{A}}$ along F .*

Proof This can easily be verified directly. With the notation above consider the pushout

$$\begin{array}{ccccc} \mathcal{A} & \xlongequal{\quad} & \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ & & \downarrow qp & & \downarrow U \\ & & \mathcal{Q} & \xrightarrow{V} & \mathcal{C} \end{array}$$

where U is an epimorphism, since so is qp ; this takes already care of the uniqueness condition. If a functor $W: \mathcal{B} \longrightarrow \mathcal{Y}$ is such that $WF = WFid_{\mathcal{A}}$ is trivial, then WF factors uniquely through the $\mathcal{Z}$ -cokernel qp of $id_{\mathcal{A}}$ and thus further through the pushout.

This easy argument is a special instance of a known result. From what we have seen above, it follows that **Cat** is a multipointed category with respect to the full subcategory of trivial categories, in the sense of Grandis and Janelidze (see [21]). The result follows then from their Proposition 1.3. $\square$

Remark 5.4 *It is possible to construct the “universal coproduct-preserving stable category” associated with the pretorsion theory described in Theorem 4.1. Indeed, this fact has been observed in Section 6 of [7].*

6 Appendix: Sketch of the proof of 3.2

This appendix presents a sketch of the proof of Proposition 3.2, which appears as Statement 1.1 in [23] and whose full proof can be found under Satz 4.4 in [8].

Proposition 6.1 *Let $\mathcal{B}$ be a small category and M the free monoid on the set $\text{Ar}(\mathcal{B})$ of arrows of $\mathcal{B}$. An element of M , that is a finite sequence of elements of $\text{Ar}(\mathcal{B})$, is reduced when it does not contain any identity morphism of $\mathcal{B}$, nor any consecutive pair of arrows which are composable in $\mathcal{B}$.*

Consider the smallest congruence on M identifying a sequence containing an identity with the sequence obtained when dropping that identity, and a sequence containing a pair of consecutive composable morphisms in $\mathcal{B}$ with the sequence where this pair is replaced by the corresponding composite.

Then every element of M is equivalent to a unique reduced element.

Proof As a matter of convention, let us use Latin letters for the arrows of $\mathcal{B}$ and Greek letters for arbitrary elements of M . The existence of a reduced element obtained from $\alpha \in M$, in the way indicated above, is obvious. The point is to prove the uniqueness.

An element of M is a finite sequence α of elements of $\text{Ar}(\mathcal{B})$; we shall often consider its length $L(\alpha) \in \mathbb{N}$. We shall also write $R \subseteq M$ for the set of reduced elements of M .

The first step of the proof is to observe, by induction on the length of β , that the following formulae define inductively a mapping

$$\Phi: \text{Ar}(\mathcal{B}) \times R \longrightarrow R,$$

where $\Phi(a, \beta)$ is such that

1. $\Phi(a, \beta) = \Phi(ab, \nu)$ when $\beta = b\nu$, with $\nu \in R$, while $b \in \text{Ar}(\mathcal{B})$ is composable with a in $\mathcal{B}$;
2. $\Phi(a, \beta) = \beta$ when a is an identity arrow in $\mathcal{B}$ and Case 1 does not apply;
3. $\Phi(a, \beta) = a\beta$ otherwise.

The second step of the proof is, again by induction on the length of ν , to define a binary operation

$$\star: M \times R \longrightarrow R$$

such that, when writing $()$ for the empty sequence (the unit of M), one has

$$() \star \nu = \nu, \quad a\beta \star \nu = \Phi(a, \beta \star \nu).$$

One considers then the equivalence relation S on M defined by

$$S = \{(\alpha, \beta) \mid \forall \nu \in R \quad \alpha \star \nu = \beta \star \nu\}$$

and proves, by induction on the length of α , that

$$\alpha \star (\beta \star \nu) = \alpha\beta \star \nu, \quad \alpha, \beta \in M, \quad \nu \in R.$$

This equality, together with the fact that $\gamma \star \nu \in R$ holds for every $\gamma \in R$, proves that S is a congruence on M . The equality implies also, via a new induction on the length of ν as appearing in the definition of S , that

$$((a, b), ab) \in S, \quad (\text{id}_B, ()) \in S, \quad \text{for all } a, b \text{ composable in } \mathcal{B}, \quad B \in \mathcal{B}.$$

By yet another induction on the length of ν , one observes that $\nu \star () = \nu$ holds for every $\nu \in R$. Then, given $\alpha \in M$ and $\nu, \eta \in R$, by definition of S , we obtain the implication

$$(\alpha, \nu) \in S \text{ and } (\alpha, \eta) \in S \implies \nu = \nu \star () = \eta \star () = \eta.$$

