A quick refresher on exposure and residence times in a well-mixed domain

Eric Deleersnijder, 12 May 2020

The exposure time of a collection of particles of a constituent of seawater (or aggregate of constituents) in a given domain is the mass-weighted average of the times they will spend in the domain of interest. Let $m(t)$ represent the mass of the particles under consideration that are present in the domain of interest at time t. Then, their mean exposure time is

$$\Theta = \lim_{\delta t \to 0}\left[\frac{m(0)}{m(0)}\delta t + \frac{m(\delta t)}{m(0)}\delta t + \frac{m(2\delta t)}{m(0)}\delta t + \ldots\right] = \frac{1}{m(0)}\int_0^\infty m(t)\,dt \qquad (1)$$

This formula is valid irrespective of the time evolution of mass $m(t)$ — provided the integral converges toward a finite value.

If the domain is well mixed, then the concentration (defined as a mass fraction) of the constituent under consideration is approximately homogeneous over the domain and, hence, is

$$C(t) \approx \frac{m(t)}{\rho V} \qquad (2)$$

where constant ρ is the reference density of seawater (Boussinesq approximation), whilst V denotes the volume of the domain, which is assumed to exhibit negligible variations over time.

Let Q ($\mathrm{m^3\,s^{-1}}$) denote the outgoing volumetric flow rate of seawater and assume it is approximately constant at the timescales of interest. Then, the outgoing mass flux of seawater (i.e. the aggregate of all the constituents) is ρQ ($\mathrm{kg\,s^{-1}}$) and the outgoing mass flux of the constituent under study ($\mathrm{kg\,s^{-1}}$) is

$$\Phi^{out}(t) = C(t)\rho Q \qquad (3)$$

This is because the concentration used herein is defined as a mass fraction. Then, combining (2) and (3) yields

$$\Phi^{out}(t) = \frac{m(t)}{V/Q} \qquad (4)$$

If the constituent under consideration is passive or inert and if there is no incoming flux of it, then its mass satisfies the following ordinary differential equation

$$\frac{dm}{dt} = -\,\Phi^{out} = -\frac{m}{V/Q} \qquad (5)$$

Its solution reads

$$m(t) = m(0)\exp\left(-\frac{t}{V/Q}\right) \qquad (6)$$

Substituting (6) into (1) leads to

$$\Theta = \frac{1}{m(0)}\int_0^\infty m(0)\exp\left(-\frac{t}{V/Q}\right)dt = \frac{V}{Q} = T_h \qquad (7)$$

where timescale $T_h = V/Q$ characterizes the hydrodynamics of the domain, specifically the timescale associated with the transport from the domain towards its environment.

Clearly, formula (7) applies to any constituent or aggregate of constituents, including the water originally present in the domain. The concentration of the latter is equal to unity at $t=0$ and decreases exponentially at time progresses, for it is progressively replaced by water originating from the environment of the domain, namely the renewing water. In other words, the mass of the original water, $m_o(t)$, and that of the renewing water, $m_r(t)$, are

$$m_o(t) = \rho V e^{-t/T_h} \quad , \quad m_r(t) = \rho V\left(1-e^{-t/T_h}\right) = \rho V - m_o(t) \tag{8}$$

with $m_o(t)+m_r(t)=\rho V$, $m_o(\infty)=0$ and $m_r(\infty)=\rho Q$.

The above considerations apply to constituents that leave the domain once and for all after crossing an open boundary for the first time. Therefore, exposure time (7) is equivalent to the strict residence time.

The mass of a constituent undergoing a first-order decay process characterised by (constant) mean life T_d obeys a modified version of (5), i.e.

$$\frac{dm}{dt} = -\frac{m}{T_h} - \frac{m}{T_d} \tag{9}$$

The corresponding mass is

$$m(t) = m(0)\, e^{-t/T^*} \tag{10}$$

with

$$T^* = \frac{T_h T_d}{T_h + T_d} \le \min(T_h, T_d) \tag{11}$$

Substituting (10) into (1) unsurprisingly leads to

$$\Theta = T^* \tag{12}$$

If the particles under consideration are not discarded at the moment they leave the domain for the first time, then they might re-enter the domain, requiring mass budget equation (9) to be transformed to

$$\frac{dm}{dt} = \Phi^{in} - \frac{m}{T^*} \tag{13}$$

where Φ^{in} ($\mathrm{kg\,s^{-1}}$) is the mass flux of re-entering particles. The solution of (13) reads

$$m(t) = m(0)e^{-t/T^*} + \int_0^t \Phi^{in}(t')e^{-(t-t')/T^*}\,dt' \tag{14}$$

so that the exposure time is

$$\Theta = T^* + \frac{1}{m(0)}\int_0^\infty\int_0^{t'} \Phi^{in}(t'')e^{-(t'-t'')/T^*}\,dt''\,dt' \ge T^* \tag{15}$$

For highly idealised studies, it may be convenient to assume that the incoming flux is proportional to the outgoing one, i.e.

$$\Phi^{in}(t) = r\frac{m(t)}{T_d} \tag{16}$$

where r is the relevant coefficient of proportionality ($0 \le r < 1$). Then, the exposure time reads

$$\Theta = \frac{T_h T_d}{T_h + (1-r) T_d} \ge T^* \tag{17}$$

The propensity of particles to return into the domain after leaving it for the first time may be estimated as follows:

$$\frac{\Theta(r \ne 0) - \Theta(r = 0)}{\Theta(r \ne 0)} = \frac{r T_d}{T_h + T_d} \tag{18}$$

Unsurprisingly, this dimensionless parameter satisfies

$$0 \le \frac{r T_d}{T_h + T_d} \le 1 \tag{19}$$

and is equal to r if there is no first-order decay ($T_d \to \infty$).

There is nothing novel in the present working note. The developments above may be viewed as a concise and consistent reformulation of results established in the following references:

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