Université catholique de Louvain

Faculté des Sciences

Institute de Recherche en
Mathématique et Physique

DOUBLE SCALING LIMITS OF TOEPLITZ, HANKEL AND FREDHOLM DETERMINANTS

Benjamin E. Fahs

Promoteur:
Prof. Tom Claeys

Thèse présentée en vue de l'obtention
du grade de docteur en sciences

11th of August 2017

Jury de thèse:

Prof. Tom Claeys, Université catholique de Louvain, *Promoteur*
Prof. Philippe Ruelle, Université catholique de Louvain, *Secrétaire*
Prof. Igor Krasovsky, Imperial College London
Prof. Maurice Duits, KTH Royal Institute of Technology
Prof. Marino Gran, Université catholique de Louvain, *Président*

Acknowledgements

During my PhD I have felt that there are two big differences compared with undergraduate mathematics: the size of the problems one works on are much larger and one often finds that undergraduate exercises which fit onto a couple of pages are explained by "it is easily seen that" in a mathematical publication; furthermore someone might want to read what one has written during the PhD, and so one needs to spend more time presenting the material carefully, instead of jotting it quickly down to prove that one can understand the concept, which was my approach to undergraduate mathematics.

I have been fortunate enough to have guidance from Tom Claeys and Igor Krasovsky, who have shown me each of their experienced approaches to tackle these large scale projects which comprise a PhD thesis, and who have beaten out my undergraduate habits with tenacity, emphasizing a focus on presentation. They have suggested excellent projects, with first a softer introduction on characteristic polynomials of random matrices, followed by a somewhat more challenging project on gap probabilities in the eigenvalue distribution of random matrices. On the way we have had interesting mathematical discussions, and we have experienced the winding paths of research, where the first attempt is not entirely successful and some backtracking is required to get back on the road. It has been a fulfilling experience, and so I would like to thank both Tom and Igor for the opportunity to work with them.

We have had a sizeable group in Louvain-la-Neuve, partially thanks to the ERC grant CRAMIS, and I would like to thank Christophe Charlier, Antoine Doeraene, Martin Venker, Thorsten Neuschel, Manuela Girotti, Max Atkin and Stefano Romano for creating a friendly atmosphere to work in, and the mathematical discussions we have had.

I would like to thank the jury of my thesis for corrections and suggestions to the thesis, and for a number of interesting questions.

Finally, I would like to thank my girlfriend Sei Howe for suggesting improvements to the presentation of my work, and thank both Sei, my family and my friends for their love and support through the ups and downs of the last four years.

Contents

Chapter 1

Introduction

1.1 Introduction

1.1.1 Preliminaries

We study the asymptotics of certain Hankel, Toeplitz and Fredholm determinants. While these are objects from the field of pure mathematics, we are motivated by the statistical properties of random matrix ensembles and certain physical models. All of the topics above were introduced in the 20th century and we give a historical overview of their initial developments, starting with the determinants, followed by the random matrices and physical models.

Toeplitz determinants find their roots in the work of Otto Toeplitz and Gábor Szegő. In 1908, O.Toeplitz [87, 88] defined Toeplitz matrices T as square matrices with the property $T_{ij} = T_{i-j}$, i.e. the matrix entries are constant along the diagonals, and he studied quadratic forms $\sum_{i,j} x_i y_j T_{ij}$. Following this he proved [89] an interesting relation between the coefficients ϕ_j of the Taylor series $\sum_{j=0}^{\infty} \phi_j z^j$ of non-negative, analytic functions ϕ on the interior of the unit disc in the complex plane and the eigenvalues of certain related Toeplitz matrices.

Szegő on the other hand studied the large n asymptotics of Toeplitz determinants

$$D_n(f) = \det(f_{j-k})_{j,k=1}^{n}, \qquad f_j = \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} \frac{d\theta}{2\pi},$$

where f is called the symbol of D_n. In 1915, Szegő [85] proved that if f is strictly positive and continuous on the unit circle then

$$\log D_n(f) = n \int_{-\pi}^{\pi} \log f(e^{i\theta}) \frac{d\theta}{2\pi} + o(n), \tag{1.1}$$

as $n \to \infty$. This is known as either Szegő's First Theorem or simply Szegő's Theorem. Thirty years later, Lars Onsager and Bruria Kaufman discovered that $D_n(f)$ appeared in the field of statistical physics in the theory of the Ising model [68], and that the more precise asymptotics of $D_n(f)$ were required to calculate the long range correlations. Szegő was thus prompted to improve the above formula by calculating the subleading $o(n)$ term. See [30] for an interesting history of Toeplitz determinants and the Ising model. Szegő found that if f is a positive and $C^{1+\epsilon}$ function on the unit circle C, for $\epsilon > 0$, then

$$\log D_n(f) = n \int_0^{2\pi} \log f(e^{i\theta}) \frac{d\theta}{2\pi} + \sum_{j=1}^{\infty} j (\log f)_j (\log f)_{-j} + o(1), \quad (1.2)$$

as $n \to \infty$. This was generalized in [56, 60, 66], and of these the most general result is due to Johansson [66] in 1988, which states that (1.2) holds for all complex valued, integrable functions f such that

$$\sum_{j=-\infty}^{\infty} |j| |(\log f)_j|^2 < \infty. \quad (1.3)$$

The asymptotic expansion (1.2) is known as the strong Szegő limit theorem.

The formula (1.2) raised a new question: namely what happens when f is not sufficiently smooth, so that (1.3) does not hold. Two examples of such functions are $f_\alpha(z) = |z-1|^\alpha$ and $f_\beta(z) = z^\beta$, for $\operatorname{Re}\alpha > -1$ and $\beta \in \mathbb{C}$. For such symbols we have that as $j \to \infty$,

$$|(\log f_\alpha)_j|, \ |(\log f_\beta)_j| = \mathcal{O}\left(\frac{1}{j}\right), \quad \alpha, \beta \neq 0, \quad (1.4)$$

and so the sum in (1.3) diverges. The functions f_α and f_β are archetypes of a class of functions with Fisher-Hartwig (FH) singularities. Another example where (1.3) is false, is the indicator function on sets $J \subset C$

$$\chi_J(z) = \begin{cases} 1 & \text{for } z \in J, \\ 0 & \text{for } z \in C \setminus J. \end{cases} \quad (1.5)$$

If J is a disjoint union of arcs on the circle, then $|(\log \chi_J)_j| = \mathcal{O}(1/j)$ as $j \to \infty$.

Toeplitz determinants are closely related to Hankel determinants, where the Hankel determinant with symbol w is defined by

$$H_n(w) = \det(w_{j+k})_{j,k=0}^{n-1}, \qquad w_j = \int_{\mathbb{R}} w(x) x^j dx,$$

for $w(x)x^j$ integrable for all $j \in \mathbb{N}$. One studies the large n asymptotics of Hankel determinants, and it is possible to define smooth symbols and FH singularities in a similar manner to Toeplitz determinants, see Section 1.1.6.

Asymptotics of Toeplitz and Hankel determinants with FH singularities and symbols χ_J have attracted a large amount of interest, and are the topic of this thesis. We give an overview of results on the topic in Section 1.1.6.

Having covered the early history of determinants, we now turn to the history of random matrices and physical systems, which as mentioned above motivates the study of determinants.

At the time when Toeplitz and Szegő started their work on Toeplitz determinants, the field of probability theory was still very much in its nascence. For example, the first central limit theorem for general classes of random variables was only first discovered by Lyapunov in 1901, and the earliest mention of a random matrix was made by Wishart in 1928. Wishart studied the distribution of the sample covariance matrix

$$\widehat{\Sigma} = \frac{1}{n}\sum_{j=1}^{n} X_j X_j^T - \widetilde{X}\widetilde{X}^T, \qquad \widetilde{X} = \frac{1}{n}\sum_{j=1}^{n} X_j,$$

where $X_i \in \mathbb{R}^p$, $i = 1, \dots, n$ are independent and identically distributed (iid) with expectation 0 and variance Σ. Here p is the number of observables, and n is the number of observations. The sample covariance matrix $\widehat{\Sigma}$ is distributed according to the Wishart ensemble [1].

In today's world of large scale data collection and the subsequent need for analysis of large data sets, there are many applications of asymptotics of sample covariance matrices and Wishart ensembles for large p and/or n, including principal component analysis, finance, studies of genomics, and optimization, see [3] for a review. However, at the time, the interest generated by Wishart's work was somewhat limited [43, 55, 58]. It would take another 30 years before large n asymptotics of Wishart ensembles were studied [4] by Anderson in 1963.

Meanwhile, during the 1950s, nuclear physicists were conducting experiments where they bombarded heavy nuclei with a beam of neutrons (see e.g. [82]), and Wigner discovered a link between nuclear physics

[1] A random matrix $W \in \mathbb{R}^{p\times p}$ is distributed according to the Wishart ensemble with m degrees of freedom if

$$W = \sum_{i=1}^{m} Z_i Z_i^T \qquad Z_i \overset{\text{iid}}{\sim} \mathcal{N}(0, \Gamma), \tag{1.6}$$

for some $p \times p$ symmetric and positive-definite matrix Γ, where $p, m = 1, 2, \dots$.

and random matrix theory, which we will now describe. The nucleus can only take on certain energy levels, and if the incoming neutron has the right energy it can excite the nucleus, which results in the neutron being reflected off the nucleus at an angle. The scattering cross-section, namely the percentage of neutrons which are reflected off the nucleus, is plotted in a graph against the energies. For certain energies, known as scattering resonances, the scattering cross-section is elevated in peaks. There are hundreds of such peaks, and one would like to know their location, but the problem is too complicated to analytically solve. Eugene Wigner theorized that the locations of the scattering resonances could be modelled by the eigenvalues of the Gaussian Orthogonal Ensemble (GOE) [28], which was later confirmed by experimental data.

The GOE can be defined as the probability distribution on $n \times n$ matrices M where the elements $M_{i,j}$ are iid real Gaussian variables for $i \geq j$, and M is conditioned to be symmetric. It is a special example of the more general Orthogonal Ensemble. This is a class of random matrix ensembles invariant under conjugation by orthogonal matrices. This invariance under conjugation was essential to Wigner's argumentation, because it is natural that the system should remain unchanged under change of basis.

We mention one final interesting application of random matrices. There is a (conjectural) connection between the eigenvalues of the Gaussian Unitary Ensemble (GUE) and the zeros of the Riemann zeta function along the critical line, which was first noticed by Montgomery [79] in 1972. Matrices M distributed according to the GUE have real iid Gaussian entries on the main diagonal M_{ii}, and complex iid Gaussian entries for M_{ij} where $i > j$. The GUE is a special example of the Unitary Ensemble, which are Hermitian matrices characterized by the requirement that the ensemble is invariant under conjugation by unitary matrices, see the next section for details.

In Section 1.1.3 and 1.1.6 we will see that the probability distribution satisfied by the eigenvalues of the Unitary Ensemble forms a determinantal point process, and the correlation functions converge to those of the sine process in the local scaling limit in the bulk of the spectrum. This motivates the study of the sine process $\mathbb{P}_{\text{sin}}$, which is the topic of Chapter 2. For bounded and integrable functions K on $\mathbb{R}^2$, define the Fredholm determinant of K on bounded sets A by

$$\det(I-K)_A = 1+\sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int_{A^j} \det(K(x_i,x_k))_{i,k=1}^{j} dx_1 \dots dx_j. \quad (1.7)$$

By Hadamard's inequality, the right hand side (RHS) of (1.7) can be seen to converge. Let $K_s(x,y) = \frac{\sin s(x-y)}{\pi(x-y)}$, and denote by $P_s(A)$ the probability of $\mathbb{P}_{\sin}$ having no eigenvalues in $\frac{s}{\pi}A$. Then

$$P_s(A) = \det(I - K_s)_A,$$

and in Chapter 2, we derive an asymptotic formula for $P_s(A)$ as $s \to \infty$, where A is the union of two intervals with endpoints which are scaled in a particular way with s. Chapter 2 is based on [42], written in collaboration with Igor Krasovsky.

Let H_n be an $n \times n$ random matrix distributed according to the unitary ensemble, and denote the characteristic polynomial of H_n by

$$P_n(x) = \det(H_n - Ix).$$

In Chapter 3, we derive an asymptotic formula for certain moments of $P_n(x)$ in the bulk of the spectrum as $n \to \infty$. These results can also be viewed as an asymptotic result on Hankel determinants with FH singularities of type α. Chapter 3 is based on [18], written in collaboration with Tom Claeys. The moments of $P_n(x)$ which we consider in Chapter 3, appear in the study of impenetrable bosons in a harmonic well (see [47]), as well as in the study of Gaussian multiplicative chaos (see [9]).

The remaining part of the introduction is structured as follows. In Section 1 we give an introduction to properties of eigenvalues of Unitary Ensembles, determinantal point processes, and Toeplitz, Hankel and Fredholm determinants and show how these topics are connected. Sections 1.1.2 to 1.1.5 are standard material, see [27] for an introduction. In Section 1.1.6, we discuss some known results about asymptotics of Toeplitz and Hankel determinants, and give one new result. In Section 2 we describe the main results of Chapter 2, and in Section 3 we describe the main results of Chapter 3.

1.1.2 Unitary Ensembles

The Unitary Ensemble is defined by the probability measure

$$\frac{1}{Z_n} e^{-n\mathrm{Tr}V(M)} dM, \tag{1.8}$$

where $dM = \prod_{1\le i\le n} dM_{ii} \prod_{1\le i<j\le n} d\mathrm{Re}\, M_{ij} d\mathrm{Im}\, M_{ij}$ is the Lebesgue measure on $\mathbb{R}^{n^2}$ space of $n \times n$ Hermitian matrices M, and Z_n is the normalizing constant such that the measure integrates to 1 over the probability space. The potential V is a real analytic function on $\mathbb{R}$ with sufficient growth as $x \to \infty$:

$$\lim_{x\to\pm\infty} \frac{V(x)}{\log(x^2+1)} = +\infty. \tag{1.9}$$

As mentioned in the preliminaries, the Unitary Ensemble is invariant under conjugation by unitary matrices. As a result, one may show that the eigenvalues of M (1.8) induce the following probability measure on $\mathbb{R}^n$ [27]:

$$\nu_n(x_1,\dots,x_n)dx_1\dots dx_n = \frac{1}{\widehat{Z}_n}\prod_{1\le i<j\le n}(x_j-x_i)^2\prod_{j=1}^n w_n(x_j)dx_j,$$
$$w_n(x) = e^{-nV(x)}, \quad (1.10)$$

with normalizing constant

$$\widehat{Z}_n = \int_{\mathbb{R}^n}\prod_{1\le i<j\le n}(x_j-x_i)^2\prod_{j=1}^n w_n(x_j)dx_j, \tag{1.11}$$

which is known as the partition function. The term $(x_j-x_i)^2$ in (1.10) can be interpreted as repulsion because it reduces the probability that particles are close together, while V is known as the confining potential (or as the external field) because $e^{-nV(x)}$ reduces the probability that the absolute value of the eigenvalues are large. Thus the repulsion and confining potential are counteracting each other.

1.1.3 Determinantal point processes and the correlation kernel

A set $\chi\subset\mathbb{R}$ is called a configuration if it has no accumulation of points, i.e. $\#(\chi\cap B)<\infty$ for any bounded set $B\subset\mathbb{R}$. A (locally finite) point process is a probability measure on the space of configurations. If $\mathbb{P}$ is a point process on $\mathbb{R}$, we define the k-point correlation function $\rho_k:\mathbb{R}^k\to[0,\infty)$ by the requirement that for any mutually disjoint Borel sets $A_1,\dots,A_j$ and positive integers $k_1,\dots,k_j$ such that $\sum_{i=1}^j k_i = k$, we have

$$\mathbb{E}\left(\prod_{i=1}^j\frac{\#(A_i\cap\mathbb{P})!}{(\#(A_i\cap\mathbb{P})-k_i)!}\right) = \int_{A_1^{k_1}\times A_2^{k_2}\times\ldots\times A_j^{k_j}}\rho_k(x_1,\dots,x_k)dx_1\dots dx_k, \tag{1.12}$$

where $\mathbb{E}$ denotes the expectation, we let $0!=1$, and when $(\#(A_i\cap\mathbb{P})-k_i)!<0$ we replace $\frac{\#(A_i\cap\mathbb{P})!}{(\#(A_i\cap\mathbb{P})-k_i)!}$ by 0.

Setting $j=1$ in (1.12), the expected number of ordered k-tuples on a set A is given by

$$\begin{aligned}
&\mathbb{E}(\#\text{ ordered } k\text{-tuples in } A) \\
&= \frac{1}{k!}\int_{A^k} \rho_k(x_1,\ldots,x_k)dx_1\ldots dx_k \\
&= \sum_{j=0}^{\infty}\binom{k+j}{k}\text{Prob}(\#(A\cap\mathbb{P})=k+j).
\end{aligned} \tag{1.13}$$

For distinct points $x_1,\ldots,x_k$ the correlation functions satisfy

$$\lim_{\epsilon\to 0}\frac{\text{Prob}(\#(x_j,x_j+\epsilon)=1 \text{ for } j=1,\ldots,k)}{\epsilon^k} = \rho_k(x_1,\ldots,x_k). \tag{1.14}$$

A determinantal point process is a point process with a correlation kernel $K(x,y)$ such that for all $k\in\mathbb{N}_{\geq 1}$:

$$\rho_k(x_1,\ldots,x_k) = \det(K(x_i,x_j))_{i,j=1}^k. \tag{1.15}$$

Remark 1.1.1. Note that the kernel K is not unique, since the kernels $K(x,y)$ and $K(x,y)\frac{f(x)}{f(y)}$ determine the same determinantal point process for any integrable non-vanishing function f.

Thus the correlation kernel determines the point process, and all statistical quantities can be expressed in terms of it. For instance, by considering (1.13), it is easy to verify that if $\mathbb{P}$ is a determinantal point process with correlation kernel K, then for any set $A\subset\mathbb{R}$,

$$\text{Prob}(\mathbb{P}\cap A=\emptyset) = \det(I-K)_A, \tag{1.16}$$

where $\det(I-K)_A$ is a Fredholm determinant, defined in (1.7).

Orthogonal polynomials on the real line

We now construct an explicit determinantal point process in terms of orthogonal polynomials on the real line. Let w be a non-negative function on $\mathbb{R}$ such that $x^j w(x)$ is integrable on $\mathbb{R}$ for all $j\in\mathbb{N}$, which we call the weight. Consider the following probability measure on $\mathbb{R}^n$

$$\begin{aligned}
\nu_n(x_1,\ldots,x_n)dx_1\ldots dx_n &= \frac{1}{\widehat{Z}_n}\prod_{1\leq i<j\leq n}(x_j-x_i)^2\prod_{j=1}^n w(x_j)dx_j, \\
\widehat{Z}_n = \widehat{Z}_n(w) &= \int_{\mathbb{R}^n}\prod_{1\leq i<j\leq n}(x_j-x_i)^2\prod_{j=1}^n w(x_j)dx_j.
\end{aligned} \tag{1.17}$$

Since ν_n is symmetric with respect to the n particles, it is easily verified using (1.14) that the correlation functions ρ_k associated with the point process are given by

$$\rho_k(x_1,\ldots,x_k) \\ = \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} \nu_n(x_1,\ldots,x_n)dx_{k+1}dx_{k+2}\ldots dx_n. \quad (1.18)$$

Let $p_j(x) = \kappa_j x^j + \gamma_j x^{j-1} + \ldots$ be polynomials of order $j \in \mathbb{N}$, orthonormal with respect to w, characterized by

$$\int_{\mathbb{R}} p_j(x)p_k(x)w(x)dx = \delta_{j,k} = \begin{cases} 0 & \text{for } j \neq k, \\ 1 & \text{for } j = k, \end{cases} \qquad \kappa_j > 0. \quad (1.19)$$

and define

$$K_n(x,y) = \sqrt{w(x)w(y)}\sum_{j=0}^{n-1} p_j(x)p_j(y). \quad (1.20)$$

Then the following is a standard result (see e.g. [27]).

Proposition 1.1.1. *The probability distribution* (1.17) *is a determinantal point process with correlation kernel* K_n*.*

An immediate consequence of the proposition, by (1.16) and (1.17), is that

$$\det(I - K_n)_A = \frac{\widehat{Z}_n(\chi_A w)}{\widehat{Z}_n(w)}, \quad (1.21)$$

where χ_A is the indicator function on A, and where $(\chi_A w)(z) = \chi_A(z)w(z)$.

Proof. It is useful to temporarily relabel the indices of x_j by defining $y_0, \ldots y_{n-1}$ such that $y_j = x_{j+1}$. The repulsive term in (1.17) is a Vandermonde determinant:

$$\prod_{0\leq i<j\leq n-1} (y_j - y_i) = \det(y_i^j)_{i,j=0}^{n-1}. \quad (1.22)$$

By standard row and column operations,

$$\begin{aligned} \left(\det\left(y_i^j\sqrt{w(y_i)}\right)_{i,j=0}^{n-1}\right)^2 &= \left(\det\left(\kappa_j^{-1}p_j(y_i)\sqrt{w(y_i)}\right)_{i,j=0}^{n-1}\right)^2 \\ &= \left(\prod_{j=0}^{n-1}\kappa_j^{-2}\right)\det(K_n(y_i,y_j))_{i,j=0}^{n-1}. \end{aligned} \quad (1.23)$$

Substituting (1.22) into (1.17), and applying (1.23), it follows that

$$\nu_n(x_1,\dots,x_n) = \frac{1}{\widehat{Z}_n}\left(\prod_{j=0}^{n-1}\kappa_j^{-2}\right)\det K_n(x_i,x_j)_{i,j=1}^n. \tag{1.24}$$

The kernel K_n satisfies

$$\int_{\mathbb{R}} K_n(x,x)dx = n, \qquad \int_{\mathbb{R}} K_n(x,y)K_n(y,z)dy = K_n(x,z), \tag{1.25}$$

where the second property is that of a reproducing kernel. By (1.25), it follows that

$$\int_{\mathbb{R}} \det K_n(x_i,x_j)_{i,j=1}^l dx_l = (n+1-l)\det K_n(x_i,x_j)_{i,j=1}^{l-1}, \tag{1.26}$$

for $l = 2,\dots,n$. This implies that

$$\int_{\mathbb{R}^n} \det K_n(x_i,x_j)_{i,j=1}^n dx_1 \dots dx_n = n!\,, \tag{1.27}$$

and so by (1.24), we have

$$\widehat{Z}_n = n!\prod_{j=0}^{n-1}\kappa_j^{-2}, \qquad \nu_n(x_1,\dots,x_n) = \frac{1}{n!}\det[K_n(x_i,x_j)]_{i,j=1}^n. \tag{1.28}$$

By substituting the form for ν_n from (1.28) into (1.18) and iterating (1.26), it follows that ρ_k has the form (1.15) with kernel K_n, which proves the result. $\square$

Using the orthogonality of the polynomials p_j it is readily verified that they satisfy a three term recurrence relation:

$$xp_j(x) = \frac{\kappa_j}{\kappa_{j+1}}p_{j+1}(x) + \left(\frac{\gamma_j}{\kappa_j} - \frac{\gamma_{j+1}}{\kappa_{j+1}}\right)p_j(x) + \frac{\kappa_{j-1}}{\kappa_j}p_{j-1}(x), \tag{1.29}$$

for $j \geq 1$. Applying (1.29) to the RHS of (1.20) (multiplied by $(x-y)$) one obtains the Christoffel-Darboux identity:

$$K_n(x,y) = \sqrt{w(x)w(y)}\frac{\kappa_{n-1}}{\kappa_n}\frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}. \tag{1.30}$$

We denote the j-th orthogonal polynomial associated with the weight w_n from (1.10) by $p_j^{(n)}$ and the correlation kernel associated with w_n by K_n.

Orthogonal polynomials on the unit circle

In the previous section we considered orthogonal polynomials (OPs) on the real line. Here we consider OPs on the unit circle instead. The theories of OPs on the unit circle and OPs on the real line are similar, and we will here give parallel results to those found in the previous section, but we will forego the proofs.

Let f be a non-negative, integrable function on the unit circle C, which we call the weight. Consider the probability measure

$$\begin{aligned} &\frac{1}{\widehat{Z}_n^C (2\pi)^n} \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^n f(e^{i\theta_j}) d\theta_j, \quad \theta_j \in [0,2\pi)^n, \\ &\widehat{Z}_n^C = \widehat{Z}_n^C(f) = \frac{1}{(2\pi)^n} \int_{C^n} \prod_{1\le i<j\le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^n f(e^{i\theta_j}) d\theta_j. \end{aligned} \tag{1.31}$$

It is well known that the Circular Unitary Ensemble (CUE), which is the Haar measure on the Unitary group U(n) consisting of $n \times n$ unitary matrices, has eigenvalues on the unit circle distributed according to (1.31) with $f(z) = 1$ for $z \in C$.

For general $f \ge 0$, one may show that the distribution (1.31) is a determinantal point process on $[0, 2\pi)$ with correlation kernel

$$K_n^C(\mu,\theta) = \frac{1}{2\pi}\sqrt{f(e^{i\mu})f(e^{i\theta})} \sum_{j=0}^{n-1} \phi_j(e^{i\mu})\overline{\phi_j(e^{i\theta})},$$

where ϕ_j are orthogonal polynomials $\phi_j(z) = \pi_j z^j + \dots$, with $\pi_j \ge 0$, such that

$$\int_0^{2\pi} \phi_j(e^{i\theta})\overline{\phi_k(e^{i\theta})} f(e^{i\theta}) \frac{d\theta}{2\pi} = \delta_{jk}, \quad j,k \in \mathbb{N}. \tag{1.32}$$

By the Christoffel-Darboux formula on the unit circle, K_n^C also has the following useful form for $\mu, \theta \in [0, 2\pi)$

$$\begin{aligned} K_n^C(\mu,\theta) &= \frac{1}{2\pi}\sqrt{f(e^{i\mu})f(e^{i\theta})} \\ &\quad \times \frac{e^{in(\mu-\theta)}\phi_n(e^{i\theta})\overline{\phi_n(e^{i\mu})} - \phi_n(e^{i\mu})\overline{\phi_n(e^{i\theta})}}{1-e^{i(\mu-\theta)}}. \end{aligned} \tag{1.33}$$

In the example of the eigenvalues of the CUE where $f = 1$, one has $\phi_j(z) = z^j$ and so

$$K_n^{\mathrm{CUE}}(\mu,\theta) = \frac{1}{2\pi}\frac{1-e^{in(\mu-\theta)}}{1-e^{i(\mu-\theta)}} = \frac{1}{2\pi} e^{i(n-1)(\mu-\theta)} \frac{\sin\frac{n(\mu-\theta)}{2}}{\sin\frac{\mu-\theta}{2}}. \tag{1.34}$$

Similarly to Section 1.1.3, one may show that for general $f \geq 0$

$$\widehat{Z}_n^C = n! \prod_{j=0}^{n-1} \pi_j^{-2}. \tag{1.35}$$

1.1.4 Hankel and Toeplitz determinants

Recall that for w non-negative on $\mathbb{R}$, and f non-negative on C, define the Hankel determinant H_n and Toeplitz determinant D_n by

$$\begin{aligned} H_n(w) &= \det\left(w_{j+k}\right)_{j,k=0}^{n-1}, \qquad w_j = \int_{\mathbb{R}} x^j w(x) dx, \\ D_n(f) &= \det\left(f_{j-k}\right)_{j,k=1}^{n}, \qquad f_j = \int_0^{2\pi} e^{-ij\theta} f(e^{i\theta}) \frac{d\theta}{2\pi}, \end{aligned} \tag{1.36}$$

provided w_j, f_j exist. The following well-known relation between $\widehat{Z}_n, \widehat{Z}_n^C$ and H_n, D_n respectively gives the connection between gap probabilities and Hankel/Toeplitz determinants, and the connection to Fredholm determinants through formula (1.21), thus binding all the topics of the present thesis together.

Proposition 1.1.2. *The partition function* (1.17) *on the real line and the partition function* (1.31) *on the unit circle are given in terms of Hankel and Toeplitz determinants respectively:*

$$\widehat{Z}_n = n!\, H_n(w), \qquad \widehat{Z}_n^C = n!\, D_n(f). \tag{1.37}$$

Proof. We prove the proposition for Hankel determinants – the proof in the case of Toeplitz determinants is similar. As previously, we temporarily relabel the indices of x_j by defining $y_0, \dots, y_{n-1}$ such that $y_j = x_{j+1}$.

Substituting (1.22) into (1.17) we obtain

$$\widehat{Z}_n = \int_{\mathbb{R}^n} \det(y_i^j)_{i,j=0}^{n-1} \left(\sum_{\sigma \in S_n} \mathrm{sgn}\sigma \prod_{j=0}^{n-1} y_{\sigma(j)}^j \right) \prod_{j=0}^{n-1} w(y_j) dy_j. \tag{1.38}$$

For any permutation $\sigma \in S_n$, the following is easily obtained by permuting the indices of the variables y_j:

$$\begin{aligned} \mathrm{sgn}\sigma \int_{\mathbb{R}^n} \det(y_i^j)_{i,j=0}^{n-1} \left(\prod_{j=0}^{n-1} y_{\sigma(j)}^j \right) \prod_{j=0}^{n-1} w(y_j) dy_j \\ = \int_{\mathbb{R}^n} \det(y_i^j)_{i,j=0}^{n-1} \left(\prod_{j=0}^{n-1} y_j^j \right) \prod_{j=0}^{n-1} w(y_j) dy_j. \end{aligned} \tag{1.39}$$

The RHS of (1.39) is equal to $H_n(w)$, and by substituting it into the RHS of (1.38), we obtain formula (1.37) $\square$

1.1.5 Large n asymptotics of eigenvalues in the macroscopic scale and the equilibrium measure

Classically, one divides the study of eigenvalues with probability density (1.10) into two viewpoints or scales: the macroscopic and microscopic scales. In the macroscopic viewpoint one takes a bird's eye view of the distribution to study the density of eigenvalues on sets of fixed size as $n \to \infty$. In the microscopic view studies eigenvalue correlations on small intervals of shrinking size where one expects to find only a finite number of eigenvalues as $n \to \infty$, which is known as the local scaling.

Recall $w_n(x) = e^{-nV(x)}$. For typical eigenvalue configurations $x_1^{(n)}, \ldots, x_n^{(n)}$ distributed according to (1.10), the joint probability density function $\nu_n(x_1, \ldots, x_n)$ is relatively "large", and

$$-\frac{1}{n^2} \log \nu_n(x_1^{(n)}, \ldots, x_n^{(n)}) = \frac{1}{n^2} \sum_{1 \le i < j \le n} 2 \log \frac{1}{|x_j^{(n)} - x_i^{(n)}|} + \frac{1}{n} \sum_{j=1}^{n} V(x_j^{(n)}) \tag{1.40}$$

is "small". Now consider the limit $n \to \infty$ in (1.40). The equilibrium measure μ_V is characterized [27, 33] as the unique measure which minimizes

$$I_V(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x), \tag{1.41}$$

over the space of Borel probability measures μ on $\mathbb{R}$. It is compactly supported, and for V real analytic as is our case, it is supported on a finite union of intervals and has continuous density ψ_V, namely $d\mu_V(x) = \psi_V(x)dx$.

The set $\{x : \psi_V(x) > 0\}$ is known as the bulk of the spectrum. Let K_n be the correlation kernel (1.20) associated with the weight w_n for the eigenvalues (1.10). Then it is well known that the density

$$\rho_1^{(n)}(x) = K_n(x,x) = n\psi_V(x) + \mathcal{O}(1)$$

as $n \to \infty$, for $x \in \operatorname{supp}\mu_V$. Furthermore, for any function $f \in C_0(\mathbb{R})$, one has that almost surely

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f\left(x_j^{(n)}\right) = \int_{\mathbb{R}} f(x) d\mu_V(x),$$

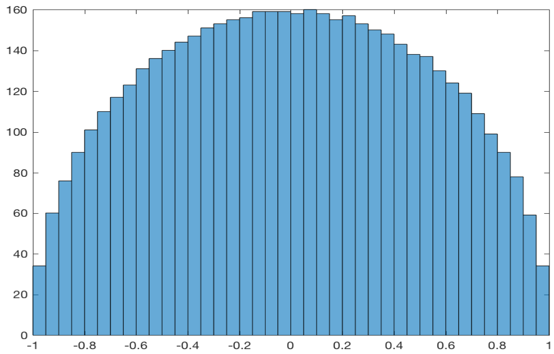

Figure 1.1: Bar plot of GUE (5000×5000)

where $\{x_j^{(n)}\}_{j=1}^n$ is the set of eigenvalues distributed according to (1.10)[2].

As an example, we mention the Gaussian Unitary Ensemble (GUE), which has potential $V(x) = 2x^2$. Then the equilibrium density μ_{GUE} is given by the semi-circle law, namely it is supported on $(-1, 1)$ and has density ψ_{GUE} which is given by

$$\psi_{\text{GUE}}(x) = \frac{2}{\pi}\sqrt{1 - x^2}. \tag{1.42}$$

In Figure 1.1 we have plotted the eigenvalues of a 5000×5000 realization of the Gaussian Unitary ensemble in a bar chart.

The equilibrium measure can equivalently be determined as the unique measure satisfying the Euler-Lagrange variational conditions (see e.g. [83])

$$2\int \log|x - y| d\mu_V(y) = V(x) - \ell, \quad \text{for } x \in \operatorname{supp}\mu_V, \tag{1.43}$$

$$2\int \log|x - y| d\mu_V(y) \leq V(x) - \ell, \quad \text{for } x \in \mathbb{R} \setminus \operatorname{supp}\mu_V, \tag{1.44}$$

for some real constant ℓ depending on V.

Throughout the rest of the introduction we will assume that μ_V is "one-cut" and "regular". This means that μ_V is supported on a single

[2]Here this means that for any $f \in C_0(\mathbb{R})$ and independent sets $\{x_j^{(n)}\}_{j=1}^n$ each distributed according to (1.10), one has for any $\epsilon > 0$ that,

$$\lim_{N\to\infty} \prod_{n=N}^{\infty} \text{Prob}\left(\left|\frac{1}{n}\sum_{j=1}^{n} f(x_j^{(n)}) - \int_{\mathbb{R}} f(x) d\mu_V(x)\right| < \epsilon\right) = 1.$$

interval (a,b), and the density ψ_V can be written in the form

$$\psi_V(x) = \sqrt{(b-x)(x-a)}h(x) \tag{1.45}$$

for $x \in (a,b)$, where h is a real analytic function on $\mathbb{R}$ and $h(x) > 0$ for $x \in [a,b]$. Furthermore, in addition to the Euler-Lagrange conditions (1.43)-(1.44), the inequality (1.44) is strict for $x \in \mathbb{R} \setminus [a,b]$.

1.1.6 Large n asymptotics of Toeplitz and Hankel determinants

It is of interest to compute the asymptotics of $D_n(f)$ and $H_n(w)$ as $n \to \infty$. Such asymptotics depend on the class of the weight in question, and we give an overview of relevant results.

Analytic weights

In the introduction we presented the Szegő strong limit theorem (1.2) which gives asymptotics for $D_n(f)$ for sufficiently smooth f satisfying (1.3).

For Hankel determinants $H_n(e^{-nV})$, the situation is in general more complicated, and not yet fully understood. In the simplest case where $V(x) = 2x^2$, we have the identity

$$\log H_n(e^{-2nx^2}) = \frac{n}{2}\log(2\pi) - \frac{n^2}{2}\log(4n) + \sum_{j=1}^{n-1}\log(j!\,), \tag{1.46}$$

and so as $n \to \infty$,

$$\begin{aligned}\log H_n(e^{-2nx^2}) = -n^2\left(\frac{3}{4} + \log 2\right)& \\ + n\log 2\pi - \frac{1}{12}\log n + \zeta'(-1) + \mathcal{O}(n^{-1}),&\end{aligned} \tag{1.47}$$

where ζ' denotes the derivative of the Riemann zeta-function.

In [84, 75] (see also [76]) it was proved that

$$\log H_n(e^{-nV(x)}) = -I_V(\mu_V)n^2 + n\log 2\pi + o(n) \tag{1.48}$$

as $n \to \infty$ for very general classes of potentials V.

The further sub-leading terms of (1.48) have been verified rigorously up to $o(1)$ for potentials V which have one-cut and two-cut regular equilibrium measures [3] – For results where V is real analytic and the equilibrium measure is one-cut regular, see [11, 40], for two-cut regular equilibrium measures and real analytic potentials, see [21]. In the two-cut case the subleading terms have oscillations given by elliptic θ-functions. Asymptotics of Hankel determinants $H_n(e^{-nV(x)})$ have also been studied in the physics literature [1, 12, 49, 41, 17], and the expansion for $\log H_n(e^{-nV(x)})$ has been conjectured up to an additive constant for very general potentials V.

In the case where the symbol is given by $e^{h(x)-nV(x)}$, the following asymptotics were found by Johansson [67] in 1998:

$$\log \frac{H_n\left(e^{h(x)-nV(x)}\right)}{H_n\left(e^{-nV(x)}\right)} = n \int h(x) d\mu_V(x) + C(h) + o(1), \tag{1.50}$$

as $n \to \infty$, where $C(h)$ is an explicit quadratic functional of h, see [67]. The function h is real valued, and in the context of this thesis an important restriction on the function h is that it is bounded and continuous on the support of the equilibrium measure $[a, b]$, excluding FH singularities which we define below. Expansion (1.50) holds for even polynomial potentials V which are one-cut regular, see [66] for conditions on h.

As mentioned in the preliminaries, the topic of this thesis is the large n asymptotics of Toeplitz and Hankel determinants which are not smooth and for which the above asymptotics do not hold, namely symbols which are indicator functions on the unit circle and symbols with Fisher-Hartwig singularities (see below).

Indicator functions

An example of such a symbol is the indicator function χ_J, where J is an arc the unit circle. A motivation for studying Toeplitz determinants

[3] An equilibrium measure is two-cut regular if if μ_V is supported on two disjoint intervals $(a_1, b_1) \cup (a_2, b_2)$, and the density ψ_V can be written in the form

$$\psi_V(x) = \prod_{j=1,2} \sqrt{(b_j - x)(x - a_j)} h(x) \tag{1.49}$$

for $x \in (a, b)$, where h is a real analytic function on $\mathbb{R}$ and $h(x) > 0$ for $x \in [a_1, b_1] \cup [a_2, b_2]$. Furthermore, in addition to the Euler-Lagrange conditions (1.43)-(1.44), the inequality (1.44) is strict for $x \in \mathbb{R} \setminus [a_1, b_1] \cup [a_2, b_2]$.

with such symbols is that

$$\begin{aligned}\frac{1}{n!}\widehat{Z}_n^C(f=\chi_J) &= D_n(\chi_J)\\ &= \text{Prob(A CUE matrix has no eigenvalues in } C\setminus J), \quad (1.51)\end{aligned}$$

where C is the unit circle, which is evident from (1.37) and (1.31). Let $J=J_2$ be a single arc of the unit circle

$$J=J_2=\left\{e^{i\theta}|\theta\in(-\pi,\theta_2)\cup(\theta_1,\pi]\right\},\quad -\pi<\theta_2<0<\theta_1<\pi. \quad (1.52)$$

The asymptotics of $D_n(\chi_{J_2})$ for a fixed arc J_2 were found by Widom [92] in 1971:

$$\log D_n(\chi_{J_2^{(n)}}) = n^2\log\cos\frac{\theta_1-\theta_2}{4}-\frac{1}{4}\log\left(n\sin\frac{\theta_1-\theta_2}{4}\right)+c_0+r_1, \quad (1.53)$$

where $r_1=\mathcal{O}(1/n)$ a $n\to\infty$, for fixed $-\pi<\theta_2<0<\theta_1<\pi$. The constant c_0 is given by

$$c_0=\frac{1}{12}\log 2+3\zeta'(-1). \quad (1.54)$$

In [31, 71], Widom's result was extended to the case of $J_2=J_2^{(n)}$ varying with n such that $|\theta_1-\theta_2|\to 0$ sufficiently slowly. Namely, the authors proved that (1.53) holds as $n\to\infty$, uniformly for $\frac{s_0}{n}\le\frac{\theta_1-\theta_2}{2}\le\pi-\epsilon$, for $\epsilon>0$ and with s_0 sufficiently large, with the following error term:

$$r_1=\mathcal{O}\left(\frac{1}{n\sin\frac{\theta_1-\theta_2}{4}}\right). \quad (1.55)$$

In Chapter 2 we consider the asymptotics of $D_n(\chi_J)$, where $J=J^{(n)}$ is the union of 2 disjoint arcs, and the arclengths are scaled with the parameter n in a specific way. This is used to prove a result on Fredholm determinants.

As a warm-up, we now present our first new (to the extent of the author's knowledge) result in this thesis, which is a very simple identity which allows one to generalize (1.53) to Toeplitz determinants supported on multiple arcs with a particular symmetry. Fix $m\in\mathbb{N}\setminus\{0\}$, let $0<\gamma<\pi/m$ and denote

$$\begin{aligned} I(\gamma) &= \{e^{i\theta}: -\gamma<\theta<\gamma\},\\ I_m(\gamma) &= \cup_{j=0}^{m-1}\{ze^{2\pi ij/m}: z\in I(\gamma)\}.\end{aligned} \quad (1.56)$$

From the multiple integral representation of $\widehat{Z}_n^C$ from (1.31), it is clear that the Toeplitz determinant is invariant under rotation of the circle, so for any $\mu \in \mathbb{R}$ we have $D_n(f(z)) = D_n(f(ze^{i\mu}))$, and thus $D_n(\chi_{I(\gamma)}) = D_n(\chi_{J_2})$ when $2\gamma = \theta_1 - \theta_2$, meaning that formula (1.53) applies. The set $I_m(\gamma)$ is composed of m disjoint arcs, such that $e^{2\pi ij/m} I_m(\gamma) = I_m(\gamma)$ for $j = 0, 1, 2, \ldots, m-1$. We use this rotational invariance to prove the following exact identity relating the Toeplitz determinant supported on $I_m(\gamma)$ to the Toeplitz determinant supported on $I(m\gamma)$.

Proposition 1.1.3 (Toeplitz determinant identity)**.** *Fix $m \in \mathbb{N} \setminus \{0\}$, and let $0 < \gamma < \pi/m$. Let $\phi_j(z) = \pi_j z^j + \ldots$ be orthonormal on $I(m\gamma)$, i.e.*

$$\int_{I(m\gamma)} \phi_j(z)\overline{\phi_k(z)} \frac{dz}{2\pi iz} = \delta_{jk}. \tag{1.57}$$

If $n \in \mathbb{N}$ and

$$n = n_1 m + n_2, \quad n_2 \in \{0, 1, 2, \ldots, m-1\}, \tag{1.58}$$

then

$$\begin{aligned} D_n(\chi_{I_m(\gamma)}) &= D_{n_1}(\chi_{I(m\gamma)})^m \pi_{n_1}^{-2n_2} && (1.59)\\ &= D_{n_1}(\chi_{I(m\gamma)})^{m-n_2} D_{n_1+1}(\chi_{I(m\gamma)})^{n_2}. && (1.60)\end{aligned}$$

Furthermore, as $n, n_1 \to \infty$,

$$\begin{aligned}\log D_n(\chi_{I_m(\gamma)}) = mn_1^2 \log\sin\frac{m\gamma}{2} + (2n_1+1)n_2 \log\sin\frac{m\gamma}{2} \\ - \frac{m}{4}\log\left(n_1\cos\frac{m\gamma}{2}\right) + mc_0 + \mathcal{O}\left(\frac{1}{n}\right),\end{aligned} \tag{1.61}$$

for fixed m, γ.

Proof. Equality between the RHS of (1.59) and (1.60) is immediate from (1.35), (1.37). Formula (1.61) is obtained by substituting Widom's asymptotics from (1.53) into (1.60).

It remains to prove (1.59). By the orthogonality of the polynomials ϕ_j on $I(m\gamma)$ we have, taking $z = y^m$,

$$m\int_{I(\gamma)} \phi_j(y^m)\overline{\phi_k(y^m)} \frac{dy}{2\pi iy} = \delta_{jk}. \tag{1.62}$$

Then taking $y \mapsto e^{2\pi ij/m} y$ for $j = 0, 1, 2, \ldots, m-1$ we have

$$\int_{I_m(\gamma)} \phi_j(y^m)\overline{\phi_k(y^m)} \frac{dy}{2\pi iy} = \delta_{jk}. \tag{1.63}$$

Note that

$$\begin{aligned}\int_{I_m(\gamma)} z^k \frac{dz}{2\pi iz} &= \sum_{j=0}^{m-1} (e^{2\pi ij/m})^k \int_{I(\gamma)} z^k \frac{dz}{2\pi iz} \\ &= \begin{cases} 0 & \text{if } k \not\equiv 0 \mod m, \\ m \int_{I(\gamma)} z^k \frac{dz}{2\pi iz} & \text{if } k \equiv 0 \mod m. \end{cases}\end{aligned} \tag{1.64}$$

Define $\psi_j(y) = \phi_{j_1}(y^m)y^{j_2}$ where $j = j_1 m + j_2$ and $j_2 \in \{0, 1, 2, \dots, m-1\}$. Then, similarly denoting $j' = j_1' m + j_2'$, it follows that if $j_2 \neq j_2'$, then all the monomials in $\psi_j(y)\overline{\psi_{j'}(y)}$ have powers which are non-zero modulo m, and thus by (1.64) we have

$$\int_{I_m(\gamma)} \psi_j(y)\overline{\psi_{j'}(y)} \frac{dy}{2\pi iy} = 0. \tag{1.65}$$

If $j_2 = j_2'$, then by (1.63)

$$\int_{I_m(\gamma)} \psi_j(y)\overline{\psi_{j'}(y)} \frac{dy}{2\pi iy} = \delta_{j_1, j_1'}. \tag{1.66}$$

Thus ψ_j are orthonormal on $I_m(\gamma)$, i.e.

$$\int_{I_m(\gamma)} \psi_j(z)\overline{\psi_k(z)} \frac{dz}{2\pi iz} = \delta_{jk}. \tag{1.67}$$

By standard theory,

$$D_n(\chi_{I_m(\gamma)}) = \prod_{j=0}^{n-1} \eta_j^{-2}, \tag{1.68}$$

where $\psi_j(z) = \eta_j z^j + \dots$. It follows from the definition of ψ_j that

$$\eta_j = \pi_{j_1}, \tag{1.69}$$

and plugging this into (1.68) we have proven the identity (1.59). □

Fisher-Hartwig singularities

Another class of Toeplitz determinants which do not satisfy the Szegő asymptotics, are Toeplitz determinants with Fisher-Hartwig singularities. A Toeplitz determinant with a Fisher-Hartwig singularity at each point $z_j = e^{i\theta_j} \in C$ for $j = 1, \dots, k$ has a symbol of the following form:

$$\begin{aligned} f_{\alpha,\beta}(z) &= e^{V(z)} \prod_{j=1}^{k} \left(\frac{z}{z_j}\right)^{\beta_j} |z - z_j|^{2\alpha_j} \omega_j(z), \\ \omega_j(z) &= \begin{cases} e^{i\pi\beta_j} & 0 \le \arg \, < \theta_j, \\ e^{-i\pi\beta_j} & \theta_j \le \arg z < 2\pi, \end{cases} \end{aligned} \tag{1.70}$$

where $\operatorname{Re}\alpha_j > -1/2$ and $\beta_j \in \mathbb{C}$. As mentioned in the preliminaries, Toeplitz determinants with symbol (1.70) appear, among other places, in the study of the long range correlations of the Ising model [30]. The large n asymptotics of Toeplitz determinants with such symbols have been extensively studied [13, 95, 8, 39, 29]. In [29] the authors were able to find an asymptotic expansion of Toeplitz determinants with any fixed number of Fisher-Hartwig singularities, at fixed positions, for all parameters z_j, α_j, β_j. In [19], the authors studied the transition of asymptotic regime between the Szegő asymptotics and the case of a single Fisher-Hartwig singularity, while in [20] the transition between one and two Fisher-Hartwig singularities was studied. It is also possible to study the transition between Fisher-Hartig symbols of type β and the indicator function, see [15, 16].

