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MEAN VALUE THEOREMS FOR TANGENTIAL DERIVATIVES

Abstract

In this note, we present a simple example showing that the tangential derivative of a continuous function can vanish everywhere along a smooth curve without implying that the function is constant along the curve. We also give some sufficient smoothness conditions on both elements that prevent the previous situation from happening.

Preliminaries

Let us briefly precise our notations and introduce the problem.

Let $n \geqslant 2$ be an integer. For an (extended) integer $0 \leqslant k \leqslant \infty$ we give $C^k([0,1];\mathbb{R}^n)$ and $C^k(\mathbb{R}^n)$ their usual meanings. Given $\alpha \geqslant 0$ and $x \in C^1([0,1];\mathbb{R}^n)$ we say that $x \in C^{1,\alpha}([0,1];\mathbb{R}^n)$ in case

$$H_\alpha(x) := \sup\left\{\frac{|\dot{x}(s)-\dot{x}(t)|}{|s-t|^\alpha} : 0 \leqslant s \neq t \leqslant 1\right\} < \infty.$$

Given $\phi : \mathbb{R}^n \to \mathbb{R}$ and $r > 0$ we let

$$\operatorname{osc}(\phi, r) := \sup\{|\phi(x) - \phi(y)| : x, y \in \mathbb{R}^n; |x-y| \leqslant r\}.$$

Given $\phi : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^n \setminus \{0\}$, we denote by $D\phi(x,u)$ the directional derivative of x in the direction of u, i.e. the number (if it exists)

$$D\phi(x,u) := \frac{1}{|u|}\lim_{h\to 0}\frac{\phi(x+tu)-\phi(x)}{t}.$$

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We call $x \in C^1([0,1];\mathbb{R}^n)$ a parametrized arc in case x is injective and satisfies $\dot{x}(t) \neq 0$ for $0 \leqslant t \leqslant 1$.

In this note, we are interested in the following question: given a parametrized arc $x \in C^1([0,1];\mathbb{R}^n)$ and $\phi \in C^0(\mathbb{R}^n)$, does the vanishing of $D\phi(x(t),\dot{x}(t))$ for each $0 \leqslant t \leqslant 1$ imply that ϕ is constant along x?

We show in Example 4 that the answer to the above question is negative, in general. We also give sufficient conditions on ϕ and f that prevent this situation from happening.

1 A mean value formula under additional hypotheses

The following Theorem yields a mean value formula for tangential derivatives, under additional hypotheses on f and ϕ.

Theorem 1. *Fix $\alpha \geqslant 0$, $\phi \in C^0(\mathbb{R}^n)$ and a parametrized arc $x \in C^{1,\alpha}([0,1];\mathbb{R}^n)$. Assume that the following conditions are satisfied:*

1. $\operatorname{osc}(\phi, r) = o(r^{1/1+\alpha})$*;*
2. $D\phi(x(t),\dot{x}(t)) = 0$ *for each* $0 \leqslant t \leqslant 1$.

Then $\phi(x(1)) = \phi(x(0))$.

Remark 2. In case $\alpha = 0$, one can replace assumption 1. by the weaker assumption that ϕ be Lipschitz in $\mathbb{R}^n$.

SKETCH OF THE PROOF. Fix $0 \leqslant t \leqslant 1$, $\varepsilon > 0$, let $M := \sup_{0 \leqslant t \leqslant 1} |\dot{x}(t)|$ and choose $\delta(t) > 0$ such that

(1) $|\phi(x) - \phi(y)| \leqslant \varepsilon |x-y|^{1/1+\alpha}$,

(2) $|\phi(x(t) + h\dot{x}(t)) - \phi(x(t))| \leqslant \varepsilon M |h|$

hold in case $|x-y| \leqslant \delta$ and $|h| \leqslant \delta$, respectively. Moreover, assume that $0 \leqslant a \leqslant b \leqslant 1$ are close enough to t for the following to hold:

(A) $H_\alpha(x)(b-a)^{1+\alpha} \leqslant \delta(t)$;

(B) $(b-a) \leqslant \delta(t)$.

