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Optimization flows landing on the Stiefel manifold^{*}

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Abstract: We study a continuous-time system that solves the optimization problem over the set of orthogonal matrices, which is also known as the Stiefel manifold. The resulting optimization flow follows a path that is not always on the manifold but eventually lands on the manifold. We introduce a generalized Stiefel manifold to which we extend the canonical metric of the Stiefel manifold. We show that the vector field of the proposed flow can be interpreted as the sum of a Riemannian gradient on a generalized Stiefel manifold and a normal vector. Moreover, we prove that the proposed flow globally converges to the set of critical points, and any local minimum and isolated critical point is asymptotically stable.

Keywords: Stiefel manifold; Landing flow; Canonical metric; Riemannian gradient; Asymptotic stability

AMS subject classifications: 37N40; 90C48

1. INTRODUCTION

Consider the optimization problem

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times p}} \quad & f(X) \\ \text{s. t.} \quad & X^\top X = I_p, \end{aligned} \quad (1)$$

where $p \leq n$, I_p denotes the $p \times p$ identity matrix, the objective function $f : \mathbb{R}^{n \times p} \rightarrow \mathbb{R}$ is continuously differentiable, and the orthogonality constraints define the *Stiefel manifold*, that is

$$\text{St}(p, n) := \{X \in \mathbb{R}^{n \times p} : X^\top X = I_p\}.$$

Optimization over orthogonal matrices as posed in (1) appears in many practical applications, such as, orthogonal procrustes problem (Eldén and Park, 1999), blind source separation (Joho and Mathis, 2002), linear eigenvalue problem (Golub and Van Loan, 2013), principal component analysis (Grubišić and Pietersz, 2007) and its sparse variant (Chen et al., 2020), electronic structure calculations (Gao et al., 2020); see Edelman et al. (1998) and Wen and Yin (2013) for a more complete list of applications. It is an instance of minimization over Riemannian manifolds for which many standard Euclidean algorithms have been extended (Absil et al., 2008; Hu et al., 2020; Boumal, 2022). These methods are feasible, in that they

follow a sequence of iterates that preserve the manifold constraint.

Continuous-time systems have been used for solving matrix and optimization problems; see, e.g., Brockett (1991), Chu (1994), Mahony et al. (1996), Absil (2006), and Bournez and Pouly (2021). Recently, Ablin and Peyré (2021) proposed a continuous-time flow called the landing flow, along with its discretization called the landing algorithm, that solves problem (1) in the special case of the orthogonal manifold (that is, when $n = p$). The main advantage of the landing algorithm is that the individual iterates do not need to satisfy the manifold constraint, and therefore it alleviates the need to compute retractions that, depending on the objective function f , may be the computational bottleneck in optimization on the Stiefel manifold. It is similar to an approach proposed in Gao et al. (2019) for solving (1), which utilizes an augmented Lagrangian update that allows for a parallel implementation.

In this paper, we extend the landing flow to solve the optimization problem (1) over the Stiefel manifold; see Fig. 1 for an illustration. By considering a generalization of the Stiefel manifold and constructing a specific Riemannian metric, we give the landing flow a geometric interpretation in the sense of the Riemannian gradient and the normal vector. In addition, we prove the global convergence to the set of critical points for the landing flow and we study the stability of the equilibria.

This paper is organized as follows. After introducing the notation in §2, we propose in §3 the landing flow on the the Stiefel manifold. In §4, a geometric interpretation of the landing flow is given. The convergence of the landing

^{*} This work was supported by the Fonds de la Recherche Scientifique – FNRS and the Fonds Wetenschappelijk Onderzoek – Vlaanderen under EOS Project no. 30468160. Bin Gao was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) via the collaborative research centre 1450–431460824, In-Sight, University of Münster, and via Germany’s Excellence Strategy EXC 2044–390685587, Mathematics Münster: Dynamics–Geometry–Structure. Simon Vary is a beneficiary of the FSR Incoming Post-doctoral Fellowship.

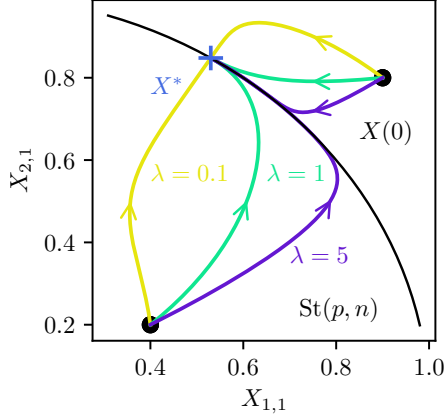


Fig. 1. Landing flows on $\text{St}(1, 2)$ to minimize a linear function with different parameters λ and two initial points.

flow and the stability of the equilibria are analyzed in §5. Finally, the conclusion is drawn in §6.