Similarly, one may consider Hankel determinants with Fisher-Hartwig singularities. A Hankel determinant with a Fisher-Hartwig singularity at the points $y_j \in \mathbb{R}$ for $j = 1, \dots, k$ has symbol

$$w_n(x) = e^{-nV(x)} \prod_{j=1}^{k} |x - y_j|^{2\alpha_j} \omega_j(x),$$
$$\omega_j(x) = \begin{cases} e^{i\pi\beta_j} & \text{for } x > y_j, \\ e^{-i\pi\beta_j} & \text{for } x \leq y_j, \end{cases} \tag{1.71}$$

where $\operatorname{Re}\alpha_j > -1/2$ and $\beta_j \in \mathbb{C}$. Here, a Fisher-Hartwig singularity is of interest if it lies in the support of the equilibrium measure μ_V or at the edge-points. We restrict ourselves to the discussion of singularities in the support μ_V. Hankel determinants with Fisher-Hartwig singularities are connected to the one-dimensional gas of impenetrable bosons, and also other physical models [46, 47, 50, 53]. Asymptotics for such Hankel determinants with fixed number of α-singularities (where $\beta = 0$) were found for integer α_j in [14, 46, 54], and for general α_j in the case of quadratic potentials [70], and finally generalized to one-cut regular potentials in [9], which we present in Section 1.3. Asymptotics of Hankel determinants with quadratic potentials with a single β-singularity (with $\alpha = 0$) were found in [61]. In Chapter 3, we consider the large n asymptotics of Hankel determinants, in particular the transition from analytic potentials to a single FH singularity, and also the transition from one FH singularity to two FH singularities, for potentials V with one-cut regular equilibrium measures.

1.2 Microscopic scale and gap probabilities

We now turn to the study of the statistical properties of the eigenvalues on small intervals with sizes of order $\mathcal{O}(1/n)$ as $n \to \infty$. We will give an overview of results from Chapter 2, which is based on [42] which was written in collaboration with Igor Krasovsky.

1.2.1 Sine kernel universality

Consider the eigenvalues distributed according to (1.10) with correlation kernel K_n. Assume a real-analytic potential V satisfying (1.9). It was proven in [34, 35], that uniformly for u, v in compact subsets of $\mathbb{R}$, and x such that $\psi_V(x) > 0$, one has

$$\lim_{n\to\infty} \frac{1}{\psi_V(x)n} K_n\left(x + \frac{u}{\psi_V(x)n}, x + \frac{v}{\psi_V(x)n}\right) = \mathbb{K}^{\sin}(u,v) = \frac{\sin \pi(u-v)}{\pi(u-v)}. \tag{1.72}$$

This is known as sine kernel universality, because the result is, up to rescaling by $\psi_V(x)$, independent of V, and the sine process with correlation kernel $\mathbb{K}^{\sin}$ is known as the bulk scaling limit.

Formula (1.72) also holds if K_n is replaced by the correlation kernel K_n^{CUE} associated with the CUE, in which case the formula is easy to prove. Then (1.72) follows from (1.34) with Remark 1.1.1 and the fact that the limiting density is $1/2\pi$.

In Figure 1.2, we consider the same realization of a 5000×5000 GUE matrix as in Figure 1.1, and we zoom in on an interval of size 0.01 at the origin which contains 30 eigenvalues, which are plotted in blue. At this scale, the density ψ_{GUE} in (1.42) can almost be viewed as constant, and the distribution should be similar to that of the sine process. There is repulsion between the points, which gives the result that they are somewhat regularly spaced, and indirectly leads to large gaps between the eigenvalues being unlikely. The points which are plotted in red, are 30 independent points distributed according to the uniform distribution (or equivalently, the Poisson process conditioned to have 30 points on the interval). The distribution of the blue points is more regular than that of the red points, among the red points there is for example a large gap on the right side. This will be formalized mathematically, see Remark 1.2.1.

Gap probabilities of the eigenvalues (1.10) **in the local scaling**

We use (1.73) to motivate the study of the gap probabilities of the sine process, through studying the Fredholm determinant of the sine kernel, which is the topic of this section and Chapter 2. However, for this motivation to be well founded, we need the following to hold true

$$\lim_{n\to\infty} \mathrm{Prob}\left(n\psi_V(x)\left(x_j^{(n)}-x\right) \notin A \text{ for all } j=1,\dots,n\right) \\ = \det(I-\mathbb{K}^{\sin})_A. \quad (1.73)$$

Formula (1.73) is proven by using (1.16) and (1.72), and the following proposition which states that convergence of sequences of kernels implies convergence of Fredholm determinants under certain conditions.

Lemma 1.2.1. *Let A be a compact set, and consider a sequence of kernels $K_n(x,y)$ on A^2, which is uniformly bounded in n, and converges pointwise to $K(x,y)$. Then, as $n\to\infty$,*

$$\det(I-K_n)_A \to \det(I-K)_A. \quad (1.74)$$

Proof. Consider the truncated sum

$$\det(I-K)_A^M = 1+\sum_{k=1}^{M}\frac{(-1)^k}{k!}\int_{A^k}\det(K(y_i,y_j))_{i,j=1}^k dy_1\dots dy_k.$$

By Hadamard's inequality and the uniform boundedness of K_n, it follows that for any $\epsilon>0$ there exists $M\in\mathbb{N}$ such that $m>M$ implies that

$$|\det(I-K_n)_A-\det(I-K)_A| \\ \leq |\det(I-K_n)_A^m-\det(I-K)_A^m|+\epsilon. \quad (1.75)$$

For fixed m,

$$|\det(I-K_n)_A^m-\det(I-K)_A^m| \to 0 \quad (1.76)$$

as $n\to\infty$. Thus (1.74) follows from (1.75)-(1.76). □

1.2.2 The Fredholm determinant of the sine kernel

We parametrize the sine kernel as follows. Let

$$K_s(x,y)=\frac{\sin s(x-y)}{\pi(x-y)}.$$

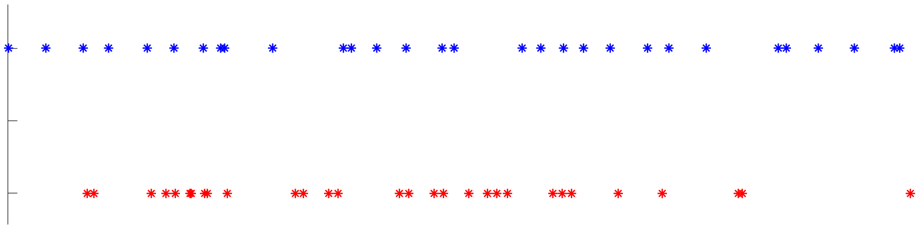

Figure 1.2: Bulk scaling limit vs. independent uniform points.

and denote the Fredholm determinant of K_s on a set A by

$$P_s(A) = \det(I - K_s)_A. \tag{1.77}$$

We will study $P_s(A)$ as $s \to \infty$. First we consider $P_s(A_0)$ where A_0 is the single interval

$$A_0 = (\alpha, \beta).$$

As a function of s it is transcendental and satisfies:

$$s\frac{d}{ds}\log P_s(A_0) = \sigma_0(-2i(\beta-\alpha)s),$$

where, denoting $\frac{d}{ds}$ by $'$, the function $\sigma_\rho(s)$ satisfies the Jimbo-Miwa-Okamoto Painlevé V differential equation [65]:

$$(s\sigma_\rho'')^2 = (\sigma_\rho - s\sigma_\rho')(\sigma_\rho - s\sigma_\rho' + 4\sigma_\rho'(\sigma_\rho' + \rho)). \tag{1.78}$$

The most general form of the Painlevé V differential equation has 4 parameters, we include the parameter ρ because it will be referred to in Section 1.3. As $s \to \infty$, the Fredholm determinant on the single interval has asymptotics

$$\begin{aligned} \log P_s(A_0) = -\frac{(\beta-\alpha)^2 s^2}{8} - \frac{1}{4}\log s& \\ -\frac{1}{4}\log\frac{\beta-\alpha}{2} + c_0 + \mathcal{O}(s^{-1})&, \\ c_0 = \frac{1}{12}\log 2 + 3\zeta'(-1)&, \end{aligned} \tag{1.79}$$

where ζ is the Riemann zeta-function.

The two leading terms in (1.79) were conjectured by des Cloizeaux and Mehta [23] in 1973, while Dyson was able to identify the constant c_0 in [37] in 1976. The constant c_0 became known as the Widom-Dyson constant. Formula (1.79) was rigorously confirmed in a series of works.

Widom [93] was able to verify the leading term, while Deift, Its, Zhou [32] calculated the leading term and the logarithmic term. The constant c_0 was independently proven by Krasovsky [71] and Ehrhardt [38], with a third proof by Deift, Its, Krasovsky, Zhou [31].

Remark 1.2.1. The point process with correlation kernel K_s has an average particle spacing of $\frac{s}{\pi}$, and (1.79) implies that the probability of a single gap $(-1,1)$ in the particle distribution is $e^{-\frac{s^2}{2}(1+o(1))}$ as $s \to \infty$. It is natural to compare this to the probability of a gap $(-1,1)$ in the particle distribution of a rate $\frac{s}{\pi}$ Poisson process, which is given exactly by $e^{-s/\pi}$. Thus, asymptotically, large gaps are more improbable for the sine process than the Poisson process.

Now we turn to study the large s asymptotics of $\det(I-K_s)_A$ when A is the union of 2 disjoint intervals

$$A = A_1 \bigcup A_2, \quad A_1 = (\alpha, -\nu),\ A_2 = (\nu, \beta). \tag{1.80}$$

Since the sine process is invariant under translation, such a representation of the 2 intervals is not a restriction. For fixed $\nu > 0$, this problem was considered in [94, 32]. It is also of interest to let $\nu \to 0$ as $s \to \infty$, and this is the topic of Chapter 2, with the important restriction that $s\nu \log \nu^{-1} \to 0$ as $s \to \infty, \nu \to 0$. We first present the results in [94, 32] for fixed ν, and then present our own results for $\nu \to 0$.

In 1995, Widom [94] proved that as $s \to \infty$,

$$\frac{d}{ds} \log P_s(A) = C_1 s + C_2(s) + o(1)$$

where C_1 is an explicit constant while $C_2(s)$ is a certain oscillating function given by a Jacobi inversion problem. In 1997, Deift, Its and Zhou [32] were able to improve on this, and we will present their result. Define the θ-function of the third kind:

$$\theta(z) = \sum_{m \in \mathbb{Z}} e^{2\pi i z m + \pi i \tau m^2}, \qquad \mathrm{Im}\,(\tau) > 0.$$

It satisfies the periodicity properties:

$$\begin{aligned} \theta(z) &= \theta(z+1), \\ \theta(z \pm \tau) &= e^{\mp 2\pi i z - \pi i \tau} \theta(z), \end{aligned}$$

and is even: $\theta(z) = \theta(-z)$. Let

$$r(z) = ((z-\alpha)(z+\nu)(z-\nu)(z-\beta))^{1/2}, \tag{1.81}$$

with branch cuts on A and such that $r(z) \sim z^2$ as $z \to \infty$. Define uniquely the monic polynomial $q(z) = z^2 + q_1 z + q_0$ of degree 2 such that

$$\int_{A_j} \frac{q(z)dz}{r_+(z)} = 0,$$

for $j = 1, 2$, where, for definiteness, $r_+(z)$ is the limit of $r(z + i\epsilon)$ as $\epsilon \to 0_+$. Then q/r has no residue at infinity. Thus, as $z \to \infty$, q/r has the form:

$$\frac{q(z)}{r(z)} = 1 + \frac{G_1}{z^2} + \mathcal{O}(z^{-1}), \tag{1.82}$$

which we take as a definition for G_1. Then, as $s \to \infty$ and for fixed α, β, ν, we have

$$\log P_s(A) = -G_1 s^2 + G_2 \log s + \log \theta(sV; \tau) + G_3 + \mathcal{O}(s^{-1}), \tag{1.83}$$

where the constants τ and V appearing in the argument of the θ-function are given in terms of q and r:

$$\tau = i \frac{\int_{-\nu}^{\nu} \frac{dx}{|r(x)|}}{\int_{\nu}^{\beta} \frac{dx}{|r_+(x)|}}, \qquad V = \frac{1}{\pi} \int_{-\nu}^{\nu} \frac{q(x)dx}{r(x)}. \tag{1.84}$$

The constant G_2 has an explicit expression, and we refer the reader to [32]. The constant G_3 is an unknown constant of integration. The reason it is unknown is that asymptotics were obtained for $\frac{d}{ds} \log P_s(A)$, and integrated in s to yield (1.83), but there is no clear starting point for integration s_0 where the formula for $\log P_{s_0}(A)$ would be known. We also note that in both [94] and [32] the authors obtained similar formulae for $\log \det(I - K_s)$ supported on any fixed number of intervals (and with fixed endpoints).

We now present our results from Chapter 2 where, in contrast to (1.83), we let $\nu \to 0$ as $s \to \infty$. Let $\gamma = \frac{1}{8}(\beta^{-1} - \alpha^{-1})$ and

$$\omega = \frac{s\sqrt{|\alpha\beta|}}{\log(\gamma\nu)^{-1}} > 0. \tag{1.85}$$

Clearly, ω is uniquely represented in the form

$$\omega = k + x, \quad k = 0, 1, 2, \ldots, \quad x \in [-1/2, 1/2).$$

We note that ν has the form

$$\nu = \gamma^{-1} e^{-\frac{s\sqrt{|\alpha\beta|}}{k+x}}.$$

We prove the following:

Theorem 1.2.2. *As $s \to \infty$, uniformly for $\nu \in (0, \nu_0)$, where $s\nu_0 \log \nu_0^{-1} \to 0$,*

$$\begin{aligned} \log P_s(A) &= \log P_s(A_0) + s\sqrt{|\alpha\beta|}\left(\omega - \frac{x^2}{\omega}\right) + c(k) \\ &\quad + \delta_k(x) + \mathcal{O}(\max\{s\nu_0 \log \nu_0^{-1}, 1/\log \nu_0^{-1}, 1/s\}), \\ c(k) &= \log\left(\frac{2^{2k^2-k}}{\pi^k}\frac{G(k+1)^4}{G(2k+1)}\right), \\ \delta_k(x) &= \log\left(1 + 2\pi\kappa_{k-1}^2(\gamma\nu)^{1+2x}\right) \\ &\qquad + \log\left(1 + (2\pi\kappa_k^2)^{-1}(\gamma\nu)^{1-2x}\right), \end{aligned} \tag{1.86}$$

where G is the Barnes G-function, and where κ_j is the leading coefficient of the Legendre polynomial of degree j orthonormal on the interval $[-2, 2]$, given by

$$\begin{aligned} &\kappa_j = 4^{-j-1/2}\sqrt{2j+1}\frac{(2j)!}{j!^2}, \qquad j = 1, 2, \ldots, \\ &\kappa_0 = 1/2, \qquad \kappa_{-1} = 0. \end{aligned} \tag{1.87}$$

As $s \to \infty$, uniformly for $\nu \in (\nu_1, \nu_0)$, where $s\nu_0 \log \nu_0^{-1} \to 0$ and $\frac{s}{\log \nu_1^{-1}} \to \infty$ (i.e., $k \to \infty$), formula (1.86) *reduces to*

$$\begin{aligned} \log P_s(A) = s^2 &\left(-\frac{(\beta-\alpha)^2}{8} + \frac{|\alpha\beta|}{\log(\gamma\nu)^{-1}}\right) - \frac{1}{2}\log s + \frac{1}{4}\log\log(\gamma\nu)^{-1} \\ &- x^2 \log(\gamma\nu)^{-1} + \log\left(1 + (\gamma\nu)^{1-2|x|}\right) - \frac{1}{4}\log\left(\frac{\beta-\alpha}{2}\sqrt{|\alpha\beta|}\right) \\ &+ \frac{1}{6}\log 2 + 6\zeta'(-1) + \mathcal{O}\left(\max\left\{s\nu_0 \log \nu_0^{-1}, \frac{1}{\log \nu_0^{-1}}, \frac{\log \nu_1^{-1}}{s}\right\}\right). \end{aligned} \tag{1.88}$$

Remark 1.2.2. Note that if $k = 0$ and $x \in (0, 1/2 - \epsilon)$ for $\epsilon > 0$, then (1.86) shows that $\frac{P_s(A)}{P_s(A_0)} \to 1$ as $s \to \infty$.

Remark 1.2.3. As $k \to \infty$, one has that $2\pi\kappa_k \to 1$, and by the standard asymptotics of the Barnes G-function,

$$c(k) = -\frac{1}{4}\log k + \frac{1}{12}\log 2 + 3\zeta'(-1) + \mathcal{O}(1/k^2). \tag{1.89}$$

The function $c(k)$ can alternatively be described in terms of the coefficients (1.87) of the Legendre polynomials:

$$c(k) = -\sum_{j=0}^{k-1} \log 2\pi\kappa_j^2, \quad k = 1, 2, \ldots, \qquad c(0) = 0. \tag{1.90}$$

Formula (1.90) shows that $\delta_k(x) + c(k)$ is continuous also at the points $|x| = 1/2$.

Remark 1.2.4. The rescaled sine process (with expected distance between particles π/s) is the determinantal point process with the m'th correlation function ρ_m, for $m = 1, 2, \dots$, given by

$$\rho_m(x_1, \dots, x_m) = \det(K_s(x_i, x_j))_{i,j=1}^m.$$

Consider the sine process conditioned to have no eigenvalues in A. Denote this process by $\mathbb{P}_A$ and its m'th correlation function by ρ_m^A. In Chapter 2, we show that for $x_1, \dots, x_m \in (-1, 1)$,

$$\nu^m \rho_m^A(\nu x_1, \dots, \nu x_m) \to \det((2K_{\mathrm{Leg}}(2x_i, 2x_j))_{i,j=1}^m \tag{1.91}$$

as $s \to \infty$ and $\nu \to 0$ such that $k \in \mathbb{N}$ and $|x| < 1/2$ remain fixed, where

$$K_{\mathrm{Leg}}(x, y) = \frac{\kappa_{k-1}}{\kappa_k} \frac{L_k(x)L_{k-1}(y) - L_k(y)L_{k-1}(x)}{x - y}, \tag{1.92}$$

and L_k is the Legendre polynomial of degree k, orthonormal on $[-2, 2]$:

$$\int_{-2}^{2} L_j(x)L_i(x)dx = \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases}$$

The process with kernel K_{Leg} is a k-point process. Thus we obtain from (1.91) and the first equation of (1.13) that, as $s \to \infty$ and $\nu \to 0$ such that $k \in \mathbb{N}$ and $|x| < 1/2$ remain fixed, the expected number of $(k+1)$-tuples of $\mathbb{P}_A$ on $(-\nu, \nu)$ converges to 0, while the expected number of k-tuples on the same interval converges to 1. It follows from the second equation of (1.13) that

$$\mathrm{Prob}\,(\mathbb{P}_A \text{ has } k \text{ particles in } (-\nu, \nu)) \to 1.$$

1.2.3 Method of proof

By (1.21), (1.37), and (1.73), we have that

$$D_n\left(\chi_{J^{(n)}}\right) \to \det(I - K_s)_A \tag{1.93}$$

as $n \to \infty$ for fixed s, where

$$J^{(n)} = \{e^{i\theta} : \theta \notin \frac{2s}{n}A\}. \tag{1.94}$$

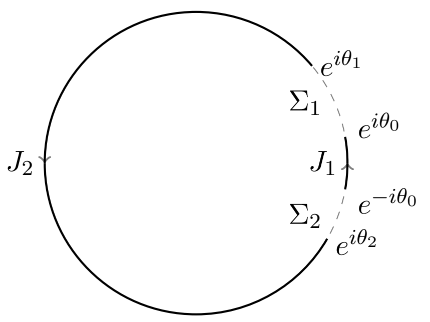

Figure 1.3: Interval J.

More generally, let $-\pi < \theta_2 < 0 < \theta_0 < \theta_1 < \pi$ and define $J = J_1 \cup J_2$ where J_1, J_2 are as in Figure 1.3:

$$J_1 = J_1(\theta_0) = \{e^{i\theta} | \theta \in (-\theta_0, \theta_0)\}, \quad J_2 = \{e^{i\theta} | \theta \in (\theta_1, \pi] \cup (-\pi, \theta_2)\}.$$

By (1.35) and (1.37), we have

$$D_n(\chi_J) = \prod_{j=0}^{n-1} (\pi_j)^{-2}, \tag{1.95}$$

where π_j are the leading coefficients of the polynomials ϕ_j on the circle orthogonal with respect to the weight χ_J as in (1.32).

We use (1.95) as a starting point to derive the following differential identity. The Toeplitz determinant $D_n(\chi_J)$ satisfies

$$\frac{\partial}{\partial \theta_0} \log D_n(\chi_J) = -\frac{1}{2\pi}[F(e^{-i\theta_0}) + F(e^{i\theta_0})],$$

where

$$F(z) = n|\phi_n(z)|^2 - 2\mathrm{Re}\left(z\overline{\phi_n(z)} \frac{d}{dz} \phi_n(z) \right).$$

Define Y in terms of the orthogonal polynomials ϕ_j

$$Y(z) = Y(z; J) = \begin{pmatrix} \chi_n^{-1} \phi_n(z) & \chi_n^{-1} \int_J \frac{\phi_n(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i \zeta^n} \\ -\chi_{n-1} z^{n-1} \overline{\phi}_{n-1}(z^{-1}) & -\chi_{n-1} \int_J \frac{\overline{\phi}_{n-1}(\zeta^{-1})}{\zeta - z} \frac{d\zeta}{2\pi i \zeta} \end{pmatrix}.$$

One can express F in terms of the elements of Y:

$$F(z) = n\chi_n^2 |Y_{11}(z)|^2 - 2\chi_n^2 \mathrm{Re}\left(z\overline{Y_{11}(z)} \frac{d}{dz} Y_{11}(z) \right). \tag{1.96}$$

The function Y uniquely solves the following Riemann-Hilbert (RH) problem

(a) $Y : \mathbb{C} \setminus J \to \mathbb{C}^{2\times 2}$ is analytic;

(b) Y possesses L^2 boundary values Y_+ and Y_- on the $+$ and $-$ side of J, respectively, related by the condition:

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} \\ 0 & 1 \end{pmatrix} \text{ for } z \in J;$$

(c) $Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

The fact that orthogonal polynomials satisfy a RH problem was first observed for polynomials orthogonal on the real line by Fokas, Its, Kitaev [45], and extended to polynomials orthogonal on the unit circle by Baik, Deift, Johansson [7].

We use the Deift-Zhou steepest descent analysis [36] to obtain asymptotics for the solution for the RH problem for $Y(z; J^{(n)})$ with $J^{(n)}$ from (1.94) as $s, n \to \infty$ and $\nu \to 0$ such that $s^3/n \to 0$ and $s\nu \log \nu^{-1} \to 0$. Subsequently we obtain asymptotics for F in the same scaling limit, and we substitute these asymptotics into the integral

$$\begin{aligned} &\log D_n(\chi_{J^{(n)}}; \nu) \\ &\quad = \log D_n(\chi_{J^{(n)}}; \nu = 0) - \frac{1}{2\pi} \int_0^{\theta_0 = \frac{2s\nu}{n}} (F(e^{-i\theta_0'}) + F(e^{i\theta_0'}))d\theta_0', \end{aligned}$$

and as a result we obtain

$$\begin{aligned} \log D_n(\chi_{J^{(n)}}; \nu) &= \log D_n(\chi_{J^{(n)}}; \nu = 0) + s\sqrt{|\alpha\beta|} \left(\omega - \frac{x^2}{\omega} \right) \\ &\quad + c(k) + \delta_k(x) + \mathcal{O}(\max\{s\nu \log \nu^{-1}, 1/\log \nu^{-1}, s^3/n, 1/s\}), \end{aligned}$$

as $s, n \to \infty$ and $\nu \to 0$ such that $s^3/n \to 0$ and $s\nu \log \nu^{-1} \to 0$. We take $n \to \infty$ and use (1.93) to prove Theorem 1.2.2.

1.3 The characteristic polynomial

In this section, we give an overview of results in Chapter 3. The chapter is based on [18], which was written in collaboration with Tom Claeys. Let H_n be distributed according to the Unitary Ensemble (1.8) with potential V. Denote by P_n the characteristic polynomial of H_n:

$$P_n(x) = \det(H_n - Ix). \tag{1.97}$$

Since H_n is a random variable, $P_n(x)$ is a random variable. It is conjectured that $P_n(x)$ shares many statistical properties with the zeros of the Riemann zeta-function along the critical line [69, 59, 24, 25, 57, 5]. In Chapter 3 we study the moments of $P_n(x)$.

Denote $z_0 = \sqrt{t}$ when t is positive and $z_0 = i\sqrt{-t}$ when t is negative, and define:

$$g_n(t) = \mathbb{E}(|P_n(-z_0)P_n(z_0)|^{\alpha}). \tag{1.98}$$

Note that if $y \in \mathbb{R}$, then one may express

$$\mathbb{E}(|P_n(y - z_0)P_n(y + z_0)|^{\alpha})$$

in terms of $g_n(t)$ by translating the potential $V(x) \mapsto V(x + y)$, so the form of $g_n(t)$ is not restrictive. The following results were first found for integer α in [14, 46, 54], obtained by Krasovsky [70] for the Gaussian Unitary Ensemble which has potential $V(x) = -2x^2$ and general $\alpha > -1/2$, and brought to its most general form with one-cut regular potentials by Berestycki, Webb and Wong in [9]. Let $\alpha, t \geq 0$, let V be a real-analytic potential satisfying (1.9) which is one-cut regular (as defined in Section 1.1.5), and assume $b - a = 2$, with $\sqrt{t}, -\sqrt{t} \in (a, b)$. Denote

$$f(x) = \frac{\alpha}{2}(V(x) + V(-x) - 2\ell).$$

Then as $n \to \infty$,

$$g_n(0) = e^{nf(0)} n^{\alpha^2} \psi_V(0)^{\alpha^2} \pi^{\alpha^2} \frac{G(1+\alpha)^2}{G(1+2\alpha)}(1 + o(1)) \tag{1.99}$$

while for $t > 0$ fixed as $n \to \infty$,

$$\begin{aligned} g_n(t) = e^{nf(\sqrt{t})} n^{\alpha^2/2} t^{-\alpha^2/4} (\psi_V(\sqrt{t})\psi_V(-\sqrt{t}))^{\alpha^2/4} \\ \times \left(\frac{\pi}{4}\right)^{\alpha^2/2} \frac{G(1+\alpha/2)^4}{G(1+\alpha)^2}(1 + o(1)). \end{aligned} \tag{1.100}$$

where G is the Barnes G-function.

Equation (1.100) is valid for $t \geq 0$ fixed as $n \to \infty$. For $t < 0$ the following was proven by Johansson [66] in 1998. When $n \to \infty$ and $t < 0$ remains fixed,

$$\log g_n(t) = n\alpha \int \log|x^2 - t| d\mu_V(x) + \mathcal{A}(t) + o(1), \tag{1.101}$$

$$\mathcal{A}(t) = -\frac{\alpha^2}{2\pi^2} \int_a^b \frac{\log|x^2 - t|}{\sqrt{(b-x)(x-a)}} \int_a^b \frac{y}{y^2 - t} \frac{\sqrt{(b-y)(y-a)}}{y - x} dy dx,$$

where the divergent integral is evaluated in the principal value sense.

We now present the results from Chapter 3 which are valid for both positive and negative t, as $n \to \infty$ and $t \to 0$.

In Chapter 3 we define two functions σ_α^+ and σ_α^- in terms of the (unique) solutions Ψ^+ and Ψ^- of two RH problems, see Sections 3 and 2 of Chapter 3 respectively. The two functions σ_α^+ and σ_α^- satisfy the Jimbo-Miwa-Okamoto σ-form of the Painlevé V equation (1.78).

The first one, $\sigma_\alpha^-(s)$, is analytic and real for $s \in (0, +\infty)$, and was proven in [19] to have the asymptotics

$$\sigma_\alpha^-(s) = \begin{cases} \alpha^2 + o(1) & s \to 0, \\ s^{-1+2\alpha} e^{-s} \frac{1}{\Gamma(\alpha)^2}(1 + \mathcal{O}(s^{-1})) & s \to +\infty, \end{cases}$$

where Γ is Euler's Γ-function. The existence of such a solution was proven in [19]. The second solution $\sigma_\alpha^+(s)$, is analytic and such that $\sigma_\alpha^+(s) + \frac{\alpha s}{2}$ is real-valued for $s \in -i\mathbb{R}_{>0}$, and was proven in [20] to have the asymptotics:

$$\sigma_\alpha^+(s) = \begin{cases} \alpha^2 + o(1) & s \to -i0^+, \\ \frac{\alpha^2}{2} - \frac{\alpha s}{2} + \mathcal{O}(|s|^{-1}) & s \to -i\infty. \end{cases}$$

For general values of $\alpha > -1/2$, $\sigma_\alpha^\pm$ are transcendental functions, but in Chapter 3 we show that there are special values of α where $\sigma_\alpha^\pm$ are algebraic functions. For σ_α^-, this is the case if α is an integer, while for σ_α^+ this is only the case if α is an even integer. For those values of α, we show that $\sigma_\alpha^\pm$ can be constructed via an explicit recursive procedure. For $\alpha = 0, 1, 2$, the expressions are simple and we have

$$\sigma_0^\pm(s) = 0, \tag{1.102}$$

$$\sigma_1^-(s) = \frac{s}{e^s - 1}, \tag{1.103}$$

$$\sigma_2^\pm(s) = s\frac{-2 + e^s(2 - 2s + s^2)}{1 - e^s(2 + s^2) + e^{2s}}. \tag{1.104}$$

In Chapter 3 we prove the following theorem.

Theorem 1.3.1. *Let V be real analytic, satisfying* (1.9) *and one-cut regular (as defined in Section 1.1.5) with* 0 *in the interior of the support $[a, b]$ of the equilibrium measure μ_V. Denote $z_0 = \sqrt{t}$ when t is positive and $z_0 = i\sqrt{-t}$ when t is negative. Define*

$$s_{n,t} = \begin{cases} -4\pi i n z_0 \psi_V(0) & \text{for } t > 0, \\ -2\pi i n \int_{-z_0}^{z_0} h(s)\left((s-a)(b-s)\right)^{1/2} ds & \text{for } t < 0, \end{cases}$$

with h defined in (1.45) *and where the square root is taken to be positive at* 0 *and analytic in a neighbourhood of* 0, *in such a way that $s_{n,t}$ is positive when $t < 0$ and negative imaginary when $t > 0$. We assume that $\alpha > -1/2$ and $t < 0$ or that $\alpha > 0$ and $t > 0$. Then $g_n(t)$ satisfies*

$$\log g_n(t) = \log g_n(0) + \int_0^{s_{n,t}} \frac{\sigma_\alpha^\pm(s) - \alpha^2}{s} ds \\ + \frac{\alpha s_{n,t}}{2} + n(f(z_0) - f(0)) + \mathcal{O}(|t|^{1/2}) + \mathcal{O}\left(n^{-1}\right), \quad (1.105)$$

with uniform error terms as $n \to \infty$ and $t \to 0$, $\pm t > 0$, where the asymptotics of $g_n(0)$ are given in (1.99).

Remark 1.3.1. When $t > 0$, the Euler-Lagrange equation (1.43) states that

$$\alpha \int_a^b \log|x^2 - t| d\mu_V(x) = f(z_0). \quad (1.106)$$

Thus, by (1.99) and (1.105), one has that for $t > 0$ and $b - a = 2$, under the assumptions of Theorem 1.3.1, that as $t \to 0$ and $n \to \infty$,

$$\log g_n(t) = n\alpha \int \log|x^2 - t| d\mu_V(x) + \alpha^2 \log n + \int_0^{s_{n,t}} \frac{\sigma_\alpha^\pm(s) - \alpha^2}{s} ds \\ + \frac{\alpha s_{n,t}}{2} + \alpha^2 \log(\pi\psi_V(0)) + \log\left(\frac{G(1+\alpha)^2}{G(1+2\alpha)}\right) + \mathcal{O}(\sqrt{|t|}) + \mathcal{O}(1/n). \quad (1.107)$$

When $t < 0$, we wish to obtain a similar formula to (1.107), but formula (1.106) no longer holds. However, we note that for some $t_0 < 0$ and $t_0 < t \leq 0$, we have

$$\int_a^b \log|x^2 - t| d\mu_V(x) = \frac{1}{2} \int_{\Gamma_1} \log(z^2 - t) d\mu_V(z) \\ + \frac{1}{2} \int_{\Gamma_2} \log(z^2 - t) d\mu_V(z) - \pi i \int_{-z_0}^{z_0} d\mu_V(z), \quad (1.108)$$

where Γ_1 and Γ_2 are as in Figure 1.4; where $\log(z^2 - t)$ is analytic on $\mathbb{C} \setminus ((-\infty, 0) \cup (-z_0, z_0))$ and is positive as $z \to +\infty$; and where $d\mu_V(z)$ is the analytic continuation of $d\mu_V(x)$ on a neighbourhood of 0. We analytically continue (1.106) in the variable z_0 to find that

$$f(z_0) = \frac{\alpha}{2} \left(\int_{\Gamma_1} \log(z^2 - t) d\mu_V(z) + \int_{\Gamma_2} \log(z^2 - t) d\mu_V(z) \right), \quad (1.109)$$

for some $t_0 < 0$ and $t_0 < t \le 0$. Thus, for $t < 0$ and $b - a = 2$, under the assumptions of Theorem 1.3.1, it follows that as $t \to 0$ and $n \to \infty$,

$$\log g_n(t) = n\alpha \int \log|x^2 - t| d\mu_V(x) + \alpha^2 \log n + \int_0^{s_{n,t}} \frac{\sigma_\alpha^{\pm}(s) - \alpha^2}{s} ds + \alpha^2 \log(\pi\psi_V(0)) + \log\left(\frac{G(1+\alpha)^2}{G(1+2\alpha)}\right) + \mathcal{O}(\sqrt{|t|}) + \mathcal{O}(1/n). \tag{1.110}$$

Remark 1.3.2. In [20] the authors obtained the following identity

$$\lim_{s \to -i\infty} \left(\int_0^s \frac{\sigma_\alpha^{+}(\widetilde{s}) - \alpha^2}{\widetilde{s}} d\widetilde{s} + \frac{\alpha s}{2} + \frac{\alpha^2}{2} \log|s| \right) = \log \frac{G\left(1 + \frac{\alpha}{2}\right)^4 G(1+2\alpha)}{G(1+\alpha)^4}. \tag{1.111}$$

It follows that for $t > 0$, upon taking $s_{n,t} \to -i\infty$ and $t \to 0$ in (1.105), the result matches formula (1.100) in the limit $t \to 0$.

We now show that for $t < 0$, upon taking $s_{n,t} \to \infty$ and $t \to 0$ in (1.110), the result matches formula (1.101) in the limit $t \to 0$. We begin by finding the asymptotics of (1.110) as $s_{n,t} \to \infty$. In [19] it is shown that

$$\int_0^s \frac{\sigma_\alpha^{-}(\widetilde{s}) - \alpha^2}{\widetilde{s}} d\widetilde{s} + \log \frac{G(1+\alpha)^2}{G(1+2\alpha)} = -\alpha^2 \log s + o(1) \tag{1.112}$$

as $s \to \infty$. Thus, by (1.110),

$$\log g_n(t) = n\alpha \int \log|x^2 - t| d\mu_V(x) - \alpha^2 \log\left(4\sqrt{|t|}\right) + o(1), \tag{1.113}$$

as $n \to \infty$, $t \to 0$ and $s_{n,t} \to \infty$. It remains to find the asymptotics of $\mathcal{A}(t)$ from (1.101) as $t \to 0$, for $t < 0$. For $x \in (a, b)$, we find by first taking a contour integral argument and then taking the limit $t \to 0$, that

$$\int_a^b \frac{y}{y^2 - t} \frac{\sqrt{(b-y)(y-a)}}{y - x} dy = \pi \left(\frac{\sqrt{|ab|}\sqrt{|t|}}{x^2 - t} - 1 \right) (1 + \mathcal{O}(\sqrt{|t|})),$$

as $t \to 0$ for $t < 0$. Thus, by the definition of $\mathcal{A}$ in (1.101),

$$\mathcal{A}(t) = \frac{\alpha^2}{2\pi} \left(\int_a^b \frac{\log|x^2| dx}{\sqrt{(b-x)(x-a)}} - \log|t| \int_{-\infty}^{\infty} \frac{dx}{x^2+1} - \int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1} dx \right) + o(1), \tag{1.114}$$

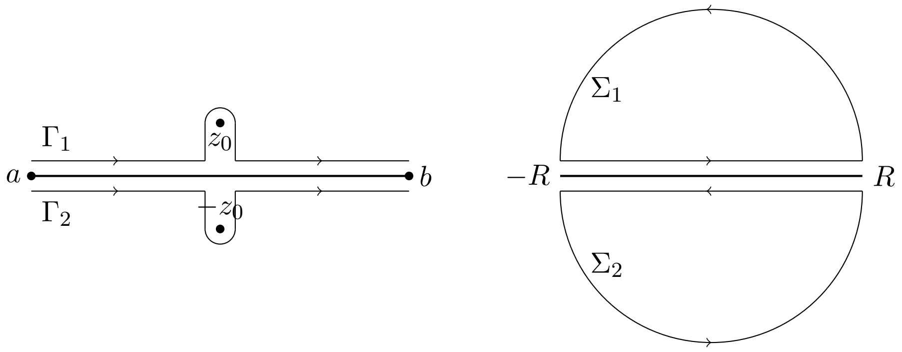

Figure 1.4: Left: Contours Γ_1 and Γ_2. Right: Contours Σ_1 and Σ_2

as $t \to 0$ for $t < 0$. By the calculus of residues

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1} dx &= \lim_{R\to\infty} \left(\int_{\Sigma_1} \frac{\log(z+i)}{z^2+1} dz - \int_{\Sigma_2} \frac{\log(z-i)}{z^2+1} dz \right) \\ &= 2\pi \log 2,\end{aligned}$$

where Σ_1 and Σ_2 are as in Figure 1.4, and $\log(z+i)$ and $\log(z-i)$ have branch cuts on $(-i, -i\infty)$ and $(i, i\infty)$ respectively, both positive as $z \to +\infty$. Similarly,

$$\begin{aligned}&\int_a^b \frac{\log|x^2| dx}{\sqrt{(b-x)(x-a)}} = \lim_{R\to\infty} \frac{1}{2} \left(\int_{\Sigma_1} \frac{i \log(z^2) dz}{r(z)} \right. \\ &\left. + \int_{\Sigma_2} \frac{i \log(z^2) dz}{r(z)} - \int_{\substack{|z|= R \\ \arg z \in (-\pi, \pi)}} \frac{i \log(z^2) dz}{r(z)} + 4\pi \int_{-\infty}^{a} \frac{dz}{r(z)} \right) \\ &= -2\pi \log 2,\end{aligned}$$

for $b - a = 2$, where $\log(z^2)$ has a branch cut on $(-\infty, 0)$ and is positive as $z \to +\infty$, and $r(z) = ((z-a)(z-b))^{1/2}$ has a branch cut on (a, b) and behaves like z as $z \to \infty$. Thus, by (1.114),

$$\mathcal{A}(t) = -\alpha^2 \log \left(4\sqrt{|t|} \right) + o(1), \tag{1.115}$$

as $t \to 0$, and we have shown that for $t < 0$, upon taking $s_{n,t} \to \infty$ in (1.110), the result matches formula (1.101) upon taking $t \to 0$.

1.3.1 Hankel determinants with Fisher-Hartwig singularities

Theorem 1.3.1 can also be stated as a result regarding the partition function of a matrix ensemble with Fisher-Hartwig singularities. Modify the probability measure on Hermitian matrices (1.8) as follows:

$$\frac{1}{Z_{n,t}}\left|\det(M^2-t)\right|^{\alpha} e^{-n\operatorname{Tr}V(M)}dM \qquad \text{for } \alpha>-1/2,\ t\in\mathbb{R}, \tag{1.116}$$

where $Z_{n,t} = Z_{n,t}(V,\alpha)$ is the normalizing constant. The measure (1.116) induces a probability measure on its real eigenvalues $x_1,\dots,x_n$, given by

$$\nu_{n,t}(x)dx = \frac{1}{\widehat{Z}_{n,t}}\prod_{1\le i<j\le n}(x_j-x_i)^2\prod_{j=1}^{n} w_{n,t}(x_j)dx_j,$$
$$w_{n,t}(x) = |x^2-t|^{\alpha}e^{-nV(x)}, \tag{1.117}$$

where $\widehat{Z}_{n,t} = \widehat{Z}_{n,t}(V,\alpha)$ is the normalizing constant. We have the following relation with $g_n(t)$:

$$g_n(t) = \frac{\widehat{Z}_{n,t}(V,\alpha)}{\widehat{Z}_n(V)} = \frac{H_n(w_{n,t})}{H_n(w_n)}, \tag{1.118}$$

and so Theorem 1.3.1 gives an expansion for $\log\widehat{Z}_{n,t}(V,\alpha)$. By comparing with the form of the symbol (1.71), $H_n(w_{n,t})$ is a Hankel determinant with Fisher-Hartwig singularities of type α:

- $\alpha\neq 0$ and $t=0$ as $n\to\infty$ corresponds to a single α-singularity at 0.
- $\alpha\neq 0$ and $t>0$ remains fixed as $n\to\infty$ corresponds to two $\alpha/2$-singularities at $-\sqrt{t}$ and $\sqrt{t}$.
- $\alpha\neq 0$ and $t<0$ remains fixed as $n\to\infty$ corresponds to an analytic potential.

By letting $t\to 0$ as $n\to\infty$ we are thus effectively studying the transition between the scaling regimes with $0,1$ and 2 α-singularities. In Section 1.1.6 we described some motivation for studying Hankel determinants with Fisher-Hartwig singularities, we list here a few applications where one is interested in this transition between regimes. Random matrix ensembles with such merging root-type singularities have applications in fractional Brownian motion with Hurst index $H=0$ [51], and in quantum chromodynamics [2, 26]. The partition function $\widehat{Z}_{n,t}$ appears in the study of impenetrable bosons in a harmonic well as the one body density matrix (up to a constant), and one needs uniform asymptotics of $\widehat{Z}_{n,t}$ in $t>0$ to determine whether Bose-Einstein condensate occurs

[47]. Finally, more recently, Berestycki, Webb and Wong used Theorem 1.3.1 to prove that the characteristic polynomial of Unitary Ensembles with one-cut regular potentials V converge in distribution to a Gaussian multiplicative chaos measure [9].

Note that the equilibrium measure associated with the modified probability measure (1.117) is given by μ_V, and thus it is the same as the equilibrium measure associated with the probability measure (1.10). This is because the addition of $-\frac{\alpha}{n^2}\sum_{j=1}^n \log|x_j^2 - t|$ leaves the large n limit in (1.40) unchanged. Thus the macroscopic distribution of the eigenvalues (1.117) remains unchanged under the addition of the term $|x^2 - t|^\alpha$ (for fixed α).

Correlation kernel

The modified probability measure (1.117) is a determinantal point process with a correlation kernel which we denote $K_{n,t}(x,y)$. The kernel $K_{n,t}(x,y)$ is defined in the same manner as in Section 1.1.3 with weight $w_{n,t}$. In Chapter 3 we also consider asymptotics of the correlation kernel $K_{n,t}$ as $n \to \infty$ and $t \to 0$ simultaneously.

If $t \neq 0$ remains fixed as $n \to \infty$, then as a slight generalization of [34, 35] $K_{n,t}$ satisfies (1.72) for $x \in [a,b] \setminus \{z_0, -z_0\}$. If $t = 0$, it was proven in [74] that uniformly for u, v in bounded subsets of $(0,\infty)$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\psi_V(0)n} K_{n,0}\left(\frac{u}{\psi_V(0)n}, \frac{v}{\psi_V(0)n}\right) &= \mathbb{K}_\alpha^{\mathrm{Bessel}}(u,v)\\ &= \pi\sqrt{uv}\frac{J_{\alpha+\frac{1}{2}}(\pi u)J_{\alpha-\frac{1}{2}}(\pi v) - J_{\alpha-\frac{1}{2}}(\pi u)J_{\alpha+\frac{1}{2}}(\pi v)}{2(u-v)},\end{aligned} \tag{1.119}$$

where J_α is the Bessel function of order α (see e.g. [81]).

In Chapter 3 we obtain large n asymptotics for the correlation kernel near the origin as $n \to \infty$ for small t. Of particular interest will be the double scaling limit where $n \to \infty$ and simultaneously $t \to 0$ in such a way that $n^2 t$ tends to a non-zero constant. This will lead us to a new family of limiting kernels which are built out of functions associated to the Painlevé V equation.

Theorem 1.3.2. *Let V be real analytic, satisfying* (1.9) *and one-cut regular with* 0 *in the interior of the support* (a,b) *of the equilibrium measure μ_V. We denote ψ_V for the density of μ_V, given by* (1.45)*, and define*

$$\tau_{n,t} = 16\pi^2\psi_V(0)^2 n^2 t, \tag{1.120}$$

for t real.

1. *As $n \to \infty$ and simultaneously $t \to 0$ in such a way that $\tau_{n,t} \to \pm\infty$, the kernel $K_{n,t}$ satisfies* (1.72) *at $x = 0$.*

2. *As $n \to \infty$ and simultaneously $t \to 0$ in such a way that $\tau_{n,t} \to 0$, the kernel $K_{n,t}$ satisfies* (1.119).

3. *As $n \to \infty$ and simultaneously $t \to 0$ in such a way that $\tau_{n,t} \to \tau \neq 0$, there exists a limiting kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ such that*

$$\lim \frac{1}{\psi_V(0)n} K_n\left(\frac{u}{\psi_V(0)n}, \frac{v}{\psi_V(0)n}\right) = \mathbb{K}_\alpha^{\mathrm{PV}}(u, v; \tau), \quad (1.121)$$

for any $u, v \in \mathbb{R}$ if $\tau < 0$, and for any $u, v \in \mathbb{R} \setminus \{\pm\sqrt{\tau}/4\}$ if $\tau > 0$.

The limits are uniform for u, v in compact subsets of $\mathbb{R}$.

The limiting kernel $\mathbb{K}_\alpha^{\mathrm{PV}}(u, v; \tau)$ has the form

$$\mathbb{K}_\alpha^{\mathrm{PV}}(u, v; \tau) = \frac{\Phi_1(\pi v; \tau)\Phi_2(\pi u; \tau) - \Phi_1(\pi u; \tau)\Phi_2(\pi v; \tau)}{2\pi i(u - v)}, \quad (1.122)$$

where the functions Φ_j, $j = 1, 2$, will be defined in Chapter 3, in terms of the unique solutions Ψ^- and Ψ^+ to two RH problems for $\tau < 0$ and $\tau > 0$ respectively.