In particular, (A) yields

$$|\phi(x(b)) - \phi[x(t) + (b-t)\dot{x}(t)]| \\ \leqslant \varepsilon \left| \int_t^b [\dot{x}(s) - \dot{x}(t)]\, ds \right|^{\frac{1}{1+\alpha}} \leqslant \varepsilon [H_\alpha(x)]^{1/1+\alpha}(b-t),$$

$$\begin{aligned}|\phi(x(a)) - \phi[x(t) - (t-a)\dot{x}(t)]|&\\ \leqslant \varepsilon \left| \int_a^t [\dot{x}(t) - \dot{x}(s)]\, ds \right|^{\frac{1}{1+\alpha}} &\leqslant \varepsilon [H_\alpha(x)]^{1/1+\alpha}(t-a).\end{aligned}$$

On the other hand,

$$\begin{aligned}|\phi[x(t) + (b-t)\dot{x}(t)] - \phi(x(t))| &\leqslant \varepsilon M(b-t);\\ |\phi[x(t) - (t-a)\dot{x}(t)] - \phi(x(t))| &\leqslant \varepsilon M(t-a).\end{aligned}$$

Defining $C := [H_\alpha(x)]^{1/1+\alpha} + M$ we get

$$\begin{aligned}|\phi(x(b)) - \phi(x(a))| &\leqslant |\phi(x(b)) - \phi[x(t) + (b-t)\dot{x}(t)]|\\ &+ |\phi[x(t) + (b-t)\dot{x}(t)] - \phi(x(t))|\\ &+ |\phi(x(t)) - \phi[x(t) - (t-a)\dot{x}(t)]|\\ &+ |\phi[x(t) - (t-a)\dot{x}(t)] - \phi(x(a))|,\\ &\leqslant C\varepsilon(b-a).\end{aligned}$$

We conclude as Cousin's lemma (see [4, Chapitre 4] for example) provides $0 = a^1 < b^1 = a^2 < b^2 = a^3 \dots b^{m-1} = a^m < b^m$ together with tags $a^j \leqslant t^j \leqslant b^j$, $1 \leqslant j \leqslant m$, such that conditions (A) and (B) are satisfied for each pair $(t^j, [a^j, b^j])$, $1 \leqslant j \leqslant m$. Hence $|\phi(x(1)) - \phi(x(0))| \leqslant \varepsilon$ and the result follows from the arbitrariness of $\varepsilon > 0$. $\square$

As we will show in the next section, additional hypotheses on ϕ and/or f are necessary for obtaining a mean value formula in this context.

2 The Helix Example

In this section, we assume $n = 3$ and we define a parametrized arc $x \in C^\infty(\mathbb{R}^3)$ by $x(t) = (\cos t, \sin t, t)$. We let $H := x([0,1])$ and for $0 \leqslant t \leqslant 1$ we let $L_t := x(t) + \langle \dot{x}(t) \rangle$ denote the tangent line to x at t. It is easy to show that $L_s \cap L_t = \emptyset$ whenever $s \neq t$.

On the tangent bundle $K := \bigcup_{0 \leqslant t \leqslant 1} L_t$ represented on Figure 1, define a function $\phi_0 : K \to \mathbb{R}$ by requiring that $\phi(\xi) = t$ for each $\xi \in L_t$. One easily shows that ϕ_0 is continuous on K and that K is closed. Hence the Tietze extension theorem yields $\phi \in C^0(\mathbb{R}^n)$ extending ϕ_0 outside K.

Remark 3. Using a more sophisticated extension result, inspired from Whitney's extension theorem [2, Section 6.2] and presented in [1, Lemma 7.3], we can extend ϕ_0 to a function $\phi \in C^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus H)$.