2. NOTATION

$\mathbb{R}_*^{n \times p}$ denotes the set of $n \times p$ matrices of full column rank. Given $X \in \mathbb{R}_*^{n \times p}$, we let X_\perp denote an $n \times (n - p)$ matrix such that $X^\top X_\perp = 0$ and $X_\perp^\top X_\perp = I_{n-p}$. The Frobenius inner product of two matrices $X, Y \in \mathbb{R}^{n \times p}$ is denoted by $\langle X, Y \rangle = \text{tr}(X^\top Y)$, where $\text{tr}(\cdot)$ denotes the matrix trace. The Frobenius norm of X is denoted by $\|X\|_F := \sqrt{\langle X, X \rangle}$.

$$\text{sym}(A) = \frac{1}{2}(A + A^\top) \quad \text{and} \quad \text{skew}(A) = \frac{1}{2}(A - A^\top)$$

are the symmetric part and the skew-symmetric part of a squared matrix A , respectively. Moreover, $\mathcal{S}_{\text{sym}}^n$ and $\mathcal{S}_{\text{skew}}^n$ denote the sets of all symmetric and skew-symmetric $n \times n$ matrices, respectively. The operator $\dim(\cdot)$ denotes the dimension of a subspace. Let \mathcal{E}_1 and \mathcal{E}_2 be two finite-dimensional vector spaces over \mathbb{R} . The Fréchet derivative of a map $F : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ at $X \in \mathcal{E}_1$ is the linear operator $DF(X) : \mathcal{E}_1 \rightarrow \mathcal{E}_2 : \xi \mapsto DF(X)[\xi]$ satisfying $F(X + \xi) = F(X) + DF(X)[\xi] + o(\|\xi\|)$.

3. LANDING FLOW ON THE STIEFEL MANIFOLD

As an extension of the $p = n$ case addressed in Ablin and Peyré (2021), we define the *landing flow* on the Stiefel manifold as the solution of a continuous-time system called the *landing system*

$$\dot{X}(t) = -\Lambda(X(t)), \quad (2)$$

where $\Lambda : \mathbb{R}^{n \times p} \rightarrow \mathbb{R}^{n \times p}$ is the *landing field*¹ defined as

$$\Lambda(X) := \psi(X)X + \lambda \nabla \mathcal{N}(X), \quad (3)$$

with the first component $\psi(X)X$ referred to as the *relative gradient*,

$$\psi(X) := 2 \text{skew}(\nabla f(X)X^\top),$$

where $\nabla f(X)$ is the Euclidean gradient of the objective f . The second component of the landing field (3) is defined by

$$\mathcal{N}(X) := \frac{1}{4} \|X^\top X - I_p\|_F^2,$$

whose set of minimizers is the Stiefel manifold, and by the regularization parameter $\lambda > 0$.

¹ In order to give a geometric interpretation to $\Lambda(X)$ later, $\psi(X)$ differs from Ablin and Peyré (2021) by a factor 2.

A crucial feature of the landing field is that we have $\nabla \mathcal{N}(X) = X(X^\top X - I_p)$ making the two components of the landing field in (3) orthogonal.

Gao et al. (2019) proposed a similar approach called PLAM for solving (1) based on the field

$$\tilde{\Lambda}(X) := \nabla f(X) - X \text{sym}(\nabla f(X)^\top X) + \lambda \nabla \mathcal{N}(X). \quad (4)$$

However, in (4), the component based on $\nabla f(X)$ is not orthogonal to $\nabla \mathcal{N}(X)$, contrary to the landing field (3).

Both fields, defined in (3) and (4), have a straightforward interpretation as a Riemannian gradient for matrices X belonging to the Stiefel manifold. To see this, notice that for $X \in \text{St}(p, n)$, the distance term $\nabla \mathcal{N}(X)$ vanishes and, as a result, $\Lambda(X)$ and $\tilde{\Lambda}(X)$ belong to the tangent space of the Stiefel manifold

$$T_X \text{St}(p, n) = \{\xi \in \mathbb{R}^{n \times p} : \xi^\top X + X^\top \xi = 0\}.$$

Moreover, they are both Riemannian gradients of the objective f on the Stiefel manifold but with respect to different Riemannian metrics. The landing field $\Lambda(X)$ corresponds to the Riemannian gradient with respect to the *canonical metric*

$$g_X^c(\xi, \zeta) := \langle \xi, (I_n - \frac{1}{2}XX^\top)\zeta \rangle \quad \text{for all } \xi, \zeta \in \mathbb{R}^{n \times p}, \quad (5)$$

while the PLAM field $\tilde{\Lambda}(X)$ corresponds to the Riemannian gradient with respect to the Euclidean metric $\langle \xi, \zeta \rangle$; see Edelman et al. (1998) for the geometry of the Stiefel manifold and a discussion on these two different metrics.