1.3.2 Method of proof

We give an outline of the proof of Theorem 1.3.1 We use the representation (1.118). By taking the derivative of the logarithm of (1.28) and considering the orthogonality of the polynomials, one can show that

$$\begin{aligned} \frac{d}{dt} \log g_n(t) &= \frac{d}{dt} \log \widehat{Z}_{n,t} = \frac{\kappa_{n-1}^{(n)}}{\kappa_n^{(n)}} \frac{\alpha}{2z_0} \left[F(z_0) - F(-z_0)\right], \\ F(z) &= \frac{\partial}{\partial z} p_{n-1}^{(n)}(z) \int_{-\infty}^{\infty} \frac{p_n^{(n)}(x) w_{n,t}(x)}{x - z} dx \\ &\qquad - \frac{\partial}{\partial z} p_n^{(n)}(z) \int_{-\infty}^{\infty} \frac{p_{n-1}^{(n)}(x) w_{n,t}(x)}{x - z} dx, \end{aligned} \quad (1.123)$$

where $p_j^{(n)}(x) = \kappa_j^{(n)} x^j + \ldots$ are orthonormal with respect to $w_{n,t}$ as in (1.19). Due to technical reasons, we only prove the above differential identity (1.123) for $t < 0, \alpha > -\frac{1}{2}$ and for $t > 0, \alpha \geq 0$. Define Y by

$$Y(z; n) = \begin{pmatrix} \frac{1}{\kappa_n^{(n)}} p_n^{(n)}(z) & \frac{1}{2\pi i \kappa_n^{(n)}} \int_{\mathbb{R}} \frac{p_n^{(n)}(x)}{x - z} w_{n,t}(x) dx \\ -2\pi i \kappa_{n-1}^{(n)} p_{n-1}^{(n)}(z) & -\kappa_{n-1}^{(n)} \int_{\mathbb{R}} \frac{p_{n-1}^{(n)}(x)}{x - z} w_{n,t}(x) dx \end{pmatrix}. \quad (1.124)$$

Substituting Y into (1.123), we obtain the following:

$$\begin{aligned}\frac{d}{dt}&\log \widehat{Z}_n(t)\\ &= -\frac{\alpha}{2z_0}\left(\left(Y^{-1}\frac{dY}{dz}\right)_{22}(z_0) - \left(Y^{-1}\frac{dY}{dz}\right)_{22}(-z_0)\right). \qquad (1.125)\end{aligned}$$

It was noticed by Fokas, Its and Kitaev [45] that Y can be characterized as the unique solution of the following RH problem.

(a) Y is analytic in $\mathbb{C} \setminus \mathbb{R}$.

(b) Y has the following jump relations on $\mathbb{R}$, except at the singularities $\pm\sqrt{t}$ in the case where $t > 0$:

$$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w_{n,t}(x) \\ 0 & 1 \end{pmatrix}.$$

(c) As $z \to \infty$,

$$Y(z) = \left(I + \mathcal{O}\left(z^{-1}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

(d) If $t > 0$ and $\alpha < 0$, $Y(z)$ has the behavior

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|z \mp \sqrt{t}|^{\alpha}) \\ \mathcal{O}(1) & \mathcal{O}(|z \mp \sqrt{t}|^{\alpha}) \end{pmatrix},$$

as $z \to \pm\sqrt{t}$. If $t > 0$ and $\alpha \geq 0$, Y is bounded near $\pm\sqrt{t}$.

Using the Deift-Zhou steepest descent analysis for RH-problems, it is possible to obtain asymptotic formulas for Y as $n \to \infty$, which we substitute into the following formula (obtained by integrating (1.125)) to obtain our results in Theorem 1.3.1:

$$\begin{aligned}\log g_n(t) &= \log g_n(0)\\ &- \int_{z=0}^{z_0} \frac{\alpha}{2z}\left(\left(Y^{-1}\frac{dY}{dz}\right)_{22}(z) - \left(Y^{-1}\frac{dY}{dz}\right)_{22}(-z)\right)dz. \qquad (1.126)\end{aligned}$$

Chapter 2

Splitting of a Gap in the Bulk of the Spectrum of Random Matrices

This chapter is based [42], a joint paper with Igor Krasovsky at Imperial College London.

2.1 Introduction

Let A be the union of m open intervals on $\mathbb{R}$, and K_s be the (trace-class) operator on $L^2(A, dx)$ given by the kernel

$$K_s(x,y) = \frac{\sin s(x-y)}{\pi(x-y)}. \tag{2.1}$$

Consider the Fredholm determinant

$$P_s(A) = \det(I - K_s)_A. \tag{2.2}$$

For a wide class of random matrix ensembles [3], in particular for the Gaussian Unitary Ensemble, $P_s(A)$ is the probability that the set $\frac{s}{\pi}A = \{\frac{s}{\pi}x : x \in A\}$ contains no eigenvalues in the bulk scaling limit where the average distance between the eigenvalues is 1. In this chapter, we are interested in the asymptotics of $P_s(A)$ as $s \to \infty$, and we study the transition between a single interval $A_0 = (\alpha, \beta)$ to the set A composed of 2 disjoint intervals

$$A = A_1 \bigcup A_2, \quad A_1 = (\alpha_1, \beta_1),\ A_2 = (\alpha_2, \beta_2). \tag{2.3}$$

Such problems have a rich history, of which we mention some relevant results. For the single interval case $A_0 = (\alpha, \beta)$,

$$\begin{aligned} \log P_s(A_0) &= -\frac{(\beta-\alpha)^2 s^2}{8} - \frac{1}{4}\log s - \frac{1}{4}\log\frac{\beta-\alpha}{2} + c_0 + \mathcal{O}(s^{-1}), \\ c_0 &= \frac{1}{12}\log 2 + 3\zeta'(-1), \end{aligned} \tag{2.4}$$

as $s \to \infty$, where ζ is the Riemann zeta-function. The leading term and logarithmic term in (2.4) were conjectured by des Cloizeaux and Mehta [23] in 1973, while the constant term c_0 remained undetermined until Dyson [37] conjectured an expression for it in 1976, relying on inverse scattering techniques and the work of Widom [92] on Toeplitz determinants (see below). The constant c_0 became known as the Widom-Dyson constant. The first rigorous confirmation of the leading term in (2.4) was given by Widom [93] in 1994. In the landmark paper of 1997, Deift, Its, Zhou [32] were able to confirm the leading term and the logarithmic term, but the proof of the constant c_0 continued to defy their techniques. Finally, two independent proofs of the constant were later given by Erhardt [38] and the second author [71], and a further third proof given in [31]. The proofs in [71], [31] use the Riemann-Hilbert (RH) methods, while [38] uses operator theory techniques.

When A is composed of any (fixed) number of intervals, the main term was found and proved by Widom [94] in 1995, where he was also able to identify the next term in the following result:

$$\frac{d}{ds}\log P_s(A) = sC_1 + C_2(s) + o(1), \tag{2.5}$$

as $s \to \infty$. The constant C_1 is explicitly computable, while $C_2(s)$ is an oscillatory function given by a Jacobi inversion problem. In [32], which was mentioned above, the authors were also able to find the full asymptotic expansion for the logarithmic derivative of the determinant on any number of intervals and describe the oscillations in terms of θ-functions. Here we describe their result when A is composed of 2 intervals as in (2.3):

$$\log P_s(A) = -G_1 s^2 + G_2 \log s + \log\theta(sV;\tau) + G_3 + \mathcal{O}(s^{-1}), \tag{2.6}$$

where

$$\theta(z) = \theta(z;\tau) = \sum_{m\in\mathbb{Z}} e^{2\pi i z m + \pi i \tau m^2} \tag{2.7}$$

is the Jacobi Theta-function of the third kind, see e.g. [91]. The constants V, τ, G_1, G_2 are given in terms of elliptic integrals and θ-functions.

Let

$$r(z) = ((z-\alpha_1)(z-\beta_1)(z-\alpha_2)(z-\beta_2))^{1/2}, \tag{2.8}$$

with branch cuts on A such that $r(z) > 0$ for $z > \beta_2$, and let $q(z)$ be the unique monic polynomial of degree 2 such that

$$\int_{A_j} \frac{q(z)dz}{r_+(z)} = 0, \qquad j = 1, 2, \tag{2.9}$$

where $r_+(z)$ is the limit of $r(z+i\epsilon)$ as $\epsilon \to 0_+$ (where the "+" side is chosen merely for definiteness). Then q/r has no residue at infinity. Hence as $z \to \infty$, q/r has the form

$$\frac{q(z)}{r(z)} = 1 + \frac{G_1}{z^2} + \mathcal{O}(z^{-3}), \tag{2.10}$$

which defines the constant G_1 appearing in (2.6).

The parameter V, τ appearing in the arguments of the θ-function in (2.6) are as follows:

$$V = \frac{1}{\pi}\int_{\beta_1}^{\alpha_2} \frac{q(x)dx}{r(x)}, \qquad \tau = i\frac{\int_{\beta_1}^{\alpha_2}\frac{dx}{|r(x)|}}{\int_{\alpha_2}^{\beta_2}\frac{dx}{|r_+(x)|}}. \tag{2.11}$$

The constant G_2 also has an explicit representation (in terms of θ-functions, see [32]).

The value of the constant G_3 is unknown. The reason that G_3 is unknown is because the method used was to find an expansion for $\frac{d}{ds}\log P_s(A)$, and then integrate it in s. There is no obvious point s_0 for the lower limit of integration where $P_{s_0}(A)$ would be known and could provide G_3.

In this chapter we begin a study of the transition between the single interval formula (2.4) and the two-interval formula (2.6). Namely, we describe the regime between a single interval and $|\alpha_2 - \beta_1|$ sufficiently small depending on s, asymptotically as $s \to \infty$. This transition resembles the study of the birth of a cut – the emergence of an extra interval of support of the limiting eigenvalue density in the Unitary Ensemble: asymptotics for the correlation kernel of the eigenvalues in that case were obtained independently by Bertola & Lee [10], Claeys [22], Mo [78].

Consider the Toeplitz determinant whose symbol $f(z)$ is the characteristic function of a subset J of the unit circle C:

$$D_n(J) = \det\left(f_{j-k}\right)_{j,k=0}^{n-1}, \quad f_k = \int_{e^{i\theta}\in J} e^{-ik\theta}\frac{d\theta}{2\pi}, \tag{2.12}$$

where integration is in the positive direction around the unit circle. The proofs of the expansion (2.4) including the constant term c_0 in [31, 71] were based on an analysis of the Toeplitz determinant $D_n(J_2)$ where J_2 is an arc of the unit circle

$$J = J_2 = \left\{ e^{i\theta} | \theta \in (-\pi, \theta_2) \cup (\theta_1, \pi] \right\}. \tag{2.13}$$

The asymptotics of $D_n(J_2)$ for a fixed arc J_2 were found by Widom [92]. In [31, 71], Widom's result was extended to the case of $J_2 = J_2^{(n)}$ varying with n such that $|\theta_1 - \theta_2| \to 0$ sufficiently slowly. Namely,

$$\log D_n(J_2^{(n)}) = n^2 \log \cos \frac{\theta_1 - \theta_2}{4} - \frac{1}{4} \log \left(n \sin \frac{\theta_1 - \theta_2}{4} \right) + c_0 + \mathcal{O}\left(\frac{1}{n \sin \frac{\theta_1 - \theta_2}{4}} \right) \tag{2.14}$$

as $n \to \infty$, uniformly for $\frac{s_0}{n} \leq \frac{\theta_1 - \theta_2}{2} \leq \pi - \epsilon$, for $\epsilon > 0$ and with s_0 sufficiently large. Asymptotics (2.4) are obtained from (2.14) by using the fact that

$$\lim_{n \to \infty} D_n(J_2^{(n)}) = \det(I - K_s)_{A_0} \tag{2.15}$$

for fixed s and by taking the limit in (2.14) as $n \to \infty$ with $\theta_1 = \frac{2s\beta}{n}$ and $\theta_2 = \frac{2s\alpha}{n}$, where α, β are fixed. The approach of the present paper is based on an analysis of $D_n(J)$ where $J = J^{(n)}$ is the union of 2 arcs $J^{(n)} = J_1^{(n)} \cup J_2^{(n)}$, with $J_1^{(n)} \subset C \setminus J_2^{(n)}$ of sufficiently small length in comparison with $C \setminus J_2^{(n)}$ (see Theorem 2.1.2 below). We obtain our results on the sine-kernel determinant by taking the limit $n \to \infty$ of $D_n(J^{(n)})$. However, we believe Theorem 2.1.2 below to be of independent interest for a future study of Toeplitz determinants with symbols supported on several arcs.

2.1.1 Results

The kernel (2.1) is translationally invariant and so we can assume the following form for A:

$$A = (\alpha, -\nu) \bigcup (\nu, \beta), \qquad \alpha < 0 < \nu < \beta. \tag{2.16}$$

In this chapter we provide the asymptotics of $\log P_s(A)$ (including the constant term) in the double scaling limit as $s \to \infty$ while $\nu \to 0$ in such a way that $s\nu \log \nu^{-1} \to 0$.

Let $\gamma = \frac{1}{8}(\beta^{-1} - \alpha^{-1})$ and

$$\omega = \frac{s\sqrt{|\alpha\beta|}}{\log(\gamma\nu)^{-1}} > 0. \tag{2.17}$$

Clearly, ω is uniquely represented in the form

$$\omega = k + x, \quad k = 0, 1, 2, \ldots, \quad x \in [-1/2, 1/2). \tag{2.18}$$

We note that ν has the form

$$\nu = \gamma^{-1} e^{-\frac{s\sqrt{|\alpha\beta|}}{k+x}}. \tag{2.19}$$

We prove the following:

Theorem 2.1.1. *As $s \to \infty$, uniformly for $\nu \in (0, \nu_0)$, where $s\nu_0 \log \nu_0^{-1} \to 0$,*

$$\begin{aligned}
\log P_s(A) &= \log P_s(A_0) + s\sqrt{|\alpha\beta|}\left(\omega - \frac{x^2}{\omega}\right) + c(k) + \delta_k(x) \\
&\qquad + \mathcal{O}(\max\{s\nu_0 \log \nu_0^{-1}, 1/\log \nu_0^{-1}, s^{-1}\}), \\
c(k) &= \log\left(\frac{2^{2k^2-k}}{\pi^k}\frac{G(k+1)^4}{G(2k+1)}\right), \\
\delta_k(x) &= \log\left(1 + 2\pi\kappa_{k-1}^2(\gamma\nu)^{1+2x}\right) + \log\left(1 + (2\pi\kappa_k^2)^{-1}(\gamma\nu)^{1-2x}\right),
\end{aligned} \tag{2.20}$$

where G is the Barnes G-function, and where κ_j is the leading coefficient of the Legendre polynomial of degree j orthonormal on the interval $[-2, 2]$, given by $\kappa_0 = 1/2$, $\kappa_{-1} = 0$,

$$\kappa_j = 4^{-j-1/2}\sqrt{2j+1}\frac{(2j)!}{j!^2}, \quad j = 1, 2, \ldots, \tag{2.21}$$

and $\log P_s(A_0)$ is given in (2.4).

As $s \to \infty$, uniformly for $\nu \in (\nu_1, \nu_0)$, where $s\nu_0 \log \nu_0^{-1} \to 0$ and $\frac{s}{\log \nu_1^{-1}} \to \infty$ (i.e., $k \to \infty$), formula (2.20) *reduces to*

$$\begin{aligned}
\log P_s(A) &= s^2\left(-\frac{(\beta-\alpha)^2}{8} + \frac{|\alpha\beta|}{\log(\gamma\nu)^{-1}}\right) - \frac{1}{2}\log s + \frac{1}{4}\log\log(\gamma\nu)^{-1} \\
&- x^2 \log(\gamma\nu)^{-1} + \log\left(1 + (\gamma\nu)^{1-2|x|}\right) - \frac{1}{4}\log\left(\frac{\beta-\alpha}{2}\sqrt{|\alpha\beta|}\right) \\
&+ \frac{1}{6}\log 2 + 6\zeta'(-1) + \mathcal{O}\left(\max\left\{s\nu_0 \log \nu_0^{-1}, \frac{1}{\log \nu_0^{-1}}, \frac{\log \nu_1^{-1}}{s}\right\}\right),
\end{aligned} \tag{2.22}$$

where ζ is the Riemann zeta-function.

Remark 2.1.1. Note that if $k = 0$, $x \in (0, 1/2 - \epsilon)$ for $\epsilon > 0$, and $s \to \infty$, then (2.20) shows that $\frac{P_s(A)}{P_s(A_0)} \to 1$.

Remark 2.1.2. The function $c(k)$ can alternatively be described in terms of the coefficients (2.21) of the Legendre polynomials:

$$c(k) = -\sum_{j=0}^{k-1} \log 2\pi\kappa_j^2, \quad k = 1, 2, \dots, \qquad c(0) = 0. \tag{2.23}$$

Formula (2.23) shows that $\delta_k(x) + c(k)$ is continuous also at the points $|x| = 1/2$.

Remark 2.1.3. The ratio $\frac{P_s(A)}{P_s(A_0)}$, where $A_0 = (\alpha, \beta)$, can be written in the following form. Let χ_A be the characteristic function of A on $\mathbb{R}$. Then

$$\begin{aligned} \frac{P_s(A)}{P_s(A_0)} &= \frac{\det(I - \chi_A K_s)_{(\alpha,\beta)}}{\det(I - K_s)_{(\alpha,\beta)}} \\ &= \det\left(I - (\chi_A - 1)K_s(I - K_s)^{-1}\right)_{(\alpha,\beta)} \\ &= \det\left(I + K_s(I - K_s)^{-1}\right)_{(-\nu,\nu)}. \end{aligned} \tag{2.24}$$

Remark 2.1.4. The rescaled sine process (with expected distance between particles π/s) is the determinantal point process with the m'th correlation function ρ_m, for $m = 1, 2, \dots$, given by

$$\rho_m(x_1, \dots, x_m) = \det(K_s(x_i, x_j))_{i,j=1}^m. \tag{2.25}$$

Consider the rescaled sine process conditioned to have no eigenvalues in A. Denote this process by $\mathbb{P}_A$ and its m'th correlation function by ρ_m^A. In Section 2.4.2, we show that for $x_1, \dots, x_m \in (-1, 1)$,

$$\nu^m \rho_m^A(\nu x_1, \dots, \nu x_m) \to \det(2K_{\text{Leg}}(2x_i, 2x_j))_{i,j=1}^m \tag{2.26}$$

as $s \to \infty$ and $\nu \to 0$ such that $k \in \mathbb{N}$ and $|x| < 1/2$ remain fixed, where

$$K_{\text{Leg}}(x, y) = \frac{\kappa_{k-1}}{\kappa_k} \frac{L_k(x)L_{k-1}(y) - L_k(y)L_{k-1}(x)}{x - y}, \tag{2.27}$$

and L_k is the Legendre polynomial of degree k, orthonormal on $[-2, 2]$:

$$\int_{-2}^{2} L_j(x)L_i(x)dx = \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases} \tag{2.28}$$

Recall that for a set $B \subset \mathbb{R}$ and a point process Λ with its m-th correlation function denoted r_m, we have

$$\begin{aligned} &\text{Expectation}(\#\text{ ordered } m\text{-tuples in } B) \\ &= \frac{1}{m!}\int_{B^m} r_m(x_1,\dots,x_m)dx_1\dots dx_m \\ &= \sum_{j=0}^{\infty}\binom{m+j}{m}\text{Prob}(\#(\Lambda\cap B)=m+j). \end{aligned} \tag{2.29}$$

The process with kernel K_{Leg} is a k-point process. Thus we obtain from (2.26) and the first equation of (2.29) that, as $s \to \infty$ and $\nu \to 0$ such that $k \in \mathbb{N}$ and $|x|< 1/2$ remain fixed, the expected number of $(k+1)$-tuples of $\mathbb{P}_A$ on $(-\nu,\nu)$ converges to 0, while the expected number of k-tuples on the same interval converges to 1. It follows from the second equation of (2.29) that

$$\text{Prob}\left(\mathbb{P}_A \text{ has } k \text{ particles in } (-\nu,\nu)\right) \to 1. \tag{2.30}$$

To summarize, the asymptotics of $\log P_s(A)$ as $s \to \infty$ depends on the value of ν, and we give an overview of the various scaling limits:

- If $\nu = 0$, the asymptotics are given by (2.4).
- If $\nu \to 0$ as $s \to \infty$, such that $s\nu\log\nu^{-1} \to 0$, the asymptotics are given by Theorem 2.1.1.
- If $\nu\log\nu^{-1}$ is of order $1/s$ or larger, the asymptotics of Theorem 2.1.1 breaks down and the transition to an asymptotic formula containing θ-functions takes place. We plan to address this in a future paper.
- If $\nu > 0$ is fixed, the asymptotics are given by the θ-function regime (2.6).

For Toeplitz determinants, we obtain the following result. Let $D_n(J)$ be given by (2.12) with $J = J^{(n)} = J_1^{(n)} \cup J_2^{(n)}$ where

$$\begin{aligned} J_1^{(n)} &= \left\{e^{i\theta} : \theta \in \left(-\frac{2s\nu}{n},\frac{2s\nu}{n}\right)\right\}, \\ J_2^{(n)} &= \left\{e^{i\theta} : \theta \in \left(-\pi,\frac{2s\alpha}{n}\right)\bigcup\left(\frac{2s\beta}{n},\pi\right]\right\}, \end{aligned} \tag{2.31}$$

with some $\alpha < 0 < \nu(s) < \beta$. Then, with the notation of Theorem 2.1.1, we have

Theorem 2.1.2. *As $s, n \to \infty$, uniformly for $\nu \in (0, \nu_0)$, where $s\nu_0 \log \nu_0^{-1} \to 0$ and $s^3/n \to 0$,*

$$\log D_n\left(J^{(n)}\right) = \log D_n\left(J_2^{(n)}\right) + s\sqrt{|\alpha\beta|}\left(\omega - \frac{x^2}{\omega}\right) + c(k) + \delta_k(x) + \mathcal{O}(\max\{s^3/n, s\nu_0 \log \nu_0^{-1}, 1/\log \nu_0^{-1}, s^{-1}\}), \tag{2.32}$$

where the expansion of $\log D_n\left(J_2^{(n)}\right)$ *is given in (2.14) with* $\theta_1 = 2s\beta/n$, $\theta_2 = 2s\alpha/n$.

We use Theorem 2.1.2 to prove Theorem 2.1.1.
Proof of Theorem 2.1.1. It is well-known that

$$|D_n(J^{(n)}) - \det(I - K_s)_A| \to 0 \tag{2.33}$$

as $n \to \infty$ for fixed s, a fact which follows from formulas (1.21), (1.37), (1.73) in Chapter 1. Taking the limit $n \to \infty$ in (2.32), we then obtain (2.20). To obtain (2.22), we substitute (2.4) for $P_s(A_0)$, and note that the standard asymptotics of the Barnes G-function $G(z+1)$ as $z \to \infty$

$$\log G(z+1) = \frac{z^2}{2}\log z - \frac{3}{4}z^2 + \frac{z}{2}\log 2\pi - \frac{1}{12}\log z + \zeta'(-1) + \mathcal{O}(z^{-2}), \tag{2.34}$$

imply that as $k \to \infty$,

$$c(k) = -\frac{1}{4}\log k + \frac{1}{12}\log 2 + 3\zeta'(-1) + \mathcal{O}(1/k^2). \tag{2.35}$$

Furthermore, we note that $2\pi\kappa_k^2 = 1 + \mathcal{O}(k^{-1})$ as $k \to \infty$. □

2.1.2 Outline of the proof of Theorem 2.1.2

It remains to prove Theorem 2.1.2.

Let $-\pi < \theta_2 < 0 < \theta_0 < \theta_1 < \pi$ and define $J = J_1 \cup J_2$ where J_1, J_2 are as in Figure 2.1:

$$J_1 = J_1(\theta_0) = \{e^{i\theta} | \theta \in (-\theta_0, \theta_0)\}, \quad J_2 = \{e^{i\theta} | \theta \in (\theta_1, \pi] \cup (-\pi, \theta_2)\}. \tag{2.36}$$

We denote the complement of J as $\Sigma = \Sigma_1 \cup \Sigma_2$ where

$$\Sigma_1 = \{e^{i\theta} | \theta \in (\theta_0, \theta_1)\}, \quad \Sigma_2 = \{e^{i\theta} | \theta \in (\theta_2, -\theta_0)\}. \tag{2.37}$$

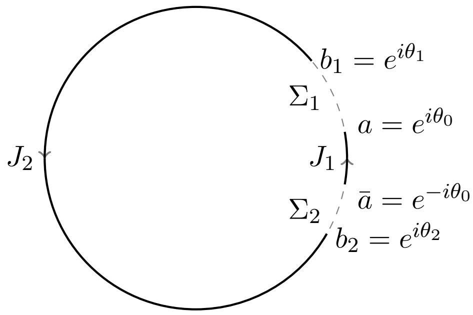

Figure 2.1: Interval J.

It follows from the integral representation for Toeplitz determinants, which follows from (1.37) in Chapter 1, that $D_j(J) > 0$ for all $j \in \mathbb{N}$. Consider the polynomials ϕ_j for $j = 0, 1, 2, \dots$ given by

$$\begin{aligned} \phi_0(z) &= \frac{1}{\sqrt{D_1(J)}}, \\ \phi_j(z) &= \frac{1}{\sqrt{D_j(J)D_{j+1}(J)}} \det \begin{pmatrix} f_0 & f_{-1} & \dots & f_{-j+1} & f_{-j} \\ f_1 & f_0 & \dots & f_{-j+2} & f_{-j+1} \\ & & \ddots & & \\ f_{j-1} & f_{j-2} & \dots & f_0 & f_{-1} \\ 1 & z & \dots & z^{j-1} & z^j \end{pmatrix} \\ &= \chi_j z^j + \dots, \quad j > 0, \end{aligned} \tag{2.38}$$

where the leading coefficient χ_j is given by

$$\chi_j = \sqrt{\frac{D_j(J)}{D_{j+1}(J)}}, \quad j = 0, 1, 2, \dots, \tag{2.39}$$

where we set $D_0(J) = 1$. The polynomials ϕ_j are orthonormal with weight 1 on J:

$$\int_J \phi_k(z)\overline{\phi_j(z)}\frac{d\theta}{2\pi} = \delta_{jk}, \quad z = e^{i\theta}, \quad j, k = 0, 1, 2, \dots. \tag{2.40}$$

Define a 2×2 matrix $Y(z) = Y_n(z)$ in terms of these orthogonal polynomials as follows:

$$Y(z) = \begin{pmatrix} \chi_n^{-1}\phi_n(z) & \chi_n^{-1}\int_J \frac{\phi_n(\zeta)}{\zeta - z}\frac{d\zeta}{2\pi i \zeta^n} \\ -\chi_{n-1}z^{n-1}\overline{\phi_{n-1}}(z^{-1}) & -\chi_{n-1}\int_J \frac{\overline{\phi_{n-1}}(\zeta^{-1})}{\zeta - z}\frac{d\zeta}{2\pi i \zeta} \end{pmatrix}. \tag{2.41}$$

Then Y is the unique solution of the following Riemann-Hilbert (RH) Problem

(a) $Y : \mathbb{C} \setminus J \to \mathbb{C}^{2\times 2}$ is analytic;

(b) Y possesses L^2 boundary values Y_+ and Y_- on the $+$ and $-$ side of J, respectively, related by the condition:

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} \\ 0 & 1 \end{pmatrix} \text{ for } z \in J;$$

(c) $Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

The fact that orthogonal polynomials satisfy a RH problem was first observed for polynomials orthogonal on the real line by Fokas, Its, Kitaev [45], and extended to polynomials orthogonal on the unit circle by Baik, Deift, Johansson [7]. The RH problem provides an efficient tool, the Deift-Zhou steepest descent method, for the asymptotic analysis of the polynomials, see e.g. [27].

In Section 2.2 we express the logarithmic derivative of the Toeplitz determinant $\frac{d}{d\theta_0} \log D_n(J)$ in terms of the polynomials ϕ_n and ϕ_{n-1}. These are, in turn, expressed in terms of Y_n. In Sections 2.3 and 2.4 we analyse the RH problem for Y_n as $n \to \infty$ in a double scaling limit where J depends on n such that $\theta_j = \frac{s}{n} u_j$ for $j = 0, 1, 2$; where $s \to \infty$ such that $s^3/n \to 0$; where $u_0 \to 0$ such that $s u_0 \log u_0^{-1} \to 0$, while u_1 and u_2 remain fixed. As a result, we obtain the asymptotics of Y_n. Substituting these into the differential identity for $\frac{d}{d\theta_0} \log D_n(J)$, and integrating with respect to θ_0, we obtain the asymptotics of $D_n(J)$, where $u_1 = 2\beta$, $u_2 = 2\alpha$, and $u_0 = u_0(\nu)$ is a function of ν, which proves Theorem 2.1.2.

2.2 Differential identity

We will now obtain the following:

Proposition 2.2.1. *(Differential identity) Let $a = e^{i\theta_0}$. The Toeplitz determinant $D_n(J)$ satisfies*

$$\frac{\partial}{\partial \theta_0} \log D_n(J) = -\frac{1}{2\pi} [F(\overline{a}) + F(a)], \tag{2.42}$$

where

$$F(z) = n\chi_n^2 |Y_{11}(z)|^2 - 2\chi_n^2 \mathrm{Re} \left(z \overline{Y_{11}(z)} \frac{d}{dz} Y_{11}(z) \right). \tag{2.43}$$

From the definition of the orthogonal polynomials it is clear that

$$D_n(J) = \prod_{j=0}^{n-1} \chi_j^{-2}. \tag{2.44}$$

The orthogonality conditions imply that

$$\begin{aligned}\frac{1}{2\pi}\int_J \frac{\partial \phi_j(z)}{\partial \theta_0}\overline{\phi_j(z)}d\theta \\ = \frac{1}{2\pi}\int_J \frac{\partial \chi_j}{\partial \theta_0}(z^j + \text{poly of deg } j-1)\overline{\phi_j(z)}d\theta = \frac{1}{\chi_j}\frac{\partial \chi_j}{\partial \theta_0},\end{aligned} \tag{2.45}$$

and similarly,

$$\frac{1}{2\pi}\int_J \phi_j(z)\frac{\partial \overline{\phi_j(z)}}{\partial \theta_0}d\theta = \frac{1}{\chi_j}\frac{\partial \chi_j}{\partial \theta_0}. \tag{2.46}$$

By (2.44)–(2.46) we obtain:

$$\begin{aligned}\frac{\partial}{\partial \theta_0}\log(D_n(J)) &= -2\sum_{j=0}^{n-1}\frac{\partial \chi_j}{\partial \theta_0}/\chi_j \\ &= -\frac{1}{2\pi}\int_J \frac{\partial}{\partial \theta_0}\left(\sum_{j=0}^{n-1}|\phi_j(z)|^2\right)d\theta.\end{aligned} \tag{2.47}$$

On the other hand, one can express F given in (2.43) in terms of the orthogonal polynomials:

$$F(z) = n|\phi_n(z)|^2 - 2\mathrm{Re}\left(z\overline{\phi_n(z)}\frac{d}{dz}\phi_n(z)\right). \tag{2.48}$$

Now the Christoffel-Darboux formula for orthogonal polynomials on the unit circle gives

$$\sum_{k=0}^{n-1}|\phi_k(z)|^2 = -F(z) \quad \text{for } z \in C, \tag{2.49}$$

(see eg. [71]), and hence (2.47) can be written as

$$\frac{\partial}{\partial \theta_0}\log D_n(J) = \frac{1}{2\pi}\int_J \frac{\partial}{\partial \theta_0}\left(F(z)\right)d\theta. \tag{2.50}$$

Since, by (2.49) and orthogonality, $\int_J F(z)\frac{d\theta}{2\pi} = n$,

$$0 = \frac{\partial}{\partial \theta_0}\left(\int_J F(z)d\theta\right) = F(\overline{a}) + F(a) + \int_J \frac{\partial}{\partial \theta_0}F(z)d\theta, \tag{2.51}$$

upon which proposititon 2.2.1 follows immediately.

2.3 Analysis of Riemann-Hilbert problem

We start by setting $\theta_0 = 0$ so that J_1 is a point, and then let J_1 develop into an arc. Throughout the rest of the paper we use the notation

$$a = e^{i\theta_0} = e^{iu_0 s/n}, \quad b_1 = e^{i\theta_1} = e^{iu_1 s/n}, \quad b_2 = e^{i\theta_2} = e^{iu_2 s/n}. \tag{2.52}$$

We let $s, n \to \infty$ such that $s^3/n \to 0$, and let $u_0 \to 0$ as $s \to \infty$ such that $su_0 \log u_0^{-1} \to 0$, while $u_2 < 0 < u_1$ remain constant. Denote $\Sigma^o = \Sigma_1^o \bigcup \Sigma_2^o$ where

$$\Sigma_1^o = \{z : 0 < \arg z < \theta_1\}, \quad \Sigma_2^o = \{z : \theta_2 < \arg z < 0\}. \tag{2.53}$$

Let g_1 be the function:

$$g_1(z) = \log \left(\frac{1}{b_1^{1/2} + b_2^{1/2}} \left(z + (b_1 b_2)^{1/2} + ((z - b_1)(z - b_2))^{1/2} \right) \right), \tag{2.54}$$

where the square root has branch cut on J_2 and is positive as $z \to +\infty$, and the logarithm $\log x$ has a branch cut for $x < 0$ and is positive for $x > 1$. At infinity,

$$g_1(z) = \log z - \log \frac{\sqrt{b_1} + \sqrt{b_2}}{2} + o(1) \quad \text{as } z \to \infty. \tag{2.55}$$

The boundary values of the function g_1 satisfy

$$g_{1,+}(z) + g_{1,-}(z) = \log z, \quad \text{for } z \in J_2, \tag{2.56}$$

and at b_1, b_2 we have

$$g_1(b_1) = \frac{1}{2} \log b_1, \quad g_1(b_2) = \frac{1}{2} \log b_2. \tag{2.57}$$

Alternatively, for $z = e^{i\theta} \in \Sigma^o$ we can write g_1 in the following form:

$$\exp\left(g_1\left(e^{i\theta}\right)\right) = e^{i\theta/2} \frac{\cos\frac{1}{2}\left(\theta - \frac{\theta_1+\theta_2}{2}\right) + \sqrt{|\sin\frac{\theta-\theta_1}{2} \sin\frac{\theta-\theta_2}{2}|}}{\cos\frac{1}{4}(\theta_1 - \theta_2)}. \tag{2.58}$$

On the $+$ and $-$ side of J_2, we have

$$\exp\left((g_1)_\pm\left(e^{i\theta}\right)\right) = e^{i\theta/2}\frac{\cos\frac{1}{2}\left(\theta - \frac{\theta_1+\theta_2}{2}\right) \mp i\sqrt{|\sin\frac{\theta-\theta_1}{2}\sin\frac{\theta-\theta_2}{2}|}}{\cos\frac{1}{4}(\theta_1-\theta_2)}, \tag{2.59}$$

from which it follows that e^{g_1} maps the $+$ side of J_2 to Σ^o and the $-$ side to J_2, and that e^{g_1} maps $\mathbb{C}\setminus J_2$ to the outside of the unit disc.

Set

$$w = \frac{n}{s}(z-1), \quad B_j = \frac{n}{s}(1-b_j) \tag{2.60}$$

for $j = 1, 2$. Note that B_1 and B_2 remain bounded as $s/n \to 0$. Then

$$\begin{aligned} g_1(z) &= g_1(1) + \log\left(1 + \frac{s}{n}\frac{\left(1-\frac{s}{n}B_1\right)^{1/2} + \left(1-\frac{s}{n}B_2\right)^{1/2}}{e^{g_1(1)}}wH(w)\right), \\ H(w) &= \frac{1}{w}\left(w + ((w+B_1)(w+B_2))^{1/2} - (B_1B_2)^{1/2}\right), \end{aligned} \tag{2.61}$$

where $H(w)$ is analytic in w at the point 0. Thus g_1 has the following expansion in w at the point $w = 0$

$$g_1(z) = g_1(1) + \frac{s}{n}(c_1 w + \mathcal{O}(w^2)), \tag{2.62}$$

where

$$\begin{aligned} g_1(1) &= \log\left(\frac{\cos(\theta_1+\theta_2)/4 + \sqrt{|\sin\theta_1/2||\sin\theta_2/2|}}{\cos(\theta_1-\theta_2)/4}\right) \\ &= \frac{s\sqrt{|u_1u_2|}}{2n}\left(1 + \mathcal{O}\left(\frac{s}{n}\right)\right), \\ c_1 &= \frac{1}{\sqrt{(1-b_1)(1-b_2)}}\left(1 - \frac{\sqrt{b_1}+\sqrt{b_2}}{2e^{g_1(1)}}\right) \\ &= \left(\frac{1}{2} + \frac{i}{4}(u_1^{-1}+u_2^{-1})\sqrt{|u_1u_2|}\right)\left(1 + \mathcal{O}\left(\frac{s}{n}\right)\right), \end{aligned} \tag{2.63}$$

as $\frac{s}{n} \to 0$.

Define

$$r(z) = ((z-b_1)(z-b_2))^{1/2}, \tag{2.64}$$

where the square root has branch cut on J_2, and is positive as $z \to +\infty$. Let

$$h(z) = r(z) \int_{\Sigma_2^o} \frac{d\xi}{r(\xi)(\xi - z)}, \tag{2.65}$$

where integration is taken in counter-clockwise direction. It is easily verified by differentiation that

$$\begin{aligned} &- r(z) \int_{\widetilde{C}}^{t} \frac{d\xi}{r(\xi)(\xi - z)} \\ &= \log\left(\frac{2r^2(z) + (2z - b_1 - b_2)(t - z) + 2r(z)r(t)}{t - z}\right) + C(z, \tilde{C}), \end{aligned} \tag{2.66}$$

for any constant $\widetilde{C}$ and some function $C(z)$. Writing

$$h(z) = r(z) \int_{b_2}^{1} \frac{d\xi}{r(\xi)(\xi - z)}, \tag{2.67}$$

we see that (2.66) can be applied, and it follows that

$$\begin{aligned} h(z) = &\log \frac{b_2 - b_1}{2}(z - 1) \\ &- \log\left(z\left(1 - \frac{b_1 + b_2}{2}\right) + b_1 b_2 - \frac{b_1 + b_2}{2} + r(z)r(1)\right). \end{aligned} \tag{2.68}$$

The function h has a logarithmic singularity at $z = 1$ and a jump on $\Sigma_2^o \bigcup J_2$, such that

$$\begin{aligned} &h_+ - h_- = \begin{cases} 0 & \text{for } z \in \Sigma_1^o, \\ 2\pi i & \text{for } z \in \Sigma_2^o, \end{cases} \\ &h_+ + h_- = 0 \quad \text{for } z \in J_2. \end{aligned} \tag{2.69}$$

The jump conditions (2.69) also imply that

$$\begin{aligned} h(b_1) &= 0 \\ h_+(b_2) &= -h_-(b_2) = \pi i. \end{aligned} \tag{2.70}$$

As $z \to \infty$,

$$h(z) \to \log \frac{b_2 - b_1}{\left((1 - b_1)^{1/2} + (1 - b_2)^{1/2}\right)^2} \equiv h(\infty). \tag{2.71}$$

On the interval Σ^o we can alternatively write h in the following form:

$$\exp(h(z)) = \frac{\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta}{2}}{\cos\frac{\theta-\theta_1-\theta_2}{2} - \cos\frac{\theta_1-\theta_2}{2}\cos\frac{\theta}{2} + 2\sqrt{|\sin\frac{\theta-\theta_1}{2}\sin\frac{\theta-\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|}}. \tag{2.72}$$

With the notation of (2.60) we can write

$$h(z) = \log w\frac{B_1 - B_2}{(B_1+B_2)w + 2B_1B_2 + 2\sqrt{(w+B_1)(w+B_2)}\sqrt{B_1B_2}}. \tag{2.73}$$

Then we can expand h at the point $z = 1$:

$$h(z) = \log w + c_0' + c_1' w + \mathcal{O}(w^2), \tag{2.74}$$

where

$$\begin{aligned} c_0' &= -\log 4\frac{B_1B_2}{B_1-B_2} = \left(-\frac{\pi i}{2} + \log\frac{(u_1^{-1} - u_2^{-1})}{4}\right) + \mathcal{O}\left(\frac{s}{n}\right), \\ c_1' &= -\frac{B_1+B_2}{2B_1B_2} = -\frac{i}{2}(u_1^{-1} + u_2^{-1}) + \mathcal{O}\left(\frac{s}{n}\right), \end{aligned} \tag{2.75}$$

as $\frac{s}{n} \to 0$.

We define the g-function by:

$$g(z) = g_1(z) + \frac{\Omega}{2\pi}h(z), \tag{2.76}$$

where $\Omega > 0$ is a constant yet to be fixed. The jump conditions for g_1 and h imply that

$$\begin{aligned} g_+ + g_- &= \log z &&\text{for } z \in J_2, \\ g_+ - g_- &= 0 &&\text{for } z \in \Sigma_1^o, \\ g_+ - g_- &= i\Omega &&\text{for } z \in \Sigma_2^o. \end{aligned} \tag{2.77}$$

We define the local variable ζ on a disc U_0 containing the interval J_1 (but not the points b_1, b_2), by

$$\zeta(z) = e^{\frac{2\pi}{\Omega}\left(g(z) - \frac{1}{2}\log z\right)} = e^{h(z) + \frac{2\pi}{\Omega}\left(g_1(z) - \frac{1}{2}\log z\right)}. \tag{2.78}$$

The jump conditions for g imply that the function ζ is analytic in U_0. The precise radius of U_0 will be determined later on by requiring that the mapping ζ be conformal on U_0.

Since e^{g_1} maps $\mathbb{C} \setminus J_2$ to the exterior of the unit disc, we have

$$e^{g_1(e^{\pm i\theta_0}) \mp i\theta_0/2} > 1. \tag{2.79}$$

For $u_0 < \epsilon$ with some $\epsilon > 0$, it follows from (2.72) that

$$e^{h(e^{\pm i\theta_0})} \in \begin{cases} (0,1) & \text{for "+"}, \\ (-1,0) & \text{for "−"}. \end{cases} \tag{2.80}$$

Now consider, as a function of Ω,

$$\zeta(a) - \zeta(\overline{a}). \tag{2.81}$$

By (2.79) and (2.80) it follows that if we let $\Omega = +\infty$ in (2.78) then (2.81) is smaller than 2, and if we instead set $\Omega = +0$ then (2.81) is equal to $+\infty$. Since (2.82) is monotone in Ω, there exists, for $u_0 < \epsilon$, a unique value for $\Omega > 0$ such that

$$\zeta(a) - \zeta(\overline{a}) = 4. \tag{2.82}$$

We define Ω so that ζ satisfies (2.82). From (2.58) and (2.72), it follows that $\zeta(\Sigma^o) \subset \mathbb{R}$. By (2.62) and (2.74) we have the following expansion at the point $z = 1$:

$$\begin{aligned} \zeta(z) =& w\zeta_0 \left(1 + \zeta_1 w + \mathcal{O}\left(\left(\frac{sw}{n\Omega}\right)^2\right)\right), \qquad w = \frac{n}{s}(z-1), \\ &\zeta_0 = e^{c_0' + \frac{2\pi}{\Omega} g_1(1)}, \\ &\zeta_1 = c_1' + \frac{2\pi s}{n\Omega}(c_1 - 1/2). \end{aligned} \tag{2.83}$$

In what follows, it will be apparent that $\Omega \to 0$ in the limit $s, n \to \infty$ and $u_0, s/n \to 0$. Moreover, by (2.63) and (2.75),

$$\begin{aligned} \zeta_0 &= -\frac{i}{4}(u_1^{-1} - u_2^{-1}) e^{\frac{\pi}{\Omega}\frac{s\sqrt{|u_1 u_2|}}{n}(1+\mathcal{O}(s/n))}(1 + \mathcal{O}(s/n)) \\ \zeta_1 &= \frac{2\pi s}{n\Omega}(c_1 - 1/2) - \frac{i}{2}(u_1^{-1} + u_2^{-1}) + \mathcal{O}(s/n). \end{aligned} \tag{2.84}$$

Substituting these expansions into (2.83), which we in turn substitute into (2.82), we obtain

$$u_0\left(1 + \mathcal{O}\left(\frac{1}{\Omega^2}\frac{u_0^2 s^2}{n^2}\right)\right) = \frac{8}{u_1^{-1} - u_2^{-1}} e^{-\frac{2\pi}{n\Omega}\frac{s\sqrt{|u_1 u_2|}}{2}(1+\mathcal{O}(s/n))}(1+\mathcal{O}(s/n)), \tag{2.85}$$

or upon taking the logarithm,

$$\log\left(u_0\frac{u_1^{-1}-u_2^{-1}}{8}\right)^{-1} = \frac{\pi s\sqrt{|u_1u_2|}}{\Omega n}\left(1+\mathcal{O}(s/n)+\mathcal{O}(\Omega)+\mathcal{O}\left(\frac{1}{\Omega}\frac{u_0^2 s}{n}\right)\right). \tag{2.86}$$

Therefore,

$$\Omega = \frac{\pi s\sqrt{|u_1u_2|}}{n}\Big/\log\frac{8}{(u_1^{-1}-u_2^{-1})u_0}\left(1+\mathcal{O}(s/n)+\mathcal{O}\left(u_0^2\log u_0^{-1}\right)\right). \tag{2.87}$$

Using the definition of g_1, h and ζ in (2.54), (2.68), (2.78), and the expansion of Ω in (2.87), it is easily seen that there are constants $m_1 < m_2$ independent of s, n, u_0 such that $\zeta'(z)$ has at least one zero in the set

$$\left\{z : \frac{sm_1}{n\log u_0^{-1}} < |z-1| < \frac{sm_2}{n\log u_0^{-1}}\right\}. \tag{2.88}$$

By (2.84) and expansion (2.87), m_1 may be chosen such that

$$\left|\zeta_1 w + \mathcal{O}\left(\left(\frac{sw}{n\Omega}\right)^2\right)\right| < 1 \tag{2.89}$$

as $\frac{sw}{n\Omega}\to 0$ and so ζ is conformal on the following set

$$\left\{z : |z-1| < \frac{sm_1}{n\log u_0^{-1}}\right\}, \tag{2.90}$$

for $u_0, s/n < \epsilon$ for some fixed $\epsilon > 0$. Thus we define U_0 to be the set (2.90).

We define $\widetilde{g}$ as

$$\begin{aligned}\widetilde{g} &= \lim_{z\to\infty} e^{g(z)-\log z}\\ &= \frac{2}{b_1^{1/2}+b_2^{1/2}}\left(\frac{b_2-b_1}{\left((1-b_1)^{1/2}+(1-b_2)^{1/2}\right)^2}\right)^{\frac{\Omega}{2\pi}},\end{aligned} \tag{2.91}$$

and T as

$$T(z) = \widetilde{g}^{n\sigma_3}Y(z)e^{-ng(z)\sigma_3}. \tag{2.92}$$

It follows by (2.77) that T satisfies the following RH problem:

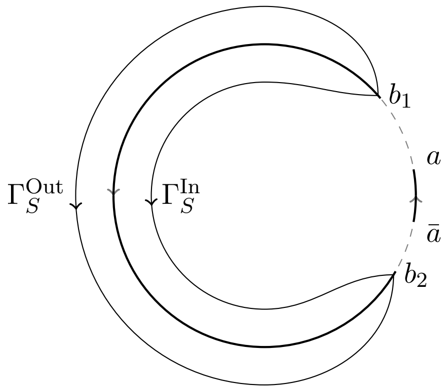

Figure 2.2: Contour Γ_S.

(a) $T : \mathbb{C} \setminus C \to \mathbb{C}^{2\times 2}$ is analytic.

(b) T has the following jumps on C:

$$\begin{aligned}
T_+ &= T_- \begin{pmatrix} e^{n(g_- - g_+)(z)} & 1 \\ 0 & e^{n(g_+ - g_-)(z)} \end{pmatrix} && \text{for } z \in J_2, \\
T_+ &= T_- e^{-in\Omega\sigma_3} && \text{for } z \in \Sigma_2. \\
T_+ &= T_- \begin{pmatrix} e^{-in\Omega} & z^{-n} e^{n(g_+ + g_-)} \\ 0 & e^{in\Omega} \end{pmatrix} && \text{for } z \in \Sigma_2^o \bigcap J_1, \\
T_+ &= T_- \begin{pmatrix} 1 & z^{-n} e^{2ng} \\ 0 & 1 \end{pmatrix} && \text{for } z \in \Sigma_1^o \bigcap J_1. \\
T_+ &= T_- && \text{for } z \in \Sigma_1.
\end{aligned}$$

(c) As $z \to \infty$,

$$T(z) = I + \mathcal{O}(z^{-1}).$$

The jump of T on J_2 factorizes as

$$\begin{aligned}
&\begin{pmatrix} e^{n(g_- - g_+)(z)} & 1 \\ 0 & e^{n(g_+ - g_-)(z)} \end{pmatrix} \\
&\qquad = \begin{pmatrix} 1 & 0 \\ e^{n(g_+ - g_-)(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{n(g_- - g_+)(z)} & 1 \end{pmatrix}. \quad (2.93)
\end{aligned}$$

Define

$$\phi(z) = 2g(z) - \log(z), \quad (2.94)$$

Then the jumps of e^{ϕ} are induced by g and we obtain by (2.77) that for $z \in J_2$.

$$\begin{aligned}\exp(\phi_+(z)) &= \exp[(g_+(z) + g_-(z) - \log(z)) + g_+(z) - g_-(z)] \\ &= \exp(g_+(z) - g_-(z)),\end{aligned} \tag{2.95}$$

and similarly

$$\exp(\phi_-(z)) = \exp(g_-(z) - g_+(z)). \tag{2.96}$$

We proceed to open the lenses around J_2 as in Figure 2.2.