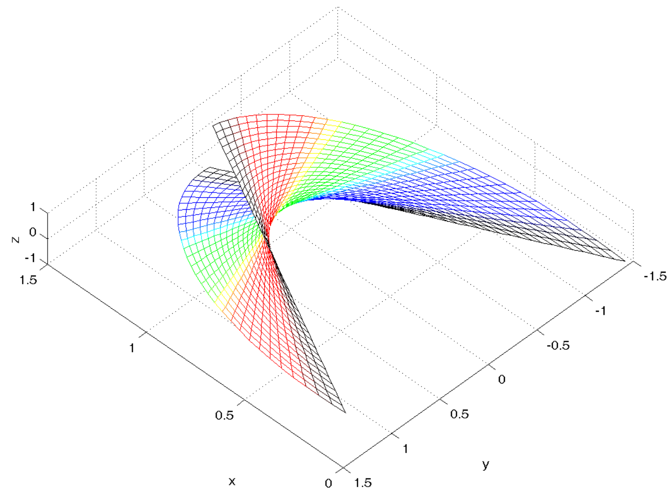

Figure 1: A part of surface K

On the other hand, for any $0 \leqslant t \leqslant 1$ we obviously get $D\phi(x(t), \dot{x}(t)) = 0$ as ϕ_0 is constant on L_t. However, it is clear that $\phi(x(0)) = 0 < 1 = \phi(x(1))$.

We can hence summarize the preceding information in the following:

Example 4. There exists a function $\phi \in C(\mathbb{R}^3)$ together with a parametrized arc $x \in C^\infty([0,1], \mathbb{R}^3)$ for which the following conditions are fullfilled:

1. $D\phi(x(t), \dot{x}(t)) = 0$ for each $0 \leqslant t \leqslant 1$,

2. $\phi(x(t)) = t$ for each $0 \leqslant t \leqslant 1$.

In particular we have $D\phi(x(t), \dot{x}(t)) = 0$ for each $0 \leqslant t \leqslant 1$ yet $\phi(x(0)) < \phi(x(1))$.

Remark 5. Let x and ϕ be those appearing in the preceding example. For $0 \leqslant s, t \leqslant 1$ compute

$$|\dot{x}(s) - \dot{x}(t)| = 2|\sin \frac{1}{2}(s-t)|,$$

from which it follows that $x \in C^{1,\alpha}(\mathbb{R}^3)$ if and only if $\alpha \leqslant 1$. According to Theorem 1, we hence get

$$\limsup_{r\to 0} \frac{\operatorname{osc}(\phi, r)}{r^{1/2}} > 0.$$

In the plane, one cannot provide an analoguous to Example 4 as we will show it in the next section.

3 The planar case

We assume from now that $x \in C^2([0,1];\mathbb{R}^2)$ is a parametrized arc satisfying $\ddot{x}(t)\ddot{x}(t) < |\ddot{x}(t)||\dot{x}(t)|$ for each $0 \leqslant t \leqslant 1$. For $0 \leqslant t \leqslant 1$ we also let $L_t := x(t) + \langle \dot{x}(t)\rangle$ denote the tangent line to x at t and we use the notation $[0,1]-t := \{h \in \mathbb{R} : t+h \in [0,1]\}$.

A first geometrical fact is summarized in the following lemma.

Lemma 6. *For $0 \leqslant t \leqslant 1$ and $h \in [0,1]-t$ we have $L_t \cap L_{t+h} = \{x_{t,h}\}$ and, for h sufficiently small*

$$\max\{|x(t) - x_{t,h}|, |x(t+h) - x_{t,h}|\} \leqslant M|h|,$$

where $M := \sup_{0\leqslant t\leqslant 1} |\dot{x}(t)|$.

A first step towards a mean value formula in our case is given in the following:

Proposition 7. *Assume that $\phi \in C^0(\mathbb{R}^2)$ is such that for each $0 \leqslant t \leqslant 1$, we have $D\phi(x(t), \dot{x}(t)) = 0$. Then $\dot{F}(t) = 0$ for a.e. $0 \leqslant t \leqslant 1$ where $F := \phi \circ x$.*

PROOF. For $0 \leqslant t \leqslant 1$ and $k \in \mathbb{N}$ define

$$\delta_k(t) := \sup\{\delta > 0 : |F(t+h) - F(t)| \leqslant 2^{-k-1}|h| \text{ for } |h| \leqslant \delta\},$$

and observe that this yields a measurable map $\delta_k : [0,1] \to \mathbb{R}_+$. For $k, l \in \mathbb{N}$ define a set

$$E_{k,l} := \{0 \leqslant t \leqslant 1 : \delta_k(t) \geqslant 2^{-l}\},$$

denote by $F_{k,l}$ the set of all its Lebesgue density points and observe that $F := \bigcap_{k\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} F_{k,l}$ has full measure in [0,1].