However, for a general matrix $X \in \mathbb{R}_*^{n \times p}$, which does not need to belong to the Stiefel manifold, the Riemannian interpretation of the landing field in (3) is not immediate. In the following section, we consider a certain generalization of the Stiefel manifold that allows us to derive a Riemannian gradient interpretation of the landing field for a general $X \in \mathbb{R}_*^{n \times p}$.

4. INTERPRETATION OF THE LANDING FLOW

We give a geometric interpretation of $\Lambda(X) = \psi(X)X + \lambda \nabla \mathcal{N}(X)$ for all $X \in \mathbb{R}_*^{n \times p}$ such that (i) $\psi(X)X$ is the Riemannian gradient of f on a Riemannian submanifold of $\mathbb{R}_*^{n \times p}$ with a specifically constructed metric and (ii) $\nabla \mathcal{N}(X)$ belongs to its normal space.

First, we generalize the Stiefel manifold $\text{St}(p, n)$ to the following set

$$\text{St}_M(p, n) := \{Y \in \mathbb{R}^{n \times p} : Y^\top Y = M\},$$

where $M \in \mathbb{R}^{p \times p}$ is a given symmetric positive-definite matrix. In particular, $\text{St}_M(p, n)$ reduces to the Stiefel manifold $\text{St}(p, n)$ when $M = I_p$.

Consider the linear map

$$\Phi : \mathbb{R}^{n \times p} \rightarrow \mathbb{R}^{n \times p} : X \rightarrow Y = XM^{\frac{1}{2}}, \quad (6)$$

which is a diffeomorphism of $\mathbb{R}_*^{n \times p}$ onto itself since M is symmetric positive definite, and it maps $\text{St}(p, n)$ to $\text{St}_M(p, n)$. As a consequence, the following proposition shows that $\text{St}_M(p, n)$ is a submanifold of $\mathbb{R}_*^{n \times p}$.

Proposition 1. $\text{St}_M(p, n)$ is a closed embedded submanifold of $\mathbb{R}_*^{n \times p}$ with dimension $np - p(p + 1)/2$ when $M \in \mathbb{R}^{p \times p}$ is a symmetric positive-definite matrix.

Proof. In the view of Φ , since $\text{St}(p, n)$ is a closed set, we have that $\text{St}_M(p, n)$ is also closed. Moreover, if $\text{St}(p, n)$ is locally a ϕ -coordinate slice, then $\text{St}_M(p, n)$ is locally a $\phi \circ \Phi^{-1}$ -coordinate slice; hence the submanifold property is preserved and by [Absil et al. \(2008, Prop. 3.3.2\)](#), the set $\text{St}_M(p, n)$ is an embedded submanifold with $\dim(\text{St}_M(p, n)) = \dim(\text{St}(p, n)) = np - p(p+1)/2$. \square

4.1 Riemannian geometry of $\text{St}_M(p, n)$

We first characterize the tangent space of $\text{St}_M(p, n)$.

Proposition 2. The tangent space of $\text{St}_M(p, n)$ at $Y \in \text{St}_M(p, n)$ can be parameterized in the following ways

$$\text{T}_Y \text{St}_M(p, n) = \{\xi \in \mathbb{R}^{n \times p} : \xi^\top Y + Y^\top \xi = 0\} \quad (7a)$$

$$= \{Y(Y^\top Y)^{-1} \Omega + Y_\perp K : \Omega \in \mathcal{S}_{\text{skew}}^p, K \in \mathbb{R}^{(n-p) \times p}\} \quad (7b)$$

$$= \{WY : W \in \mathcal{S}_{\text{skew}}^n\} \quad (7c)$$

$$= \{\Phi(\zeta) : \zeta \in \text{T}_{\Phi^{-1}(Y)} \text{St}(p, n)\}. \quad (7d)$$

Proof. The first form can be obtained as in [Absil et al. \(2008, §3.3.2\)](#), where I_p has to be replaced by M and $X\hat{Z}$ by $YM^{-1}\hat{Z}$.

The second formulation (7b) can be verified by plugging into (7a) and from the fact that its dimension $(p(p-1)/2 + (n-p)p)$ agrees with $\text{T}_Y \text{St}_M(p, n)$.