Proposition 2.3.1. *For z on the edges of the lense $\Gamma_S^{in} \cup \Gamma_S^{out}$ in Figure 2.2 such that $|z - b_1|, |z - b_2| > \epsilon s/n$ for some fixed $\epsilon > 0$, there exists a constant $C > 0$ independant of s, n, z such that*

$$e^{-ng(z)} < e^{-sC}, \tag{2.97}$$

as $s, n \to \infty$ and for u_0 sufficiently small.

Proof. Since e^{g_1} sends $\mathbb{C} \setminus J_2$ to the outside of the unit disc, it is clear that for $|z| < 1$ we have

$$|e^{2g_1(z) - \log(z)}| > 1. \tag{2.98}$$

Let $g_1^{(-)}$ denote the function defined as in (2.54), but with $+((z-b_1)(z-b_2))^{1/2}$ replaced with $-((z-b_1)(z-b_2))^{1/2}$ Then $e^{g_1^{(-)}}$ maps $\mathbb{C} \setminus J_2$ to the inside of the circle and for $z \in \mathbb{C} \setminus J_2$ we have the relation

$$e^{g_1(z) + g_1^{(-)}(z)} = z. \tag{2.99}$$

It follows that if $|z| > 1$, then

$$|e^{2g_1(z) - \log z}| = |e^{\log z - 2g_1^{(-)}(z)}| > 1. \tag{2.100}$$

Using (2.98), (2.100), the definition of g, and the fact that $\Omega = \mathcal{O}(s/(n \log u_0^{-1}))$, as $s/(n \log u_0^{-1}) \to 0$, it follows that $e^{-\phi(z)}$ lies in interior of the unit disc for z sufficiently close to the interval J_2, and in particular that

$$e^{-n\phi(z)} = \mathcal{O}\left(e^{-cn}\right), \tag{2.101}$$

uniformly for z on the lense that is opened around J_2 in Figure 2.2, except near the endpoints b_1 and b_2, for some constant $c > 0$.

Consider $h(z)$ and $g_1(z)$ at $z = b_1$. Let $w_1 = \frac{n}{s}(z - b_1)$. From (2.68) we have, with $B_j = \frac{n}{s}(1 - b_j)$,

$$h(z) = \log \frac{B_1 - B_2}{2}(w_1 - B_1)$$
$$-\log\left(w_1 \frac{B_1 + B_2}{2} + \frac{B_2 - B_1}{2} B_1 + (w_1(w_1 + B_2 - B_1)B_1 B_2)^{1/2}\right).$$

Thus it follows from (2.69) that $h(z)/w_1^{1/2}$ is analytic at $w_1 = 0$, and we have

$$h(z) = -w_1^{1/2} \frac{2}{(B_2 - B_1)^{1/2}} \left(\frac{B_2}{B_1}\right)^{1/2} (1 + \mathcal{O}(w_1)). \tag{2.102}$$

Likewise, we let $w_2 = \frac{n}{s}(z - b_2)$. Then at the point $w_2 = 0$ we have the expansion

$$h(z) = \pm\pi i - w_2^{1/2} \frac{2}{(B_1 - B_2)^{1/2}} \left(\frac{B_1}{B_2}\right)^{1/2} (1 + \mathcal{O}(w_2)), \tag{2.103}$$

where $\pm$ means $+$ on the $+$side and $-$ on the $-$side of the unit circle C (so the jumps agree with (2.69)).

We evaluate g_1 using the definition (2.54):

$$g_1(z) = \log \frac{b_1 + (b_1 b_2)^{1/2} + \frac{s}{n}((w_1(w_1 + B_2 - B_1))^{1/2} + w_1)}{b_1^{1/2} + b_2^{1/2}}. \tag{2.104}$$

From (2.104) and (2.56), we have that $(g_1(z) - \frac{\log z}{2})/w_1^{1/2}$ is analytic at $w_1 = 0$, and that at the point $w_1 = 0$ we have the expansion

$$g_1(z) - \frac{\log z}{2} = \frac{s}{n} w_1^{1/2} \frac{(B_2 - B_1)^{1/2}}{b_1 + (b_1 b_2)^{1/2}} (1 + \mathcal{O}(w_1)). \tag{2.105}$$

Likewise we expand g_1 at the point $w_2 = 0$:

$$g_1(z) - \frac{\log z}{2} = \frac{s}{n} w_2^{1/2} \frac{(B_1 - B_2)^{1/2}}{b_2 + (b_1 b_2)^{1/2}} (1 + \mathcal{O}(w_2)). \tag{2.106}$$

Consider now a neighbourhood of b_1. The error terms in (2.102), (2.105) are uniform for $0 < s/n < \delta$ for some sufficiently small δ. By (2.102), (2.105) and the fact that $\Omega = \mathcal{O}\left(\frac{s}{n \log u_0^{-1}}\right)$, it follows that there is a constant $C_1 > 0$ independant of s, n, z for u_0 sufficiently small such that

$$\mathrm{Re}\,(g) > w_1^{1/2} C_1 s/n \tag{2.107}$$

as $z \to b_1, s/n \to 0$. Thus from (2.107) it follows that there exists $\epsilon, C > 0$ such that for all $|z - b_1| > \epsilon s/n$ we have

$$e^{-ng(z)} < e^{-sC} \tag{2.108}$$

as $n, s \to \infty$, and for u_0 sufficiently small. The same may be shown at the point b_2, concluding the proof. $\square$

Let

$$S(z) = \begin{cases} T(z) & \text{for } z \text{ outside the lenses,} \\ T(z)\begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)} & 1 \end{pmatrix} & \text{for } z \text{ inside the lense and outside the unit disc,} \\ T(z)\begin{pmatrix} 1 & 0 \\ -e^{-n\phi(z)} & 1 \end{pmatrix} & \text{for } z \text{ inside the lense and inside the unit disc,} \end{cases} \tag{2.109}$$

Then S satisfies the following RH problem:

(a) $S : \mathbb{C} \setminus \Gamma_S \to \mathbb{C}^{2\times 2}$ is analytic, where $\Gamma_S = C \cup \Gamma_S^{in} \cup \Gamma_S^{out}$ as shown in Figure 2.2.

(b) On $\Gamma_S \setminus \{a, \overline{a}, b_1, b_2\}$, S has the following jumps:

$$\begin{aligned}
S_+ &= S_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} && \text{for } z \in J_2, \\
S_+ &= S_- \begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)} & 1 \end{pmatrix} && \text{for } z \in \Gamma_S^{in} \cup \Gamma_S^{out}, \\
S_+ &= S_- e^{-in\Omega\sigma_3} && \text{for } z \in \Sigma_2. \\
S_+ &= S_- \begin{pmatrix} e^{-in\Omega} & z^{-n}e^{n(g_+ + g_-)} \\ 0 & e^{in\Omega} \end{pmatrix} && \text{for } z \in (\overline{a}, 1) = J_1 \bigcap \Sigma_2^o, \\
S_+ &= S_- \begin{pmatrix} 1 & z^{-n}e^{2ng} \\ 0 & 1 \end{pmatrix} && \text{for } z \in (1, a) = J_1 \bigcap \Sigma_1^o. \\
S_+ &= S_- && \text{for } z \in \Sigma_1.
\end{aligned}$$

(c) As $z \to \infty$,

$$S(z) = I + \mathcal{O}\left(z^{-1}\right). \tag{2.110}$$

2.3.1 Main parametrix

In the region $\mathbb{C} \setminus (U_0 \cup U_1 \cup U_2)$, we approximate the RH problem for S by a main parametrix M, which satisfies the RH problem:

(a) $M : \mathbb{C} \setminus \overline{\{J_2 \bigcup \Sigma_2^o\}} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) On J_2 and Σ_2^o, M has the following jumps:

$$M_+(z) = M_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{for } z \in J_2,$$
$$M_+(z) = M_-(z) e^{-in\Omega\sigma_3} \qquad \text{for } z \in \Sigma_2^o.$$

(c) As $z \to \infty$,

$$M(z) = I + \mathcal{O}\left(z^{-1}\right).$$

A solution to the RH problem for M is given by

$$M(z) = \left(I + \frac{F}{z-1}\right) D^{-1}(\infty) \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix} D(z), \tag{2.111}$$

where F is a constant matrix and

$$\begin{aligned} \gamma_1(z) &= \frac{1}{2}\left(\left(\frac{z-b_1}{z-b_2}\right)^{1/4} + \left(\frac{z-b_2}{z-b_1}\right)^{1/4}\right) \\ \gamma_2(z) &= \frac{1}{2i}\left(\left(\frac{z-b_1}{z-b_2}\right)^{1/4} - \left(\frac{z-b_2}{z-b_1}\right)^{1/4}\right) \end{aligned} \tag{2.112}$$

with branch cuts on J_2 and such that $\gamma_1(z) \to 1$ and $\gamma_2(z) \to 0$ as $z \to \infty$. For $y \in \mathbb{R}$, let $\langle y \rangle$ be defined such that

$$\langle y \rangle \in [-1/2, 1/2), \quad y - \langle y \rangle \in \mathbb{Z}. \tag{2.113}$$

Then D is given by

$$D(z) = \exp\left(-\left\langle \frac{n\Omega}{2\pi} \right\rangle h(z)\sigma_3\right), \tag{2.114}$$

where it follows from the jumps of h (2.69) that D is analytic for $z \in \mathbb{C} \setminus (\overline{\Sigma_2^o \bigcup J_2})$, and

$$\begin{aligned} D_-^{-1} D_+ &= \exp\left(-2\pi i \left\langle \frac{n\Omega}{2\pi}\right\rangle \sigma_3\right) && \text{for } z \in \Sigma_2^o, \\ D_- D_+ &= I && \text{for } z \in J_2. \end{aligned} \tag{2.115}$$

The function M defined in (2.111) will solve the RH problem for M with any constant matrix F, which we will define later in (2.128). The reason for the prefactor $I + \frac{F}{z-1}$ in (2.111), which does not affect the jump conditions for M, will become apparent later on.

2.3.2 Model RH problem Φ

Consider the following RH problem for $\Phi(\zeta;k)$, where $k \in \mathbb{N}$:

(a) $\Phi(\zeta) : \mathbb{C} \setminus [\eta_1, \eta_2] \to \mathbb{C}^{2\times 2}$ is analytic for given $\eta_1 < \eta_2$.

(b) Φ has L^2 boundary values for $\zeta \in (\eta_1, \eta_2)$ satisfying

$$\Phi_+(\zeta;k) = \Phi_-(\zeta;k) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(c) As $\zeta \to \infty$,

$$\Phi(\zeta;k) = \left(I + \frac{\Phi_1}{\zeta} + \frac{\Phi_2}{\zeta^2} + \mathcal{O}\left(\zeta^{-3}\right) \right) \zeta^{k\sigma_3}. \tag{2.116}$$

It is well-known by the standard theory and is easy to verify that the unique solution to this RH problem is given by

$$\begin{aligned} \Phi(\zeta;0) &= \begin{pmatrix} 1 & \frac{1}{2\pi i}\log\left(\frac{\zeta-\eta_2}{\zeta-\eta_1}\right) \\ 0 & 1 \end{pmatrix} \\ \Phi(\zeta;k) &= \begin{pmatrix} \frac{1}{\kappa_k} L_k(\zeta) & \frac{1}{2\pi i \kappa_k} \int_{\eta_1}^{\eta_2} \frac{L_k(x)}{x-\zeta} dx \\ -2\pi i \kappa_{k-1} L_{k-1}(\zeta) & -\kappa_{k-1} \int_{\eta_1}^{\eta_2} \frac{L_{k-1}(x)}{x-\zeta} dx \end{pmatrix} \end{aligned} \tag{2.117}$$

for $k \geq 1$, where L_k are the Legendre polynomials of degree k with positive leading coefficients, orthonormal on (η_1, η_2):

$$\int_{\eta_1}^{\eta_2} L_k(\zeta) L_j(\zeta) d\zeta = \delta_{jk} = \begin{cases} 0 & \text{for } j \neq k, \\ 1 & \text{for } j = k, \end{cases} \tag{2.118}$$

and we denote the first 3 leading coefficients as follows:

$$L_k(\zeta) = \kappa_k \zeta^k + \mu_k \zeta^{k-1} + \nu_k \zeta^{k-2} + \dots. \tag{2.119}$$

Writing the large ζ expansion of (2.117) and using orthogonality in the second column, we obtain that

$$\begin{aligned} \Phi_1 &= \begin{pmatrix} \frac{\mu_k}{\kappa_k} & -\frac{1}{2\pi i}\kappa_k^{-2} \\ -2\pi i \kappa_{k-1}^2 & -\frac{\mu_k}{\kappa_k} \end{pmatrix} && \text{for } k \geq 1, \\ \Phi_2 &= \begin{pmatrix} \frac{\nu_k}{\kappa_k} & \frac{\mu_{k+1}}{2\pi i \kappa_{k+1} \kappa_k^2} \\ -2\pi i \kappa_{k-1} \mu_{k-1} & \frac{1}{\kappa_k \kappa_{k+1}} (\mu_k \mu_{k+1} - \nu_{k+1} \kappa_k) \end{pmatrix} && \text{for } k \geq 2. \end{aligned} \tag{2.120}$$

When $k=0$ we have

$$\Phi_1 = \begin{pmatrix} 0 & -\frac{\eta_2-\eta_1}{2\pi i} \\ 0 & 0 \end{pmatrix}. \tag{2.121}$$

It is well known that L_k has the explicit representation for $k \geq 0$:

$$\begin{aligned} L_k(\zeta) &= \sqrt{\frac{2k+1}{\eta_2-\eta_1}} \sum_{j=0}^{k} \binom{k}{j}\binom{k+j}{j}\left(\frac{\zeta-\eta_2}{\eta_2-\eta_1}\right)^j \\ &= \sqrt{\frac{2k+1}{\eta_2-\eta_1}} \sum_{j=0}^{k} \binom{k}{j}\binom{k+j}{j}\left(\frac{\zeta-\eta_1}{\eta_2-\eta_1}\right)^j (-1)^{k-j}, \end{aligned} \tag{2.122}$$

where $\binom{0}{0} = 1$. As a consequence, the coefficients in (2.119) are given by

$$\begin{aligned} \kappa_k &= (\eta_2-\eta_1)^{-k-1/2}\sqrt{2k+1}\binom{2k}{k}, \\ \mu_k &= -(\eta_2-\eta_1)^{-k-1/2}\sqrt{2k+1}\binom{2k}{k}\frac{k}{2}(\eta_1+\eta_2), \\ \nu_k &= (\eta_2-\eta_1)^{-k-1/2}\sqrt{2k+1}\binom{2k-2}{k-2} \\ &\qquad \frac{k}{2}\left(k(\eta_1+\eta_2)^2 - \left(\eta_1^2+\eta_2^2\right)\right), \end{aligned} \tag{2.123}$$

for $k \geq 0, 1, 2$ respectively. From (2.117), (2.122), (2.123), it follows that for $k \geq 1$,

$$\begin{aligned} \Phi_{11}(\zeta) &= (-1)^k \frac{(\eta_2-\eta_1)^k}{\binom{2k}{k}} \\ &\qquad \times \left(1 - \frac{k(k+1)}{\eta_2-\eta_1}(\zeta-\eta_1) + \mathcal{O}((\zeta-\eta_1)^2)\right) \\ \Phi_{21}(\zeta) &= (-1)^k 2\pi i(2k-1)(\eta_2-\eta_1)^{-k}\binom{2k-2}{k-1} \\ &\qquad \times \left(1 - \frac{k(k-1)}{\eta_2-\eta_1}(\zeta-\eta_1) + \mathcal{O}((\zeta-\eta_1)^2)\right), \end{aligned} \tag{2.124}$$

as $\zeta \to \eta_1$, while as $\zeta \to \eta_2$,

$$\begin{aligned} \Phi_{11}(\zeta) &= \frac{(\eta_2-\eta_1)^k}{\binom{2k}{k}}\left(1 + \frac{k(k+1)}{\eta_2-\eta_1}(\zeta-\eta_2) + \mathcal{O}((\zeta-\eta_2)^2)\right) \\ \Phi_{21}(\zeta) &= -2\pi i(2k-1)(\eta_2-\eta_1)^{-k}\binom{2k-2}{k-1} \\ &\qquad \times \left(1 + \frac{k(k-1)}{\eta_2-\eta_1}(\zeta-\eta_2) + \mathcal{O}((\zeta-\eta_2)^2)\right). \end{aligned} \tag{2.125}$$

2.3.3 Local parametrix at 1

Recall that U_0 defined by (2.90) is an open disc containing J_1 and that as $s, n \to \infty, u_0 \to 0$ the radius of U_0 is of length $\epsilon s/(n \log u_0^{-1})$ for some $\epsilon > 0$. On U_0 we defined a local variable ζ in (2.78). We define the local parametrix P on U_0 by

$$P(z) = E(z)\Phi\left(\zeta(z); k\right) e^{-n\left(g(z)-\frac{\log(z)}{2}\right)\sigma_3},$$
$$k = \frac{\Omega n}{2\pi} - x, \quad x = \left\langle \frac{\Omega n}{2\pi} \right\rangle, \quad (2.126)$$

where Φ is given by (2.117), and where E is an analytic function on U_0:

$$E(z) = \left(I + \frac{F}{z-1}\right) D(\infty)^{-1}$$
$$\times \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix} B(z) \left(I - \frac{X}{\zeta(z)}\right), \quad (2.127)$$
$$B(z) = e^{\frac{2\pi}{\Omega}x\left(g_1(z)-\frac{1}{2}\log z\right)\sigma_3},$$

with constant matrices F and X defined below. From (2.58) and (2.72) we see that $\zeta(J_1) \subset \mathbb{R}$. We let $\eta_1 = \zeta(\overline{a})$ and $\eta_2 = \zeta(a)$. Then $\zeta(J_1) = (\eta_1, \eta_2)$, and so $P(z)$ has a jump on J_1 induced by that of Φ on (η_1, η_2). F is a constant, nilpotent matrix

$$F = \widetilde{h}^{-\sigma_3} \begin{pmatrix} f & \psi f \\ -f/\psi & -f \end{pmatrix} \widetilde{h}^{\sigma_3}, \quad (2.128)$$

$$\widetilde{h} = D_{11}(\infty) = \exp(-xh(\infty)) = \left(\frac{(1-b_1)^{1/2} + (1-b_2)^{1/2}}{b_2 - b_1} \right)^{-x}, \quad (2.129)$$

$h(\infty)$ was defined in (2.71), and where

$$\psi = \begin{cases} -\frac{\gamma_1(1)}{\gamma_2(1)} & \text{for } 0 \le x < 1/2, \\ \frac{\gamma_2(1)}{\gamma_1(1)} & \text{for } -1/2 \le x < 0, \end{cases}$$
$$f = \begin{cases} -\frac{(1-b_1)^{1/2}(1-b_2)^{1/2}\rho}{1+\rho} & \text{for } 0 \le x < 1/2, \\ -\frac{(1-b_1)^{1/2}(1-b_2)^{1/2}\rho}{1-\rho} & \text{for } -1/2 \le x < 0, \end{cases}$$
$$\rho = \begin{cases} \frac{1}{2\pi\kappa_k^2} \exp\left[\frac{2\pi}{\Omega} g_1(1)\left(-1+2x\right)\right] & \text{for } 0 \le x < 1/2, \\ -2\pi\kappa_{k-1}^2 \exp\left[\frac{2\pi}{\Omega} g_1(1)\left(-1-2x\right)\right] & \text{for } -1/2 \le x < 0, \end{cases} \quad (2.130)$$

and X is a constant matrix given in terms of elements of (2.120)

$$X = \begin{cases} \Phi_{1,12} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{for } 0 \le x < 1/2, \\ \Phi_{1,21} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{for } -1/2 \le x < 0. \end{cases} \tag{2.131}$$

The factor $I - X/\zeta(z)$ in $E(z)$ is needed to cancel the would be $u_0^{1-2|x|} \log u_0^{-1}$ non-smallness in the matrix elements of $\Delta_1^{(1)}$ originating from $\Phi_{1,12} B_{11}^2$ for $0 \le x < 1/2$ and $\Phi_{1,21} B_{11}^{-2}$ for $-1/2 \le x < 0$ (see Proposition 2.3.2 below) so that P and M match to the main order on the boundary ∂U_0 for all $x \in [-1/2, 1/2)$. This factor, however, has a pole at $z = 1$, but we need $E(z)$ to be analytic in U_0. As is easy to verify, the analyticity of $E(z)$ (i.e. the absence of a pole at 1) is achieved by choosing F as defined in (2.128).

Proposition 2.3.2. *As $u_0 \to 0$ and $s, n \to \infty$, we have the following matching condition uniformly on the boundary ∂U:*

$$\begin{aligned} P(z) M^{-1}(z) = I + \Delta_1^{(1)}(z) + \Delta_2^{(1)}(z) + \Xi^{(1)} \\ + \mathcal{O}(\widetilde{e} u_0^{-2|x|} (u_0 \log u_0^{-1})^3 + \widetilde{e}(s^2 u_0^4 (\log u_0^{-1})^2), \end{aligned} \tag{2.132}$$

where

$$\widetilde{e} = \left(1 + u_0^{1-2|x|} \log u_0^{-1}\right)^2, \tag{2.133}$$

and $\Delta_1^{(1)}$, $\Delta_2^{(1)}$ and $\Xi^{(1)}$ are given by (2.140) below. We have, uniformly for z on the boundary ∂U,

$$\begin{aligned} \Delta_1^{(1)}(z) &= \mathcal{O}(\widetilde{e} u_0^{1+2|x|} \log u_0^{-1} + \widetilde{e} s u_0^2 \log u_0^{-1}), \\ \Delta_2^{(1)}(z) &= \mathcal{O}\left(\widetilde{e} u_0^2 (\log u_0^{-1})^2 \left(\frac{s}{\log u_0^{-1}} + u_0^{1-2|x|} \log u_0^{-1}\right)\right), \\ \Xi^{(1)} &= \mathcal{O}\left(\widetilde{e} s u_0^{-2|x|} u_0^3 (\log u_0^{-1})^2\right). \end{aligned} \tag{2.134}$$

Proof. First, assume that k is bounded. Since

$$\zeta^k(z) e^{-n\left(g(z) - \frac{\log z}{2}\right)} = e^{-\frac{2\pi x}{\Omega}\left(g(z) - \frac{\log z}{2}\right)}, \qquad x = \frac{n\Omega}{2\pi} - k \tag{2.135}$$

we have on the boundary ∂U that (recall (2.76), (2.111), (2.116), (2.127))

$$P(z)M^{-1}(z) = E(z)\left(I + \frac{\Phi_1}{\zeta} + \frac{\Phi_2}{\zeta^2} + \mathcal{O}(\zeta^{-3})\right)\left(I - \frac{X}{\zeta(z)}\right)E^{-1}(z). \tag{2.136}$$

It follows from (2.63), (2.87), (2.130) that

$$\begin{aligned} \rho &= \mathcal{O}\left(u_0^{1-2|x|}\right), \\ f, F &= \mathcal{O}\left(s u_0^{1-2|x|}/n\right), \end{aligned} \tag{2.137}$$

as $u_0, s/n \to 0$ and $s, n \to \infty$. Denote

$$\widetilde{E}(z) = \left(I + \frac{F}{z-1}\right) D(\infty)^{-1} \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix}. \tag{2.138}$$

From (2.137), the boundedness of $\widetilde{h}, \widetilde{h}^{-1}, \psi, \psi^{-1}, \gamma_1(z), \gamma_2(z)$ for $z \in \partial U_0$, and the fact that the radius of $U_0 = \epsilon \frac{s}{n \log u_0^{-1}}$ for some $\epsilon > 0$, we have

$$\widetilde{E}(z) = \mathcal{O}(\sqrt{e}) \tag{2.139}$$

as $u_0 \to 0$ and $s, n \to \infty$, uniformly for $z \in \partial U_0$. From (2.136) and (2.138) it follows that $\Delta_1^{(1)}, \Delta_2^{(1)}$ take the form

$$\begin{aligned} \Delta_1^{(1)} &= \begin{cases} \zeta^{-1}\widetilde{E}(z)B(z)\begin{pmatrix} \Phi_{1,11} & 0 \\ \Phi_{1,21} & \Phi_{1,22} \end{pmatrix} B^{-1}(z)\widetilde{E}^{-1}(z) & 0 \le x < \frac{1}{2}, \\ \zeta^{-1}\widetilde{E}(z)B(z)\begin{pmatrix} \Phi_{1,11} & \Phi_{1,12} \\ 0 & \Phi_{1,22} \end{pmatrix} B^{-1}(z)\widetilde{E}^{-1}(z) & -\frac{1}{2} \le x < 0, \end{cases} \\ \Delta_2^{(1)} &= \zeta^{-2}(z)\widetilde{E}(z)B(z)(\Phi_2 - X\Phi_1)B^{-1}(z)\widetilde{E}^{-1}(z), \\ \Xi^{(1)} &= -\zeta^{-3}(z)\widetilde{E}(z)B(z)X\Phi_2 B^{-1}(z)\widetilde{E}^{-1}(z). \end{aligned} \tag{2.140}$$

From (2.62)–(2.63), and recalling (2.87) and the fact that $w = \mathcal{O}\left(\frac{1}{\log u_0^{-1}}\right)$ as $u_0 \to 0$, it follows that

$$e^{\frac{2pi}{\Omega}x\left(g_1(z) - \frac{1}{2}\log z\right)} = \mathcal{O}\left(u_0^{-1}\right) \tag{2.141}$$

as $u_0 \to 0$, uniformly on the boundary ∂U_0. Similarly, substituting (2.84) into (2.83) and recalling (2.87), we have

$$\zeta(z) = \mathcal{O}\left(\frac{1}{u_0 \log u_0^{-1}}\right) \tag{2.142}$$

as $u_0 \to 0$, uniformly on the boundary ∂U_0. From (2.83) and (2.84) we have

$$\zeta(a) + \zeta(\overline{a}) = \mathcal{O}(u_0 \log u_0^{-1}), \tag{2.143}$$

as $u_0 \to 0$. Using (2.143) it follows from (2.120) and (2.123) that

$$\begin{aligned} &\Phi_{1,11}, \Phi_{1,22}, \Phi_{2,12}, \Phi_{2,21} = \mathcal{O}\left(u_0 \log u_0^{-1}\right) \\ &\Phi_{1,12}, \Phi_{1,21}, \Phi_{2,11}, \Phi_{2,22} = \mathcal{O}\left(1\right) \end{aligned} \tag{2.144}$$

as $u_0 \to 0$ (for finite k). Combining (2.139)–(2.144) the proposition is proven for bounded k.

Now consider $k \to \infty$. From Stirling's formula we have

$$2\pi\kappa_k^2 \to 1 \tag{2.145}$$

as $k \to \infty$. Thus (2.137) holds uniformly for $k \in \mathbb{N}$. We study the particular double scaling limit where $k, \zeta \to \infty$, and from (2.85), (2.143) we have that $\eta_1 + \eta_2 \to 0$ in such a manner that $k/\zeta, k(\eta_1 + \eta_2) = \mathcal{O}(u_0 s) \to 0$. Thus using (2.145) we find that as $k \to \infty$

$$\Phi_1 = \begin{pmatrix} -\frac{k}{2}(\eta_1+\eta_2) & i + \mathcal{O}(k^{-1}) \\ -i + \mathcal{O}(k^{-1}) & \frac{k}{2}(\eta_1+\eta_2) \end{pmatrix}. \tag{2.146}$$

We also find that as $k \to \infty$ and $(\eta_1 + \eta_2) \to 0$ such that $(\eta_1 + \eta_2)k \to 0$

$$\Phi_2 = \begin{pmatrix} -\frac{k}{8}(\eta_1^2+\eta_2^2) + \mathcal{O}(1) & \frac{ik}{2}(\eta_1+\eta_2) + \mathcal{O}(\eta_1+\eta_2) \\ \frac{ik}{2}(\eta_1+\eta_2) + \mathcal{O}(\eta_1+\eta_2) & \frac{k}{8}(\eta_1^2+\eta_2^2) + \mathcal{O}(1) \end{pmatrix}. \tag{2.147}$$

Thus we know the large k, ζ behaviour of $\zeta^{-1}\Phi_1$, $\zeta^{-2}\Phi_2$, and upon substituting into (2.140) this yields (2.134). It remains to calculate the error terms of order $\zeta^{-3}\Phi_3$ and higher, and in particular establish their behaviour as $k \to \infty$ with ζ. We rely here on the work by Kuijlaars, McLaughlin, Van Assche and Vanlessen in [73] where the authors found uniform error terms for the Legendre polynomials L_k as $k \to \infty$. In the remaining part of the proof of the proposition, we let $\widehat{Y}, \widehat{R}, \widehat{N}$ denote the functions Y, R, N found in [73]. We compare $\widehat{Y}$ to Φ in (2.117) in the present paper:

$$\Phi(\zeta) = 2^{k\sigma_3}\widehat{Y}(y(\zeta)), \quad y(\zeta) = \frac{1}{2}\left(z - \frac{\eta_1+\eta_2}{2}\right) \tag{2.148}$$

where the parameter n in [73] is set to be k here. For y bounded away from $[-1, 1]$ it follows from equations (3.1), (4.2), (5.5), (7.1) in [73] that

$$\begin{aligned} \widehat{Y}(y) &= 2^{-k\sigma_3}\widehat{R}(y)\widehat{N}(y)y^{k\sigma_3}\left(1 + \left(1 - y^{-2}\right)^{1/2}\right)^{k\sigma_3}, \\ \widehat{N}(y) &= \begin{pmatrix} \frac{1}{2}(a(y) + 1/a(y)) & \frac{1}{2i}(a(y) - 1/a(y)) \\ -\frac{1}{2i}(a(y) - 1/a(y)) & \frac{1}{2}(a(y) + 1/a(y)) \end{pmatrix}, \end{aligned} \tag{2.149}$$

where $a(y) = \left(\frac{y-1}{y+1}\right)^{1/4}$. By the form of $\widehat{N}$ in (2.149) above and formula (8.11) in [73] it is clear that

$$\widehat{R}(y(\zeta);k)\widehat{N}(y(\zeta)) = I + \frac{\chi_1}{\zeta} + \frac{\chi_2}{\zeta^2} + \mathcal{O}(\zeta^{-3})$$

as $\zeta \to \infty$, where χ_1 and χ_2 are bounded for $k \in \mathbb{N}$ and the $\mathcal{O}(\zeta^{-3})$ term is uniform for $k \in \mathbb{N}$. As $\zeta, k \to \infty$ and $\eta_1 + \eta_2 \to 0$ such that $k/\zeta \to 0$, $k(\eta_1 + \eta_2) \to 0$, we have

$$\begin{aligned}\left(1 + \left(1 - y^{-2}\right)^{1/2}\right)^{\pm k} &= 1 \mp k\zeta^{-1}\left(\frac{\eta_1 + \eta_2}{2} + \zeta^{-1}\right) \\ &\quad + \mathcal{O}\left(k^2|\zeta|^{-2}(|\eta_1 + \eta_2| + |\zeta|^{-1})^2\right). \quad (2.150)\end{aligned}$$

It follows from (2.149)-(2.150) that as $\zeta, k \to \infty$ and $\eta_1 + \eta_2 \to 0$ such that $k/\zeta \to 0$, $k(\eta_1 + \eta_2) \to 0$,

$$\begin{aligned}\Phi(\zeta;k) &= \left(I + \chi_1/\zeta + \chi_2/\zeta^2 + \mathcal{O}\left(\zeta^{-3}\right)\right)\zeta^{k\sigma_3} \\ &\times \begin{pmatrix} 1 - k\zeta^{-1}\left(\frac{\eta_1+\eta_2}{2} + \zeta^{-1}\right) + \mathcal{O}\left(k^2|\zeta|^{-2}(|\eta_1 + \eta_2| + |\zeta|^{-1})^2\right) & 0 \\ 0 & 1 + k\zeta^{-1}\left(\frac{\eta_1+\eta_2}{2} + \zeta^{-1}\right) + \mathcal{O}\left(k^2\zeta^{-2}(|\eta_1 + \eta_2| + |\zeta|^{-1}))^2\right) \end{pmatrix},\end{aligned}$$

where, in particular, χ_1 and χ_2 are bounded for $k \in \mathbb{N}$ and the $\mathcal{O}(\zeta^{-3})$ term is uniform for $k \in \mathbb{N}$. By comparison with the form of (2.116) it follows that as $\zeta, k \to \infty$ and $\eta_1 + \eta_2 \to 0$ such that $k/\zeta \to 0$, $k(\eta_1 + \eta_2) \to 0$,

$$\begin{aligned}&\Phi(\zeta;k)\zeta^{-k\sigma_3} - I - \Phi_1/\zeta - \Phi_2/\zeta^2 = \\ &\mathcal{O}(\zeta^{-3}) + \begin{pmatrix} \mathcal{O}(k^2(\zeta^{-4} + |\eta_1 + \eta_2|\zeta^{-3})) & 0 \\ 0 & \mathcal{O}(k^2(\zeta^{-4} + |\eta_1 + \eta_2|\zeta^{-3})) \end{pmatrix}.\end{aligned}$$

□

2.3.4 Model RH problem Ψ

The following RH problem has a solution in terms of Bessel functions.

(a) $\Psi : \mathbb{C} \setminus \overline{\Gamma_\Psi} \to \mathbb{C}^{2\times 2}$ is analytic, where $\Gamma_\Psi = \mathbb{R}^- \cup \Gamma_\Psi^\pm$, with $\Gamma_\Psi^\pm = \{xe^{\pm\frac{2\pi}{3}i} : x \in \mathbb{R}^+\}$, and with orientation taken in the direction of increasing real part.

(b) Ψ has continuous boundary values Ψ_+, Ψ_- on Γ_Ψ satisfying the following jump conditions:

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } \zeta \in \mathbb{R}^-, \tag{2.151}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{for } \zeta \in \Gamma_\Psi^\pm. \tag{2.152}$$

(c) As $\zeta \to \infty$, Ψ has the following asymptotics:

$$\Psi(\zeta) = \left(\pi \zeta^{\frac{1}{2}}\right)^{-\frac{\sigma_3}{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \times \left(I + \frac{1}{8\sqrt{\zeta}} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} + \mathcal{O}\left(\zeta^{-1}\right) \right) e^{\zeta^{\frac{1}{2}}\sigma_3}. \tag{2.153}$$

(d) As $\zeta \to 0$, the behaviour of Ψ is

$$\Psi(\zeta) = \mathcal{O}(\log|\zeta|). \tag{2.154}$$

This RH problem has a solution given in [73], in terms of Bessel functions. For definitions and properties of Bessel functions see [81]. For $|\arg \zeta| < 2\pi/3$, we have

$$\Psi(\zeta) = \begin{pmatrix} I_0(\zeta^{1/2}) & \frac{i}{\pi} K_0(\zeta^{1/2}) \\ \pi i \zeta^{1/2} I_0'(\zeta^{1/2}) & -\zeta^{1/2} K_0'(\zeta^{1/2}) \end{pmatrix}.$$

For $2\pi/3 < \arg \zeta < \pi$ the solution is given by

$$\Psi(\zeta) = \frac{1}{2} \begin{pmatrix} H_0^{(1)}(e^{\frac{\pi i}{2}} \zeta^{1/2}) & H_0^{(2)}(e^{\frac{\pi i}{2}} \zeta^{1/2}) \\ \pi \zeta^{1/2} \left(H_0^{(1)}\right)'(e^{\frac{\pi i}{2}} \zeta^{1/2}) & \pi \zeta^{1/2} \left(H_0^{(2)}\right)'(e^{\frac{\pi i}{2}} \zeta^{1/2}) \end{pmatrix}.$$

For $-\pi < \arg \zeta < -2\pi/3$ it is defined as

$$\Psi(\zeta) = \frac{1}{2} \begin{pmatrix} H_0^{(2)}(e^{\frac{\pi i}{2}} \zeta^{1/2}) & -H_0^{(1)}(e^{\frac{\pi i}{2}} \zeta^{1/2}) \\ -\pi \zeta^{1/2} \left(H_0^{(2)}\right)'(e^{\frac{\pi i}{2}} \zeta^{1/2}) & \pi \zeta^{1/2} \left(H_0^{(1)}\right)'(e^{\frac{\pi i}{2}} \zeta^{1/2}) \end{pmatrix}.$$

We have the following useful asymptotics as $\zeta \to 0$ for I_0 and K_0:

$$I_0(\zeta) = 1 + \frac{\zeta^2}{4} + \frac{\zeta^4}{64} + \mathcal{O}(\zeta^6), \tag{2.155}$$

$$K_0(\zeta) = -\log \frac{\zeta}{2} \left(1 + \frac{\zeta^2}{4} + \frac{\zeta^4}{64} + \mathcal{O}(\zeta^6) \right). \tag{2.156}$$

2.3.5 Local parametrix at b_1 and b_2

Let U_1 and U_2 be discs of radius $\frac{\epsilon s}{n}$ for some fixed but sufficiently small $\epsilon > 0$, centered at b_1 and b_2 respectively. Recalling $w_j = \frac{n}{s}(z - b_j)$ for $j = 1, 2$, we have $|w_j| = \epsilon$ on ∂U_j. For $z \in U_1$, define

$$\zeta_1(z) = \frac{n^2}{4}\phi(z)^2, \tag{2.157}$$

where ϕ was defined in (2.94). Recall the notation $B_j = \frac{n}{s}(1 - b_j)$ for $j = 1, 2$. By (2.102) and (2.105) we have the following expansion of $\zeta_1(z)$ for w_1 in a neighbourhood of 0:

$$\begin{aligned} \zeta_1(z) &= s^2\zeta_{1,0}w_1\left(1 + \mathcal{O}(w_1)\right) \\ \zeta_{1,0} &= \frac{B_2 - B_1}{(b_1 + (b_1b_2)^{1/2})^2}\left(1 - \frac{n\Omega}{\pi s}\frac{b_1 + (b_1b_2)^{1/2}}{B_2 - B_1}\left(\frac{B_2}{B_1}\right)^{1/2}\right)^2, \end{aligned} \tag{2.158}$$

and by considering (2.77) in addition, one verifies that ζ_1 is analytic on U_1.

Recall from (2.57), (2.70) that $\phi_\pm(b_2) = \pm\Omega i$ and define

$$\widetilde{\phi}(z) = \begin{cases} \phi(z) - \Omega i & \text{for } z \in U_2 \text{ and } z \in \mathcal{D}, \\ \phi(z) + \Omega i & \text{for } z \in U_2 \text{ and } z \notin \mathcal{D}. \end{cases} \tag{2.159}$$

where $\mathcal{D}$ denotes the unit disc. Then $\widetilde{\phi} : U \setminus J_2 \to \mathbb{C}$ is analytic, with a square root singularity at b_2. We define the local variable

$$\zeta_2(z) = \frac{n^2}{4}\widetilde{\phi}^2(z),$$

which is analytic on U_2. Then, by (2.103) and (2.106), $\zeta_2(z)$ has the following expansion at $w_2 = 0$:

$$\begin{aligned} \zeta_2(z) &= s^2\zeta_{2,0}w_2\left(1 + \mathcal{O}(w_2)\right) \\ \zeta_{2,0} &= \frac{B_1 - B_2}{(b_2 + (b_1b_2)^{1/2})^2}\left(1 - \frac{n\Omega}{\pi s}\frac{b_2 + (b_1b_2)^{1/2}}{B_1 - B_2}\left(\frac{B_1}{B_2}\right)^{1/2}\right)^2, \end{aligned} \tag{2.160}$$

and by considering (2.77) in addition, one verifies that ζ_2 is analytic on U_2.

The local parametrix is given by

$$\begin{aligned} P_j(z) &= E_j(z)\sigma_3^j\Psi(\zeta_j(z))\sigma_3^j e^{-\frac{n}{2}\phi(z)\sigma_3} \\ E_j(z) &= M(z)e^{\pm\frac{n}{2}\phi_+(b_j)\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & (-1)^{j+1}\,i \\ (-1)^{j+1}\,i & 1 \end{pmatrix}\left(\pi\zeta_j(z)^{1/2}\right)^{\frac{1}{2}\sigma_3}, \end{aligned} \tag{2.161}$$

on the $\pm$ side of the contour C, where $\phi_+(b_1) = 0$ and $\phi_+(b_2) = \Omega i$. As a consequence of the expansions of ζ_j above, we have

$$\zeta_{j,-}^{-1/4}\zeta_{j,+}^{1/4} = (-1)^j i, \tag{2.162}$$

and recalling the definition of M in (2.111), one may verify that E_j is analytic on U_j. Recalling the jumps of ϕ in (2.95)–(2.96) and jumps of g in (2.77), one verifies that the jumps of P_j match those of S on U_j.

Since, recalling (2.130), $F = \mathcal{O}(s/n)$ as $s/n \to 0$ while $D(\infty)$ remains bounded, we have that E_j is uniformly bounded on ∂U_j, and it follows that uniformly for $z \in \partial U_j$ we have the following matching condition of M and P_j

$$\left(P_j M^{-1}\right)(z) = I + \Delta_1^{(b_j)}(z) + \mathcal{O}(1/s^2), \quad \Delta_1^{(b_j)}(z) = \mathcal{O}(1/s), \tag{2.163}$$

as $s \to \infty$. A simple calculation yields

$$\begin{aligned}\Delta_1^{(b_1)}(z) = {}& \frac{(B_2 - B_1)^{1/2}}{16s\sqrt{\zeta_{1,0}}w_1}\left(I + \frac{F}{b_1 - 1}\right) \\ & \times D(\infty)^{-1}\begin{pmatrix}1 & i \\ i & -1\end{pmatrix} D(\infty)\left(I - \frac{F}{b_1 - 1}\right) + \mathcal{O}(1)\end{aligned} \tag{2.164}$$

as $z \to b_1$, where the $\mathcal{O}(1)$ part is analytic on U_1. Similarly, as $z \to b_2$ we have:

$$\begin{aligned}\Delta_1^{(b_2)}(z) = {}& \frac{(B_1 - B_2)^{1/2}}{16s\sqrt{\zeta_{2,0}}w_2}\left(I + \frac{F}{b_2 - 1}\right) \\ & \times D(\infty)^{-1}\begin{pmatrix}1 & -i \\ -i & -1\end{pmatrix} D(\infty)\left(I - \frac{F}{b_2 - 1}\right) + \mathcal{O}(1),\end{aligned} \tag{2.165}$$

again with $\mathcal{O}(1)$ analytic.

2.3.6 Small norm RH problem

We define R as follows:

$$R(z) = \begin{cases} SM^{-1} & \text{for } z \in \mathbb{C} \setminus \overline{\left(\cup_{j=0}^2 U_j\right)}, \\ SP^{-1} & \text{for } z \in U_0, \\ SP_j^{-1} & \text{for } z \in U_j \text{ where } j = 1, 2. \end{cases} \tag{2.166}$$

Using standard small norm analysis, it follows from Proposition 2.3.2, (2.164)-(2.165) and the fact that the contour lengths are $\partial U_j =$

$\mathcal{O}(s/n)$ for $j=1,2$ as $s/n \to 0$ and $\partial U_0 = \mathcal{O}(s/n(\log u_0^{-1}))$ as $s/n, u_0 \to 0$ that given $\epsilon > 0$,

$$R(z) = I + \mathcal{O}(1/n), \tag{2.167}$$

uniformly for $|z-1| > \epsilon$.

If $z \in U_0$ then it follows from Proposition 2.3.2, and (2.164)-(2.165) that

$$\begin{aligned} R(z) &= I + R_1(z) + R_2(z) \\ &\quad + \mathcal{O}(||R_1||\,||R_2|| + \widetilde{e} u_0^{-2|x|}(u_0 \log u_0^{-1})^3 + \widetilde{e}(s^2 u_0^4 (\log u_0^{-1})^2)), \end{aligned}$$

where $||R_j||$ is the largest element of R_j in absolute value for $j=1,2$, and where the matrices R_j are given by

$$\begin{aligned} R_1(z) &= \int_{\partial U_0} \frac{\Delta_1^{(1)}(u)}{(u-z)} \frac{du}{2\pi i} + \sum_{j=1,2} \int_{\partial U_j} \frac{\Delta_1^{(b_j)}(u)}{(u-z)} \frac{du}{2\pi i}, \\ R_2(z) &= \int_{\partial U_0} \frac{R_{1-}(u)\Delta_1^{(1)}(u) + \Delta_2^{(1)}(u) + \Xi^{(1)}(u)}{u-z} \frac{du}{2\pi i} \\ &\quad + \sum_{j=1,2} \int_{\partial U_j} \frac{R_{1-}(u)\Delta_1^{(b_j)}(u) + \Delta_2^{(b_j)}(u)}{u-z} \frac{du}{2\pi i}, \end{aligned} \tag{2.168}$$

with clockwise orientation taken in the integrals.

2.4 Asymptotic analysis of the differential identity and correlation functions

2.4.1 Asymptotics of χ_n

From (2.41) we have

$$\chi_{n-1}^2 = -(Y_n)_{21}(0). \tag{2.169}$$

By the transformations $Y = \widetilde{g}^{-n\sigma_3} T e^{ng\sigma_3}$ and $T = S = RM$ at $z=0$, (see (2.92), (2.109), (2.166)) and recalling that χ_n is positive, we find from (2.169) that

$$\left|\chi_{n-1}^2 \widetilde{g}^{-2n}\right| = \left|\widetilde{g}^{-n} e^{ng(0)} (R(0)M(0))_{21}\right|. \tag{2.170}$$

From the definition of g in (2.76) and $\widetilde{g}$ in (2.91) it follows by computing $g_1(0)$, $h(0)$ in (2.54), (2.68) that

$$\left|\widetilde{g}^{-1} e^{g(0)}\right| = \left| \frac{1 - \frac{b_1+b_2}{2} + (b_1 b_2)^{1/4}\sqrt{|(1-b_1)(1-b_2)|}}{1 - \frac{b_1^{-1}+b_2^{-1}}{2} + (b_1 b_2)^{-1/4}\sqrt{|(1-b_1)(1-b_2)|}} \right|^{\Omega/2\pi} = 1,$$

so that

$$\left|\chi_{n-1}^2 \widetilde{g}^{-2n}\right| = \left|(R(0)M(0))_{21}\right|. \tag{2.171}$$

By (2.167)

$$R(0) = I + \mathcal{O}(1/n),$$

as $n \to \infty$. Furthermore, we note that $F = \mathcal{O}(s/n)$ and that $\gamma_2(0) = -1 + \mathcal{O}(s/n)$ as $s/n \to 0$, and substitute this into the definition of M in (2.111) to find

$$\begin{aligned}(R(0)M(0))_{21} &= -(I + \mathcal{O}(s/n))e^{-\left\langle \frac{n\Omega}{2\pi}\right\rangle (h(\infty)+h(0))} \\ &= -(I + \mathcal{O}(s/n))|e^{-\left\langle \frac{n\Omega}{2\pi}\right\rangle h(\infty)}|^2,\end{aligned} \tag{2.172}$$

as $s/n \to 0$. Substituting (2.172) into (2.171) and recalling the notation (2.130) it follows that

$$\left|\widetilde{g}^{-2n}\chi_{n-1}^2\right| = (1 + \mathcal{O}(s/n))\left|\widetilde{h}\right|^2, \tag{2.173}$$

as $s/n \to 0$. We note that $\widetilde{g} = 1 + \mathcal{O}(s/n(\log u_0^{-1}))$ as $s/n(\log u_0^{-1}) \to 0$, and thus we have

$$\left|\widetilde{g}^{-2n}\chi_n^2\right| = (1 + \mathcal{O}(s/n))\left|\widetilde{h}\right|^2, \tag{2.174}$$

as $s/n \to 0$.