This proves the uniqueness condition in the statement. $\square$

References

- [1] **E. Artin**, *The Free product of groups*, Amer. J. of Math. **99** (1947), 1–4
- [2] **M. Barr**, *Exact categories*, Springer Lecture Notes in Math. **236** (1971), 1–120
- [3] **M. Bednarczyk, A. Borzyszkowski, W. Pawłowski**, *Generalized congruences, epimorphisms in Cat*, Theory and Applications of Categories 5:266-280 (1999)
- [4] **F. Borceux**, *Handbook of Categorical Algebra I: Basic Category Theory*, Cambridge University Press (1994)
- [5] **F. Borceux, F. Campanini, M. Gran**, *The stable category of internal preorders in a pretopos I: general theory*, J. Pure Applied Algebra **226** (2022) 106997
- [6] **F. Borceux, F. Campanini, M. Gran**, *The stable category of internal preorders in a pretopos II: the universal property*, Annali Mat. Pura Appl. (2022) <https://doi.org/10.1007/s10231-022-01222-w>
- [7] **F. Borceux, F. Campanini, M. Gran**, *Pretorsion theories in extensive categories*, preprint (2022), <https://arxiv.org/abs/2205.11054>
- [8] **R. Börger**, *Kongruenzrelationen auf Kategorien*, Master's thesis (*Diplomarbeit*), Münster (1977), available at <https://tholen.mathstats.yorku.ca/files/2022/05/Bo%CC%88rgerThesis2.pdf?x32402>
- [9] **D. Bourn and M. Gran**, *Torsion theories in homological categories*, J. Algebra, 305, 2006, 18-47
- [10] **G.A. Lopez Cafaggi**, *Torsion Theories in Simplicial Groups: preradicals and homology*, Thesis, Université catholique de Louvain (2022)
- [11] **M.M. Clementino, D. Dikranjan, W. Tholen**, *Torsion theories and radicals in normal categories*, J. Algebra, 305 (2006) 98–129.
- [12] **M. Duckerts-Antoine**, *Fundamental group functors in descent-exact homological categories*, Adv. Math., 310 (2017), 64-120.

- [13] **C. Ehresmann**, *Sur une notion générale de cohomologie*, C. R. Acad. Sci. Paris **259** (1964) 2050–2053.
- [14] **A. Facchini, C.A. Finocchiaro**, *Pretorsion theories, stable category and preordered sets*, Ann. Mat. Pura. Appl. (4)199 (3) (2020) 1073–1089.
- [15] **A. Facchini, C.A. Finocchiaro and M. Gran**, *Pretorsion theories in general categories*, J. Pure Appl. Algebra **225** (2) (2021) 106503
- [16] **A. Facchini, C.A. Finocchiaro and M. Gran**, *A new Galois structure in the category of internal preorders*, Theory Appl. Categories **35** No. 11 (2020) 326–349
- [17] **A. Facchini, L. Heidari Zadeh**, *An extension of properties of symmetric group to monoids and a pretorsion theory on a category of mappings*, J. Algebra Appl. 18(12) (2019) 1950234
- [18] **P. Gabriel, M. Zisman**, *Calculus of fractions and homotopy theory*, Springer 1967
- [19] **M. Gran, G. Kadjo and J. Vecruijsse**, *A torsion theory in the category of cocommutative Hopf algebras*, App. Categ. Structures Vol. 24 (3) (2016) 269–282
- [20] **M. Gran and A. Michel**, *Torsion theories and coverings of preordered groups*, Algebra Universalis 82 (2021) Paper No. 22
- [21] **M. Grandis, G. Janelidze**, *From torsion theories to closure operators and factorization systems*, Categories and General Algebraic Structures with Applications **12** (1) (2019) pp. 89–121.
- [22] **R. Henrard, S. Kvamme, A.-C. van Roosmalen**, *Auslander’s formula and correspondence for exact categories*, Adv. Math. 401 (2022) 108296
- [23] **J. Isbell**, *Epimorphisms and Dominions III*, Amer. J. of Math. **90** (1968), 1025–1030
- [24] **G. Janelidze and W. Tholen**, *Characterization of torsion theories in general categories*, in: Categories in Algebra, Geometry and Mathematical Physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, 249–256

- [25] **S. Mac Lane**, *Categories for the working mathematician*, Springer (1971)
- [26] **N. Martins-Ferreira, D. Rodelo and T. Van der Linden**, *An observation on n -permutability*, Bull. Belgian Math. Soc. **21** (2014), 223–230
- [27] **J. Mersch**, *Le problème du quotient dans les catégories*, Mémoire Soc. Roy. Sc. Liège (1965)
- [28] **B. Pareigis**, *Categories and functors*, Academic Press (1970)
- [29] **G. Peschke and W. Tholen**, *Diagram, fibrations and the decomposition of colimits*, <https://arxiv.org/pdf/2006.10890> (2020)
- [30] **G. Peschke and T. Van der Linden**, *The Yoneda isomorphism commutes with homology*, J. Pure Appl. Algebra **220** (2) (2016) 495–517
- [31] **J. Rosický and W. Tholen**, *Factorization, fibration and torsion*, J. Homotopy Relat. Struct. Vol. 2 (2) (2007) 295–314
- [32] **B.L. van der Waerden**, *Free products of groups*, Amer. J. of Math. **70** (1948), 527–528
- [33] **J. Xarez**, *A pretorsion theory for the category of small categories*, Cahiers Top. Géom. Diff. Catég. **LXIII** (1) (2021) 25–34

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