The third formulation in (7c) can be also verified by (7a) and by its dimension. Specifically, we have

$$\{WY : W \in \mathcal{S}_{\text{skew}}^n\} \subseteq \text{T}_Y \text{St}_M(p, n).$$

Since $Y \in \mathbb{R}_*^{n \times p}$, there exists an orthogonal matrix $P \in \mathbb{R}^{n \times n}$ such that $P^\top Y = [I_p \ 0]^\top$. Let $B := P^\top W P = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. It turns out that

$$\begin{aligned} & \dim\{WY : W \in \mathcal{S}_{\text{skew}}^n\} \\ &= \dim\{P^\top W P P^\top Y : W \in \mathcal{S}_{\text{skew}}^n\} \\ &= \dim\{B \begin{bmatrix} I_p \\ 0 \end{bmatrix} : W \in \mathcal{S}_{\text{skew}}^n\} \\ &= \dim\{\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} : B_{11} \in \mathcal{S}_{\text{skew}}^p, B_{21} \in \mathbb{R}^{(n-p) \times p}\} \\ &= \frac{1}{2}p(p-1) + (n-p)p = np - \frac{1}{2}p(p+1), \end{aligned}$$

which agrees with the dimension of $\text{T}_Y \text{St}_M(p, n)$. The first equality comes from the fact that multiplying a subspace by an invertible matrix does not change its dimension.

The final formulation (7d) follows from (7a). \square

Note that (7c) is an over-parameterization of $\text{T}_Y \text{St}_M(p, n)$ since $\dim(\mathcal{S}_{\text{skew}}^n) = n(n-1)/2$ and the dimension of the tangent space is only $p(p-1)/2 + (n-p)p$.

Given $Y \in \text{St}_M(p, n)$, let $X = \Phi^{-1}(Y) = YM^{-\frac{1}{2}} \in \text{St}(p, n)$. By using the canonical metric (5), we construct the following metric on $\mathbb{R}_*^{n \times p}$ for all $\xi, \zeta \in \mathbb{R}^{n \times p}$,

$$\begin{aligned} g_Y(\xi, \zeta) &:= g_{\Phi^{-1}(Y)}^c(\Phi^{-1}(\xi), \Phi^{-1}(\zeta)) \\ &= \langle \xi, (I_n - \frac{1}{2}YM^{-1}Y^\top)\zeta M^{-1} \rangle \\ &= \langle \xi, (I_n - \frac{1}{2}Y(Y^\top Y)^{-1}Y^\top)\zeta(Y^\top Y)^{-1} \rangle. \quad (8) \end{aligned}$$

Since Φ is a diffeomorphism of $\mathbb{R}_*^{n \times p}$ onto itself and g^c is a well-defined Riemannian metric, it follows that

g is also a well-defined Riemannian metric on $\mathbb{R}_*^{n \times p}$. Hence, $(\text{St}_M(p, n), g)$ is a Riemannian submanifold of $(\mathbb{R}_*^{n \times p}, g)$. In particular, the metric g reduces to the canonical metric when $M = I_p$, which implies that we generalize the canonical metric to all manifolds $\text{St}_M(p, n)$. Furthermore, by construction, Φ is actually an isometry between the manifolds $(\text{St}(p, n), g^c)$ and $(\text{St}_M(p, n), g)$.

The normal space at $Y \in \text{St}_M(p, n)$ with respect to g is defined as the set of matrices $N \in \mathbb{R}^{n \times p}$ such that

$$g_Y(N, \xi) = 0 \quad \text{for all } \xi \in \text{T}_Y \text{St}_M(p, n).$$

The following proposition gives the explicit form of any normal vector.

Proposition 3. The normal space of $(\text{St}_M(p, n), g)$ at $Y \in \text{St}_M(p, n)$ admits the following expression,

$$\text{N}_Y \text{St}_M(p, n) := \{Y(Y^\top Y)^{-1}S : S \in \mathcal{S}_{\text{sym}}^p\}. \quad (9)$$