2.4.2 Convergence of correlation functions

Let $H_n(x, y)$ be the kernel built out of the orthogonal polynomials on J

$$H_n(x, y) = \frac{s}{\pi n}\sum_{j=0}^{n-1} \phi_j^{(n)}\left(\exp\left(\frac{2sxi}{n}\right)\right)\overline{\phi_j^{(n)}\left(\exp\left(\frac{2syi}{n}\right)\right)}. \tag{2.175}$$

By the Christoffel-Darboux formula, H_n also has the following useful form

$$H_n(y_1, y_2) = \frac{s}{\pi n}\frac{z_1^n z_2^{-n}\phi_n^{(n)}(z_2)\overline{\phi_n^{(n)}(z_1)} - \overline{\phi_n^{(n)}(z_2)}\phi_n^{(n)}(z_1)}{1 - z_2^{-1}z_1}, \tag{2.176}$$

where $z_j = \exp\left(\frac{2sy_j i}{n}\right)$ for $j = 1, 2$. Let $\widehat{K}_n$ be defined similarly, but for the special case where $J = C$, namely:

$$\widehat{K}_n(y_1, y_2) = \frac{s}{\pi n}\frac{z_1^{n/2}z_2^{-n/2} - z_2^{n/2}z_1^{-n/2}}{1 - z_2^{-1}z_1}. \tag{2.177}$$

Let $\rho_m^{(n)}$ be the m'th correlation function of the determinantal point process with correlation kernel $\widehat{K}_n$, and let $\rho_m^{(n,A)}$ be the m'th correlation function of the same process conditioned to have no points in $A = (\alpha, -\nu) \cup (\nu, \beta)$. Then

$$\begin{aligned} \rho_m^{(n)}(x_1, \ldots, x_m) &= \det(\widehat{K}_n(x_i, x_j))_{i,j=1}^m, \\ \rho_m^{(n,A)}(x_1, \ldots, x_m) &= \det(H_n(x_i, x_j))_{i,j=1}^m. \end{aligned} \tag{2.178}$$

The two correlation functions are also related as follows:

$$\rho_m^{(n,A)}(x_1, \ldots, x_m) = \frac{\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{A^j} \rho_{j+m}^{(n)}(x_1, \ldots, x_{j+m}) dx_{m+1} \ldots dx_{j+m}}{D_n(J^{(n)})}. \tag{2.179}$$

Similarly, we can write ρ_m^A in terms of ρ_m (both defined in Remark 2.1.4):

$$\rho_m^A(x_1, \ldots, x_m) = \frac{\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{A^j} \rho_{j+m}(x_1, \ldots, x_{j+m}) dx_{m+1} \ldots dx_{j+m}}{\det(I - K_s)_A}. \tag{2.180}$$

The infinite sums (2.179) and (2.180) can be seen to converge for fixed s by Hadamard's inequality. Since

$$\left| \widehat{K}_n(x, y) - K_s(x, y) \right| = \mathcal{O}(1/n), \tag{2.181}$$

as $n \to \infty$, it follows by formulas (2.180) and (2.179) that

$$|\rho_m^{(n,A)}(x_1, \ldots, x_m) - \rho_m^A(x_1, \ldots, x_m)| \to 0, \tag{2.182}$$

as $n \to \infty$ for fixed s (similarly to convergence of the determinants (2.33)).

By the definition of Y in (2.41) and the formula for H_n in (2.176) we have

$$H_n(y_1, y_2) = \frac{s\chi_n^2}{\pi n(1 - z_1 z_2^{-1})} (z_1^n z_2^{-n} Y_{11}(z_2) \overline{Y_{11}(z_1)} - \overline{Y_{11}(z_2)} Y_{11}(z_1)),$$

where $z_j = \exp\left(\frac{2sy_j i}{n}\right)$ for $j = 1, 2$. For the asymptotics of the correlation kernel we are less ambitious and choose not to proceed with all the detail in last section, and work with $|x|$ bounded away from $1/2$ as $n \to \infty$. Since the intention of F and X was to obtain uniform asymptotics up to the points $|x| = 1/2$, we can let $F, X = 0$ in M and P when we consider $|x|$ bounded away from $1/2$. Then, in place of Proposition 2.3.2, we have $PM^{-1} = I + o(1)$ as $u_0 \to 0$ and $s, n \to \infty$ such that $s/n \to 0$ and $k \in \mathbb{N}, |x| < 1/2$ remain fixed, uniformly on the boundary ∂U_0. Thus $R = I + o(1)$ in the same limit, and tracing back the transformations $Y \to T = S = RP$ we have, by (2.92), (2.109), (2.166), that for $z \in J_1$:

$$\begin{aligned} Y_{11}(z) &= \widetilde{g}^{-n}(RP)_{11}(z)e^{ng(z)} \\ &= \widetilde{g}^{-n} z^{n/2} \widetilde{h}^{-1} \left(\gamma_1 B_{11}(1)\Phi_{11}(\zeta(z)) - \gamma_2 B_{22}(1)\Phi_{21}(\zeta(z))\right)(1 + o(1)), \end{aligned}$$

where $\widetilde{g}$ is given in (2.91), and $\widetilde{h}$ in (2.129). Thus it follows by (2.117), (2.174), the fact that $B(z)$ is real to the main order and that

$$\gamma_1 \overline{\gamma_2} = -1/2 + o(1) \tag{2.183}$$

as $s/n \to 0$, that as $u_0 \to 0$ and $s, n \to \infty$ such that $s/n \to 0$, while $k \in \mathbb{N}$ and $|x| < 1/2$ remain fixed we have

$$\begin{aligned} H_n(y_1, y_2) = \frac{\kappa_{k-1}}{\kappa_k (y_1 - y_2)} \Big[& L_k\left(2\zeta_0 y_1\right) L_{k-1}\left(2\zeta_0 y_2\right) \\ & - L_k\left(2\zeta_0 y_2\right) L_{k-1}\left(2\zeta_0 y_1\right) \Big] (1 + o(1)). \end{aligned} \tag{2.184}$$

Since $\eta_1 = -2 + \mathcal{O}(u_0)$ and $\eta_2 = 2 + \mathcal{O}(u_0)$ as $u_0 \to 0$, it follows by continuity of the polynomials that $L_k^{(\eta_1,\eta_2)}$ can be replaced by $L_k^{(-2,2)}$ in (2.184) without modifying the error term. Similarly, by (2.85), $2|\zeta_0|$ can be replaced by $4u_0^{-1}$ without modifying the error terms. Thus, combining (2.184) and (2.182), we prove formula (1.91) in Remark 2.1.4.

2.4.3 Expansion of differential identity

In this section we start by writing the differential identity in a more convenient form, and find an expansion for it as $s, n \to \infty$ and $u_0 \to 0$ such that $su_0 \log u_0^{-1} \to 0$ and $s/n \to 0$, before proceeding to integrate it in Section 2.4.4. Throughout the rest of the paper, the implicit constants in $\mathcal{O}(\dots)$ are independent of s, u_0, n. For example, if we write $\mathcal{O}(\frac{u_0}{s} + u_0^2)$, then in particular it is uniform in n, and if we write $\mathcal{O}(1)$, it means this expression is bounded in the double scaling limit described above.

Write the parametrix P in (2.126) in U_0 by grouping the factors as follows

$$P(z) = A(z)B(z)C(z)e^{-n\left(g(z) - \frac{1}{2}\log z\right)\sigma_3}, \tag{2.185}$$

where $A(z)$ and $C(z)$ are by

$$
\begin{aligned}
A(z) &= \left(I + \frac{F}{z-1}\right) \widetilde{h}^{-\sigma_3} \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix} \\
C(z) &= \left(I - \frac{X}{\zeta(z)}\right) \Phi(\zeta(z))
\end{aligned}
\tag{2.186}
$$

By the transformations $Y = \widetilde{g}^{-n\sigma_3} T e^{ng\sigma_3}$ and $T = S = RM$ at $z = 1$, (see (2.92), (2.109), (2.166)) we have for $z \in U_0$

$$
Y_{11}(z) = \widetilde{g}^{-n} z^{n/2} \left[RABC\right]_{11}, \tag{2.187}
$$

where

$$
\begin{aligned}
A(z) &= A_1(z) + \frac{A_2(z)}{z-1}, \\
A_1(z) &= \widetilde{h}^{-\sigma_3} \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix}, \\
A_2(z) &= f \widetilde{h}^{-\sigma_3} \begin{pmatrix} \gamma_1(z) + \gamma_2(z)\psi & -\gamma_2(z) + \gamma_1(z)\psi \\ -\gamma_2(z) - \gamma_1(z)/\psi & -\gamma_1(z) + \gamma_2(z)/\psi \end{pmatrix},
\end{aligned}
\tag{2.188}
$$

and

$$
C(z) = \begin{cases} \begin{pmatrix} \Phi_{11}(\zeta) - \frac{\Phi_{1,12}}{\zeta}\Phi_{21}(\zeta) & * \\ \Phi_{21}(\zeta) & * \end{pmatrix} & \text{for } 0 \le x < 1/2, \\ \begin{pmatrix} \Phi_{11}(\zeta) & * \\ \Phi_{21}(\zeta) - \frac{\Phi_{1,21}}{\zeta}\Phi_{11}(\zeta) & * \end{pmatrix} & \text{for } -1/2 \le x < 0. \end{cases}
\tag{2.189}
$$

The expression for C in (2.189) is valid for $k \ge 1$, while for $k = 0$, we have

$$
C(z) = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}. \tag{2.190}
$$

It follows that

$$
Y'_{11}(z) = \widetilde{g}^{-n} z^{n/2} \left[\frac{n}{2z} RABC + R'ABC + RA'BC + RAB'C + RABC'\right]_{11}, \tag{2.191}
$$

where we suppress dependency on the variable z on the right hand side, and where $'$ denotes differentiation with respect to z. Substituting (2.174), (2.187), (2.191) into (2.42) we find that

$$
\begin{aligned}
F(z) = -2|\widetilde{h}|^2 \mathrm{Re} \Big[z\overline{(RABC)_{11}} \Big(& R'ABC + RA'BC \\
& + RAB'C + RABC' \Big)_{11} \Big] (1 + \mathcal{O}(s/n)),
\end{aligned}
\tag{2.192}
$$

for $z \in U_0$. We would now like to evaluate $F(a)$ and $F(\overline{a})$. Since $\zeta\left(e^{i\theta}\right)$ is real for real θ on U_0, it follows that $\frac{d}{d\theta}\zeta\left(e^{i\theta}\right)$ is real. Consider the entries of C and recall that $\Phi_{11}(x)$ is real for $x \in \mathbb{R}$, and that $\Phi_{21}(x)$ and $\Phi_{1,12}$ are purely imaginary. By (2.189), $C_{11}(e^{i\theta})$ is real and so $z\frac{d}{dz}C_{11}(e^{i\theta})$ is purely imaginary, while $C_{21}(e^{i\theta})$ is purely imaginary and $z\frac{d}{dz}C_{21}(e^{i\theta})$ is real. From (2.58), we recall that $B_{jj}\left(e^{i\theta}\right)$ is real for $j = 1, 2$. Thus $\overline{B_{11}}B_{22} = \overline{B_{22}}B_{11} = 1$. From these observations, we find that

$$\begin{aligned}
&\mathrm{Re}\left[z\overline{(RABC)_{11}}\left(RABC'\right)_{11}\right] \\
&\qquad = z(C_{11}C_{21}' - C_{11}'C_{21})\mathrm{Re}\left[\overline{(RA)_{11}}\left(RA\right)_{12}\right], \\
&\mathrm{Re}\left[z\overline{(RABC)_{11}}\left(RAB'C\right)_{11}\right] \\
&\qquad = z(B_{11}B_{22}' - B_{11}'B_{22})C_{11}C_{21}\mathrm{Re}\left[\overline{(RA)_{11}}\left(RA\right)_{12}\right].
\end{aligned} \tag{2.193}$$

When $k = 0$, both expressions in (2.193) are equal to 0.

Evaluation of (2.193)

Using the expansion for ζ from (2.83)–(2.84), and the fact that $\zeta(a) - \zeta(\overline{a}) = 4$ from (2.82), we find that

$$\begin{aligned}
\zeta(e^{\pm i\theta_0}) &= \pm 2\left(1 \pm \zeta_1 iu_0 + \mathcal{O}\left(u_0\frac{s}{n} + u_0^2(\log u_0^{-1})^2\right)\right), \\
\left.\frac{d}{dz}\zeta(z)\right|_{z=e^{\pm i\theta_0}} &= -\frac{2in}{szu_0}\left(1 \pm 2\zeta_1 iu_0 + \mathcal{O}\left(u_0\frac{s}{n} + u_0^2(\log u_0^{-1})^2\right)\right).
\end{aligned} \tag{2.194}$$

We substitute the expansion of ζ from (2.83) into (2.189), and recall (2.120)-(2.124), to find

$$\begin{aligned}
C_{11}(e^{\pm i\theta_0}) &= \begin{cases} \Phi_{11}(\zeta(e^{\pm i\theta_0}))\frac{k+1}{2k+1}(1 \pm iu_0\zeta_1 + \mathcal{O}(r_1)) & \text{for } 0 \le x < 1/2, \\ \Phi_{11}(\zeta(e^{\pm i\theta_0})) & \text{for } -1/2 \le x < 0, \end{cases} \\
C_{21}(e^{\pm i\theta_0}) &= \begin{cases} \Phi_{21}(\zeta(e^{\pm i\theta_0})) & \text{for } 0 \le x < 1/2, \\ \Phi_{21}(\zeta(e^{\pm i\theta_0}))\frac{k-1}{2k-1}(1 \pm iu_0\zeta_1 + \mathcal{O}(r_1)) & \text{for } -1/2 \le x < 0, \end{cases} \\
r_1 &= u_0(\log u_0^{-1})^2/s + (u_0\log u_0^{-1})^2 + su_0/n.
\end{aligned} \tag{2.195}$$

Using the expression

$$\Phi_\zeta'(\eta_j) = \begin{pmatrix} (-1)^j\frac{k(k+1)}{4}\Phi_{11}(\eta_j) & * \\ (-1)^j\frac{k(k-1)}{4}\Phi_{21}(\eta_j) & * \end{pmatrix} \quad j = 1, 2. \tag{2.196}$$

which follows from (2.124), we compute the following:

$$z\left(C_{11}(z)\frac{d}{dz}C_{21}(z)-\frac{d}{dz}C_{11}(z)C_{21}(z)\right)\Bigg|_{z=e^{\pi i\theta_0}}= \\ =\begin{cases}\frac{2\pi k^2 n}{su_0}\left(\frac{k+1}{2k+1}\pm 2i\zeta_1 u_0\right)(1+\mathcal{O}(r_1)) & \text{for } 0\le x<1/2,\\ \frac{2\pi k^2 n}{su_0}\left(\frac{k-1}{2k-1}\pm 2i\zeta_1 u_0\right)(1+\mathcal{O}(r_1)) & \text{for } -1/2\le x<0.\end{cases} \tag{2.197}$$

We also have

$$\big(B_{11}(z)B_{22}'(z)-B_{11}'B_{22}(z)\big)\Big|_{z=e^{\pm i\theta_0}}=\mathcal{O}(n\log u_0^{-1}/s), \tag{2.198}$$

where the derivative is taken with respect to z. We now evaluate RA. Let K denote the constant

$$K=\frac{n}{s}\frac{u_1^{-1}-u_2^{-1}}{4}. \tag{2.199}$$

We have the derivatives of $\gamma_1(e^{i\theta})$ and $\gamma_2(e^{i\theta})$ with respect to θ evaluated at $\theta=0$:

$$\begin{aligned}
\frac{d}{d\theta}\gamma_1(1)&=-iK\gamma_2(1)(1+\mathcal{O}(s/n))\\
\frac{d}{d\theta}\gamma_2(1)&=iK\gamma_1(1)(1+\mathcal{O}(s/n))\\
\frac{d^2}{d\theta^2}\gamma_1(1)&=\frac{n^2}{s^2 2^4}\left(\gamma_1(1)(u_1^{-2}+u_2^{-2}-2u_1^{-1}u_2^{-1})-4i\gamma_2(1)(u_1^{-2}-u_2^{-2})\right)\\
&\qquad\times(1+\mathcal{O}(s/n))\\
\frac{d^2}{d\theta^2}\gamma_2(1)&=\frac{n^2}{s^2 2^4}\left(\gamma_2(1)(u_1^{-2}+u_2^{-2}-2u_1^{-1}u_2^{-1})+4i\gamma_1(1)(u_1^{-2}-u_2^{-2})\right)\\
&\qquad\times(1+\mathcal{O}(s/n)),\\
\frac{d^3}{d\theta^3}\gamma_1(1)&=\frac{n^3}{s^3}\Big(\frac{\gamma_1(1)}{8}(u_1^{-1}-u_2^{-1})(u_1^{-2}-u_2^{-2})\\
&\quad-\frac{3i}{2^6}\gamma_2(1)(11(u_1^{-3}-u_2^{-3})+u_1^{-1}u_2^{-2}-u_2^{-1}u_1^{-2})\Big)(1+\mathcal{O}(s/n)),\\
\frac{d^3}{d\theta^3}\gamma_2(1)&=\frac{n^3}{s^3}\Big(\frac{\gamma_2(1)}{8}(u_1^{-1}-u_2^{-1})(u_1^{-2}-u_2^{-2})\\
&\quad+\frac{3i}{2^6}\gamma_1(1)(11(u_1^{-3}-u_2^{-3})+u_1^{-1}u_2^{-2}-u_2^{-1}u_1^{-2})\Big)(1+\mathcal{O}(s/n)).
\end{aligned} \tag{2.200}$$

Let x_1 and x_2 denote the following functions:

$$\begin{aligned}
x_1(z)&=\widetilde{h}^{-1}R_{11}(z)\gamma_1(1)+\widetilde{h}R_{12}(z)\gamma_2(1),\\
x_2(z)&=-\widetilde{h}^{-1}R_{11}(z)\gamma_2(1)+\widetilde{h}R_{12}(z)\gamma_1(1).
\end{aligned} \tag{2.201}$$

Then, using (2.188), (2.200) and (2.201), expand A. When $0 \leq x < 1/2$ we have,

$$
\begin{aligned}
(RA)_{11}(e^{i\theta}) = &\Bigg[x_1(z)\left(1 - Kf\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right)\left(1 + \frac{n\theta}{2s}(u_1^{-1} + u_2^{-1})\right)\right) \\
&+ iKx_2(z)\theta + \frac{n^2\theta^2}{s^2 2^5}\Bigg(x_1(z)(u_1^{-1} - u_2^{-1})^2 - \frac{nf}{4s}x_1(z)\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right) \\
&\times (11(u_1^{-3} - u_2^{-3}) + u_1^{-1}u_2^{-2} - u_1^{-2}u_2^{-1}) + 4ix_2(z)(u_1^{-2} - u_2^{-2})\Bigg)\Bigg] \\
&\times (1 + \mathcal{O}(s/n)) + \mathcal{O}\left(|\theta|^3 n^3/s^3\right), \\
(RA)_{12}(e^{i\theta}) = &\Bigg[\frac{ifx_1(z)}{\theta}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right) + x_2(z) + ix_1(z)K\theta\Bigg(- 1 \\
&+ K\frac{f}{2}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right)\Bigg) + \frac{n^2\theta^2}{s^2 2^5}\Bigg(x_2(z)(u_1^{-1} - u_2^{-1})^2 - 4ix_1(z)(u_1^{-2} - u_2^{-2}) \\
&- \frac{2fnx_1(z)}{3s}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right)(u_1^{-3} + u_2^{-3} - u_1^{-1}u_2^{-2} - u_1^{-2}u_2^{-1})\Bigg)\Bigg] \\
&\times (1 + \mathcal{O}(s/n)) + \mathcal{O}(|\theta|^3 n^3/s^3),
\end{aligned}
\tag{2.202}
$$

where we denote $\gamma_j = \gamma_j(1)$ for $j = 1, 2$.

When $-1/2 \leq x < 0$,

$$
\begin{aligned}
(RA)_{11}(e^{i\theta}) = &\Bigg[\frac{ifx_2(z)}{\theta}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right) + x_1(z) + ix_2(z)K\theta\Bigg(1 \\
&+ K\frac{f}{2}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right)\Bigg) + \frac{n^2\theta^2}{s^2 2^5}\Bigg(x_1(z)(u_1^{-1} - u_2^{-1})^2 + 4ix_2(z)(u_1^{-2} - u_2^{-2}) \\
&- \frac{2fnx_2(z)}{3s}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right)(u_1^{-3} + u_2^{-3} + u_1^{-1}u_2^{-2} + u_1^{-2}u_2^{-1})\Bigg)\Bigg] \\
&\times (1 + \mathcal{O}(s/n)) + \mathcal{O}(|\theta|^3 n^3/s^3),
\end{aligned}
\tag{2.203}
$$

$$
\begin{aligned}
(RA)_{12}(e^{i\theta}) = &\Bigg[x_2(z)\left(1+Kf\left(\frac{\gamma_1}{\gamma_2}+\frac{\gamma_2}{\gamma_1}\right)\left(1+\frac{n\theta}{2s}(u_1^{-1}+u_2^{-1})\right)\right)\\
&-iKx_1(z)\theta+\frac{n^2\theta^2}{2^5s^2}\Bigg(x_2(z)(u_1^{-1}-u_2^{-1})^2+\frac{nf}{4s}x_2(z)\left(\frac{\gamma_1}{\gamma_2}+\frac{\gamma_2}{\gamma_1}\right)\\
&\times(11(u_1^{-3}-u_2^{-3})+u_1^{-1}u_2^{-2}-u_1^{-2}u_2^{-1})-4ix_1(z)(u_1^{-2}-u_2^{-2})\Bigg)\Bigg]\\
&\times(1+\mathcal{O}(s/n))+\mathcal{O}(|\theta|^3n^3/s^3).
\end{aligned}
$$

We note that

$$
\begin{aligned}
|\gamma_1|^2,\,|\gamma_2|^2 &= \frac{1}{4}\left(\left|\frac{u_1}{u_2}\right|^{1/2}+\left|\frac{u_2}{u_1}\right|^{1/2}\right)(1+\mathcal{O}(s/n))\\
\overline{\gamma_1}/\overline{\gamma_2},\,\gamma_2/\gamma_1 &= \frac{-2\sqrt{|u_1u_2|}-i(u_1+u_2)}{u_1-u_2}(1+\mathcal{O}(s/n)).
\end{aligned}
\tag{2.204}
$$

Recalling (2.130), (2.112), (2.199), it is readily checked that

$$
f\left(\frac{\gamma_1}{\gamma_2}+\frac{\gamma_2}{\gamma_1}\right)\in\mathbb{R},
\tag{2.205}
$$

and that as $n,s\to\infty, s/n\to 0$,

$$
f\left(\frac{\gamma_1}{\gamma_2}+\frac{\gamma_2}{\gamma_1}\right)K=\begin{cases}\frac{\rho}{1+\rho}(1+\mathcal{O}(s/n)) & \text{for } 0\le x<1/2,\\ \frac{\rho}{1-\rho}(1+\mathcal{O}(s/n)) & \text{for } -1/2\le x<0.\end{cases}
\tag{2.206}
$$

From (2.202), (2.203), (2.205)–(2.206) it follows that

$$
\begin{aligned}
&\mathrm{Re}\left[\overline{(RA)_{11}(z)}(RA)_{12}(z)\right]\\
&\qquad=\mathrm{Re}\left[\overline{x_1(z)}x_2(z)\right]\left(1+\mathcal{O}(s/n+u_0^2)\right).
\end{aligned}
\tag{2.207}
$$

Thus, from (2.193) and (2.197),

$$
\begin{aligned}
F_0(e^{\pm i\theta_0}) &= \mathrm{Re}\left[z\overline{(RABC)_{11}}\left(RABC'\right)_{11}\right]\\
&= \mathrm{Re}\left[\overline{x_1(e^{\pm i\theta_0})}x_2(e^{\pm i\theta_0})\right]\\
&\times\begin{cases}\frac{2\pi k^2 n}{su_0}\left(\frac{k+1}{2k+1}\pm i\zeta_1u_0\right)(1+\mathcal{O}(r_1)), & 0\le x<1/2,\\ \frac{2\pi k^2 n}{su_0}\left(\frac{k-1}{2k-1}\pm i\zeta_1u_0\right)(1+\mathcal{O}(r_1)), & -1/2\le x<0,\end{cases}
\end{aligned}
\tag{2.208}
$$

where r_1 was defined in (2.195).

Proposition 2.4.1. *We have*

$$\begin{aligned}\operatorname{Re}\left[\overline{x_1}(e^{\pm i\theta_0})x_2(e^{\pm i\theta_0})\right] &= \frac{|\widetilde{h}|^{-2}}{2}\\ &+ \mathcal{O}\left(s/n + \left(\widetilde{e}u_0^2 s\log u_0^{-1} + \widetilde{e}u_0^{1+2|x|}\log u_0^{-1} + s^{-1}\right)^2\right),\end{aligned} \tag{2.209}$$

and $|\widetilde{h}|^{-1} = \mathcal{O}(1)$*, where* $\widetilde{h}$ *was given in* (2.129) *and* $\widetilde{e}$ *in* (2.133)*.*

The main term of Proposition 2.4.1 is easy to calculate from (2.201) and (2.112), but we defer the rest of the proof to Section 2.4.5.

From (2.124), (2.195) we obtain that

$$C_{11}(e^{\pm i\theta_0}), C_{21}(e^{\pm i\theta_0}) = \mathcal{O}(\sqrt{k}). \tag{2.210}$$

Recall that $k = \mathcal{O}(s/\log u_0^{-1})$. Combining (2.193), (2.198), (2.207), (2.210), and using Proposition 2.4.1 gives us

$$\operatorname{Re}\left[z\overline{(RABC)_{11}}(RAB'C)_{11}\right] = \mathcal{O}(n). \tag{2.211}$$

Evaluation of (2.192)

Suppressing z dependence, we write

$$\begin{aligned}&\operatorname{Re}\left[z\overline{(RABC)_{11}}(RA'BC)_{11}(z)\right] = F_1 + F_2 + F_3 + F_4,\\ &F_1 = B_{11}^2C_{11}^2\operatorname{Re}\left[z\overline{(RA)_{11}}(RA')_{11}\right],\\ &F_2 = \operatorname{Re}\left[zC_{11}\overline{(RA)_{11}}C_{21}(RA')_{12}\right],\\ &F_3 = -\operatorname{Re}\left[zC_{11}(RA')_{11}C_{21}\overline{(RA)_{12}}\right],\\ &F_4 = -B_{22}^2C_{21}^2\operatorname{Re}\left[z\overline{(RA)_{12}}(RA')_{12}\right].\end{aligned} \tag{2.212}$$

Recall that $K = \mathcal{O}(n/s)$, and that $\theta_0 = u_0\frac{s}{n}$. From (2.202) and (2.203) we obtain that for $k \geq 1$

$$\begin{aligned}&\frac{F_1(e^{\pm i\theta_0})}{B_{11}^2C_{11}^2} = \begin{cases}\operatorname{Re}\left[\overline{x_1}x_2\right]\left[\frac{K}{1+\rho}\right] + \mathcal{O}\left(u_0 n/s + 1\right) & \text{for } 0 \leq x < 1/2,\\ -\operatorname{Re}\left[\overline{x_1}x_2\right]\left[\frac{f}{\theta_0^2}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right)\right] + \mathcal{O}(n/s) & \text{for } -1/2 \leq x < 0,\end{cases}\\ &F_2(e^{\pm i\theta_0}) + F_3(e^{\pm i\theta_0}) = \pm iC_{11}C_{21}\operatorname{Re}\left[\overline{x_1}x_2\right]\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right)\left[\frac{2fK}{\theta_0} + \mathcal{O}(n/s)\right],\\ &\frac{F_4(e^{\pm i\theta_0})}{B_{22}^2C_{21}^2} = \begin{cases}\operatorname{Re}\left[\overline{x_1}x_2\right]\left[\frac{f}{\theta_0^2}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right)\right] + \mathcal{O}(n/s) & \text{for } 0 \leq x < 1/2,\\ \operatorname{Re}\left[\overline{x_1}x_2\right]\left[\frac{K}{1-\rho}\right] + \mathcal{O}\left(u_0 n/s + 1\right) & \text{for } -1/2 \leq x < 0.\end{cases}\end{aligned} \tag{2.213}$$

When $k = 0$, we have $F_2, F_3, F_4 = 0$, while F_1 is as in (2.213) for $0 < x < 1/2$.

From (2.87) we have that $\Omega = \mathcal{O}\left(\frac{s}{n \log u_0^{-1}}\right)$, and thus substituting $k + x = \frac{n\Omega}{2\pi}$ into (2.85) we have

$$\frac{8}{u_1^{-1} - u_2^{-1}} e^{-\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \\ = u_0 \left(1 + \mathcal{O}\left(\frac{s \log u_0^{-1}}{n} + u_0^2 (\log u_0^{-1})^2\right)\right). \quad (2.214)$$

Recalling the expansion of g_1 in (2.62)–(2.63) and the definition of B in (2.127), and substituting the values of κ_j and Φ from (2.123)–(2.124) into the expansion of C from (2.195), we obtain that for $k \geq 1$, and $z = e^{\pm i\theta_0}$,

$$\begin{aligned}
(B_{11}^2 C_{11}^2) &= \begin{cases} e^{2x\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \frac{(k+1)^2}{4(2k+1)\kappa_k^2} \left(1 + \mathcal{O}(r_2)\right) & \text{for } 0 \leq x < 1/2, \\ e^{2x\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \frac{(2k+1)}{4\kappa_k^2} \left(1 + \mathcal{O}(r_2)\right) & \text{for } -1/2 \leq x < 0, \end{cases} \\
(B_{22}^2 C_{21}^2) &= \begin{cases} -e^{-2x\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \pi^2 \kappa_{k-1}^2 (2k-1) \left(1 + \mathcal{O}(r_2)\right), & 0 \leq x < \frac{1}{2}, \\ -e^{-2x\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \frac{\pi^2 \kappa_{k-1}^2 (k-1)^2}{(2k-1)} \left(1 + \mathcal{O}(r_2)\right), & -\frac{1}{2} \leq x < 0, \end{cases} \\
(C_{11} C_{21}) &= \begin{cases} \mp \pi i k \frac{k+1}{2k+1} \left(1 + \mathcal{O}(u_0 \log u_0^{-1} + s/n)\right) & \text{for } 0 \leq x < 1/2, \\ \mp \pi i k \frac{k-1}{2k-1} \left(1 + \mathcal{O}(u_0 \log u_0^{-1} + s/n)\right) & \text{for } -1/2 \leq x < 0, \end{cases} \\
r_2 &= u_0 \log u_0^{-1} + \log u_0^{-1} s/n.
\end{aligned} \quad (2.215)$$

When $k = 0$ (and $0 < x < 1/2$) we have

$$(B_{11}^2 C_{11}^2)(e^{\pm i\theta_0}) = e^{s\sqrt{|u_1 u_2|}} (1 + \mathcal{O}(u_0 \log u_0^{-1} + s^2/n)). \quad (2.216)$$

Substituting (2.206), (2.209), (2.215) into (2.213) we find that

$$F_1 = \begin{cases} \frac{|\widetilde{h}|^{-2}}{8}\frac{K}{1+\rho}\frac{(k+1)^2}{(2k+1)\kappa_k^2}e^{2x\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} & 0 \le x < \dfrac{1}{2}, \\ \quad \times(1+\mathcal{O}(r_2+s^{-2})) & \\ -\frac{|\widetilde{h}|^{-2}}{8}\frac{\rho}{\theta_0^2K(1-\rho)}\frac{(2k+1)}{\kappa_k^2}e^{2x\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} & -\dfrac{1}{2} \le x < 0, \\ \quad \times(1+\mathcal{O}(r_2+s^{-2})) & \end{cases}$$

$$F_2+F_3 = \begin{cases} |\widetilde{h}|^{-2}\frac{\rho}{1+\rho}\frac{1}{\theta_0}\frac{\pi k(k+1)}{2k+1} & 0 \le x < \dfrac{1}{2}, \\ \times(1+\mathcal{O}(u_0\log u_0^{-1}+\frac{s}{n}+u_0^{2|x|}+s^{-2})) & \\ |\widetilde{h}|^{-2}\frac{\rho}{1-\rho}\frac{1}{\theta_0}\frac{\pi k(k-1)}{2k-1} & -\dfrac{1}{2} \le x < 0, \\ \times(1+\mathcal{O}(u_0\log u_0^{-1}+\frac{s}{n}+u_0^{2|x|}+s^{-2})) & \end{cases}$$

$$F_4 = \begin{cases} -\frac{|\widetilde{h}|^{-2}\rho\pi^2\kappa_{k-1}^2(2k-1)}{2(1+\rho)K\theta_0^2}e^{-2x\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} & 0 \le x < \dfrac{1}{2}, \\ \quad \times(1+\mathcal{O}(r_2+s^{-2})) & \\ -\frac{|\widetilde{h}|^{-2}K}{2(1-\rho)}\frac{\pi^2\kappa_{k-1}^2(k-1)^2}{(2k-1)}e^{-2x\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} & -\dfrac{1}{2} \le x < 0, \\ \quad \times(1+\mathcal{O}(r_2+s^{-2})) & \end{cases} \tag{2.217}$$

when evaluated at $e^{\pm i\theta_0}$. When $k=0$ (and $0<x<1/2$), we have $F_2, F_3, F_4 = 0$, while F_1 is given by

$$F_1(e^{\pm i\theta_0}) = \frac{|\widetilde{h}|^{-2}K}{2(1+\rho)}e^{s\sqrt{|u_1u_2|}}(1+\mathcal{O}(u_0\log u_0^{-1}+s^2/n+s^{-2})). \tag{2.218}$$

Finally we find the order of the term which includes R' in (2.192). Using the equation for R_1 in (2.168), it is readily seen that

$$\begin{aligned} \left|\frac{d}{dz}R(e^{\pm i\theta_0})\right| &\le \left(\sup_{u\in\partial U_0}|\Delta_1^{(1)}(u)|\right)\int_{\partial U_0}\frac{du}{2\pi|u-e^{\pm i\theta_0}|^2} \\ &+ \sum_{j=1,2}\left(\sup_{u\in\partial U_j}|\Delta_1^{(b_j)}(u)|\right)\int_{\partial U_j}\frac{du}{2\pi|u-e^{\pm i\theta_0}|^2}. \end{aligned} \tag{2.219}$$

We recall that the radius of U_0 is of size $\mathcal{O}(s/n\log u_0^{-1})$, and that the radius of U_j is of sized $\mathcal{O}(s/n)$ for $j=1,2$. Thus

$$\begin{aligned} &\int_{\partial U_0}\frac{du}{2\pi|u-e^{\pm i\theta_0}|^2} = \mathcal{O}\left(\frac{n}{s\log u_0^{-1}}\right), \\ &\int_{\partial U_j}\frac{du}{2\pi|u-e^{\pm i\theta_0}|^2} = \mathcal{O}(n/s) \quad \text{for } j=1,2. \end{aligned}$$

Substituting the asymptotics for $\Delta_1^{(1)}$ from (2.134) and $\Delta_1^{(b_1)}$, $\Delta_1^{(b_2)}$ from (2.163) into (2.219), it follows that

$$\frac{d}{dz}R(e^{\pm i\theta_0}) = \mathcal{O}(\widetilde{e}u_0^{1+2|x|}(\log u_0^{-1})^2 n/s + \widetilde{e}n(u_0\log u_0^{-1})^2 + n/s^2). \quad (2.220)$$

Thus, we have from (2.201), since $\widetilde{h}$, $\widetilde{h}^{-1}$, $\gamma_j(1) = \mathcal{O}(1)$, that also

$$\frac{d}{dz}x_1(e^{\pm i\theta_0}) = \mathcal{O}(\widetilde{e}u_0^{1+2|x|}(\log u_0^{-1})^2 n/s + \widetilde{e}n(u_0\log u_0^{-1})^2 + n/s^2). \quad (2.221)$$

The formula for $R'A$ is given by (2.202), (2.203) but with x_1, x_2 replaced by the derivatives x_1', x_2'. Recall that

$$\begin{gathered} C_{11}, C_{21} = \mathcal{O}\left(\sqrt{k}\right) = \mathcal{O}\left(\sqrt{s/\log u_0^{-1}}\right), \quad K = \mathcal{O}(n/s), \\ B_{11} = \mathcal{O}(u_0^{-x}), \quad B_{22} = \mathcal{O}(u_0^{x}), \quad f = \mathcal{O}\left(\frac{s}{n}u_0^{1-2|x|}\right). \end{gathered} \quad (2.222)$$

From (2.221) we obtain that

$$\begin{aligned} &\operatorname{Re}\left[\left(\overline{(RABC)_{11}}(R'ABC)_{11}\right)(e^{\pm i\theta_0})\right] \\ &\quad = \mathcal{O}(\widetilde{e}nu_0\log u_0^{-1} + \widetilde{e}nsu_0^{2-2|x|}\log u_0^{-1} + nu_0^{-2|x|}/(s\log u_0^{-1})). \end{aligned} \quad (2.223)$$

By substituting (2.208), (2.211), (2.212), (2.223) into the definition of $F(z)$ from (2.192), and substituting the resulting expression into the expression for the differential identity (2.42), we obtain the following proposition.

Proposition 2.4.2. *We have the following asymptotics for* $\log D_n(J)$, *as* $u_0 \to 0$ *and* $s, n \to \infty$ *such that* $su_0 \to 0$ *and* $s/n \to 0$*:*

$$\begin{aligned} &\log D_n(J) = \log D_n(J_2) \\ &+ \frac{|\widetilde{h}|^2}{\pi}\int_0^{\theta_0}\left[\left(\sum_{j=0}^4 F_j(e^{i\theta}) + F_j(e^{-i\theta})\right)(1+\mathcal{O}(s/n))\right. \\ &\left. + \mathcal{O}(n + ns(1+u^{1-2|x|}\log u^{-1})^2 u^{2-2|x|}\log u^{-1} + nu^{-2|x|}/(s\log u^{-1}))\right] d\theta, \end{aligned}$$

where the variable of integration $\theta = \frac{s}{n}u$, *and where the asymptotics of* $F_0(z)$ *are given in* (2.208) *and the asymptotics of* F_1, F_2, F_3, F_4 *are given in* (2.217).

2.4.4 Integration of differential identity

We evaluate the integral in Proposition 2.4.2 asymptotically to prove Theorem 2.1.2. Using (2.214), (2.130) we find that

$$\begin{aligned}
\theta_0 = \theta_0(k;x) &= \frac{su_0}{n} \\
&= \frac{8s}{(u_1^{-1}-u_2^{-1})n} e^{-\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} \left(1+\mathcal{O}\left(\frac{s}{n}\log u_0^{-1} + u_0^2(\log u_0^{-1})^2\right)\right) \\
\frac{d\theta_0}{dx} &= \frac{s\theta_0\sqrt{|u_1u_2|}}{2(k+x)^2}\left(1+\mathcal{O}\left(\frac{s}{n}\log u_0^{-1} + u_0^2(\log u_0^{-1})^2\right)\right), \\
\rho &= \begin{cases} \frac{1}{2\pi\kappa_k^2} e^{\frac{s\sqrt{|u_1u_2|}}{2(k+x)}(-1+2x)}\left(1+\mathcal{O}\left(\frac{s}{n}\log u_0^{-1}\right)\right) & \text{for } 0 \le x < \dfrac{1}{2}, \\ -2\pi\kappa_{k-1}^2 e^{\frac{s\sqrt{|u_1u_2|}}{2(k+x)}(-1-2x)}\left(1+\mathcal{O}\left(\frac{s}{n}\log u_0^{-1}\right)\right) & \text{for } -\dfrac{1}{2} \le x < 0. \end{cases}
\end{aligned} \tag{2.224}$$

Letting k in the expression for θ_0 in (2.224) be fixed, we integrate in θ_0, denoting

$$\int_{\theta_0(k,-1/2)}^{\theta_0(k,1/2)} * \, d\theta = \int_{x=-1/2}^{x=1/2} * \, d\theta_0. \tag{2.225}$$

Note that by (2.123) with $\eta_2 - \eta_1 = 4$,

$$\frac{\kappa_{k-1}^2}{\kappa_k^2} = \frac{4k^2}{(2k+1)(2k-1)}. \tag{2.226}$$

We integrate F_1 from (2.217), changing the variable of integration using (2.224) and recalling K from (2.199) and ρ from (2.224) to find that for $k \ge 1$,

$$
\begin{aligned}
&\frac{|\widetilde{h}|^2}{\pi}\int_{x=-1/2}^{x=1/2} F_1(e^{\pm i\theta_0})d\theta_0 \\
&= \frac{(k+1)^2}{2(2k+1)}\int_0^{1/2}\left(\frac{1}{2\pi\kappa_k^2}e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right)\frac{s\sqrt{|u_1u_2|}}{2(k+x)^2}\frac{(1+\mathcal{O}(r_2+s^{-2}))dx}{1+\frac{1}{2\pi\kappa_k^2}e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}} \\
&+\frac{(2k+1)\kappa_{k-1}^2}{8\kappa_k^2}\int_{-1/2}^{0}\frac{s\sqrt{|u_1u_2|}}{2(k+x)^2}\left(1-\frac{2\pi\kappa_{k-1}^2e^{-(2x+1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}}{1+2\pi\kappa_{k-1}^2e^{-(2x+1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times(1+\mathcal{O}(r_2+s^{-2}))dx \\
&=\left[\frac{(k+1)^2}{2(2k+1)^2}\log\left(1+\frac{1}{2\pi\kappa_k^2}e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right)\right]_{x=0}^{1/2}+\left[\frac{k^2}{2(2k-1)^2}\times\right. \\
&\left.\times\log\left(1+2\pi\kappa_{k-1}^2e^{-(2x+1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right)-\frac{sk^2\sqrt{|u_1u_2|}}{4(2k-1)(k+x)}\right]_{x=-1/2}^{0} \\
&+\mathcal{O}\left(\max_{x\in[-1/2,1/2)}\left[r_2\log u_0^{-1}+s^{-2}\log u_0^{-1}\right]\right) \\
&=\frac{(k+1)^2}{2(2k+1)^2}\log\left(1+\frac{1}{2\pi\kappa_k^2}\right)-\frac{k^2}{2(2k-1)^2}\log\left(1+2\pi\kappa_{k-1}^2\right) \\
&+s\sqrt{|u_1u_2|}\frac{k}{4(2k-1)^2}+\mathcal{O}\left(\max_{x\in[-1/2,1/2)}\left[r_2\log u_0^{-1}+s^{-2}\log u_0^{-1}\right]\right),
\end{aligned}
\tag{2.227}
$$

where r_2 is given in (2.215), $u_0 = u_0(k,x)$.