Given $t \in F$, find increasing sequences (k_i), (l_i) such that $t \in F_{k_i,l_i}$ for each i and observe that

$$\frac{|F(t+h) - F(t)|}{|h|} \leqslant M \left[\frac{|\phi(x(t+h)) - \phi(x_{t,h})|}{|x(t+h) - x_{t,h}|} + \frac{|\phi(x(t)) - \phi(x_{t,h})|}{|x(t) - x_{t,h}|} \right].$$

where $M := \sup_{0\leqslant t\leqslant 1} |\dot{x}(t)|$. We now compute

$$\overline{\lim_{\substack{h\to 0\\ t+h\in F_{k_i,l_i}}}} \frac{|F(t+h)-F(t)|}{|h|} \leqslant 2^{-k_i} M.$$

It is then easy to show now that $\dot{F}(t) = 0$ for a.e. $0 \leqslant t \leqslant 1$ using [3, Theorem 3.1.8] and [6, Theorem 6.6.8]. $\square$

Given a function $F : [0,1] \to \mathbb{R}$ having bounded variation, we denote by V^*F the associated variational measure as defined in [7]. Another measure-theoretical argument yields the following:

Proposition 8. *Assume that $\phi \in C^0(\mathbb{R}^2)$ and that for each $0 \leqslant t \leqslant 1$ we have $D\phi(x(t), \dot{x}(t)) = 0$. If moreover $F := \phi \circ x$ has bounded variation, then $\dot{F}(t) = 0$ for all $0 \leqslant t \leqslant 1$ not belonging to a V^*F-negligible set.*

Proof. Define for $k, l \in \mathbb{N}$ a measurable function δ_k and a set $E_{k,l}$ as in the proof of Proposition 7 and denote by $D_{k,l}$ the (countable) set of all isolated points in $E_{k,l}$. Let $D := \bigcup_{k,l\in\mathbb{N}} D_{k,l}$. Let also B denote the set of $0 \leqslant t \leqslant 1$ at which $\dot{F}(t)$ does not exist (finite or infinite). According to de la Vallée Poussin's theorem [7, Chapter IV, Theorem 9.6], B is V^*F-negligible. Observe that $V^*F(B \cup D) = 0$ (recall that V^*F cannot concentrate on points for F is continuous).

Given $0 \leqslant t \leqslant 1$ satisfying $t \notin B \cup D$ and given $k, l \in \mathbb{N}$ for which $t \in E_{k,l}$, observe that t is an accumulation point of $E_{k,l}$. As in the proof of Proposition 7, infer

$$|\dot{F}(t)| = \lim_{\substack{h\to 0\\ t+h\in E_{k,l}}} \left| \frac{F(t+h)-F(t)}{h} \right| \leqslant 2^{-k} M$$

where $M := \sup_{0\leqslant t\leqslant 1} |\dot{x}(t)|$. Yet k can be as large as we wish since we have $[0,1] = \bigcup_{l\in\mathbb{N}} E_{k,l}$ for each $k \in \mathbb{N}$. $\square$

We can now state and prove the final version of our mean value formula in the plane.

Theorem 9. *Assume that $\phi \in C^0(\mathbb{R}^2)$ is such that $D\phi(x(t), \dot{x}(t)) = 0$ for each $0 \leqslant t \leqslant 1$. If moreover $F := \phi \circ x$ has bounded variation, then F is constant.*

Proof. This is an immediate corollary of the preceding proposition as de la Vallée Poussin's theorem yields

$$V^*F([0,1]) = \int_0^1 \dot{F}(t)\, dt = 0.$$

The proof is complete. □

The material in the present summary is mostly taken in the forthcoming paper [5].

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