Proof. For any $S \in \mathcal{S}_{\text{sym}}^p$ and $W \in \mathcal{S}_{\text{skew}}^n$, it holds that

$$\begin{aligned} & g_Y(WY, Y(Y^\top Y)^{-1}S) \\ &= \langle WY, (I_n - \frac{1}{2}Y(Y^\top Y)^{-1}Y^\top)Y(Y^\top Y)^{-1}S(Y^\top Y)^{-1} \rangle \\ &= \langle Y^\top WY, \frac{1}{2}(Y^\top Y)^{-1}S(Y^\top Y)^{-1} \rangle = 0, \end{aligned}$$

where the first equality is the definition of the metric g , the second equality can be verified by expanding the sum with the identity and one of the inverses canceling the $Y^\top Y$ term, and the last equality follows from $Y^\top WY$ being skew-symmetric and therefore being orthogonal to the symmetric matrix on the right side of the inner product. According to (7c), and counting the dimension of $\text{T}_Y \text{St}_M(p, n)$ and $\{Y(Y^\top Y)^{-1}S : S \in \mathcal{S}_{\text{sym}}^p\}$ (i.e., $p(p+1)/2$), it yields the result. \square

4.2 Riemannian gradient of f on $(\text{St}_{X^\top X}(p, n), g)$

Let $X \in \mathbb{R}_*^{n \times p}$, by definition of $\text{St}_M(p, n)$ and the fact that X is of full rank, we have that $X \in \text{St}_{X^\top X}(p, n)$. Recall, the Riemannian gradient of f with respect to the metric g , denoted as $\text{grad}f(X)$, is uniquely defined to be an element of $\text{T}_X \text{St}_{X^\top X}(p, n)$ such that

$$g_X(\text{grad}f(X), \xi) = \text{D}f(X)[\xi] = \langle \nabla f(X), \xi \rangle \quad (10)$$

holds for all $\xi \in \text{T}_X \text{St}_{X^\top X}(p, n)$. The definition allows us to derive the following proposition giving a Riemannian interpretation to the relative gradient $\psi(X)X$ in the landing field (3).

Proposition 4. The Riemannian gradient of a function f on $(\text{St}_{X^\top X}(p, n), g)$ has the following form

$$\text{grad}f(X) = \psi(X)X,$$

where $\psi(X) = 2 \text{skew}(\nabla f(X)X^\top)$.

Proof. Let $W \in \mathcal{S}_{\text{skew}}^n$ be any skew-symmetric matrix. In view of (7c), WX is an arbitrary element of the tangent space $\text{T}_X \text{St}_{X^\top X}(p, n)$. We have

$$\begin{aligned}
& g_X(\psi(X)X, WX) \\
&= \langle \psi(X)X, (I_n - \frac{1}{2}X(X^\top X)^{-1}X^\top)WX(X^\top X)^{-1} \rangle \\
&= \langle (I_n - \frac{1}{2}X(X^\top X)^{-1}X^\top)\psi(X)X(X^\top X)^{-1}X^\top, W \rangle \\
&= \langle \nabla f(X)X^\top - \text{sym}(X(X^\top X)^{-1}X^\top \nabla f(X)X^\top), W \rangle \\
&= \langle \nabla f(X)X^\top, W \rangle \\
&= \langle \nabla f(X), WX \rangle,
\end{aligned}$$

where in the second equality we rearranged the terms in the inner product, the third equality can be verified by expanding the sum with the identity and writing out the definition of $\psi(X)$, the fourth equality comes from the fact that symmetric matrices are orthogonal to skew-symmetric matrices, and in the last fifth equality we move X^\top to the right side of the inner product. According to the definition (10), it yields the result. \square

The above Proposition 4 and the expression of the normal space in Proposition 3 give a clear interpretation of both components of the landing field $\Lambda(X) = \psi(X)X + \lambda \nabla \mathcal{N}(X)$. Specifically, $\psi(X)X$ is the Riemannian gradient of f on the submanifold $(\text{St}_{X^\top X}(p, n), g)$, and

$$\nabla \mathcal{N}(X) = X(X^\top X)^{-1}((X^\top X)^2 - X^\top X)$$

belongs to the normal space $N_X \text{St}_{X^\top X}(p, n)$. Subsequently, $\Lambda(X)$ is the linear combination of the two orthogonal fields in the tangent and the normal space of $(\text{St}_{X^\top X}(p, n), g)$; see Fig. 2 for a geometric illustration. The orthogonal property will have important consequences in the next section where we analyze convergence of the landing flow.

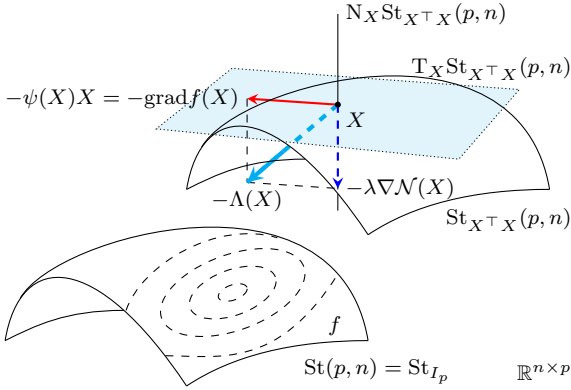


Fig. 2. Geometric interpretation of the orthogonal components of the landing field.