When $k=0$ we have for $x \in [0,1/2)$

$$
\begin{aligned}
\frac{|\widetilde{h}|^2}{\pi}\int_{x=0}^{x=x} F_1(e^{\pm i\theta_0})d\theta_0 &= \frac{1}{2}\log\left(1+\frac{2}{\pi}e^{-(2x+1)\frac{s\sqrt{|u_1u_2|}}{2x}}\right) \\
&+\mathcal{O}\left(su_0^{1-2x}\left(u_0\log u_0^{-1}+s^2/n+s^{-2}\right)\right),
\end{aligned}
\tag{2.228}
$$

where, in the $\mathcal{O}$ error term, $u_0 = u_0(k=0,x)$. Similarly, we integrate

F_4 for $k \geq 1$:

$$\begin{aligned}
&\frac{|\widetilde{h}|^2}{\pi}\int_{-1/2}^{1/2} F_4(e^{\pm i\theta_0})d\theta_0 \\
&= \left[\frac{(k-1)^2}{2(2k-1)^2}\log\left(1+2\pi\kappa_{k-1}^2 e^{(-2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right)\right]_{x=-1/2}^{0} \\
&+ \left[\frac{k^2}{2(2k+1)^2}\log\left(1+\frac{1}{2\pi\kappa_k^2}e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right)+\frac{sk^2\sqrt{|u_1u_2|}}{4(2k+1)(k+x)}\right]_{x=0}^{1/2} \\
&+\mathcal{O}\left(\max_{x\in[-1/2,1/2)}\left[r_2\log u_0^{-1}+s^{-2}\log u_0^{-1}\right]\right) \\
&= -\frac{(k-1)^2}{2(2k-1)^2}\log\left(1+2\pi\kappa_{k-1}^2\right)+\frac{k^2\log\left(1+\frac{1}{2\pi\kappa_k^2}\right)}{2(2k+1)^2}-\frac{sk\sqrt{|u_1u_2|}}{4(2k+1)^2} \\
&+\mathcal{O}\left(\max_{x\in[-1/2,1/2)}\left[r_2\log u_0^{-1}+s^{-2}\log u_0^{-1}\right]\right).
\end{aligned} \tag{2.229}$$

When $k=0$ and $x\in[0,1/2)$, we have $F_4=0$. Thus, for $k\geq 1$,

$$\begin{aligned}
&\frac{|\widetilde{h}|^2}{\pi}\int_{-1/2}^{1/2}(F_1+F_4)(e^{\pm i\theta_0})d\theta_0 = -\frac{k^2+(k-1)^2}{2(2k-1)^2}\log\left(1+2\pi\kappa_{k-1}^2\right) \\
&+\frac{k^2+(k+1)^2}{2(2k+1)^2}\log\left(1+\frac{1}{2\pi\kappa_k^2}\right)-\frac{sk\sqrt{|u_1u_2|}}{4}\left(\frac{1}{(2k+1)^2}-\frac{1}{(2k-1)^2}\right) \\
&+\mathcal{O}\left(\max_{x\in[-1/2,1/2)}\left[u_0(\log u_0^{-1})^2+\frac{s}{n}(\log u_0^{-1})^2+s^{-2}\log u_0^{-1}\right]\right).
\end{aligned} \tag{2.230}$$

If $-1/2<x<1/2$, then for $k\geq 1$

$$\begin{aligned}
&\frac{|\widetilde{h}|^2}{\pi}\int_{-1/2}^{x}(F_1+F_4)(e^{\pm i\theta_0})d\theta_0 \\
&= -\frac{k^2+(k-1)^2}{2(2k-1)^2}\log(1+2\pi\kappa_{k-1}^2)-\frac{s}{4}\sqrt{|u_1u_2|}w_1(x)+r_3,
\end{aligned} \tag{2.231}$$

$$w_1(x)=\begin{cases}-\frac{k}{(2k-1)^2}+\frac{k}{2k+1}\frac{x}{k+x} & \text{for } 0\leq x<1/2,\\ -\frac{k^2}{(2k-1)^2}\frac{1+2x}{k+x} & \text{for } -1/2\leq x<0,\end{cases}$$

where $r_3=o(1)$. We keep the term r_3 in (2.231) as it is not uniform in x, and is not small as x approaches $\pm 1/2$.

$$
\begin{aligned}
r_3 = & \frac{k^2+(k+1)^2}{2(2k+1)^2} \log\left(1 + \frac{1}{2\pi\kappa_k^2} e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right) + \\
& + \frac{k^2+(k-1)^2}{2(2k-1)^2} \log\left(1 + 2\pi\kappa_{k-1}^2 e^{(-2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right) \\
& + \mathcal{O}\left(\max_{x\in[-1/2,1/2)} \left[u_0(\log u_0^{-1})^2 + \frac{s}{n}(\log u_0^{-1})^2 + s^{-2}\log u_0^{-1}\right]\right).
\end{aligned}
\tag{2.232}
$$

Similarly but simpler, using (2.217) and (2.224), we find that for $k \geq 1$,

$$
\begin{aligned}
& \frac{|\widetilde{h}|^2}{\pi} \int_{-1/2}^{x} (F_2+F_3)(e^{\pm i\theta_0}) d\theta_0 \\
= & \begin{cases} -\frac{k(k-1)}{(2k-1)^2} \log\left(1+2\pi\kappa_{k-1}^2\right) + r_4 & \text{for } |x|< 1/2, \\ -\frac{k(k-1)}{(2k-1)^2} \log\left(1+2\pi\kappa_{k-1}^2\right) + \frac{k(k+1)}{(2k+1)^2} \log\left(1+\frac{1}{2\pi\kappa_k^2}\right) & \text{for } x = 1/2, \\ \qquad +\mathcal{O}\left(\max_{x\in[-1/2,1/2)} \left[u_0 \log u_0^{-1} + \frac{1}{s} + \frac{s}{n}\right]\right) & \end{cases}
\end{aligned}
\tag{2.233}
$$

where $r_4 = o(1)$. When $|x|< 1/2$ we again keep track of the error term

$$
\begin{aligned}
r_4 = & \frac{k(k+1)}{(2k+1)^2} \log\left(1 + \frac{1}{2\pi\kappa_k^2} e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right) \\
& + \frac{k(k-1)}{(2k-1)^2} \log\left(1 + 2\pi\kappa_{k-1}^2 e^{(-2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right) \\
& + \mathcal{O}\left(\max_{x\in[-1/2,1/2)} \left[u_0 \log u_0^{-1} + \frac{1}{s} + \frac{s}{n}\right]\right).
\end{aligned}
\tag{2.234}
$$

When $k = 0$, we have $F_2, F_3, F_4 = 0$, and thus the integral of $F_1 + F_2 + F_3 + F_4$ over $[x = 0, x = x_0 < 1/2]$ for $k = 0$ is given by (2.228). For any $k \geq 1$ and $-1/2 \leq x < 1/2$, combining (2.228), (2.230), (2.231), (2.233)

$$
\begin{aligned}
& \frac{|\widetilde{h}|^2}{\pi} \int_0^{\theta_0} (F_1+F_2+F_3+F_4)(e^{\pm i\theta}) d\theta = -\frac{s}{4}\sqrt{|u_1u_2|} w_1(x) \\
& - \frac{1}{2}\sum_{j=0}^{k-1} \left(\log 2\pi\kappa_j^2 + \frac{s}{2}\sqrt{|u_1u_2|}\left(\frac{j}{(2j+1)^2} - \frac{j}{(2j-1)^2}\right)\right) \\
& + \frac{1}{2}\delta_k(x) + \mathcal{O}\left(su_0 \log u_0^{-1} + s^3/n + 1/\log u_0^{-1} + 1/s\right), \\
& \delta_k(x) = \begin{cases} \log(1+2\pi\kappa_{k-1}^2) & \text{for } x = -1/2 \\ o(1) & \text{for } |x|< 1/2. \end{cases}
\end{aligned}
\tag{2.235}
$$

By (2.232) and (2.234), it follows that for $|x|< 1/2$, $\delta_k(x)$ is given explicitly as

$$\delta_k(x) = \log\left(1 + 2\pi\kappa_{k-1}^2 e^{-(2x+1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right) + \log\left(1 + (2\pi\kappa_k^2)^{-1} e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}\right). \quad (2.236)$$

Furthermore, by (2.208) (which holds for all $k \geq 0$) and Proposition 2.4.1, we find that

$$\begin{aligned}
&\frac{|\widetilde{h}|^2}{\pi}\int_0^{\theta_0}\left(F_0\left(e^{i\theta_0}\right) + F_0\left(e^{-i\theta}\right)\right)d\theta \\
&= s\sqrt{|u_1u_2|}\left(w_2(x) + \sum_{j=1}^{k-1}\frac{j(j+1)}{(2j+1)^2} + \frac{j(j-1)}{(2j-1)^2}\right) \\
&+ \mathcal{O}\left(\frac{s^3}{n} + su_0\log u_0^{-1} + \frac{1}{\log u_0^{-1}}\right) \\
&w_2(x) = \begin{cases} \frac{k(k-1)}{(2k-1)^2} + \frac{k(k+1)}{2k+1}\frac{x}{k+x} & \text{for } 0 \leq x < 1/2 \\ \frac{k^2(k-1)}{(2k-1)^2}\frac{1+2x}{k+x} & \text{for } -1/2 \leq x < 0, \end{cases}
\end{aligned} \quad (2.237)$$

for $k \geq 1$ and $F_0 = 0$ for $k = 0$.

Proof of Theorem 2.1.1

We sum together (2.235), (2.237), and substitute the result into Proposition 2.4.2, to find that

$$\log D_n(J) = \log D_n(J_2) + s\sqrt{|u_1u_2|}\left(w_3(x) + \sum_{j=1}^{k-1}\frac{2j^2}{4j^2-1}\right) - \sum_{j=0}^{k-1}\log 2\pi\kappa_j^2 + \delta_k(x) + \mathcal{O}(1/s + su_0\log u_0^{-1} + s^3/n + 1/\log u_0^{-1}).$$

$$w_3 = -\frac{w_1}{2} + w_2 = \frac{k^2}{2(2k-1)}\frac{1+2x}{k+x}, \quad (2.238)$$

where $\sum_{j=1}^{0} = 0$. The first sum can be evaluated by noting that

$$\sum_{j=1}^{k-1}\frac{2j^2}{4j^2-1} = \frac{k(k-1)}{2k-1}, \quad (2.239)$$

and, it follows that

$$s\sqrt{|u_1u_2|}\left(w_3+\sum_{j=1}^{k-1}\frac{2j^2}{4j^2-1}\right)=\frac{s}{2}\sqrt{|u_1u_2|}\left(\omega-\frac{x^2}{\omega}\right), \qquad (2.240)$$

where $\omega=k+x$. The sum with the leading coefficients κ_j is given by

$$\prod_{j=0}^{k-1}\kappa_j^2=4^{-k^2}\frac{G(2k+1)}{G(k+1)^4}, \qquad (2.241)$$

where G is the Barnes G-function. By substituting (2.240), (2.241) into (2.238) we find that

$$\begin{aligned}&\log D_n\left(J(u_0)\right)=\log D_n\left(J_2\right)+\frac{s}{2}\sqrt{|u_1u_2|}\left(\omega-\frac{x^2}{\omega}\right)\\&+c(k)+\delta_k(x)+\mathcal{O}(s^3/n+su_0\log u_0^{-1}+1/\log u_0^{-1}\}).\end{aligned} \qquad (2.242)$$

Now define $\alpha=u_2/2$, $\beta=u_1/2$, and

$$\nu=\frac{8}{\beta^{-1}-\alpha^{-1}}e^{-\frac{s\sqrt{|\alpha\beta|}}{\omega}}. \qquad (2.243)$$

Then, by (2.224),

$$\frac{u_0}{2}=\nu\left(1+\mathcal{O}\left(\frac{s}{n}\log\nu^{-1}+\nu^2(\log\nu^{-1})^2\right)\right). \qquad (2.244)$$

It is an easy exercise using (2.244), the continuity of w,x as functions of u_0, and (2.242) to show that

$$\log D_n(\nu)=\log D_n(u_0)+\mathcal{O}(u_0+su_0\log u_0^{-1}+s^3/n+1/\log u_0^{-1}). \qquad (2.245)$$

Substituting the asymptotics from (2.242) into (2.245), and using uniformity of the error terms, we prove Theorem 2.1.2.

2.4.5 Proof of Proposition 2.4.1

We consider the small norm matrices and prove Proposition 2.4.1. From (2.201) it follows that

$$\begin{aligned}&\mathrm{Re}\left[(\overline{x_1}x_2)(e^{\pm i\theta_0})\right]=-|\widetilde{h}|^{-2}\mathrm{Re}\left[\overline{\gamma_1}\gamma_2\right]\Big(1+2\mathrm{Re}\,R_{1,11}(e^{\pm i\theta_0})\\&+2\mathrm{Re}\,R_{2,11}(e^{\pm i\theta_0})\Big)+\mathcal{O}\left(\left(|\gamma_1|^2-|\gamma_2|^2\right)+(|R_{1,11}|+|R_{12}|)^2+||R_3||\right),\end{aligned} \qquad (2.246)$$

where $||\,.\,||$ denotes the value of the largest element of the matrix in absolute value. From (2.204) we have

$$\begin{aligned}\operatorname{Re}[\overline{\gamma_1}\gamma_2] &= -1/2+\mathcal{O}(s/n),\\ |\gamma_1|^2-|\gamma_2|^2 &= \mathcal{O}(s/n).\end{aligned} \tag{2.247}$$

It follows from Proposition 2.3.2, (2.164)–(2.165), (2.168) that

$$\begin{aligned}R_1(e^{\pm i\theta_0}) &= \mathcal{O}(s^{-1}+\widetilde{e}u_0^{1+2|x|}\log u_0^{-1}+\widetilde{e}su_0^2\log u_0^{-1}),\\ R_2(e^{\pm i\theta_0}) &= \mathcal{O}((s^{-1}+\widetilde{e}u_0^{1+2|x|}\log u_0^{-1})^2+\widetilde{e}su_0^2\log u_0^{-1}),\\ R_3(e^{\pm i\theta_0}) &= \mathcal{O}(||R_1||\,||R_2||+(u_0\log u_0^{-1})^3).\end{aligned} \tag{2.248}$$

From (2.246)–(2.248), it follows that

$$\begin{aligned}\operatorname{Re}[(\overline{x_1}x_2)(e^{\pm i\theta_0})] &= \frac{1}{2}|\widetilde{h}|^{-2}\Big(1+2\operatorname{Re}R_{1,11}(e^{\pm i\theta_0})+2\operatorname{Re}R_{2,11}(e^{\pm i\theta_0})\Big)\\ &+\mathcal{O}\left(\left(s^{-1}+\widetilde{e}u_0^{1+2|x|}\log u_0^{-1}+\widetilde{e}su_0^2\log u_0^{-1}\right)^2+s/n\right).\end{aligned} \tag{2.249}$$

We will evaluate $\operatorname{Re}R_{1,11}$ and $\operatorname{Re}R_{2,11}$ to prove Proposition 2.4.1.

We recall from (2.168) that R_1 is a sum of 3 terms. The first term is an integral of $\Delta_1^{(1)}$, and the two other terms are integrals of $\Delta_1^{(b_1)}$ and $\Delta_1^{(b_2)}$. We first evaluate the contribution from the terms $\Delta_1^{(b_1)}$ and $\Delta_1^{(b_2)}$.

Contribution to R_1 from $\Delta_1^{(b_1)}$ and $\Delta_1^{(b_2)}$

It follows from (2.164) and (2.165) that

$$\frac{\Delta_{1,11}^{(b_1)}(z)}{b_1-e^{\pm i\theta_0}}=\frac{C_{\Delta,1}}{z-b_1}+\mathcal{O}(1),\qquad \frac{\Delta_{2,11}^{(b_1)}(z)}{b_2-e^{\pm i\theta_0}}=\frac{C_{\Delta,1}}{z-b_2}+\mathcal{O}(1) \tag{2.250}$$

as $z\to b_1$ and $z\to b_2$ respectively, and that the matrices $C_{\Delta,1}$ and C_{Δ_2} are given as follows as $s\to\infty$

$$C_{\Delta,1}=-\frac{\left[i+\frac{fi}{\theta_1}(\psi+\psi^{-1})+\frac{f^2}{\theta_1^2}\left(2i+\left(\frac{\gamma_1}{\gamma_2}-\frac{\gamma_2}{\gamma_1}\right)\right)\right]}{8s(u_1\mp u_0)\left(1-4\frac{k+x}{s(u_1-u_2)}\left|\frac{u_1}{u_2}\right|^{1/2}\right)}(1+\mathcal{O}(s/n),)$$

$$C_{\Delta,2}=\frac{\left[-i+\frac{fi}{\theta_1}(\psi+\psi^{-1})+\frac{f^2}{\theta_1^2}\left(-2i+\left(\frac{\gamma_1}{\gamma_2}-\frac{\gamma_2}{\gamma_1}\right)\right)\right]}{8s(u_2\mp u_0)\left(1-4\frac{k+x}{s(u_1-u_2)}\left|\frac{u_1}{u_2}\right|^{1/2}\right)}(1+\mathcal{O}(s/n)). \tag{2.251}$$

We recall that f is real to the main order and from (2.204) we have

$$\operatorname{Re}\left(\frac{\gamma_1}{\gamma_2}-\frac{\gamma_2}{\gamma_1}\right)=\mathcal{O}(s/n). \tag{2.252}$$

Thus the interior of the bracket [ ] in (2.251) is imaginary to main order, we can calculate the residue in the integral of (2.250) to find:

$$\operatorname{Re}\left[\sum_{j=1,2}\int_{\partial U_j}\frac{\Delta_{1,11}^{(b_j)}(u)}{u-e^{\pm i\theta_0}}\frac{du}{2\pi i}\right]=\mathcal{O}(s/n). \tag{2.253}$$

Contribution to R_1 from $\Delta_1^{(1)}$

Denote

$$\begin{aligned} y_0(z)&=-2i(\gamma_1^2-\gamma_2^2)=-i\left(\left(\frac{z-b_2}{z-b_1}\right)^{1/2}+\left(\frac{z-b_1}{z-b_2}\right)^{1/2}\right),\\ x_0(z)&=-4\gamma_1\gamma_2=-i\left(\left(\frac{z-b_2}{z-b_1}\right)^{1/2}-\left(\frac{z-b_1}{z-b_2}\right)^{1/2}\right),\end{aligned} \tag{2.254}$$

with branch cuts on J_2 such that the square root is positive as $z\to\infty$. Our goal is to evaluate the terms in (2.140), and given A we denote

$$LA=\widetilde{E}(z)B(z)AB^{-1}(z)\widetilde{E}^{-1}(z). \tag{2.255}$$

Define $D(z)$ and $E^{(\pm)}(z)$ by

$$D(z)=L\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad E^{(+)}(z)=L\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad E^{(-)}(z)=L\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Recalling the definition of B in (2.127), the definition of F in (2.128), and the definition of $\widetilde{E}$ in (2.138), we find that

$$\begin{aligned} D(z)&=\frac{i}{2}y_0(z)-\frac{x_0(z)f}{2(z-1)}(\psi+\psi^{-1})\\ &\quad-\frac{f^2}{(z-1)^2}\left(iy_0(z)+\frac{x_0(z)}{2}\left(-\psi+\psi^{-1}\right)\right),\\ E^{(\pm)}(z)&=\frac{1}{4}\Bigg(x_0(z)+\frac{f}{z-1}\left(iy_0(z)(\psi+\psi^{-1})\pm 2(-\psi+\psi^{-1})\right)\\ &\quad-\frac{f^2}{(z-1)^2}\left(2x_0(z)+iy_0(z)\left(\psi-\psi^{-1}\right)\mp 2(\psi+\psi^{-1})\right)\Bigg).\end{aligned} \tag{2.256}$$

Then from (2.140) it follows that

$$\Delta_{1,11}^{(1)}(z) = \begin{cases} \zeta^{-1}(z)\left(\Phi_{1,11}D(z) + B_{22}^2(z)\Phi_{1,21}E^{(-)}(z)\right), & 0 \le x < 1/2, \\ \zeta^{-1}(z)\left(\Phi_{1,11}D(z) + B_{11}^2(z)\Phi_{1,12}E^{(+)}(z)\right), & -1/2 \le x < 0. \end{cases} \tag{2.257}$$

From (2.130) and (2.204) we have

$$\operatorname{Im}\left(\psi + \psi^{-1}\right) = \mathcal{O}(s/n), \quad \operatorname{Re}\left(\psi - \psi^{-1}\right) = \mathcal{O}(s/n). \tag{2.258}$$

From (2.254) we see that

$$\begin{aligned} \operatorname{Im}\left(x_0\left(e^{i\theta}\right)\right) &= \mathcal{O}(s/n), \qquad & \operatorname{Im}\left(\frac{d^j}{d\theta^j}x_0\left(e^{i\theta}\right)\right) &= \mathcal{O}(n^{j-1}/s^{j-1}), \\ \operatorname{Im}\left(y_0\left(e^{i\theta}\right)\right) &= \mathcal{O}(s/n), \qquad & \operatorname{Im}\left(\frac{d^j}{d\theta^j}y_0\left(e^{i\theta}\right)\right) &= \mathcal{O}(n^{j-1}/s^{j-1}), \end{aligned} \tag{2.259}$$

for $e^{i\theta} \in U_0 \cap C$. Write (2.257) in the form

$$\Delta_1^{(1)}(z) = \frac{\Delta_{1,-3}^{(1)}(z)}{(z-1)^3} + \frac{\Delta_{1,-2}^{(1)}(z)}{(z-1)^2} + \frac{\Delta_{1,-1}^{(1)}(z)}{(z-1)}, \tag{2.260}$$

where $\Delta_{1,-j}^{(1)}$ are analytic functions in z in U_0. Then a calculation of residues gives the following expansion as $z \to 1$:

$$\begin{aligned} \int_{\partial U_0} \frac{\Delta_1^{(1)}(u)}{u-z}\frac{du}{2\pi i} &= \frac{1}{6}\frac{d^3}{dz^3}\left(\Delta_{1,-3}^{(1)}\right)(1) + \\ &\frac{1}{2}\frac{d^2}{dz^2}\left(\Delta_{1,-2}^{(1)}\right)(1) + \frac{d}{dz}\left(\Delta_{1,-1}^{(1)}\right)(1) + \mathcal{O}(z-1). \end{aligned} \tag{2.261}$$

We note that ζ is real on J_1, and recall the expansion of ζ in (2.83)–(2.84). We also note that $\operatorname{Im} f = \mathcal{O}(s^2/n^2)$, and that $\Phi_{1,11}$ is real but that $\Phi_{1,12}$ and $\Phi_{1,21}$ are imaginary. Combining with (2.257)–(2.261), we conclude that

$$\operatorname{Re}\left(\int_{\partial U_0} \frac{\Delta_1^{(1)}(u)}{u - e^{\pm i\theta_0}}\frac{du}{2\pi i}\right) = \mathcal{O}(s/n). \tag{2.262}$$

As a consequence of (2.253) and (2.262) we have

$$\operatorname{Re} R_{1,11}(e^{\pm i\theta_0}) = \mathcal{O}(s/n). \tag{2.263}$$

Order of $R_2(e^{\pm i\theta_0})$

From (2.256) and (2.140) we have

$$\Delta^{(1)}_{2,11}(z) = \begin{cases} \zeta^{-2}(z)\Big[\Phi_{2,11}D(z) + B^2_{11}(z)(\Phi_{2,12} \\ \qquad -\Phi_{1,12}\Phi_{1,22})E^{(+)}(z) + \mathcal{O}(1)\Big] & \text{for } 0 \le x < 1/2, \\ \zeta^{-2}(z)\Big[\Phi_{2,11}D(z) + B^2_{22}(z)(\Phi_{2,21} \\ \qquad -\Phi_{1,21}\Phi_{1,11})E^{(-)}(z) + \mathcal{O}(1)\Big] & \text{for } -1/2 \le x < 0, \end{cases}$$

$$\Xi^{(1)}(z) = \begin{cases} -\zeta^{-3}(z)\left[B^2_{11}(z)\Phi_{1,12}\Phi_{2,22}E^{(+)}(z) + \mathcal{O}(1)\right], & 0 \le x < \dfrac{1}{2}, \\ -\zeta^{-3}(z)\left[B^2_{22}(z)\Phi_{1,21}\Phi_{2,11}E^{(-)}(z) + \mathcal{O}(1)\right], & -\dfrac{1}{2} \le x < 0. \end{cases}$$

By inspection of the signs of each element, it follows that

$$\operatorname{Re}\left(\int_{\partial U_0} \frac{\Delta^{(1)}_{2,11}(u) + \Xi^{(1)}(u)}{u - e^{\pm i\theta_0}} \frac{du}{2\pi i}\right) = \mathcal{O}(s/n). \tag{2.264}$$

The remaining contributions to $R_{2,11}$, defined in (2.140), are calculated using rougher estimates from (2.248). Thus it follows that

$$\operatorname{Re} R_{2,11} = \mathcal{O}\left(s/n + (\widetilde{e}u_0^{1+2|x|}\log u_0^{-1} + \widetilde{e}su_0^2\log u_0^{-1} + s^{-1})^2\right). \tag{2.265}$$

Substituting (2.263) and (2.265) into (2.249) yields Proposition 2.4.1.

Chapter 3

Random matrices with merging singularities and the Painlevé V equation

This chapter is based on [18], a joint work with Tom Claeys at Université catholique de Louvain.

3.1 Introduction

We consider a random matrix model on $n \times n$ Hermitian matrices, defined by a probability measure of the form

$$\frac{1}{Z_n} \left|\det\left(M^2 - tI\right)\right|^\alpha e^{-n \operatorname{Tr} V(M)} dM \qquad \text{for } \alpha > -1/2,\ t \in \mathbb{R}. \quad (3.1)$$

Here $dM = \prod_{1 \le i \le n} dM_{ii} \prod_{1 \le i < j \le n} d\mathrm{Re}\, M_{ij} d\mathrm{Im}\, M_{ij}$ is the Lebesgue measure on $n \times n$ Hermitian matrices and $Z_n = Z_n(t, \alpha, V)$ is a normalizing constant. The confining potential $V : \mathbb{R} \to \mathbb{R}$ is a real analytic function which increases sufficiently fast asymptotically, namely

$$\lim_{x \to \pm\infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty. \quad (3.2)$$

The measure (3.1) is invariant under unitary conjugation and induces a probability measure on the eigenvalues x_i, $i = 1, 2, \ldots, n$, given by

$$\frac{1}{\widehat{Z}_n} \prod_{1 \le i < j \le n} (x_j - x_i)^2 \prod_{j=1}^{n} w_n(x_j) dx_j, \qquad w_n(x) = e^{-nV(x)} |x^2 - t|^\alpha, \quad (3.3)$$

where

$$\widehat{Z}_n = \widehat{Z}_n(t,\alpha,V) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\leq i<j\leq n} (x_j - x_i)^2 \prod_{j=1}^{n} w_n(x_j)dx_j \quad (3.4)$$

is the partition function associated to the ensemble. It can alternatively be written as the Hankel determinant

$$\widehat{Z}_n(t,\alpha,V) = n! \det\left(\int_{-\infty}^{+\infty} x^{i+j} w_n(x)dx\right)_{i,j=0,\ldots,n-1}. \quad (3.5)$$

While α is kept fixed and does not depend on the matrix size n, we will be particularly interested in double scaling limits where t is n-dependent and approaches 0 as $n \to \infty$. In such a case, the weight function w_n has the feature that it has two merging singularities approaching 0 as $n \to \infty$. If $t > 0$, the two singularities $\pm\sqrt{t}$ are real; if $t < 0$, the singularities $\pm i\sqrt{-t}$ are purely imaginary. Random matrix ensembles with such merging root-type singularities have applications in fractional Brownian motion with Hurst index $H = 0$ [50], and in quantum chromodynamics [2, 26]. The partition function $\widehat{Z}_n$ appears in the study of impenetrable bosons in a harmonic well as the one body density matrix (up to a constant) [47].

Although the singularities have no effect on the macroscopic limiting density of the eigenvalues as $n \to \infty$, they do have an effect on the partition function $\widehat{Z}_n$ and on the microscopic correlations between eigenvalues near the origin. We will obtain asymptotics for the partition function and for the eigenvalue correlation kernel near the origin, in double scaling limits where $n \to \infty$ and $t \to 0$ simultaneously. The asymptotics for the partition function are described in terms of solutions to the fifth Painlevé equation; the limit of the correlation kernel can also be expressed in terms of functions related to the Painlevé V equation. As $t \to 0$ at a fast rate, the limiting kernel degenerates to a Bessel kernel, and as $t \to 0$ at a slow rate, we recover the sine kernel.

3.1.1 Statement of results

The macroscopic large n behavior of the eigenvalues is described by the limit of the mean eigenvalue distribution as $n \to \infty$, which we denote by μ_V, and which is independent of t and α, for t, α bounded. It is characterized [27, 33] as the unique equilibrium measure μ_V which minimizes

$$\iint \log\frac{1}{|x-y|} d\mu(x)d\mu(y) + \int V(x)d\mu(x), \quad (3.6)$$

over the space of Borel probability measures μ on $\mathbb{R}$. The equilibrium measure is alternatively characterized by the Euler-Lagrange variational conditions [83]

$$2\int \log|x-y|d\mu_V(y) = V(x) - \ell, \qquad \text{for } x \in \operatorname{supp}\mu_V, \tag{3.7}$$

$$2\int \log|x-y|d\mu_V(y) \le V(x) - \ell, \qquad \text{for } x \in \mathbb{R}\setminus \operatorname{supp}\mu_V, \tag{3.8}$$

for some constant ℓ depending on V.

Throughout the paper we will assume that V is such that the equilibrium measure is supported on a single interval $[a,b]$. Then, the equilibrium measure can be written in the form [33]

$$d\mu_V(x) = \psi_V(x)dx = \sqrt{(b-x)(x-a)}\, h(x)dx, \qquad x \in [a,b], \tag{3.9}$$

with h a real analytic function. We require V to be such that the following generic conditions hold: the density ψ_V is strictly positive on (a,b), it vanishes like a square root at the endpoints a,b (in other words, $h(x) > 0$ for $x \in [a,b]$), and the variational inequality (3.8) is strict on $\mathbb{R}\setminus[a,b]$. If all those assumptions hold, we say that V is one-cut regular (see [72] for a general classification of singularities). In addition, we require zero to be contained in the interior of the support of μ_V, i.e. $a < 0 < b$.

If zero were outside of the support of μ_V, no eigenvalues are expected near the origin for large n, and the singularities of the weight will not play a significant role for large n. The case where 0 is an edge point of the support of V is also interesting, but will not be studied here - we refer to [62] for results about the local eigenvalue correlations when a singularity lies near a soft edge, and to [97] when a singularity approaches a hard edge in a modified Jacobi ensemble.

Painlevé V functions

In order to describe the asymptotic behavior in the matrix model (3.1) for n large and t small, we need a special 1-parameter form of the more general 4-parameter Jimbo-Miwa-Okamoto σ-form of the fifth Painlevé equation (see [63, 64], and [48] for the explicit relation to the usual Painlevé V equation), which is given by

$$\begin{aligned}(s\sigma''(s))^2 = (\sigma(s) - s\sigma'(s) + 2(\sigma'(s))^2 \\ + 2\alpha\sigma'(s))^2 - 4(\sigma'(s))^2(\sigma'(s)+\alpha)^2.\end{aligned} \tag{3.10}$$

For $\alpha > -1/2$, we are interested in two particular solutions σ_α^+ and σ_α^- to this equation. The first one, $\sigma_\alpha^-(s)$, is analytic and real for $s \in (0, +\infty)$, and it has the asymptotics

$$\sigma_\alpha^-(s) = \begin{cases} \alpha^2 + o(1) & s \to 0, \\ s^{-1+2\alpha} e^{-s} \frac{1}{\Gamma(\alpha)^2} (1 + \mathcal{O}(s^{-1})) & s \to +\infty, \end{cases} \tag{3.11}$$

where Γ is Euler's Γ-function. The existence of such a solution was proved in [19]. The second solution $\sigma_\alpha^+(s)$ is analytic and such that $\sigma_\alpha^+(s) + \frac{\alpha s}{2}$ is real-valued for $s \in -i\mathbb{R}_{>0}$, satisfying in addition:

$$\sigma_\alpha^+(s) = \begin{cases} \alpha^2 + o(1) & s \to -i0^+, \\ \frac{\alpha^2}{2} - \frac{\alpha s}{2} + \mathcal{O}(|s|^{-1}) & s \to -i\infty. \end{cases}$$

Its existence was proved in [20]. It should be noted that uniqueness of the solutions $\sigma_\alpha^\pm$ satisfying the above asymptotic conditions was not proven in [19, 20]. Those Painlevé solutions σ_α^- and σ_α^+ can be characterized in terms of Riemann-Hilbert (RH) problems; we will do this in Section 3.2 for σ_α^- and in Section 3.3 for σ_α^+. For general values of $\alpha > -1/2$, $\sigma_\alpha^\pm$ are transcendental functions, but there are special values of α where they are algebraic. For σ_α^-, this is the case if α is an integer, while for σ_α^+ only if α is an even integer. For those values of α, $\sigma_\alpha^\pm$ can be constructed via a recursive procedure, involving Schlesinger type transformations [44], which we will explain in detail in Section 3.2.2 and Section 3.3.2.

For $\alpha = 0, 1, 2$, the expressions are simple and we have

$$\sigma_0^\pm(s) = 0, \tag{3.12}$$

$$\sigma_1^-(s) = \frac{s}{e^s - 1}, \tag{3.13}$$

$$\sigma_2^\pm(s) = s \frac{-2 + e^s(2 - 2s + s^2)}{1 - e^s(2 + s^2) + e^{2s}}. \tag{3.14}$$

Asymptotics for the partition function

Theorem 3.1.1. *Let V be real analytic, satisfying* (3.2) *and one-cut regular with* 0 *in the interior of the support* $[a, b]$ *of the equilibrium measure μ_V. Denote $z_0 = \sqrt{t}$ when t is positive and $z_0 = i\sqrt{-t}$ when t is negative. Define*

$$s_{n,t} = -2\pi i n \int_{-z_0}^{z_0} h(s) \left((s - a)(b - s)\right)^{1/2} ds,$$

with h defined in (3.9) *and where the square root is taken to be positive at* 0 *and analytic in a neighbourhood of* 0*, in such a way that $s_{n,t}$ is positive when $t < 0$ and negative imaginary when $t > 0$. We assume that $\alpha > -1/2$ and $t < 0$ or that $\alpha > 0$ and $t > 0$. Then, the partition function $\widehat{Z}_n(t)$ defined in (3.4) satisfies*

$$\log \widehat{Z}_n(t,\alpha,V) = \log \widehat{Z}_n(0,\alpha,V) + \int_0^{s_{n,t}} \frac{\sigma_\alpha^\pm(s) - \alpha^2}{s} ds + \frac{\alpha s_{n,t}}{2} + \frac{n\alpha}{2}(V(z_0) + V(-z_0) - 2V(0)) + \mathcal{O}(|t|^{1/2}) + \mathcal{O}\left(n^{-1}\right), \quad (3.15)$$

with uniform error terms as $n \to \infty$ and $t \to 0$, $\pm t > 0$.

Remark 3.1.1. If we set $\widehat{s}_{n,t} = -4\pi i n z_0 \psi_V(0)$, we have $\widehat{s}_{n,t} = s_{n,t}(1 + \mathcal{O}(|t|))$ as $n \to \infty$, $t \to 0$. If $t > 0$, it follows from equation (3.42) below that one can replace $s_{n,t}$ by $\widehat{s}_{n,t}$ in (3.15) without modifying the error terms. Doing the same when $t < 0$ induces an error term of order $\mathcal{O}(n|t|^{3/2})$.

Remark 3.1.2. We believe that the expansion (3.15), with error term $o(1)$, holds for $t > 0$ and $-1/2 < \alpha < 0$ as well, but because of technical obstacles, we do not prove the result in this case.

Remark 3.1.3. The asymptotic behavior for $\widehat{Z}_n(t,\alpha,V)$ is described in terms of the Painlevé V solutions $\sigma_\alpha^\pm$ and in terms of the partition function $\widehat{Z}_n(0,\alpha,V)$, which corresponds to a weight function which has only one singularity at 0. Asymptotics for $\widehat{Z}_n(0,\alpha,V)$ were obtained in [70] in the case where V is quadratic, an important special case which we will return to later in this section. For general V, we have not been able to find explicit asymptotic expansions in the literature for $\log \widehat{Z}_n(0,\alpha,V)$, up to decaying terms as $n \to \infty$.

Remark 3.1.4. By (3.12)-(3.14), the integral appearing in (3.15) can be computed explicitly for $\alpha = 0, 1, 2$. We have

$$\int_0^{s_{n,t}} \frac{\sigma_0^\pm(s)}{s} ds = 0, \quad (3.16)$$

$$\int_0^{s_{n,t}} \frac{\sigma_1^-(s) - 1}{s} ds = \log\left(2\sinh\frac{s_{n,t}}{2}\right) - \frac{s_{n,t}}{2} - \log s_{n,t}, \quad (3.17)$$

$$\int_0^{s_{n,t}} \frac{\sigma_2^\pm(s) - 4}{s} ds = \log\left(4\sinh^2\frac{s_{n,t}}{2} - s_{n,t}^2\right) - s_{n,t} - 4\log s_{n,t} + \log 12. \quad (3.18)$$

Remark 3.1.5. Our proofs of Theorem 3.1.1 and Theorem 3.1.2 are based on an asymptotic analysis of the orthogonal polynomials on the real line defined by (3.19), using RH methods [27, 34, 35]. The asymptotic analysis for those orthogonal polynomials shows similarities to the one performed in [19, 20] (following the general method from [29]) for orthogonal polynomials on the unit circle with respect to weights with merging or emerging singularities, where they were used to obtain asymptotics for Toeplitz determinants. Nevertheless, there is no known way to directly derive the asymptotics for the Hankel determinant (3.5), with respect to the n-dependent weight w_n supported on the full real line, from the asymptotics for Toeplitz determinants.

Asymptotics for the correlation kernel near 0

Denote by $p_j^{(n)}$ the degree j orthonormal polynomials with respect to the weight $w_n(x)$ (defined in (3.3)), characterized by

$$\int_{\mathbb{R}} p_j^{(n)}(x)p_k^{(n)} w_n(x)dx = \delta_{jk} = \begin{cases} 0 & \text{for } j \neq k, \\ 1 & \text{for } j = k, \end{cases} \tag{3.19}$$

with positive leading coefficient $\kappa_j^{(n)}$. The eigenvalue density (3.3) characterizes a determinantal point process with correlation kernel given by

$$K_n(x,y) = \sqrt{w_n(x)w_n(y)} \sum_{j=0}^{n-1} p_j^{(n)}(x)p_j^{(n)}(y), \tag{3.20}$$

or alternatively

$$K_n(x,y) = \sqrt{w_n(x)w_n(y)} \frac{\kappa_{n-1}^{(n)}}{\kappa_n^{(n)}} \frac{p_n^{(n)}(x)p_{n-1}^{(n)}(y) - p_{n-1}^{(n)}(x)p_n^{(n)}(y)}{x-y}, \tag{3.21}$$

by the Christoffel-Darboux formula. Note that the orthogonal polynomials and the correlation kernel K_n depend on the parameter t in the weight w_n. This chapter is concerned with the microscopic large n behavior of the eigenvalues near 0. We will study the scaled correlation kernel $\frac{1}{cn}K_n\left(\frac{u}{cn}, \frac{v}{cn}\right)$ for a suitable choice of $c > 0$, and obtain asymptotics for it as $n \to \infty$ and $t \to 0$. We will see that a transition in the asymptotics for the kernel takes place if t is of the order n^{-2}, or equivalently if the distance between the two singularities in the weight is of the order n^{-1}, which is of the same order as the typical distance between consecutive eigenvalues.

Local scaling limits near the origin for the correlation kernel K_n are well understood for $t \neq 0$ fixed and for $t = 0$, as $n \to \infty$. If $t \neq 0$ is independent of n, then, as a slight generalization of results obtained in [34, 35], we have for $u, v \in \mathbb{R}$ that

$$\lim_{n\to\infty} \frac{1}{\psi_V(0)n} K_n\left(\frac{u}{\psi_V(0)n}, \frac{v}{\psi_V(0)n}\right) = \mathbb{K}^{\sin}(u,v) = \frac{\sin \pi(u-v)}{\pi(u-v)}. \tag{3.22}$$

The convergence is uniform for u, v in compact subsets of $\mathbb{R}$. If $t = 0$, it was proved in [74] that for u, v in bounded subsets of $(0, \infty)$ that

$$\lim_{n\to\infty} \frac{1}{\psi_V(0)n} K_n\left(\frac{u}{\psi_V(0)n}, \frac{v}{\psi_V(0)n}\right) = \mathbb{K}^{\mathrm{Bessel}}_\alpha(u,v) = \pi\sqrt{uv}\frac{J_{\alpha+\frac{1}{2}}(\pi u)J_{\alpha-\frac{1}{2}}(\pi v) - J_{\alpha-\frac{1}{2}}(\pi u)J_{\alpha+\frac{1}{2}}(\pi v)}{2(u-v)}, \tag{3.23}$$

where J_ν is the Bessel function of order ν (see [81] for a reference on Bessel functions and other special functions which appear throughout the paper). The convergence is uniform for u, v in compact subsets of $(0, \infty)$.

We obtain large n asymptotics for the correlation kernel near the origin as $n \to \infty$ for t small. Of particular interest will be the double scaling limit where $n \to \infty$ and simultaneously $t \to 0$ in such a way that $n^2 t$ tends to a non-zero constant. This will lead us to a new family of limiting kernels which are built out of functions associated to the Painlevé V equation.

Theorem 3.1.2. *Let V be real analytic, satisfying* (3.2) *and one-cut regular with* 0 *in the interior of the support* $[a, b]$ *of the equilibrium measure μ_V. We denote ψ_V for the density of μ_V, given by* (3.9)*, and define*

$$\tau_{n,t} = 16\pi^2 \psi_V(0)^2 n^2 t, \tag{3.24}$$

for t real.

1. *As $n \to \infty$ and simultaneously $t \to 0$ in such a way that $\tau_{n,t} \to \pm\infty$, we have*

$$\lim \frac{1}{\psi_V(0)n} K_n\left(\frac{u}{\psi_V(0)n}, \frac{v}{\psi_V(0)n}\right) = \mathbb{K}^{\sin}(u,v), \tag{3.25}$$

for any $u, v \in \mathbb{R}$, with $\mathbb{K}^{\sin}$ as in (3.22)*.*

2. *As $n \to \infty$ and simultaneously $t \to 0$ in such a way that $\tau_{n,t} \to 0$, we have*

$$\lim \frac{1}{\psi_V(0)n} K_n\left(\frac{u}{\psi_V(0)n}, \frac{v}{\psi_V(0)n}\right) = \mathbb{K}_\alpha^{\mathrm{Bessel}}(u,v), \qquad (3.26)$$

for any $u, v \in \mathbb{R} \setminus \{0\}$, with $\mathbb{K}_\alpha^{\mathrm{Bessel}}$ as in (3.23).

3. *As $n \to \infty$ and simultaneously $t \to 0$ in such a way that $\tau_{n,t} \to \tau \neq 0$, there exists a limiting kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ such that*

$$\lim \frac{1}{\psi_V(0)n} K_n\left(\frac{u}{\psi_V(0)n}, \frac{v}{\psi_V(0)n}\right) = \mathbb{K}_\alpha^{\mathrm{PV}}(u,v;\tau), \qquad (3.27)$$

for any $u, v \in \mathbb{R}$ if $\tau < 0$, and for any $u, v \in \mathbb{R} \setminus \{\pm\sqrt{\tau}/4\}$ if $\tau > 0$.

The limits are uniform for u, v in compact subsets of $\mathbb{R}$.

Remark 3.1.6. The limiting kernel $\mathbb{K}_\alpha^{\mathrm{PV}}(u,v;\tau)$ has the form

$$\mathbb{K}_\alpha^{\mathrm{PV}}(u,v;\tau) = \frac{\Phi_1(\pi v;\tau)\Phi_2(\pi u;\tau) - \Phi_1(\pi u;\tau)\Phi_2(\pi v;\tau)}{2\pi i(u-v)}, \qquad (3.28)$$

and the functions Φ_j, $j = 1, 2$, will be defined in Section 3.2 for $\tau < 0$ and in Section 3.3 for $\tau > 0$. The easiest way to define Φ_1 and Φ_2 is via a RH problem which appeared in [19, 20]. Alternatively, they can be characterized as special solutions to a Lax pair which is related to the fifth Painlevé equation. For $\tau < 0$, $\Phi_1(u;\tau)$ and $\Phi_2(u;\tau)$ are analytic as functions of u in a neighborhood of $\mathbb{R}$; for $\tau > 0$, they are analytic functions of u in a neighborhood of $\mathbb{R} \setminus \{\pm\frac{\sqrt{\tau}}{4}\}$; they are analytic as functions of τ for τ in a neighborhood of $\mathbb{R} \setminus \{0\}$ and continuous in τ at 0. For $\tau = 0$, the limiting kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ is a special case of a large family of limiting kernels which appeared in [6].

Remark 3.1.7. In general, the kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ is a transcendental function of u, v, τ. However, for $\tau < 0$ and $\alpha \in \mathbb{N}$, and for $\tau > 0$ and α even, the kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ is algebraic and can be constructed explicitly by a recursive procedure. We will present this procedure in Section 3.2.2 for $\tau < 0$ and in Section 3.3.2 for $\tau > 0$.

Remark 3.1.8. The limiting kernel $\mathbb{K}_\alpha^{\mathrm{PV}}(u,v;\tau)$ degenerates to the sine and Bessel kernels for large and small values of τ. Indeed, we will show that

$$\lim_{\tau \to \pm\infty} \mathbb{K}_\alpha^{\mathrm{PV}}(u,v;\tau) = \mathbb{K}^{\sin}(u,v), \qquad u, v \in \mathbb{R}, \qquad (3.29)$$

and that

$$\lim_{\tau\to 0}\mathbb{K}_\alpha^{\mathrm{PV}}(u,v;\tau)=\mathbb{K}_\alpha^{\mathrm{Bessel}}(u,v),\qquad u,v\in\mathbb{R}\setminus\{0\}. \tag{3.30}$$

This implies that (3.25), (3.26), and (3.27) are consistent: letting $\tau\to\pm\infty$ in (3.27), we recover (3.25), and letting $\tau\to 0$, we get (3.26).

3.1.2 Consequences and applications

Asymptotics for Toeplitz determinants

In [19], large n asymptotics were obtained for Toeplitz determinants

$$D_n(f)=\det(f_{j-k})_{j,k=0}^{n-1},\quad f_j=\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\theta})e^{-ij\theta}d\theta \tag{3.31}$$

with an emerging Fisher-Hartwig singularity, depending on parameters α,β. In the special case $\beta=0$, the weight has the form

$$f_t(z)=(z-e^t)^\alpha(z-e^{-t})^\alpha z^{-\alpha}e^{-\pi i\alpha}e^{V(z)},\quad z=e^{i\theta},\ \theta\in[0,2\pi)$$

for $\operatorname{Re}\alpha>-\frac{1}{2}$, $t>0$, and where $V(z)=\sum_{k=-\infty}^{+\infty}V_kz^k$ is analytic on an annulus containing the unit circle. The singularities $e^{\pm t}$ approach the unit circle as $t\to 0$, and form a Fisher-Hartwig type singularity if $t=0$. Asymptotics for Toeplitz determinants with weight functions of this form were also obtained in the context of the 2d Ising model [96], see [30] for a review of Toeplitz determinants and the Ising model. Theorem 1.1 in [19] states that

$$\begin{aligned}\log D_n(f_t)&=nV_0+\alpha nt+\sum_{k=1}^{\infty}k\left(V_k-\alpha\frac{e^{-tk}}{k}\right)\left(V_{-k}-\alpha\frac{e^{-tk}}{k}\right)\\&+\log\frac{G(1+\alpha)^2}{G(1+2\alpha)}+\alpha^2\log 2nt+\int_0^{2nt}\frac{\sigma_\alpha^-(x)-\alpha^2}{x}dx+o(1),\end{aligned} \tag{3.32}$$

as $n\to\infty$ and simultaneously $t\to 0$ where G is Barnes's G-function. If $\beta=0$ and $\alpha=1,2$, we can substitute (3.17) and (3.18), here with $s_{n,t}=2nt$, and obtain the more explicit asymptotic expansions

$$\log D_n(f_t)=nV_0+\sum_{k=1}^{\infty}kV_kV_{-k}-\sum_{k=1}^{\infty}(V_k+V_{-k})e^{-tk}+\log\frac{\sinh nt}{\sinh t}+t+o(1),$$

for $\alpha=1$, and

$$\begin{aligned}\log D_n(f_t)=nV_0+\sum_{k=1}^{\infty}kV_kV_{-k}-2\sum_{k=1}^{\infty}(V_k+V_{-k})e^{-tk}+4t\\-2\log(2\sinh^2 t)+\log(\sinh^2 nt-n^2t^2)+o(1),\end{aligned} \tag{3.33}$$

for $\alpha = 2$, as $n \to \infty$ and $t \to 0$. With more effort, we can obtain explicit asymptotic expansions for any integer α.

In [20], asymptotics were obtained for Toeplitz determinants $D_n(f_t)$ with merging Fisher-Hartwig singularities, depending on parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$. Setting $\alpha_1 = \alpha_2 = \alpha/2$ and $\beta_1 = \beta_2 = 0$, the weight has the form

$$f_t(z) = e^{V(z)}|z - e^{it}|^{\alpha}|z - e^{i(2\pi - t)}|^{\alpha},$$

where $\operatorname{Re}\alpha > -\frac{1}{2}$, $t \in (0, \pi)$, and with V again analytic on an annulus containing the unit circle. Then, the weight function has two Fisher-Hartwig type singularities if $t > 0$ which merge to a single one as $t \to 0$. In Theorem 1.5 of [20], the following result is stated:

$$\log D_n(f_t) = \log D_n(f_0) + \int_0^{-2int} \frac{1}{s}\left(\sigma_{CK}(s) - \frac{\alpha^2}{2}\right) ds \\ - \frac{\alpha^2}{2}\log\frac{\sin t}{t} - \frac{\alpha}{2}(V(e^{it}) + V(e^{-it}) - 2V(1)) + o(1) \quad (3.34)$$

as $n \to \infty$ and $t \to 0$. The function σ_{CK} is related to σ_α^+ by the formula

$$\sigma_{CK}(s) = \sigma_\alpha^+(s) - \frac{\alpha^2}{2} + \frac{\alpha s}{2}. \quad (3.35)$$

For $\alpha = 2$, we can substitute (3.18) (with $s_{n,t} = -2int$) and obtain

$$\log D_n(f_t) = nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} - 2\log 2t\sin t + \log\left(n^2t^2 - \sin^2 nt\right) \\ - \left(V\left(e^{it}\right) + V\left(e^{-it}\right) - 2V_0\right) + o(1),$$

as $n \to \infty$ and $t \to 0$. With more effort, we can obtain explicit asymptotic expansions for any even α.