5. CONVERGENCE OF THE LANDING FLOW

In this section, we establish convergence analyses for the solutions of the landing system (2), denoted as $\varphi_t(X_0)$ for a starting point $X_0 \in \mathbb{R}_*^{n \times p}$ and for all $t \geq 0$. The proof consists of two parts, firstly by the convergence of $X(t)^\top X(t)$ to I_p and secondly by the convergence of $X(t)$ to the set of critical points of f relative to $\text{St}(p, n)$.

Standing assumption: ∇f is locally Lipschitz continuous.

We show that the solutions of the landing system exist and are unique, thus making the landing flow well defined.

Proposition 5. (Existence and uniqueness). For the landing system (2) starting at $X_0 \in \mathbb{R}_*^{n \times p}$, there exists a unique solution $t \mapsto \varphi_t(X_0)$ defined for all $t \geq 0$ such that $\varphi_0(X_0) = X_0$. Moreover, we have that $\mathcal{N}(\varphi_t(X_0))$ is nonincreasing and bounded, $\mathcal{N}(\varphi_t(X_0)) \leq \mathcal{N}(X_0)$ for all $t \geq 0$.

Proof. Differentiating $\mathcal{N}(X(t))$ with respect to t gives

$$\begin{aligned}
\frac{d}{dt} \mathcal{N}(X(t)) &= \langle \dot{X}(t), \nabla \mathcal{N}(X(t)) \rangle \\
&= -\langle \psi(X(t))X(t) + \lambda \nabla \mathcal{N}(X(t)), \nabla \mathcal{N}(X(t)) \rangle \\
&= -\lambda \|\nabla \mathcal{N}(X(t))\|_F^2 \leq 0,
\end{aligned}$$

where the second equality comes from the definition of the landing field and the last equality is the consequence of $\psi(X)X$ being orthogonal to $\nabla \mathcal{N}(X(t))$. Hence, $\mathcal{N}(X(t))$ is nonincreasing and each solution of the landing system remains in a compact set. By ∇f being locally Lipschitz, we have that $\Lambda(X)$ is also locally Lipschitz. By the Picard–Lindelöf theorem, the landing system has a unique solution. \square

It is worth noting that Proposition 5 holds for any $\lambda > 0$. This is due to the orthogonality of the two components of the landing field. By contrast with the landing field, the components of the PLAM field $\tilde{\Lambda}$ defined in (4) do not satisfy the orthogonal property, and as a consequence, the existence of its flow requires a lower threshold on $\lambda > \lambda_0 > 0$ (Gao et al., 2019).

Since $\mathcal{N}(\varphi_t(X_0))$ is nonincreasing and the set of minimizers of \mathcal{N} is the Stiefel manifold, it follows that the Stiefel manifold is an invariant of the landing flow. Recall also that, on the Stiefel manifold, the landing flow reduces to the Riemannian gradient flow with respect to the canonical metric.

5.1 Convergence of $X^\top X$ to I_p

The following result shows that the landing flow $\varphi_t(X_0)$ converges to $\text{St}(p, n)$ as $t \rightarrow +\infty$ for any $X_0 \in \mathbb{R}_*^{n \times p}$.

Proposition 6. (Convergence to the Stiefel manifold). For all $X_0 \in \mathbb{R}_*^{n \times p}$, we have that $\varphi_t(X_0) \in \mathbb{R}_*^{n \times p}$ for all $t > 0$ and

$$\lim_{t \rightarrow +\infty} \mathcal{N}(\varphi_t(X_0)) = 0.$$

Proof. Let $\chi(t) := X(t)^\top X(t)$ with $X(t)$ following the dynamics of the landing system (2). Differentiating $\chi(t)$ with respect to t yields

$$\begin{aligned}
\dot{\chi}(t) &= \dot{X}(t)^\top X(t) + X(t)^\top \dot{X}(t) \\
&= -2\lambda \chi(t) (\chi(t) - I_p)
\end{aligned}$$

By the right hand side being a matrix polynomial function of a symmetric matrix $\chi(t)$, we have that $\chi(t)$ has constant eigenvectors for all $t \geq 0$ and its eigenvalues $\{\chi_i\}_{i=1}^p$ follow $\dot{\chi}_i(t) = -2\lambda \chi_i(t)(\chi_i(t) - 1)$. The solution of the ODE for the eigenvalues can be computed explicitly as

$$\chi_i(t) = \frac{\chi_i(0)e^{2\lambda t}}{\chi_i(0)(e^{2\lambda t} - 1) + 1}.$$

Since $\lambda > 0$ and $\chi_i(0) > 0$ because $X_0 \in \mathbb{R}_*^{n \times p}$ is of full rank, we have that $\lim_{t \rightarrow +\infty} \chi_i(t) = 1$, i.e., all eigenvalues of $\chi(t)$ converge to 1. Hence $\chi(t)$ converges to I_p , and thus $\mathcal{N}(\varphi_t(X_0))$ converges to 0. \square

5.2 Convergence of the landing flow

Let \mathcal{C} denote the set of critical points of f relative to $\text{St}(p, n)$. Since $\psi(X)X$ is the Riemannian gradient on $\text{St}(p, n)$ with the canonical metric when $X \in \text{St}(p, n)$, according to [Absil et al. \(2008, §4.1\)](#), it follows that

$$\mathcal{C} = \{X^* \in \text{St}(p, n) : \psi(X^*)X^* = 0\}.$$

Moreover, by the orthogonality of the two terms in the landing field Λ , we have that

$$X^* \in \mathcal{C} \quad \text{if and only if} \quad \Lambda(X^*) = 0.$$

Recall that the ω -limit set of a trajectory φ_t is the set of points φ^* for which there exists a sequence $\{t_n\}$ with $\lim_{n \rightarrow +\infty} t_n = +\infty$ such that $\lim_{n \rightarrow +\infty} \varphi_{t_n} = \varphi^*$; see e.g., [Khalil \(1996\)](#).

Next, we show that the landing flow converges to the set of critical points of f on $\text{St}(p, n)$.

Theorem 7. (Convergence to the set of critical points). For all $X_0 \in \mathbb{R}_*^{n \times p}$, the ω -limit points of $\varphi_t(X_0)$ belong to \mathcal{C} ; in other words, the solution of the landing system (2) converges to the set of critical points of f relative to the Stiefel manifold $\text{St}(n, p)$.

Proof. Let $b \geq a \geq 0$ and

$$\mathcal{N}^{-1}([a, b]) := \{X \in \mathbb{R}_*^{n \times p} : a \leq \mathcal{N}(X) \leq b\}.$$

Let $\epsilon > 0$ and

$$B_\epsilon(\mathcal{C}) := \bigcup_{X^* \in \mathcal{C}} B_\epsilon(X^*)$$

where $B_\epsilon(X^*) := \{X \in \mathbb{R}_*^{n \times p} : \|X - X^*\|_{\mathbb{F}} < \epsilon\}$.

We claim (to be proven in the next paragraph) that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\max_{X \in \mathcal{N}^{-1}([0, \delta]) \setminus B_\epsilon(\mathcal{C})} \text{D}f(X)[- \Lambda(X)] < 0.$$

According to [Proposition 6](#), the solution $\varphi_t(X_0)$ eventually stays in $\mathcal{N}^{-1}([0, \delta])$. As a result, $\varphi_t(X_0)$ converges to $B_\epsilon(\mathcal{C})$; otherwise, due to the above claim, $\lim_{t \rightarrow +\infty} f(\varphi_t(X_0)) = -\infty$, which is impossible since f is continuous and $\mathcal{N}^{-1}([0, \delta])$ is compact. Since the convergence of $\varphi_t(X_0)$ to $B_\epsilon(\mathcal{C})$ holds for all $\epsilon > 0$, it follows that $\varphi_t(X_0)$ converges to \mathcal{C} .

We now show by contradiction that the claim in the first part of the proof is true. Suppose the statement is not true, that is, there exists $\delta_k > 0$ monotonically decreasing with $\lim_{k \rightarrow +\infty} \delta_k = 0$ and $X_k \in \mathcal{N}^{-1}([0, \delta_k]) \setminus B_\epsilon(\mathcal{C})$ such that

$$\text{D}f(X_k)[- \Lambda(X_k)] \geq 0.$$

Since $\mathcal{N}^{-1}([0, \delta_0]) \setminus B_\epsilon(\mathcal{C})$ is compact, the sequence $\{X_k\}$ has a convergent subsequence. Let \tilde{X} be its limit. On the one hand, by continuity of $X \mapsto \text{D}f(X)[- \Lambda(X)]$, we have $\text{D}f(\tilde{X})[- \Lambda(\tilde{X})] \geq 0$. On the other hand, $\tilde{X} \in \text{St}(p, n)$ since $\mathcal{N}(X_k) \leq \delta_k$ and $\lim_{k \rightarrow +\infty} \delta_k = 0$, and moreover $\tilde{X} \notin \mathcal{C}$ since it is at least a distance ϵ away from the critical points. Hence we have $\text{D}f(\tilde{X})[- \Lambda(\tilde{X})] < 0$, a contradiction. \square

5.3 Stability of the equilibria

We investigate the stability of the equilibria of the landing system (2) for minimizing f relative to $\text{St}(p, n)$. Note

that [Absil \(2004\)](#) considered the continuous-time flows on quotient spaces by using a similar idea.