Extreme values of GUE characteristic polynomials

Let H be a random $n \times n$ GUE matrix, normalized such that the joint probability distribution of the eigenvalues is given by

$$\frac{1}{\widehat{Z}_n^{\mathrm{GUE}}} \prod_{1 \le i < j \le n} (x_j - x_i)^2 \prod_{j=1}^{n} e^{-2nx_j^2} dx_j,$$

where $\widehat{Z}_n^{\mathrm{GUE}} = Z_n(\alpha = 0, V(x) = 2x^2)$ is the normalizing constant. The limiting mean eigenvalue density is then supported on $[-1, 1]$ as $n \to \infty$. Define the (random) characteristic polynomial

$$P_n(x) = \det(xI - H).$$

The average of products of powers of characteristic polynomials of the form $\mathbb{E}\left(\prod_{j=1}^{2}|P_n(u_j)|^{\alpha}\right)$, given $u_1, u_2 \in \mathbb{R}$, can be expressed as

$$\mathbb{E}\left(\prod_{j=1}^{2}|P_n(u_j)|^{\alpha}\right) = \frac{1}{\widehat{Z}_n^{\mathrm{GUE}}}\int_{\mathbb{R}^n}\prod_{1\le i<j\le n}(x_j-x_i)^2 \times \prod_{j=1}^{n}|x_j-u_1|^{\alpha}|x_j-u_2|^{\alpha}e^{-2nx_j^2}dx_j. \quad (3.36)$$

The following asymptotic results as $n\to\infty$ were obtained by Krasovsky [70] (see also [14, 46, 54] for α_j's integers):

$$\mathbb{E}\left(\prod_{j=1}^{k}|P_n(u_j)|^{2\alpha_j}\right) = F(n,(u_i,\alpha_i)_{i=1}^{k})\left(1+\mathcal{O}\left(\frac{\log n}{n}\right)\right), \quad (3.37)$$

where $k=1,2$ and

$$\begin{aligned} F(n,(u_i,\alpha_i)_{i=1}^{k}) &= \prod_{j=1}^{k} C(\alpha_j)(1-u_j^2)^{\alpha_j^2/2}(n/2)^{\alpha_j^2}e^{(2u_j^2-1-2\log 2)\alpha_j n} \\ &\quad\times \prod_{1\le i<j\le k}(2|u_i-u_j|)^{-2\alpha_i\alpha_j}, \\ C(\alpha) &= 2^{2\alpha^2}\frac{G(\alpha+1)^2}{G(2\alpha+1)}, \end{aligned}$$

and G is the Barnes G-function. If u_1 and u_2 approach each other, we can use Theorem 3.1.1 to obtain asymptotics for $\mathbb{E}\left(\prod_{j=1}^{2}|P_n(u_j)|^{2\alpha_j}\right)$. Indeed, by (3.36) and (3.4), it follows that

$$\mathbb{E}\left(\prod_{j=1}^{2}|P_n(u_j)|^{\alpha}\right) = \frac{\widehat{Z}_n(t=\frac{(u_1-u_2)^2}{4},\alpha,V(x)=2(x+(u_1+u_2)/2)^2)}{\widehat{Z}_n^{\mathrm{GUE}}}, \quad (3.38)$$

which is seen to hold after the simple change of variables $y_n = x_n - \frac{u_1+u_2}{2}$. Substituting (3.15) in the numerator, we obtain asymptotics for $\mathbb{E}\left(\prod_{j=1}^{2}|P_n(u_j)|^{\alpha}\right)$.

This observation can be used to obtain information about extreme values of $|P_n(x)|$ for large n. This problem was investigated in [52], and in this context the authors needed large n asymptotics for integrals of the form (see in particular Section 2 of [52])

$$I_n(\theta,\rho) = \int_{-\theta}^{\theta}\int_{-\theta}^{\theta} \mathbb{E}\left(\prod_{j=1}^{2}|P_n(u_j)|^{\alpha}\right) \times \prod_{j=1}^{2} e^{-\alpha n \lim_{k\to\infty}\left(\frac{1}{k}\mathbb{E}\log|P_k(u_j)|\right)}\rho(u_j)du_j, \quad (3.39)$$

with $\theta \in [0,1]$, ρ strictly positive and continuous on $(-\theta,\theta)$, and $\alpha > 0$. We note that one can differentiate (3.37) for $k = 1$ with respect to α and evaluate at $\alpha = 0$ to obtain

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}(\log|P_n(u)|) = u^2 - 1/2 - \log 2, \quad (3.40)$$

since $C(0) = 1$, $C'(0) = 0$. In order to obtain asymptotics for I_n as $n \to \infty$, one needs asymptotics for $\mathbb{E}\left(\prod_{j=1}^{2}|P_n(u_j)|^{\alpha}\right)$, both in the region where u_1 and u_2 are bounded away from each other (in this region, we can use Krasovsky's result (3.37)) and in the region where u_1 and u_2 are close to each other (in this region, we need to use our expansion (3.15)).

As a corollary to Theorem 3.1.1 we have

Corollary 3.1.3. *Let $0 < \theta < 1$, $\alpha > 0$, and let ρ be a strictly positive continuous function on $(-\theta,\theta)$. Then,*

$$I_n(\theta,\rho) = \begin{cases} C_1(\alpha)\, n^{\alpha^2/2}\,(1+o(1)) & \text{for } \alpha^2 < 2, \\ C_2\, n\log n\,(1+o(1)) & \text{for } \alpha^2 = 2, \\ C_3(\alpha)\, n^{\alpha^2-1}\,(1+o(1)) & \text{for } \alpha^2 > 2, \end{cases}$$

as $n \to \infty$, where

$$C_1(\alpha) = \frac{G(1+\frac{\alpha}{2})^4}{G(1+\alpha)^2}\int_{-\theta}^{\theta}\int_{-\theta}^{\theta}\frac{((1-u_1^2)(1-u_2^2))^{\alpha^2/8}}{|u_1-u_2|^{\alpha^2/2}}\rho(u_1)\rho(u_2)du_1du_2,$$

$$C_2 = 2\frac{G(1+\frac{1}{\sqrt{2}})^4}{G(1+\sqrt{2})^2}\int_{-\theta}^{\theta}(1-u^2)^{1/2}\rho(u)^2du,$$

$$C_3(\alpha) = 2^{\alpha^2}\frac{G(\alpha+1)^2}{G(2\alpha+1)}\int_{-\theta}^{\theta}(1-u^2)^{\frac{\alpha^2-1}{2}}\rho(u)^2du \times \int_0^{\infty}\exp\left(\int_0^{-2iv}\frac{\sigma_\alpha^+(s)-\alpha^2}{s}ds - i\alpha v\right)dv.$$

Remark 3.1.9. For $\alpha^2 < 2$, the main contribution in the integral $I_n(\theta,\rho)$ comes from the asymptotics (3.37) in the region of integration where u_1 and u_2 are bounded away from each other. For $\alpha^2 > 2$, the main contribution comes from the asymptotics in Theorem 3.1.1 in the region of integration where $|u_1 - u_2| = \mathcal{O}(n^{-1})$.

Proof. First, we note that (3.37) can be extended: as $n \to \infty$ and $|u_1 - u_2| \to 0$ sufficiently slowly such that $n|u_1 - u_2| \to \infty$,

$$\mathbb{E}\left(\prod_{j=1}^{2}|P_n(u_j)|^\alpha\right) = F\left(n, u_1, u_2, \frac{\alpha}{2}, \frac{\alpha}{2}\right)(1+o(1)), \tag{3.41}$$

with the error term uniform for $t_0 > |u_1 - u_2| > 2t_n$ for some $t_0 > 0$ and $t_n \to 0$ such that $nt_n \to \infty$. This follows by recalling the identity

$$\lim_{s\to -i\infty}\left(\int_0^s \frac{\sigma_\alpha^+(\widetilde{s}) - \alpha^2}{\widetilde{s}}d\widetilde{s} + \frac{\alpha s}{2} + \frac{\alpha^2}{2}\log|s|\right) = \log\frac{G\left(1+\frac{\alpha}{2}\right)^4 G(1+2\alpha)}{G(1+\alpha)^4}, \tag{3.42}$$

from [20, formula (1.26)] and applying it to (3.15), which one substitutes into (3.38).

We now split the integral in (3.39) in two parts: the integral over

$$\mathcal{A}_{\theta,t_n} = \{(u_1,u_2) : |u_1|, |u_2| < \theta, |u_1 - u_2| > 2\sqrt{t_n}\},$$

and the integral over $[-\theta,\theta]^2 \setminus \mathcal{A}_{\theta,t_n}$, for some t_n which converges sufficiently slowly to 0 as $n\to\infty$, slower than n^{-2}. To compute the contribution of the integral over $\mathcal{A}_{\theta,t_n}$, we can use (3.40), (3.37), and (3.41). For the contribution of the complement of $\mathcal{A}_{\theta,t_n}$, we substitute (3.15) in (3.38) and then, together with (3.40), in (3.39). Using (3.37) to compute the ratio $\widehat{Z}_n(0,\alpha,V)/\widehat{Z}_n^{\mathrm{GUE}}$ and changing variables $t = (u_1-u_2)^2/4$, $u = (u_1+u_2)/2$, we finally obtain

$$\begin{aligned} I_n(\theta,\rho) = \Bigg(&\left(\frac{n}{2}\right)^{\alpha^2/2} C(\alpha/2)^2 \times \iint_{\mathcal{A}_{\theta,t_n}} \frac{((1-u_1^2)(1-u_2^2))^{\alpha^2/8}}{(2|u_1-u_2|)^{\alpha^2/2}} \\ &\times \rho(u_1)\rho(u_2)du_1du_2 + 2\left(\frac{n}{2}\right)^{\alpha^2} C(\alpha)\int_{-\theta}^{\theta}(1-u^2)^{\alpha^2/2}\rho(u)^2 \\ &\times\left(\int_0^{t_n}\exp\left(\int_0^{\widehat{s}_{n,t}}\frac{\sigma_\alpha^+(s)-\alpha^2}{s}ds + \frac{\alpha\widehat{s}_{n,t}}{2}\right)\frac{dt}{\sqrt{t}}\right)du\Bigg)\times(1+o(1)), \end{aligned} \tag{3.43}$$

as $n \to \infty$, where

$$\widehat{s}_{n,t} = -4\pi i n\sqrt{t}\psi_V(0) = -8in\sqrt{t}\sqrt{1-u^2}$$

in terms of the integration variables t, u. Here we need that the error terms found in Theorem 3.1.1 are uniform in $-\theta < u < \theta$ for the potential $V(x) = 2(x+u)^2$. This is not mentioned elsewhere in this chapter but can readily be seen to hold true by inspection of the proof of Theorem 3.1.1, if $|\theta| < 1$. If $\theta = 1$, contributions from the region where the singularities approach an edge point of the support of the equilibrium measure will have to be taken into account as well, but this is outside the scope of this chapter.

From (3.42) and the small s asymptotics of σ_α^+ it follows that with $v = 4n\sqrt{1-u^2}\sqrt{t}$,

$$\begin{aligned}
&\int_0^{t_n} \exp\left(\int_0^{\widehat{s}_{n,t}} \frac{\sigma_\alpha^+(s)-\alpha^2}{s} ds + \frac{\alpha \widehat{s}_{n,t}}{2}\right) \frac{dt}{\sqrt{t}} \\
&= \frac{1}{2n\sqrt{1-u^2}} \int_0^{4n\sqrt{1-u^2}\sqrt{t_n}} \exp\left(\int_0^{-2iv} \frac{\sigma_\alpha^+(s)-\alpha^2}{s} ds - i\alpha v\right) dv \\
&= \begin{cases} o(n^{-\alpha^2/2}), & \alpha^2 < 2, \\ \frac{1}{4\sqrt{1-u^2}} \frac{G(1+\sqrt{2}/2)^4 G(1+2\sqrt{2})}{G(1+\sqrt{2})^4} n^{-1}\log n + o(n^{-1}\log n), & \alpha^2 = 2, \\ cn^{-1} + o(n^{-1}), & \alpha^2 > 2, \end{cases}
\end{aligned} \tag{3.44}$$

as $n \to \infty$, with

$$c = \frac{1}{2\sqrt{1-u^2}} \int_0^\infty \exp\left(\int_0^{-2iv} \frac{\sigma_\alpha^+(s)-\alpha^2}{s} ds - i\alpha v\right) dv.$$

Formulas (3.43) and (3.44) yield the corollary, where it is readily seen from (3.42) that the integral appearing in the constant C_3 is well defined. $\square$

3.1.3 Outline

In Section 3.2, we recall a model RH problem, introduced in [19], associated to the fifth Painlevé equation. We define the functions Φ_1 and Φ_2 in terms of the solution to the RH problem for $\tau < 0$, and describe their relation with the solution σ_α^- to the fifth Painlevé equation. For α integer, we construct the RH solution recursively. In Section 3.3, we describe a second model RH problem which was introduced in [20], we define Φ_1 and Φ_2 in terms of its solution for $\tau > 0$, and we relate

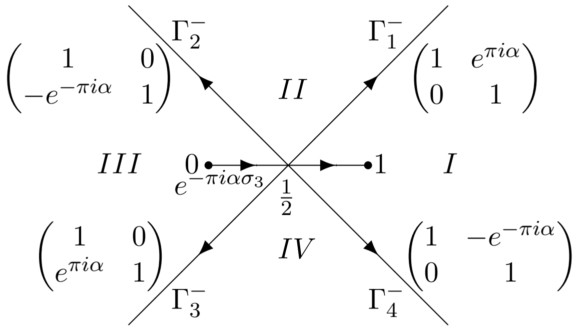

Figure 3.1: The jump contour Γ^- and the jump matrices for Ψ^-.

them to the Painlevé solution σ_α^+. For α even, we construct the RH solution recursively. In Section 3.4, we recall the standard RH problem which characterizes the orthogonal polynomials $p_j^{(n)}$, and we define the g-function needed to analyze this RH problem asymptotically. In Section 3.5, we perform a Deift/Zhou steepest descent analysis to obtain large n asymptotics for the orthogonal polynomials in the case where $t < 0$ (i.e. the case where the singularities are complex and approach 0). The construction of a local parametrix near 0, in terms of the model problem defined in Section 3.2, is crucial here. At the end of Section 3.5, we will prove Theorem 3.1.2 and Theorem 3.1.1 in the case where $t < 0$. In Section 3.6, we perform a similar asymptotic analysis in the case $t > 0$ (i.e. the case where the singularities are real and approach 0). Here the local parametrix is constructed in terms of the model problem from Section 3.3. This will allow us to prove Theorem 3.1.2 and Theorem 3.1.1 for $t > 0$.

3.2 Model RH problem for $\tau < 0$

The functions Φ_1 and Φ_2 that appeared in the limiting kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ can be defined in terms of a RH problem. We need to distinguish between the case where $\tau > 0$ and $\tau < 0$.

The first model RH problem is a special case of the one studied in [19, Section 1.3]. For our purposes, the parameter β from [19] is set to zero. The RH problem depends on parameters $\alpha > -1/2$ and $s \in \mathbb{C}$. The relevant case for us will be $s > 0$.

RH problem for Ψ^-

(a) $\Psi^- : \mathbb{C}\backslash\Gamma^- \to \mathbb{C}^{2\times 2}$ is analytic, with $\Gamma^- = \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3^- \cup \Gamma_4^- \cup [0,1]$, and

$$\Gamma_1^- = \frac{1}{2} + e^{\frac{\pi i}{4}}\mathbb{R}^+, \ \Gamma_2^- = \frac{1}{2} + e^{\frac{3\pi i}{4}}\mathbb{R}^+, \ \Gamma_3^- = \frac{1}{2} + e^{-\frac{3\pi i}{4}}\mathbb{R}^+, \ \Gamma_4^- = \frac{1}{2} + e^{-\frac{\pi i}{4}}\mathbb{R}^+,$$

oriented as in Figure 3.1.

(b) Ψ^- has continuous boundary values on $\Gamma^- \setminus \{0, \frac{1}{2}, 1\}$, which we denote by $\Psi_+^-(z)$ if the limit is taken from the left when oriented along the contour, and $\Psi_-^-(z)$ if the limit is taken from the right. We have the jump relations

$$\begin{aligned}
\Psi_+^-(z) &= \Psi_-^-(z) \begin{pmatrix} 1 & e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix} && \text{for } z \in \Gamma_1^-, \\
\Psi_+^-(z) &= \Psi_-^-(z) \begin{pmatrix} 1 & 0 \\ -e^{-\pi i\alpha} & 1 \end{pmatrix} && \text{for } z \in \Gamma_2^-, \\
\Psi_+^-(z) &= \Psi_-^-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix} && \text{for } z \in \Gamma_3^-, \\
\Psi_+^-(z) &= \Psi_-^-(z) \begin{pmatrix} 1 & -e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix} && \text{for } z \in \Gamma_4^-, \\
\Psi_+^-(z) &= \Psi_-^-(z) e^{-\pi i\alpha\sigma_3} && \text{for } z \in (0,1),
\end{aligned}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix.

(c) There exist q, p, r independent of z (but depending on s, α) such that $\Psi^-(z)$ has the following behavior as $z \to \infty$:

$$\Psi^-(z) = \left(I + \frac{1}{z} \begin{pmatrix} q & r \\ p & -q \end{pmatrix} + \mathcal{O}(z^{-2}) \right) \exp(-\frac{s}{2} z\sigma_3). \quad (3.45)$$

(d) The functions $G(z) := \Psi^-(z) z^{-\frac{\alpha}{2}\sigma_3}$ and $H(z) := \Psi^-(z)(z-1)^{\frac{\alpha}{2}\sigma_3}$ are bounded for z near 0 and 1 respectively, and Ψ^- is bounded near 1/2. The branches in the definitions of G, H are such that $z^{\frac{\alpha}{2}}, (z-1)^{\frac{\alpha}{2}} > 0$ on the + side of $(1, \infty)$.

For $\alpha > -1/2$, it was shown in [19, Theorem 1.8 (i)] that the RH problem for Ψ^- has a unique solution for all $s > 0$.

It will be useful to note that the following symmetry relation holds:

$$e^{\frac{s}{2}\sigma_3} \Psi^- \left(z + \frac{1}{2} \right) = \sigma_1 \Psi^- \left(-z + \frac{1}{2} \right) \sigma_1, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.46)$$

It can indeed be verified that the left and right hand side satisfy the same, uniquely solvable, RH problem.

For $\tau < 0$ and $u \in \mathbb{R}$, we define

$$\begin{pmatrix} \Phi_1(u;\tau) \\ \Phi_2(u;\tau) \end{pmatrix} = \Psi^- \left(z = -\frac{2iu}{s} + \frac{1}{2}; s \right) e^{\mp \pi i \alpha/2\sigma_3} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\text{for } \pm u > 0, \quad s = \sqrt{-\tau}. \quad (3.47)$$

One verifies using the jump conditions for Ψ^- that Φ_1 and Φ_2 are analytic in a neighborhood of the real line. In fact, the singularities $z = 0$ and $z = 1$ of Ψ^- are transformed to complex conjugate singularities $u = \pm is/4 = \pm i\sqrt{-\tau}/4$ for Φ_1 and Φ_2. We have the asymptotic behavior

$$\begin{pmatrix} \Phi_1(u;\tau) \\ \Phi_2(u;\tau) \end{pmatrix} = \left(I + \mathcal{O}(u^{-1})\right) \exp((iu - \sqrt{-\tau}/4)\sigma_3) e^{\mp \pi i \alpha/2\sigma_3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (3.48)$$

as $u \to \pm\infty$. The kernel $\mathbb{K}_\alpha^{\mathrm{PV}}(u, v; \tau)$ is then given by (3.28) for $\tau < 0$.

3.2.1 Lax pair and Painlevé V

There is a connecton between the RH problem for Ψ^- and the Painlevé V equation, which relies on the theory of isomonodromy preserving deformations [44]. We recall from [19] (in particular Theorem 1.8 (ii) and Section 4.3 in that paper) that the RH solution $\Psi = \Psi^-$ solves a system of linear differential equations

$$\Psi_z(z;s) = A(z;s)\Psi(z;s), \quad (3.49)$$
$$\Psi_s(z;s) = B(z;s)\Psi(z;s), \quad (3.50)$$

where the matrices A and B have the form

$$A(z;s) = A_\infty(s) + \frac{A_0(s)}{z} + \frac{A_1(s)}{z-1}, \quad (3.51)$$
$$B(z;s) = B_0(s) + B_1(s)z. \quad (3.52)$$

The 2×2 matrices A_0, A_1, A_∞, B_0, B_1, which are independent of z but can depend on s and α, can be parametrized as follows,

$$A_0 = \begin{pmatrix} -v + \alpha/2 & uy(v - \alpha) \\ -\frac{v}{uy} & v - \alpha/2 \end{pmatrix}, \quad (3.53)$$

$$A_1 = \begin{pmatrix} v - \alpha/2 & -y(v - \alpha) \\ \frac{v}{y} & -v + \alpha/2 \end{pmatrix}, \quad (3.54)$$

$$A_\infty = -\frac{s}{2}\sigma_3, \quad B_1 = -\frac{1}{2}\sigma_3, \quad B_0 = \frac{A_0 + A_1}{s}. \quad (3.55)$$

Here u, v, and y are functions of s and α. We have the following relation between u, v and the functions q, r, p appearing in the asymptotic expansion (3.45) for Ψ^- as $z \to \infty$,

$$v = \frac{\alpha}{2} - q - srp, \tag{3.56}$$

$$u = 1 + \frac{sp}{(1-s)p + sp'}. \tag{3.57}$$

The compatibility condition $\Psi_{sz} = \Psi_{zs}$ implies that $A_s - B_z + AB - BA = 0$, and this implies the following system of equations for u, v, y:

$$su_s = su + (\alpha - 2v)(u-1)^2, \tag{3.58}$$

$$sv_s = v(u - \frac{1}{u})(v - \alpha), \tag{3.59}$$

$$sy_s = y\left(-2v + \alpha + uv - u\alpha + \frac{v}{u} - s\right). \tag{3.60}$$

By eliminating v from the top two equations, we obtain a special case of the fifth Painlevé equation,

$$u_{ss} = \left(\frac{1}{2u} + \frac{1}{u-1}\right)u_s^2 - \frac{1}{s}u_s + \frac{(1-u)^2}{s^2}\left(\frac{\alpha^2}{2}(u - \frac{1}{u})\right) + \frac{u}{s} - \frac{u(u+1)}{2(u-1)}. \tag{3.61}$$

If we define $\sigma(s)$ as

$$\sigma(s) = \int_s^{+\infty} v(\xi) d\xi \tag{3.62}$$

(it was shown in [19] that v decays rapidly as $s \to +\infty$ so that this integral converges), then σ solves (3.10).

3.2.2 Recursive construction of Ψ^- if $\alpha \in \mathbb{N}$

For positive integer values of α, the RH solution Ψ^- can be constructed explicitly. We will denote Ψ_α^- instead of Ψ^- in this section to emphasize the dependence on α. The crucial observation here is that the jump matrices for Ψ_α^- are periodic in α: they remain the same if we replace α by $\alpha + 2$. This fact indicates that there is a Schlesinger type transformation [80] relating Ψ_α to $\Psi_{\alpha+2}$. Even more, if we replace α by $\alpha + 1$, the jump matrices are modified in a simple way: the jumps on the diagonals $\Gamma_1^-, \Gamma_2^-, \Gamma_3^-, \Gamma_4^-$ are the same for the functions Ψ_α^- and $\sigma_3 \Psi_{\alpha+1}^- \sigma_3$, whereas they are opposite on $(0, 1)$. Based on this observation, and on the fact that the solution for $\alpha = 0$ is simple and explicit, we can recursively construct a solution for any integer α.

For positive integer α and $s \in (0, +\infty)$, define X_α as follows,

$$X_\alpha(z; s) = \begin{cases} \Psi_\alpha^-(z; s) & \text{for } z \in II, IV, \\ \Psi_\alpha^-(z; s) \begin{pmatrix} 1 & (-1)^\alpha \\ 0 & 1 \end{pmatrix} & \text{for } z \in I, \\ \Psi_\alpha^-(z; s) \begin{pmatrix} 1 & 0 \\ (-1)^\alpha & 1 \end{pmatrix} & \text{for } z \in III, \end{cases} \tag{3.63}$$

with regions I, II, III, IV as in Figure 3.1. Then, by the RH conditions for Ψ_α^-, X_α satisfies the following RH problem.

RH problem for X_α

(a) $X_\alpha : \mathbb{C} \setminus [0, 1] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) For $z \in (0, 1)$, we have the jump relation $X_{\alpha,+}(z) = (-1)^\alpha X_{\alpha,-}(z)$.

(c) As $z \to \infty$,

$$X_\alpha(z) = (I + \mathcal{O}(z^{-1})) \exp\left(-\frac{sz}{2}\sigma_3\right).$$

(d) The functions G_α and H_α defined by

$$G_\alpha(z) = X_\alpha(z) \begin{pmatrix} 1 & 0 \\ -(-1)^\alpha & 1 \end{pmatrix} z^{-\frac{\alpha}{2}\sigma_3}, \tag{3.64}$$

$$H_\alpha(z) = X_\alpha(z) \begin{pmatrix} 1 & -(-1)^\alpha \\ 0 & 1 \end{pmatrix} (z-1)^{\frac{\alpha}{2}\sigma_3}, \tag{3.65}$$

are analytic at $z = 0$ and at $z = 1$ respectively. The branches in the definitions of G_α, H_α are such that $z^{\frac{\alpha}{2}}, (z-1)^{\frac{\alpha}{2}} > 0$ on the $+$ side of $(1, \infty)$.

For $\alpha = 0$, this RH problem is easy to solve: we have

$$X_0(z) = \exp\left(-\frac{sz}{2}\sigma_3\right).$$

Define

$$W_\alpha(z) = \left(\frac{z}{z-1}\right)^{\frac{1}{2}\sigma_3} X_\alpha(z)\sigma_3 X_{\alpha+1}(z)^{-1}\sigma_3, \tag{3.66}$$

where the square root is analytic on $\mathbb{C} \setminus [0,1]$ and positive for large positive z. We will explicitly construct W_α in terms of G_α and H_α, and thus we will have the recursive relation

$$X_{\alpha+1}(z) = \sigma_3 W_\alpha(z)^{-1} \left(\frac{z}{z-1}\right)^{\frac{1}{2}\sigma_3} X_\alpha(z)\sigma_3. \tag{3.67}$$

From the jump relations for X_α, one verifies that W_α is meromorphic in z, with singularities at 0 and 1, and as $z \to \infty$, we have

$$W_\alpha(z) = I + \mathcal{O}\left(\frac{1}{z}\right). \tag{3.68}$$

By condition (d) of the RH problem for X_α, we obtain:

$$W_\alpha(z) = \left(\frac{z}{z-1}\right)^{\frac{1}{2}\sigma_3} G_\alpha(z) z^{-\frac{1}{2}\sigma_3}\sigma_3 G_{\alpha+1}(z)^{-1}\sigma_3, \tag{3.69}$$

$$W_\alpha(z) = \left(\frac{z}{z-1}\right)^{\frac{1}{2}\sigma_3} H_\alpha(z)(z-1)^{\frac{1}{2}\sigma_3}\sigma_3 H_{\alpha+1}(z)^{-1}\sigma_3, \tag{3.70}$$

for z in neighbourhoods of 0 and 1 respectively. It follows from (3.68)-(3.70) that W_α takes the form

$$W_\alpha(z) = I + \frac{P_\alpha}{z} + \frac{Q_\alpha}{z-1}, \tag{3.71}$$

for matrices P_α, Q_α which are independent of z. Moreover, it is easily verified that X_α exhibits the following symmetry,

$$X_\alpha\left(z + \frac{1}{2}\right) = e^{-\frac{s}{2}\sigma_3}\sigma_1 X_\alpha\left(-z + \frac{1}{2}\right)\sigma_1, \tag{3.72}$$

which yields the relation

$$-e^{-\frac{s}{2}\sigma_3}\sigma_1 P_\alpha \sigma_1 e^{\frac{s}{2}\sigma_3} = Q_\alpha. \tag{3.73}$$

Condition (3.69) gives us that

$$z^{\frac{1}{2}\sigma_3} G_\alpha(z)^{-1} z^{-\frac{1}{2}\sigma_3}\sigma_3 W_\alpha(z) \quad \text{is analytic in a neighborhood of 0.} \tag{3.74}$$

By substituting (3.71) into (3.74) it follows that the first row of P_α is 0, and consequently, by (3.74),

$$z^{\frac{1}{2}\sigma_3} G_\alpha(0)^{-1} z^{-\frac{1}{2}\sigma_3}\sigma_3 W_\alpha(z) \quad \text{is analytic in a neighborhood of 0.} \tag{3.75}$$

This implies

$$G_{\alpha,21}(0)(1+P_{\alpha,22}) + G_{\alpha,11}(0)P_{\alpha,21} = 0, \tag{3.76}$$

$$e^{-s}G_{\alpha,21}(0)P_{\alpha,21} + G_{\alpha,11}(0)P_{\alpha,22} = 0. \tag{3.77}$$

By the unique solvability of the RH problem for X_α, it follows that this linear system has a unique solution for $P_{\alpha,22}$ and $P_{\alpha,21}$ for any positive s.

In conclusion, since we know $X_0(z)$, we can compute $G_0(0)$ by (3.64), and from that we can compute P_0 by (3.76)-(3.77). Substituting those in (3.73), (3.71), and (3.67), we obtain $X_1(z)$. Similarly, once we know $X_1(z)$, we can compute $G_1(0)$, P_1, and X_2, and so on.

For $\alpha = 1$ and $\alpha = 2$, in this way, we obtain the expressions

$$X_1(z) = \frac{1}{(e^s-1)\sqrt{(z-1)z}} \begin{pmatrix} 1+(e^s-1)z & -1 \\ e^s & -e^s+z(e^s-1) \end{pmatrix} e^{-\frac{sz}{2}\sigma_3},$$

$$\begin{aligned} X_2(z) &= \frac{1}{(1-e^s(2+s^2)+e^{2s})(z-1)z} \\ &\times \begin{pmatrix} 1+e^s(s-1)-2z+e^sz(2-2s+s^2)+z^2-e^sz^2(2+s^2-e^s) & 1-e^s+e^ss-z(2+s+e^s(s-2)) \\ e^s(-1-s+e^s+z(2+s+e^s(-2+s))) & e^s(-1-s+e^s)+ze^s(2+2s+s^2-2e^s)+z^2+z^2e^s(-2-s^2+e^s) \end{pmatrix} e^{-\frac{sz}{2}\sigma_3}. \end{aligned} \tag{3.78}$$

Expanding X_α as $z \to \infty$, we can obtain expressions for the Painlevé V functions $q(s), r(s), p(s)$ by (3.45). In particular, for $\alpha = 0, 1, 2$, we obtain

$$q_0(s) = 0, \quad q_1(s) = \frac{1}{e^s-1} + \frac{1}{2}, \quad q_2(s) = \frac{-1+e^s(-2s+e^s)}{1-e^s(2+s^2)+e^{2s}}. \tag{3.79}$$

Using the relation (see [19, (4.109)])

$$\sigma_\alpha^-(s) = sq(s) - \frac{\alpha s}{2}, \tag{3.80}$$

we can compute σ_0^-, σ_1^-, and σ_2^-: this gives (3.12)-(3.14).

3.2.3 The kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ as $\tau \to -\infty$

We follow the steps of [19, Section 4.1] in the analysis as $\tau \to -\infty$. Define

$$\widetilde{\Psi}(z) = \Psi^-(z) \left(\frac{z-1}{z}\right)^{\frac{\alpha}{2}\sigma_3}, \tag{3.81}$$

where the power $\alpha/2$ is taken analytic except on $[0,1]$ and tending to 1 as $z \to +\infty$. The fraction cancels out the jump of Ψ^- on $(0,1)$ and the singularities at 0 and 1, and $\widetilde{\Psi}$ solves the following RH problem.

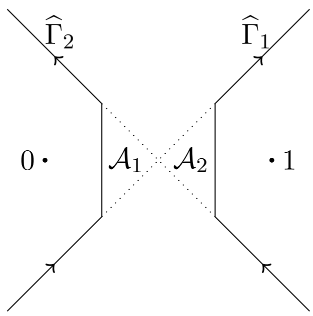

Figure 3.2: Contour $\widehat{\Gamma}$

RH problem for $\widetilde{\Psi}$

(a) $\widetilde{\Psi} : \mathbb{C} \setminus (\Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3^- \cup \Gamma_4^-)$.

(b) $\widetilde{\Psi}$ has the jump relations

$$\widetilde{\Psi}_+(z) = \widetilde{\Psi}_-(z) \begin{pmatrix} 1 & e^{\pi i\alpha} \left(\frac{z-1}{z}\right)^{-\alpha} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_1^-,$$

$$\widetilde{\Psi}_+(z) = \widetilde{\Psi}_-(z) \begin{pmatrix} 1 & 0 \\ -e^{-\pi i\alpha} \left(\frac{z-1}{z}\right)^{\alpha} & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_2^-,$$

$$\widetilde{\Psi}_+(z) = \widetilde{\Psi}_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} \left(\frac{z-1}{z}\right)^{\alpha} & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_3^-,$$

$$\widetilde{\Psi}_+(z) = \widetilde{\Psi}_-(z) \begin{pmatrix} 1 & -e^{-\pi i\alpha} \left(\frac{z-1}{z}\right)^{-\alpha} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_4^-.$$

(c) As $z \to \infty$,

$$\widetilde{\Psi}(z) = \left(I + \mathcal{O}(z^{-1})\right) \exp(-\frac{s}{2} z\sigma_3). \tag{3.82}$$

Now we will deform the jump contour in such a way that it stays away from 0, 1, and 1/2, to a contour as shown in Figure 3.2. Define

$$\widehat{\Psi}(z) = \begin{cases} e^{\frac{s}{4}\sigma_3} \widetilde{\Psi}(z) \exp\left(\frac{s}{2}\left(z - \frac{1}{2}\right)\sigma_3\right) & \text{for } z \notin \mathcal{A}_1 \cup \mathcal{A}_2, \\ e^{\frac{s}{4}\sigma_3} \widetilde{\Psi}(z) \begin{pmatrix} 1 & 0 \\ \left(\frac{1-z}{z}\right)^{\alpha} & 1 \end{pmatrix} \exp\left(\frac{s}{2}\left(z - \frac{1}{2}\right)\sigma_3\right) & \text{for } z \in \mathcal{A}_1, \\ e^{\frac{s}{4}\sigma_3} \widetilde{\Psi}(z) \begin{pmatrix} 1 & \left(\frac{1-z}{z}\right)^{-\alpha} \\ 0 & 1 \end{pmatrix} \exp\left(\frac{s}{2}\left(z - \frac{1}{2}\right)\sigma_3\right) & \text{for } z \in \mathcal{A}_2, \end{cases} \tag{3.83}$$

with $\mathcal{A}_1$ and $\mathcal{A}_2$ given in Figure 3.2. Here, $\left(\frac{1-z}{z}\right)^{\pm\alpha}$ is analytic except on $(-\infty, 0] \cup [1, +\infty)$. Then $\widehat{\Psi}$ satisfies a small norm RH problem as $s \to +\infty$.

RH problem for $\widehat{\Psi}$

(a) $\widehat{\Psi} : \mathbb{C} \setminus \widehat{\Gamma} \to \mathbb{C}^{2\times 2}$ is analytic, with $\widehat{\Gamma} = \widehat{\Gamma}_1 \cup \widehat{\Gamma}_2$ given in Figure 3.2.

(b) $\widehat{\Psi}$ has the following jumps on $\widehat{\Gamma}$:

$$\widehat{\Psi}_+(z) = \widehat{\Psi}_-(z) \begin{pmatrix} 1 & \left(\frac{1-z}{z}\right)^{-\alpha} e^{-s(z-\frac{1}{2})} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \widehat{\Gamma}_1,$$

$$\widehat{\Psi}_+(z) = \widehat{\Psi}_-(z) \begin{pmatrix} 1 & 0 \\ -\left(\frac{1-z}{z}\right)^{\alpha} e^{s(z-\frac{1}{2})} & 1 \end{pmatrix} \qquad \text{for } z \in \widehat{\Gamma}_2.$$

As $s \to +\infty$, this means that

$$\widehat{\Psi}_+(z) = \widehat{\Psi}_-(z)(I + \mathcal{O}\left(\exp\left(-cs\right)\right)$$

on $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$, for some constant $c > 0$.

(c) $\widehat{\Psi}(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

By standard small norm analysis, we have

$$\widehat{\Psi}(z) = I + \mathcal{O}\left(\frac{1}{|z|+1} e^{-cs}\right), \qquad s \to +\infty, \tag{3.84}$$

uniformly for $z \in \mathbb{C} \setminus \widehat{\Gamma}$. Inverting the transformations $\Psi^- \mapsto \widetilde{\Psi} \mapsto \widehat{\Psi}$ and using (3.47), we obtain

$$\begin{pmatrix} \Phi_1(x; -s^2) \\ \Phi_2(x; -s^2) \end{pmatrix} = \widehat{\Psi}\left(\frac{-2ix}{s} + \frac{1}{2}; s\right) \left(\frac{-4ix - s}{-4ix + s}\right)^{-\frac{\alpha}{2}\sigma_3} e^{\mp\frac{\pi i \alpha}{2}\sigma_3} e^{ix\sigma_3} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

for $\pm x > 0$. By (3.84), as $s \to +\infty$, $x \in \mathbb{R}$,

$$\begin{pmatrix} \Phi_1(\pi u; -s^2) \\ \Phi_2(\pi u; -s^2) \end{pmatrix} = \begin{pmatrix} e^{\pi i u} \\ -e^{-\pi i u} \end{pmatrix} + \mathcal{O}(s^{-1}),$$

which, by (3.28), yields the sine kernel limit as $s \to +\infty$,

$$\mathbb{K}_\alpha^{\mathrm{PV}}(u, v; -s^2) = \frac{\sin \pi(u-v)}{\pi(u-v)} + \mathcal{O}(s^{-1}),$$

and (3.29) is proved. In fact, it is clear from this derivation that (3.29) holds also as u, v tend to infinity together with s, as long as $u - v$ converges.

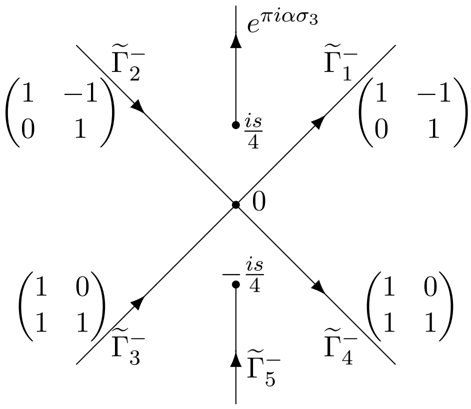

Figure 3.3: The jump contour $\widetilde{\Gamma}^-$ and the jump matrices for $\Psi^{(1)}$.

3.2.4 The kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ as $\tau \to 0$

We will now study the RH problem for Ψ^- in the limit where $s \to 0$ and prove (3.30).

Define

$$\Psi^{(1)}(z;s) = e^{\frac{s}{4}\sigma_3}\Psi^-\left(-\frac{2iz}{s}+\frac{1}{2};s\right)\chi(z), \tag{3.85}$$

where

$$\chi(z) = \begin{cases} e^{-\frac{\pi i\alpha}{2}\sigma_3} & \text{for } \operatorname{Re} z > 0, \\ e^{\frac{\pi i\alpha}{2}\sigma_3} & \text{for } \operatorname{Re} z < 0. \end{cases} \tag{3.86}$$

Then $\Psi^{(1)}$ satisfies the following RH problem.

RH problem for $\Psi^{(1)}$

(a) $\Psi^{(1)} : \mathbb{C} \setminus \widetilde{\Gamma}^- \to \mathbb{C}^{2\times 2}$ is analytic, where:

$$\widetilde{\Gamma} = \bigcup_{i=1}^{5} \widetilde{\Gamma}_i^-, \qquad \widetilde{\Gamma}_1^- = \{z : \arg z = \pi/4\},$$
$$\widetilde{\Gamma}_2^- = \{z : \arg z = 3\pi/4\}, \qquad \widetilde{\Gamma}_3^- = \{z : \arg z = 5\pi/4\},$$
$$\widetilde{\Gamma}_4^- = \{z : \arg z = 7\pi/4\}, \qquad \widetilde{\Gamma}_5^- = i\mathbb{R} \setminus [-is/4, is/4].$$

The orientation is as in Figure 3.3. Note that the orientation is different compared to the one in the jump contour for Ψ^-.

(b) $\Psi^{(1)}$ has continuous boundary values on $\widetilde{\Gamma}^- \setminus \{0\}$:

$$\Psi^{(1)}_+(z) = \Psi^{(1)}_-(z) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \widetilde{\Gamma}_1^- \cup \widetilde{\Gamma}_2^- ,$$

$$\Psi^{(1)}_+(z) = \Psi^{(1)}_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{for } z \in \widetilde{\Gamma}_3^- \cup \widetilde{\Gamma}_4^-,$$

$$\Psi^{(1)}_+(z) = \Psi^{(1)}_-(z) e^{\pi i \alpha \sigma_3} \qquad \text{for } z \in \widetilde{\Gamma}_5^- .$$

(c) $\Psi^{(1)}$ has the following behavior as $z \to \infty$:

$$\Psi^{(1)}(z) = (I + \mathcal{O}(z^{-1})) e^{iz\sigma_3} \chi(z). \tag{3.87}$$

(d0) $\Psi^{(1)}$ has the following behavior as $z \to -is/4$:

$$\Psi^{(1)}(z) = \begin{pmatrix} \mathcal{O}(|z + \frac{is}{4}|^{\alpha/2}) & \mathcal{O}(|z + \frac{is}{4}|^{-\alpha/2}) \\ \mathcal{O}(|z + \frac{is}{4}|^{\alpha/2}) & \mathcal{O}(|z + \frac{is}{4}|^{-\alpha/2}) \end{pmatrix}.$$

(d1) $\Psi^{(1)}$ has the following behavior as $z \to is/4$:

$$\Psi^{(1)}(z) = \begin{pmatrix} \mathcal{O}(|z - \frac{is}{4}|^{-\alpha/2}) & \mathcal{O}(|z - \frac{is}{4}|^{\alpha/2}) \\ \mathcal{O}(|z - \frac{is}{4}|^{-\alpha/2}) & \mathcal{O}(|z - \frac{is}{4}|^{\alpha/2}) \end{pmatrix}.$$

(e) $\Psi^{(1)}$ is bounded at 0.

Using (3.47) and (3.85), we obtain

$$\begin{pmatrix} \Phi_1(x; -s^2) \\ \Phi_2(x; -s^2) \end{pmatrix} = e^{-\frac{s}{4}\sigma_3} \Psi^{(1)}(x; s) \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.88}$$

To study $\Psi^{(1)}(z; s)$ as $s \to 0$, we need to recall results from Section 4.2.1 and Section 4.2.2 in [19]. The asymptotics as $s \to 0$ can be described as follows: we have for $|z| > \delta$ with $\delta > 0$ arbitrary that

$$\Psi^{(1)}(z; s) = \left(I + \mathcal{O}(|s \log|s||) + \mathcal{O}(s^{1+2\alpha})\right) M(z) e^{\mp \frac{\pi i \alpha}{2} \sigma_3}, \tag{3.89}$$

for $\pm \mathrm{Re}\, z > 0$. Here M satisfies the following RH conditions.

RH problem for M

(a) $M : \mathbb{C} \setminus \Gamma_M \to \mathbb{C}^{2\times 2}$ is analytic, where:

$$\Gamma_M = \cup_{i=1}^5 \Gamma_{M,i},$$
$$\Gamma_{M,1} = \{z : \arg(z) = \pi/4\}, \quad \Gamma_{M,2} = \{z : \arg(z) = 3\pi/4\},$$
$$\Gamma_{M,3} = \{z : \arg(z) = 5\pi/4\}, \quad \Gamma_{M,4} = \{z : \arg(z) = 7\pi/4\}.$$

The orientation is as in Figure 3.4.

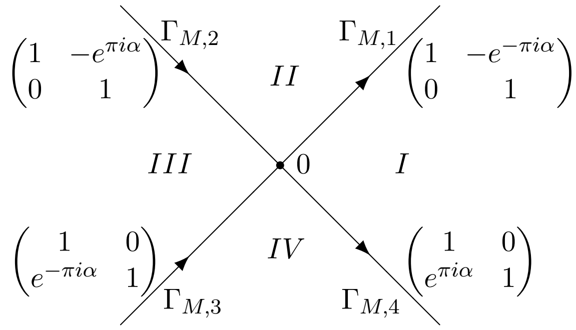

Figure 3.4: The contour Γ_M and the jump matrices for M.

(b) M has continuous boundary values on $\Gamma_M \setminus \{0\}$:

$$M_+(z) = M_-(z) \begin{pmatrix} 1 & -e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_{M,1}\ ,$$

$$M_+(z) = M_-(z) \begin{pmatrix} 1 & -e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_{M,2}\ ,$$

$$M_+(z) = M_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_{M,3},$$

$$M_+(z) = M_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_{M,4}.$$

(c) M has the following behavior as $z \to \infty$:

$$M(z) = (I + \mathcal{O}(z^{-1}))e^{iz\sigma_3}.$$

The RH problem for M has an explicit solution (which is not unique, unless one adds a condition to control the behaviour of M near 0) given in terms of confluent hypergeometric functions which is used in [19] and [20]. But in the special case we are dealing with, the confluent hypergeometric functions degenerate to Hankel and modified Bessel functions $H_\alpha^{(1)}$, $H_\alpha^{(2)}$, I_α and K_α (see [81]), and we use an explicit formula for M similar to the one given in [74, 90]. Writing

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3},$$

we have

$$M(z) = \begin{cases} \frac{z^{\frac{1}{2}}\sqrt{\pi}}{2} K \begin{pmatrix} H^{(2)}_{\alpha+\frac{1}{2}}(z) & -iH^{(1)}_{\alpha+\frac{1}{2}}(z) \\ H^{(2)}_{\alpha-\frac{1}{2}}(z) & -iH^{(1)}_{\alpha-\frac{1}{2}}(z) \end{pmatrix} e^{-\frac{\pi i}{2}(\alpha+\frac{1}{2})\sigma_3}\sigma_1\sigma_3, & z \in I, \\ K \begin{pmatrix} z^{\frac{1}{2}}\sqrt{\pi} I_{\alpha+\frac{1}{2}}(ze^{-\frac{\pi i}{2}}) & -z^{\frac{1}{2}}\frac{1}{\sqrt{\pi}} K_{\alpha+\frac{1}{2}}(ze^{-\frac{\pi i}{2}}) \\ -iz^{\frac{1}{2}}\sqrt{\pi} I_{\alpha-\frac{1}{2}}(ze^{-\frac{\pi i}{2}}) & -iz^{\frac{1}{2}}\frac{1}{\sqrt{\pi}} K_{\alpha-\frac{1}{2}}(ze^{-\frac{\pi i}{2}}) \end{pmatrix} \sigma_1\sigma_3, & z \in II, \\ (e^{\pi i}z)^{\frac{1}{2}}\frac{\sqrt{\pi}}{2} K \begin{pmatrix} -H^{(2)}_{\alpha+\frac{1}{2}}(e^{\pi i}z) & -iH^{(1)}_{\alpha+\frac{1}{2}}(e^{\pi i}z) \\ H^{(2)}_{\alpha-\frac{1}{2}}(e^{\pi i}z) & iH^{(1)}_{\alpha-\frac{1}{2}}(e^{\pi i}z) \end{pmatrix} e^{-\frac{\pi i}{2}(\alpha+\frac{1}{2})\sigma_3}, & z \in III, \\ K \begin{pmatrix} -iz^{\frac{1}{2}}\sqrt{\pi} I_{\alpha+\frac{1}{2}}(ze^{\frac{\pi i}{2}}) & -iz^{\frac{1}{2}}\frac{1}{\sqrt{\pi}} K_{\alpha+\frac{1}{2}}(ze^{\frac{\pi i}{2}}) \\ z^{\frac{1}{2}}\sqrt{\pi} I_{\alpha-\frac{1}{2}}(ze^{\frac{\pi i}{2}}) & -z^{\frac{1}{2}}\frac{1}{\sqrt{\pi}} K_{\alpha-\frac{1}{2}}(ze^{\frac{\pi i}{2}}) \end{pmatrix}, & z \in IV, \end{cases} \tag{3.90}$$

where the square roots and the Hankel and Bessel functions have branch cuts on $(-\infty, 0]$.