By definition (see e.g., [Absil and Kurdyka \(2006\)](#)), X^* is an equilibrium point of the system (2) if $\Lambda(X^*) = 0$, i.e., X^* is a critical point of f relative to $\text{St}(p, n)$. Furthermore, X^* is stable if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, $\|X(0) - X^*\| < \delta$ implies $\|X(t) - X^*\| < \epsilon$ for all $t \geq 0$. Moreover, it is asymptotically stable if it is stable and for any $\epsilon > 0$, there exists $\delta > 0$ such that, $\|X(0) - X^*\| < \delta$ implies $\lim_{t \rightarrow +\infty} X(t) = X^*$.

Next, we give a sufficient condition for asymptotic stability. The proof is based on the theory of semidefinite Lyapunov functions; see [Iggidr et al. \(1996\)](#).

Theorem 8. (Asymptotic stability). If X^* is a local minimum and isolated critical point of f relative to $\text{St}(p, n)$, then X^* is an asymptotically stable point of the landing system (2).

Proof. Consider $\mathcal{N}(X(t))$.

(i) Since X^* is a critical point, it is also an equilibrium point of (2), and $X^* \in \text{St}(p, n)$, i.e., $\mathcal{N}(X^*) = 0$. In addition, we have $\mathcal{N}(X(t)) \geq 0$.

(ii) According to the proof of [Proposition 5](#), it holds that $\frac{d}{dt} \mathcal{N}(X(t)) \leq 0$ for all $t \geq 0$.

(iii) By calculation, we have

$$\{X \in \mathbb{R}_*^{n \times p} : \frac{d}{dt} \mathcal{N}(X(t)) = 0\} = \text{St}(p, n).$$

Since X^* is a local minimum and isolated, there exists $\epsilon > 0$ such that for all $X \in B_\epsilon(X^*) \cap \text{St}(p, n)$, $f(X^*) < f(X)$ when $X \neq X^*$. Hence, starting from any point $X_0 \in B_\epsilon(X^*) \cap \text{St}(p, n)$, all solutions $\phi_t(X_0)$ for the system $\dot{X}(t) = \Lambda(X(t))$ do not belong to $B_\epsilon(X^*) \cap \text{St}(p, n)$ as $t \rightarrow +\infty$ since $\Lambda(X(t))$ is a strictly ascent direction for f (see the proof in [Theorem 7](#)); otherwise, there is $\phi_t(X_0) \in B_\epsilon(X^*) \cap \text{St}(p, n)$ for all $t \geq 0$, and $\lim_{t \rightarrow +\infty} f(\phi_t(X_0)) = +\infty$, which is impossible since $B_\epsilon(X^*) \cap \text{St}(p, n)$ is compact and f is continuous.

The above points (i)-(iii) combined with [Iggidr et al. \(1996, Theorem 2\)](#) yield the result that X^* is an asymptotically stable equilibrium. \square

Corollary 9. For all $X_0 \in \mathbb{R}_*^{n \times p}$, if X^* is a local minimum and isolated critical point of f relative to $\text{St}(p, n)$, and if X^* is an ω -limit point of $\varphi_t(X_0)$, then $\lim_{t \rightarrow +\infty} \varphi_t(X_0) = X^*$.

Proof. Since X^* is an ω -limit point, $\varphi_t(X_0)$ eventually enters any neighborhood of X^* . Since moreover, in view of [Theorem 8](#), X^* is (asymptotically) stable, it follows that $\varphi_t(X_0)$ eventually stays in any neighborhood of X^* . \square

6. CONCLUSION

We have proposed an extension of the landing flow of [Ablin and Peyré \(2021\)](#) to rectangular matrices, obtained a Riemannian gradient interpretation to the $\psi(X)X$ term of the landing field (3), and proven that the solutions of the landing system globally converge to the set of equilibria of the objective function relative to the Stiefel manifold. In future work, we will address the question of

finding a discrete-time counterpart of the landing flow that preserves its favorable convergence properties.

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