By (3.89) and (3.88), we have, for $\pm x > 0$,

$$\begin{pmatrix} \Phi_1(x;-s^2) \\ \Phi_2(x;-s^2) \end{pmatrix} = M(x;s)e^{\mp\frac{\pi i\alpha}{2}\sigma_3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mathcal{O}(|s|\log|s|) + \mathcal{O}(|s|^{1+2\alpha}),\ s \to 0.$$

Using (3.28) and substituting in (3.90), one obtains that as $\tau \to 0$:

$$\begin{aligned} &\mathbb{K}^{\mathrm{PV}}_\alpha(u,v;\tau) \\ &= \frac{\pi\sqrt{uv}}{8(u-v)}\Bigg(\left(H^{(1)}_{\alpha-\frac{1}{2}}(\pi v) + H^{(2)}_{\alpha-\frac{1}{2}}(\pi v)\right)\left(H^{(1)}_{\alpha+\frac{1}{2}}(\pi u) + H^{(2)}_{\alpha+\frac{1}{2}}(\pi u)\right) \\ &- \left(H^{(1)}_{\alpha+\frac{1}{2}}(\pi v) + H^{(2)}_{\alpha+\frac{1}{2}}(\pi v)\right)\left(H^{(1)}_{\alpha-\frac{1}{2}}(\pi u) + H^{(2)}_{\alpha-\frac{1}{2}}(\pi u)\right)\Bigg) \\ &\qquad + \mathcal{O}(|\tau|^{1/2}\log|\tau|) + \mathcal{O}(|\tau|^{\frac{1}{2}+\alpha}). \end{aligned}$$

The identity $H^{(1)}_\alpha(z) + H^{(2)}_\alpha(z) = 2J_\alpha(z)$ yields our result: as $\tau \to 0$ we have

$$\begin{aligned} \mathbb{K}^{\mathrm{PV}}_\alpha(u,v;\tau) = \frac{\pi\sqrt{uv}}{2(u-v)}&\left(J_{\alpha-\frac{1}{2}}(\pi v)J_{\alpha+\frac{1}{2}}(\pi u) - J_{\alpha+\frac{1}{2}}(\pi v)J_{\alpha-\frac{1}{2}}(\pi u)\right) \\ &+ \mathcal{O}(|\tau|^{1/2}\log|\tau|) + \mathcal{O}(|\tau|^{\frac{1}{2}+\alpha}). \end{aligned} \tag{3.91}$$

This proves (3.30).

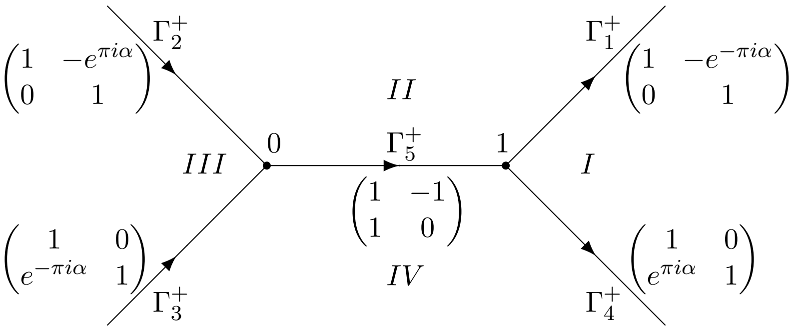

Figure 3.5: The jump contour Γ^+ and the jump matrices for Ψ^+.

3.3 Model RH problem for $\tau > 0$

The model RH problem relevant for the case $\tau > 0$ is a special case of the one found in [20] (up to a simple transformation), and differs from the one for $\tau < 0$, although there are similarities. For instance, the following RH problem, is also connected to the Painlevé V equation, and an explicit solution can be constructed for even α (but not for odd α, which was possible when $\tau < 0$). In the RH problem below, the relevant values for the parameters are $\alpha > -1/2$ and $s \in (0, -i\infty)$.

RH problem for Ψ^+

(a) $\Psi^+ : \mathbb{C}\backslash\Gamma^+ \to \mathbb{C}^{2\times 2}$ is analytic, with $\Gamma^+ = \Gamma_1^+\cup\Gamma_2^+\cup\Gamma_3^+\cup\Gamma_4^+\cup[0,1]$, and

$$\Gamma_1^+ = 1+e^{\frac{\pi i}{4}}\mathbb{R}^+,\ \Gamma_2^+ = e^{\frac{3\pi i}{4}}\mathbb{R}^+,\ \Gamma_3^+ = e^{-\frac{3\pi i}{4}}\mathbb{R}^+,\ \Gamma_4^+ = 1+e^{-\frac{\pi i}{4}}\mathbb{R}^+,$$

oriented as in Figure 3.5.

(b) Ψ^+ has continuous boundary values $\Psi_\pm^+$ on $\Gamma^+ \setminus \{0,1\}$. We have the jump relations

$$\Psi_+^+(z) = \Psi_-^+(z)\begin{pmatrix}1 & -e^{-\pi i\alpha}\\ 0 & 1\end{pmatrix} \qquad \text{for } z \in \Gamma_1^+,$$

$$\Psi_+^+(z) = \Psi_-^+(z)\begin{pmatrix}1 & -e^{\pi i\alpha}\\ 0 & 1\end{pmatrix} \qquad \text{for } z \in \Gamma_2^+,$$

$$\Psi_+^+(z) = \Psi_-^+(z)\begin{pmatrix}1 & 0\\ e^{-\pi i\alpha} & 1\end{pmatrix} \qquad \text{for } z \in \Gamma_3^+,$$

$$\Psi^+_+(z) = \Psi^+_-(z)\begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_4^+,$$
$$\Psi^+_+(z) = \Psi^+_-(z)\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{for } z \in (0,1).$$

(c) There exist q, r, p (depending on s, α but not on z) such that $\Psi^+(z)$ has the following limiting behavior as $z \to \infty$:

$$\Psi^+(z) = \left(I + \frac{1}{z}\begin{pmatrix} q & r \\ p & -q \end{pmatrix} + \mathcal{O}(z^{-2})\right)\exp\left(-\frac{s}{2}z\sigma_3\right).$$

(d) If $\alpha \notin \mathbb{N}$ then the functions

$$G(z) := \Psi^+(z)\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} z^{-\frac{\alpha}{2}\sigma_3}, \tag{3.92}$$
$$H(z) := \Psi^+(z)\begin{pmatrix} 1 & e^{-\pi i\alpha}g \\ 0 & 1 \end{pmatrix}(z-1)^{-\frac{\alpha}{2}\sigma_3} \tag{3.93}$$

are bounded for z near 0 and 1 respectively when z is in region IV, with $g = \frac{e^{\pi i\alpha}-1}{2i\sin(\pi\alpha)}$. We let $z^{\frac{\alpha}{2}}, (z-1)^{\frac{\alpha}{2}} > 0$ on the + side of $(1,\infty)$. If $\alpha \in \mathbb{N}$ then the functions

$$G(z) := \Psi^+(z)\begin{pmatrix} 1 & \frac{1-e^{\pi i\alpha}}{2\pi i}\log z \\ 0 & 1 \end{pmatrix} z^{-\frac{\alpha}{2}\sigma_3}, \tag{3.94}$$
$$H(z) := \Psi^+(z)\begin{pmatrix} 1 & \frac{e^{-\pi i\alpha}-1}{2\pi i}\log(z-1) \\ 0 & 1 \end{pmatrix}(z-1)^{-\frac{\alpha}{2}\sigma_3} \tag{3.95}$$

are bounded for z near 0 and 1 respectively when z is in region IV. We let $\log z, \log(z-1) > 0$ on the + side of $(1,\infty)$.

The above RH problem is equivalent to the one from [20, Section 3]. Indeed, if we define

$$\Psi_{CK}(\zeta; s) = e^{\frac{s}{4}\sigma_3}\Psi^+\left(\frac{i}{2}(\zeta - i); s\right), \tag{3.96}$$

then Ψ_{CK} solves the RH problem in [20, Section 3] in the case $\beta_1 = \beta_2 = 0$, $\alpha_1 = \alpha_2 = \alpha/2$.

In [20], it was proved that the RH problem for Ψ_{CK} is solvable if $\alpha > -1/2$ for every $s \in (0, -i\infty)$, and it follows that the same is true for the RH problem for Ψ^+.

For $\tau > 0$ and $u \in \mathbb{R}$, we define

$$\begin{pmatrix} \Phi_1(u;\tau) \\ \Phi_2(u;\tau) \end{pmatrix} = \Psi^+\left(z = \frac{2u}{|s|} + \frac{1}{2}; s\right)\Delta\left(\frac{2u}{|s|} + \frac{1}{2}\right), \tag{3.97}$$

where $s = -i\sqrt{\tau}$, and

$$\Delta(z) = \begin{cases} \begin{pmatrix} e^{-\frac{\pi i\alpha}{2}} \\ -e^{\frac{\pi i\alpha}{2}} \end{pmatrix} & \text{for } \operatorname{Re} z > 1, \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{for } 0 < \operatorname{Re} z < 1, \\ \begin{pmatrix} e^{\frac{\pi i\alpha}{2}} \\ -e^{-\frac{\pi i\alpha}{2}} \end{pmatrix} & \text{for } \operatorname{Re} z < 0, \end{cases} \tag{3.98}$$

and where $\Psi^+(z;s)$ has to be understood as $\Psi^+_+(z;s)$ for $z \in (0,1)$. Then, $\Phi_1(u;\tau), \Phi_2(u;\tau)$ are smooth on the real line, except at the singularities $\pm\frac{\sqrt{\tau}}{4}$. As $u \to \pm\infty$, Φ_1 and Φ_2 have the asymptotic behavior

$$\begin{pmatrix} \Phi_1(u;\tau) \\ \Phi_2(u;\tau) \end{pmatrix} = \left(I + \mathcal{O}(u^{-1})\right) \exp((iu+i\sqrt{\tau}/4)\sigma_3)\Delta\left(\frac{2u}{\sqrt{\tau}} + \frac{1}{2}\right), \tag{3.99}$$

similar to (3.48).

3.3.1 Lax pair and Painlevé V

The results about the Lax pair and the relation with the Painlevé V equation hold for Ψ^+ as well as for Ψ^-. Indeed, $\Psi = \Psi^+$ satisfies the Lax pair (3.49)-(3.50), where A and B can be parameterized as in (3.51)-(3.55). The functions v, u defined in (3.56)-(3.57) in terms of q, r, p, solve equations (3.58)-(3.59), and σ defined as in (3.62) solves the σ-form (3.10) of Painlevé V. It has to be noted that the functions $q(s), r(s), p(s), v(s), u(s), \sigma(s)$ defined here for $\tau > 0$ are in general different from their counterparts in Section 3.2 for $\tau < 0$.

3.3.2 Recursive construction for $\alpha \in 2\mathbb{N}$

Similarly to the case $\tau < 0$, we define

$$X_\alpha(z;s) = \begin{cases} \Psi^+_\alpha(z;s) & \text{for } z \in I, III, \\ \Psi^+_\alpha(z;s)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in II, \\ \Psi^+_\alpha(z;s)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } z \in IV, \end{cases} \tag{3.100}$$

with regions I, II, III, IV as in Figure 3.5. Then, it is easily verified that, for even α, X_α satisfies the RH problem for X_α stated in Section 3.2.2. It is important to note that this is not the case for α odd. It follows that the formulas for X_2 and q_2 given in (3.78) and (3.79) still hold for $\tau > 0$. By (3.80), we obtain the formula (3.14) for σ_2^+.

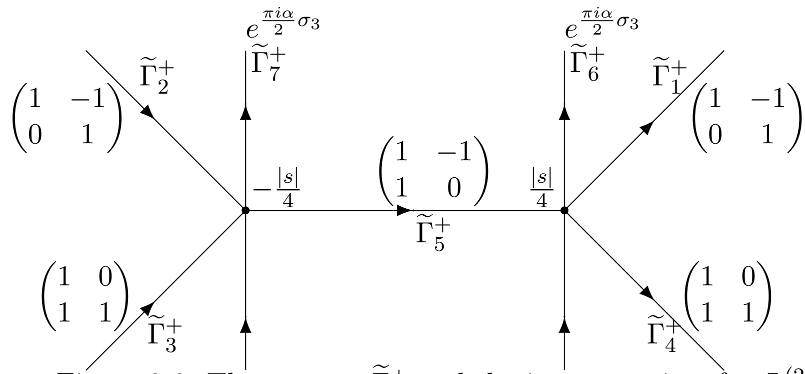

Figure 3.6: The contour $\widetilde{\Gamma}^+$ and the jump matrices for $\Psi^{(2)}$.

3.3.3 The kernel $\mathbb{K}_\alpha$ as $\tau \to +\infty$

An asymptotic analysis as $s = -i\sqrt{\tau} \to -i\infty$ of the RH problem for Ψ_{CK}, equivalent to the RH problem for Ψ^+, has been performed in [20, Section 5]. The following result can be extracted from that analysis:

$$\Psi_+^+(x;s)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} e^{\frac{sx}{2}\sigma_3} = I + \mathcal{O}\left(\frac{1}{s(|x|+1)}\right) \tag{3.101}$$

for $x \in (0,1)$, as $s \to -i\infty$. Substituting (3.101) and (3.97) in (3.28), we obtain that the Painlevé kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ converges to the sine kernel, proving (3.29) whenever $|u-v|$ remains bounded in the limit where $\tau \to +\infty$:

$$\mathbb{K}_\alpha^{\mathrm{PV}}(u,v;\tau) = \mathbb{K}^{\sin}(u,v) + \mathcal{O}\left(\tau^{-1/2}\right).$$

3.3.4 The kernel $\mathbb{K}_\alpha^{\mathrm{PV}}$ as $\tau \to 0$

Recall that the parameter s in the model problem for Ψ^+ is negative imaginary. Define

$$\Psi^{(2)}(z;s) = e^{\frac{s}{4}\sigma_3}\Psi^+\left(\frac{2z}{|s|}+\frac{1}{2};s\right)\chi(z), \tag{3.102}$$

$$\chi(z) = \begin{cases} I, & \text{for } -\frac{|s|}{4} < \operatorname{Re} z < \frac{|s|}{4},\\ e^{\frac{\pi i\alpha}{2}\sigma_3}, & \text{for } \operatorname{Re} z < -\frac{|s|}{4},\\ e^{\frac{-\pi i\alpha}{2}\sigma_3}, & \text{for } \operatorname{Re} z > \frac{|s|}{4}. \end{cases} \tag{3.103}$$

Then $\Psi^{(2)}$ satisfies the following RH problem:

RH problem for $\Psi^{(2)}$

(a) $\Psi^{(2)} : \mathbb{C} \setminus \widetilde{\Gamma}^+ \to \mathbb{C}$ is analytic, where $\widetilde{\Gamma}^+$ is as in Figure 3.6,

$$\widetilde{\Gamma}^+ = \cup_{i=1}^8 \widetilde{\Gamma}_i^+, \qquad \widetilde{\Gamma}_1^+ = \{z : \arg(z - \frac{|s|}{4}) = \frac{\pi}{4}\},$$
$$\widetilde{\Gamma}_2^+ = \{z : \arg(z + \frac{|s|}{4}) = \frac{3\pi}{4}\}, \quad \widetilde{\Gamma}_3^+ = \{z : \arg(z + \frac{|s|}{4}) = \frac{5\pi}{4}\},$$
$$\widetilde{\Gamma}_4^+ = \{z : \arg(z - \frac{|s|}{4}) = \frac{7\pi}{4}\}, \quad \widetilde{\Gamma}_5^+ = [-\frac{|s|}{4}, \frac{|s|}{4}],$$
$$\widetilde{\Gamma}_6^+ = \{z : \operatorname{Re} z = \frac{|s|}{4}\}, \qquad \widetilde{\Gamma}_7^+ = \{z : \operatorname{Re} z = -\frac{|s|}{4}\}.$$

(b) $\Psi^{(2)}$ has continuous boundary values on $\widetilde{\Gamma}^+ \setminus \{-\frac{|s|}{4}, \frac{|s|}{4}\}$ given by:

$$\Psi_+^{(2)}(z) = \Psi_-^{(2)}(z) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \widetilde{\Gamma}_1^+ \cup \widetilde{\Gamma}_2^+,$$
$$\Psi_+^{(2)}(z) = \Psi_-^{(2)}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{for } z \in \widetilde{\Gamma}_3^+ \cup \widetilde{\Gamma}_4^+ ,$$
$$\Psi_+^{(2)}(z) = \Psi_-^{(2)}(z) \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{for } z \in \widetilde{\Gamma}_5^+,$$
$$\Psi_+^{(2)}(z) = \Psi_-^{(2)}(z) e^{\frac{\pi i \alpha}{2}\sigma_3} \qquad \text{for } z \in \widetilde{\Gamma}_6^+ \cup \widetilde{\Gamma}_7^+.$$

(c) $\Psi^{(2)}(z)$ has the following asymptotic behavior as $z \to \infty$:

$$\Psi^{(2)}(z) = (I + \mathcal{O}(z^{-1}))e^{iz\sigma_3}\chi(z). \tag{3.104}$$

(d) $\Psi^{(2)}(z)$ inherits its local behavior at $-\frac{|s|}{4}$ and $\frac{|s|}{4}$ from $\Psi^+(z)$ through (3.92)-(3.95) and (3.102).

When $s \to -i0$ it was shown in [20, Section 6.2] that, for z bounded away from 0,

$$\Psi^{(2)}(z; s)\chi^{-1}(z)M^{-1}(z) = I + \epsilon(z, s), \tag{3.105}$$

where M is as in (3.90), $\epsilon(z, s) = \mathcal{O}(|s|^2) + \mathcal{O}(|s|^{2+4\alpha})$ for $2\alpha \notin \mathbb{Z}$, and $\epsilon(z, s) = \mathcal{O}(|s|\log|s|)$ for $2\alpha \in \mathbb{Z}$.

We substitute (3.105) and (3.102) into (3.97) to find that as $\tau \to 0$, $\pm x > 0$,

$$\begin{pmatrix} \Phi_1(x; -s^2) \\ \Phi_2(x; -s^2) \end{pmatrix} = M(x)e^{\mp \frac{\pi i \alpha}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \epsilon(x, s).$$

By (3.28), we obtain the Bessel kernel after similar calculations as for the case $\tau < 0$, proving (3.30).

3.4 RH problem for orthogonal polynomials

3.4.1 RH problem for orthogonal polynomials and differential identity

The correlation kernel (3.21) for the eigenvalues in the random matrix ensemble (3.1) can be expressed in terms of the solution of a RH problem which characterizes the orthogonal polynomials $p_j^{(n)}$ with respect to the weight w_n defined in (3.3).

Define Y by

$$Y(z;n) = \begin{pmatrix} \frac{1}{\kappa_n^{(n)}} p_n^{(n)}(z) & \frac{1}{2\pi i \kappa_n^{(n)}} \int_{\mathbb{R}} \frac{p_n^{(n)}(x)}{x-z} w_n(x)dx \\ -2\pi i \kappa_{n-1}^{(n)} p_{n-1}^{(n)}(z) & -\kappa_{n-1}^{(n)} \int_{\mathbb{R}} \frac{p_{n-1}^{(n)}(x)}{x-z} w_n(x)dx \end{pmatrix}, \quad (3.106)$$

where $\kappa_j^{(n)} > 0$ is the leading coefficient of the normalized orthogonal polynomial $p_j^{(n)}$. It is well-known [45] that Y can be characterized as the unique solution of the following RH problem.

RH problem for Y

(a) Y is analytic in $\mathbb{C} \setminus \mathbb{R}$.

(b) Y has the following jump relations on $\mathbb{R}$, except at the singularities $\pm\sqrt{t}$ in the case where $t > 0$:

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_n(x) \\ 0 & 1 \end{pmatrix}.$$

(c) As $z \to \infty$,

$$Y(z) = \left(I + \mathcal{O}\left(z^{-1}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

(d) If $t > 0$ and $\alpha < 0$, $Y(z)$ has the behavior

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|z \mp \sqrt{t}|^{\alpha}) \\ \mathcal{O}(1) & \mathcal{O}(|z \mp \sqrt{t}|^{\alpha}) \end{pmatrix},$$

as $z \to \pm\sqrt{t}$. If $t > 0$ and $\alpha \geq 0$, Y is bounded near $\pm\sqrt{t}$.

We can express the eigenvalue correlation kernel K_n and the logarithmic t-derivative of the partition function $\widehat{Z}_n$ exactly in terms of Y.

Proposition 3.4.1. *(1) Let K_n be the correlation kernel defined in* (3.21)*. For $x, y \in \mathbb{R}$, we have*

$$K_n(x,y) = \frac{\sqrt{w_n(x)w_n(y)}}{2\pi i} \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{x-y}. \quad (3.107)$$

(2) Let $\widehat{Z}_n(t) = \widehat{Z}_n(t,\alpha,V)$ be the partition function defined in (3.4)*. Write $z_0 = \sqrt{t}$ when $t > 0$ and $z_0 = i\sqrt{-t}$ when $t < 0$. For $t < 0, \alpha > -\frac{1}{2}$ and for $t > 0, \alpha \geq 0$, we have the differential identity*

$$\frac{d}{dt} \log \widehat{Z}_n(t) = -\frac{\alpha}{2z_0} \left(\left(Y^{-1} \frac{dY}{dz} \right)_{22} (z_0) - \left(Y^{-1} \frac{dY}{dz} \right)_{22} (-z_0) \right). \quad (3.108)$$

Proof. The formula for the correlation kernel (3.107) follows after a straightforward calculation from (3.21) and (3.106).

To obtain (3.108), we follow ideas from [70] and [20]. In the remaining part of this proof, we suppress the dependence on n in our notations and we write p_j, κ_j, and w instead of $p_j^{(n)}$, $\kappa_j^{(n)}$, and w_n. We start with the well known formula

$$\widehat{Z}_n(t) = n! \prod_{j=0}^{n-1} \kappa_j^{-2}.$$

Using the orthogonality of the polynomials and the Christoffel-Darboux formula, it follows that

$$\begin{aligned} \frac{d}{dt} \log \widehat{Z}_n(t) &= -2 \sum_{j=0}^{n-1} \frac{\partial_t \kappa_j}{\kappa_j} = -2 \sum_{j=0}^{n-1} \int_{-\infty}^{\infty} p_j(x) \partial_t (p_j(x)) w(x) dx \\ &= -\int_{-\infty}^{\infty} \partial_t \left(\sum_{j=0}^{n-1} p_j^2(x) \right) w(x) dx \\ &= -\int_{-\infty}^{\infty} \partial_t \left(\frac{\kappa_{n-1}}{\kappa_n} (\partial_x (p_n(x)) p_{n-1}(x) - p_n(x) \partial_x (p_{n-1}(x))) \right) w(x) dx. \end{aligned}$$

By orthogonality, it is straightforward to check that

$$\frac{d}{dt} \log \widehat{Z}_n(t) = -n \frac{\partial_t \kappa_{n-1}}{\kappa_{n-1}} + \frac{\kappa_{n-1}}{\kappa_n} (J_1 - J_2), \quad (3.109)$$

$$J_1 = \int_{-\infty}^{\infty} \partial_t p_n(x) \partial_x p_{n-1}(x) w(x) dx, \quad (3.110)$$

$$J_2 = \int_{-\infty}^{\infty} \partial_x p_n(x) \partial_t p_{n-1}(x) w(x) dx. \quad (3.111)$$

We start by evaluating J_1:

$$J_1 = -\int_{-\infty}^{\infty} p_n(x)\partial_t(\partial_x p_{n-1}(x)w(x))dx + \int_{-\infty}^{\infty} \partial_t(p_n(x)\partial_x p_{n-1}(x)w(x))dx. \quad (3.112)$$

The rightmost term in this expression vanishes, as one can see by taking the derivative outside the integral and using orthogonality. By the form of the weight function w given in (3.3), we obtain

$$J_1 = \frac{\alpha}{2z_0}(I_+^{(1)} - I_-^{(1)}), \quad (3.113)$$

$$I_\pm^{(1)} = \int_{-\infty}^{\infty} \frac{p_n(x)\partial_x p_{n-1}(x)w(x)}{x \mp z_0}dx. \quad (3.114)$$

Writing $\partial_x(p_m(x))|_{x=y}= p_{m,x}(y)$, we find that

$$\begin{aligned} I_\pm^{(1)} =& \int_{-\infty}^{\infty} \frac{p_n(x)(p_{n-1,x}(x) - p_{n-1,x}(\pm z_0))w(x)}{x \mp z_0}dx \quad (3.115)\\ &+ p_{n-1,x}(\pm z_0)\int_{-\infty}^{\infty} \frac{p_n(x)w(x)}{x \mp z_0}dx\\ =& p_{n-1,x}(\pm z_0)\int_{-\infty}^{\infty} \frac{p_n(x)w(x)}{x \mp z_0}dx. \end{aligned}$$

We proceed in a similar fashion for J_2 and find that

$$\begin{aligned} J_2 = -n\frac{\kappa_n \partial_t \kappa_{n-1}}{\kappa_{n-1}^2} + \frac{\alpha}{2z_0}p_{n,x}(z_0)\int_{-\infty}^{\infty} \frac{p_{n-1}(x)w(x)}{x - z_0}dx\\ - \frac{\alpha}{2z_0}p_{n,x}(-z_0)\int_{-\infty}^{\infty} \frac{p_{n-1}(x)w(x)}{x + z_0}dx. \quad (3.116) \end{aligned}$$

Substituting (3.113)–(3.116) into (3.109), and using (3.106), it is straightforward to obtain (3.108). □

Remark 3.4.1. When $t > 0$ and $\alpha < 0$, the integrals in (3.114) are not defined. Using techniques similar to those in [70], one can regularize those integrals and obtain a differential identity similar to (3.108). We refer the reader to [70, 20] for more details.

3.4.2 The g-function and normalization of the RH problem

We proceed with a transformation of the RH problem for Y to one which is normalized to the identity at infinity, and which has suitable jump conditions. This is a standard part of the Deift/Zhou steepest descent method [36] which we apply in order to get asymptotics for Y as $n \to \infty$ with t small. Define

$$T(z) = e^{\frac{n\ell}{2}\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{\frac{-n\ell}{2}\sigma_3}, \tag{3.117}$$

where g is defined by

$$g(z) = \int \log(z-x) d\mu_V(x), \tag{3.118}$$

with μ_V the equilibrium measure given by (3.9), minimizing (3.6) and satisfying the conditions (3.7)-(3.8). Clearly $g(z) = \log z + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$, and it follows that T satisfies the following RH problem.

RH problem for T

(a) T is analytic in $\mathbb{C} \setminus \mathbb{R}$.

(b) T has the following jump relation for $x \in \mathbb{R}$ (except for $x = \pm\sqrt{t}$ if $t > 0$),

$$T_+(x) = T_-(x) \begin{pmatrix} e^{-n(g_+(x)-g_-(x))} & |x^2-t|^\alpha e^{n(g_+(x)+g_-(x)-V(x)+\ell)} \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix}.$$

(c) $T(z) = I + \mathcal{O}(z^{-1})$, as $z \to \infty$.

(d) If $t > 0$, $\alpha < 0$, as $z \to \pm\sqrt{t}$, we have

$$T(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|z \mp \sqrt{t}|^\alpha) \\ \mathcal{O}(1) & \mathcal{O}(|z \mp \sqrt{t}|^\alpha) \end{pmatrix}.$$

If $t > 0$ and $\alpha \geq 0$, T is bounded near $\pm\sqrt{t}$.

By the fact that V is one-cut regular, we have (3.9) with h real analytic and positive on $[a,b]$. By (3.118) and the Euler-Lagrange conditions (3.7)-(3.8), it can be shown that g satisfies the following properties which we will need later on:

(a) $$g_+(x) - g_-(x) = \begin{cases} 0 & \text{for } x > b, \\ 2\pi i & \text{for } x < a, \\ 2\pi i \int_x^b h(\xi)((\xi-a)(b-\xi))^{\frac{1}{2}} d\xi & \text{for } x \in [a,b], \end{cases}$$

(b) $g_+(x) + g_-(x) - V(x) + \ell < 0$ for $x > b$ and $x < a$,

(c) $g_+(x) + g_-(x) - V(x) + \ell$

$$= \begin{cases} 2\pi \int_x^a h(\xi)\left((\xi-a)(\xi-b)\right)^{1/2} d\xi & \text{for } x < a, \\ 0 & \text{for } x \in [a,b], \\ -2\pi \int_b^x h(\xi)\left((\xi-a)(\xi-b)\right)^{1/2} d\xi & \text{for } x > b. \end{cases}$$

In part (a), the square root is positive for $s \in (a,b)$, and in part (c), the square roots are positive for $s > b$ and negative for $s < a$, with branch cut on $[a,b]$. The function h is analytic on a neighborhood $\mathcal{U}$ of $\mathbb{R}$, and we define the function $\phi : \mathcal{U} \setminus (-\infty, b] \to \mathbb{C}$ as

$$\phi(z) = \pi \int_b^z h(\xi)\left((\xi-a)(\xi-b)\right)^{1/2} d\xi, \tag{3.119}$$

with the square root analytic on $\mathbb{C} \setminus [a,b]$ and positive for $\xi > b$, and with the path of integration not crossing $(-\infty, b]$. By property (a) of the function g, it follows that

$$\mp(g_+(x) - g_-(x)) = 2\phi_\pm(x) \quad \text{for } x \in (a,b).$$

Combining this with property (c) for the function g we obtain

$$T_+(x) = T_-(x)\begin{pmatrix} e^{2n\phi_+(x)} & |x^2-t|^\alpha \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} \quad \text{for } x \in [a,b].$$

By property (c) of the function g it follows that

$$g_+(x) + g_-(x) - V(x) + \ell = \begin{cases} -2\phi(x) & \text{for } x > b, \\ -2\phi_+(x) - 2\pi i & \text{for } x < a. \end{cases} \tag{3.120}$$

Combined with property (a) of g it follows that

$$T_+(x) = T_-(x)\begin{pmatrix} 1 & |x^2-t|^\alpha e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix} \text{ for } x < a \text{ and for } b < x,$$

where we note that $e^{-2n\phi_+(x)} = e^{-2n\phi_-(x)}$ by (3.120).

3.5 Asymptotic analysis for T as $n \to \infty$ for $t < 0$

In what follows, we will perform further transformations on T in order to obtain a RH problem for which we can compute large n asymptotics. We first complete the analysis in the case $t < 0$.

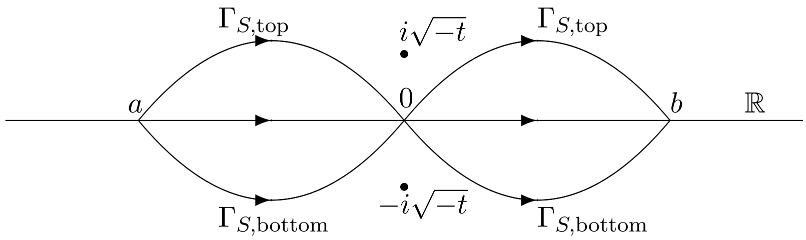

Figure 3.7: The contour Γ_S.

3.5.1 Opening of the lens

We deform the jump contour for T, which is the real line, to a lens-shaped contour around $[a, b]$ as in Figure 3.7. It is important that the singularities $\pm i\sqrt{-t}$ lie in the region outside of the lens, and that the lenses lie within the region $\mathcal{U}$ of analyticity of h. Since we want to obtain asymptotics which are valid for t arbitrary small, we choose the lenses to pass through 0, and with shape independent of t and n.

Define

$$S(z) = \begin{cases} T(z) & \text{for } z \text{ outside the lens,} \\ T(z)\begin{pmatrix} 1 & 0 \\ -\frac{e^{2n\phi(z)}}{(z^2-t)^\alpha} & 1 \end{pmatrix} & \text{for } z \text{ in the upper parts of the lens,} \\ T(z)\begin{pmatrix} 1 & 0 \\ \frac{e^{2n\phi(z)}}{(z^2-t)^\alpha} & 1 \end{pmatrix} & \text{for } z \text{ in the lower parts of the lens.} \end{cases} \tag{3.121}$$

The function $(z^2 - t)^\alpha$ is chosen with branch cuts on $(-i\infty, -i\sqrt{-t}] \cup [i\sqrt{-t}, +i\infty)$, outside the lens-shaped region, and positive for real z. It is such that $(z^2 - t)^\alpha = |z^2 - t|^\alpha$ for $z \in (a, b)$.
It follows that S solves the following RH problem.

RH problem for S

(a) S is analytic for $z \in \mathbb{C} \setminus \Gamma_S$ where $\Gamma_S = \Gamma_{S,\text{top}} \cup \Gamma_{S,\text{bottom}} \cup \mathbb{R}$ is the contour shown in Figure 3.7.

(b) S has the following jump relations on Γ_S:

$$S_+(x) = S_-(x) \begin{pmatrix} 1 & |x^2-t|^\alpha e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } x \in \mathbb{R} \setminus [a,b],$$

$$S_+(x) = S_-(x) \begin{pmatrix} 0 & |x^2-t|^\alpha \\ -\frac{1}{|x^2-t|^\alpha} & 0 \end{pmatrix}, \qquad \text{for } x \in (a,b),$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ \frac{e^{2n\phi(z)}}{(z^2-t)^\alpha} & 1 \end{pmatrix}, \qquad \text{for } z \in \Gamma_{S,\text{top}} \cup \Gamma_{S,\text{bottom}}.$$

(c) $S(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.

(d) $S(z)$ is bounded as $z \to 0$, as $z \to a$, and as $z \to b$.

3.5.2 Global parametrix for $t < 0$

The jump matrices for S tend to the identity matrix exponentially fast everywhere on Γ_S, with the exception of the interval (a,b) and small neighborhoods of a, b, and 0. We construct a global parametrix N which solves the RH problem for S, ignoring exponentially small jumps and small neighborhoods of a, b, and 0. We will show later on that the global parametrix is a good approximation of S away from $a, b, 0$.

RH problem for N

(a) $N : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) N has the following jump relation on (a,b), where the orientation is taken from left to right:

$$N_+(x) = N_-(x) \begin{pmatrix} 0 & |x^2-t|^\alpha \\ -\frac{1}{|x^2-t|^\alpha} & 0 \end{pmatrix} \qquad \text{for } x \in (a,b).$$

(c) $N(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Define

$$N(z) = D(\infty)^{-1} N_0(z) D(z), \tag{3.122}$$

where

$$N_0(z) = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \gamma(z) & 0 \\ 0 & \gamma(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1},$$

$$\gamma(z) = \left(\frac{z-b}{z-a}\right)^{1/4}, \tag{3.123}$$

with $\left(\frac{z-b}{z-a}\right)^{1/4}$ analytic on $\mathbb{C}\setminus[a,b]$ and positive for $z>b$. $D(z)$ depends on α and t and is given by

$$D(z) = d(z)^{-\sigma_3}, \tag{3.124}$$

$$d(z) = \exp\left(-\alpha\frac{((z-a)(z-b))^{1/2}}{2\pi}\left(\int_a^b \frac{\log|\xi^2-t|d\xi}{(\xi-z)\sqrt{(\xi-a)(b-\xi)}}\right)\right), \tag{3.125}$$

with $\sqrt{(z-a)(z-b)}$ analytic in $\mathbb{C}\setminus[a,b]$ and positive for $z>b$, and with $\sqrt{(\xi-a)(b-\xi)}$ positive for $\xi\in(a,b)$. Then, it is straightforward to verify that N solves the RH problem for N. Moreover, as $z\to a$ and $z\to b$, $N(z)$ has the following behavior:

$$N(z) = \mathcal{O}\left(|z-a|^{-1/4}\right) \qquad \text{as } z\to a, \tag{3.126}$$

$$N(z) = \mathcal{O}\left(|z-b|^{-1/4}\right) \qquad \text{as } z\to b. \tag{3.127}$$

To verify this, one first shows that $N_0(z)$ satisfies the conditions (3.126)-(3.127). It then remains to show that d is bounded near a and b. This can be seen by deforming the integral between a and b to a contour surrounding the interval $[a,b]$ and applying a residue argument.

3.5.3 Local parametrix at a and b

We need to construct local parametrices near the endpoints a and b. This construction, using the Airy function, is standard [27, 34, 35]. The explicit formula for the local parametrices near a and b is not relevant to later calculations, we only need that these local parametrices exist.

Let U_b (U_a) be an open set which is independent of n and which contains b (a). The following RH problems can be solved in terms of the Airy function. We do not give details about this construction here.

RH problem for $P^{(b)}$

(a) $P^{(b)}: U_b\setminus\Gamma_S\to\mathbb{C}^{2\times 2}$ is analytic on $U_b\setminus\Gamma_S$.

(b) $P^{(b)}$ has exactly the same jump relations as S has for $z\in\Gamma_S\cap U_b$.

(c) $P^{(b)}(z) = (I+\mathcal{O}(n^{-1}))N(z)$ as $n\to\infty$, uniformly for $z\in\partial U_b$ and $|t|$ sufficiently small.

RH problem for $P^{(a)}$

(a) $P^{(a)} : U_a \setminus \Gamma_S \to \mathbb{C}^{2\times 2}$ is analytic on $U_a \setminus \Gamma_S$.

(b) $P^{(a)}$ has exactly the same jump relations as S has for $z \in \Gamma_S \cap U_a$.

(c) $P^{(a)}(z) = (I + \mathcal{O}(n^{-1}))N(z)$ as $n \to \infty$, uniformly for $z \in \partial U_a$ and $|t|$ sufficiently small.

3.5.4 Local parametrix at the origin

Let U_0 be a sufficiently small disk around the origin, which is independent of n. We want to construct a local parametrix which has the same jumps as S in the disk U_0 and which matches with the global parametrix up to order $1/n$ at the boundary of U_0. This matching condition has to be valid in the double scaling limit where $n \to \infty$ and simultaneously $t \to 0$ in such a way that $n^2 t$ tends to a non-zero constant, or equivalently such that $\tau_{n,t} \to \tau < 0$, with $\tau_{n,t}$ defined in (3.24).

RH problem for P

(a) $P : U_0 \setminus \Gamma_S \to \mathbb{C}^{2\times 2}$ is analytic.

(b) P has the same jump relations as S:

$$P_-^{-1}(z)P_+(z) = S_-^{-1}(z)S_+(z), \quad \text{for } z \in U_0 \cap \Gamma_S.$$

(c) $P(z) = \left(I + \mathcal{O}\left(n^{-1}\right)\right) N(z)$ uniformly for z on the boundary ∂U_0, as $n \to \infty$ and simultaneously $t \to 0$ in such a way that $n^2 t$ tends to a non-zero constant.

We will construct a solution to the RH problem for P in terms of the model RH solution Ψ^- studied in Section 3.2.

Construction of the parametrix

Define $\lambda : U_0 \to \mathbb{C}$ by

$$\lambda(z) = -in \times \begin{cases} \phi(z) - \frac{\phi(i\sqrt{-t}) - \phi(-i\sqrt{-t})}{2} & \text{for } \operatorname{Im} z > 0, \\ -\phi(z) - \frac{\phi(i\sqrt{-t}) - \phi(-i\sqrt{-t})}{2} & \text{for } \operatorname{Im} z < 0. \end{cases} \tag{3.128}$$

By (3.119), λ is a conformal map in U_0 (if this one is chosen sufficiently small, but containing $\pm i\sqrt{-t}$), and we have

$$\lambda(i\sqrt{-t}) = -\lambda(-i\sqrt{-t}) = -\frac{in}{2}(\phi(i\sqrt{-t}) + \phi(-i\sqrt{-t})),$$

and

$$\lambda'(0) = n\pi\psi_V(0).$$

We define $s_{n,t}$ such that λ sends $i\sqrt{-t}$ to $is_{n,t}/4$ and $-i\sqrt{-t}$ to $-is_{n,t}/4$:

$$\begin{aligned} s_{n,t} = -4i\lambda(i\sqrt{-t}) &= -2n(\phi(i\sqrt{-t}) + \phi(-i\sqrt{-t})) \\ &= -4n\mathrm{Re}\,\phi(i\sqrt{-t}) > 0. \end{aligned} \tag{3.129}$$

By (3.119), we have

$$\begin{aligned} s_{n,t} &= -4\pi n\mathrm{Re}\left(\int_0^{i\sqrt{-t}} h(s)\left((s-a)(s-b)\right)^{1/2} ds\right) \\ &= 4n\pi\sqrt{-t}\psi_V(0) + \mathcal{O}(n|t|^{3/2}) = \sqrt{-\tau_{n,t}} + \mathcal{O}(n|t|^{3/2}), \end{aligned} \tag{3.130}$$

as $t \to 0$, $n \to \infty$, with $\tau_{n,t}$ as in (3.24).

There remains some freedom to choose the contour Γ_S, which has to be of the shape shown in Figure 3.7, but is otherwise arbitrary. We choose Γ_S such that the upper and lower lenses are mapped to the straight lines $\widetilde{\Gamma}_1^-, \ldots, \widetilde{\Gamma}_4^-$ (see Figure 3.3) by λ.

Recall the definition of $\Psi^{(1)}$ from (3.85). We take P of the form

$$P(z) = E(z)\Psi^{(1)}(\lambda(z); s_{n,t})W(z). \tag{3.131}$$

Here, E has to be analytic in U_0, and W takes the form

$$W(z) = \begin{cases} \left((z^2-t)^{\alpha/2}e^{-n\phi(z)}\right)^{\sigma_3}\sigma_3\sigma_1 & \text{for } \mathrm{Im}\, z > 0, \\ \left((z^2-t)^{-\alpha/2}e^{n\phi(z)}\right)^{\sigma_3} & \text{for } \mathrm{Im}\, z < 0. \end{cases} \tag{3.132}$$

It is constructed in such a way that P has the required jump conditions on $\Gamma_S \cap U_0$. The branch cuts of $(z^2-t)^{\alpha/2}$ are taken on the lines $(-i\infty, -i\sqrt{-t})$ and $(i\sqrt{-t}, i\infty)$. E is an analytic matrix-valued function on U_0 which is constructed such that $P(z)$ approximates $N(z)$ on the boundary ∂U_0:

$$E(z) = N(z)W^{-1}(z)\left(\frac{\lambda(z) - \frac{is_{n,t}}{4}}{\lambda(z) + \frac{is_{n,t}}{4}}\right)^{\frac{\alpha}{2}\sigma_3} e^{-i\lambda(z)\sigma_3}e^{\frac{\pi i\alpha}{2}\sigma_3}, \tag{3.133}$$

where we have chosen the branch cuts for z in $(i\sqrt{-t}, i\infty)$ and $(-i\sqrt{-t}, -i\infty)$ such that

$$\left(\frac{\lambda(z) - \frac{is_{n,t}}{4}}{\lambda(z) + \frac{is_{n,t}}{4}}\right)^{\frac{\alpha}{2}\sigma_3} = 1+\mathcal{O}(n^{-1}) \quad \text{for } \mathrm{Re}\,(z) > 0 \text{ and } z \in \partial U, \tag{3.134}$$

as $n \to \infty$ and $s_{n,t}$ remains bounded. Note that $N(z)W^{-1}(z)$ is continuous across the real line and that $W^{-1}\left(\frac{\lambda(z)-\frac{is_{n,t}}{4}}{\lambda(z)+\frac{is_{n,t}}{4}}\right)^{\frac{\alpha}{2}\sigma_3}$ has no singularities at $\pm i\sqrt{-t}$. No other component of E has discontinuities. It follows that E is analytic on U_0.

Proposition 3.5.1. *Let $P(z)$ be defined as in (3.131). Then, P satisfies the RH problem for P.*

Proof. Conditions (a) and (b) are satisfied by construction. We verify (c). For $z \in \partial U_0$ we have

$$P(z)N^{-1}(z) = E(z)\Psi^{(1)}(\lambda(z); s_{n,t})W(z)N^{-1}(z). \tag{3.135}$$

It follows from (3.87) that

$$\Psi^{(1)}(\lambda(z); s_{n,t}) \left(\frac{\lambda(z) - \frac{is_{n,t}}{4}}{\lambda(z) + \frac{is_{n,t}}{4}}\right)^{\frac{\alpha}{2}\sigma_3} e^{-i\lambda(z)\sigma_3} e^{\frac{\pi i\alpha}{2}\sigma_3} = I + \mathcal{O}(n^{-1}), \tag{3.136}$$

as $n \to \infty$, where $\mathcal{O}(n^{-1})$ is uniform for $s_{n,t}$ in compact subsets of $(0, \infty)$ and $z \in \partial U_0$ and branches are chosen as in (3.134). By (3.132), we find that

$$P(z)N^{-1}(z) = E(z)(I + \mathcal{O}(n^{-1}))E^{-1}(z). \tag{3.137}$$

The proposition follows if $E(z)$ is bounded uniformly on ∂U_0. By (3.133) it is clear that the only part of E which could diverge is $W^{-1}(z)e^{-i\lambda(z)\sigma_3}$, which contains $e^{n\phi(z)-i\lambda(z)}$. To deal with this we note that

$$e^{n\phi(z)-i\lambda(z)} = e^{-\frac{n}{2}(\phi(i\sqrt{-t})-\phi(-i\sqrt{-t}))}, \tag{3.138}$$

which has modulus 1 because of the symmetry $\overline{\phi(\bar{z})} = \phi(z)$. Substituting this into (3.133), one indeed finds that $E(z) = \mathcal{O}(1)$ uniformly on the boundary ∂U_0. $\square$

3.5.5 Small norm RH problem

Define $R(z)$ by

$$R(z) = \begin{cases} S(z)P(z)^{-1} & \text{for } z \in U_0, \\ S(z)P^{(a)}(z)^{-1} & \text{for } z \in U_a, \\ S(z)P^{(b)}(z)^{-1} & \text{for } z \in U_b, \\ S(z)N(z)^{-1} & \text{for } z \in \mathbb{C} \setminus (U_0 \cup U_a \cup U_b). \end{cases} \tag{3.139}$$

It follows from the conditions of the RH problems for P, $P^{(a)}$, $P^{(b)}$ and N that $R(z)$ satisfies

RH problem for R

(a) R is analytic on $\mathbb{C} \setminus \Gamma_R$ where

$$\Gamma_R = \partial U_a \cup \partial U_b \cup \partial U_0 \cup (\Gamma_S \setminus ((a,b) \cup U_0 \cup U_a \cup U_b)). \quad (3.140)$$

(b) R has the following jump relations on Γ_R:

$$R_+(z) = R_-(z)(I + \mathcal{O}(n^{-1}))$$

as $n \to \infty$, uniformly for $z \in \Gamma_R$, in the double scaling limit where $\tau_{n,t} \to \tau < 0$.

(c) $R(z)$ has the following asymptotics as $z \to \infty$:

$$R(z) = I + \mathcal{O}(z^{-1}).$$

We have that R solves a RH problem, normalized at infinity and with jump matrices that are uniformly close to the identity matrix. This type of RH problem is often called a small norm RH problem, and it implies that R itself is uniformly close to I: in the double scaling limit where $n \to \infty$ and $t \to 0$ in such a way that $\tau_{n,t} \to \tau < 0$, we have

$$R(z) = I + \mathcal{O}\left(n^{-1}\right), \quad (3.141)$$

where the error term $\mathcal{O}(n^{-1})$ is uniform for $z \in \mathbb{C} \setminus \Gamma_R$.

Remark 3.5.1. In fact, it can be shown that the estimate (3.141) holds not only if $n \to \infty$, $t \to 0$ in such a way that $\tau_{n,t} \to \tau < 0$, but also if $\tau_{n,t} \to 0$ and if $\tau_{n,t} \to -\infty$. To see this, we first recall from Proposition 3.1 in [19] that (3.136) holds uniformly for $s_{n,t} \in (0,\infty)$ (or $\tau_{n,t} < 0$) as $n \to \infty$, and it follows that the same holds for (3.137). Since E is uniformly bounded, this implies that condition (b) of the RH problem for R is uniform for $\tau_{n,t} < 0$ as $n \to \infty$, and thus (3.141) is too.

3.5.6 Proof of Theorem 3.1.2 for $t < 0$

If $\tau_{n,t} \to \tau < 0$, we have by (3.130) that $s_{n,t} \to s \in (0, +\infty)$. We have that $s_{n,t} = 4n\pi\psi_V(0)\sqrt{-t} + \mathcal{O}(n^{-1})$ as $n \to \infty$.

First, recall formula (3.107) which expresses the correlation kernel in terms of Y. Expressing Y in T by (3.117), we obtain

$$K_n(x,y) = |x^2 - t|^{\alpha/2} e^{n\left(g_+(x) + \frac{\ell - V(x)}{2}\right)} |y^2 - t|^{\alpha/2} e^{n\left(g_+(y) + \frac{\ell - V(y)}{2}\right)} \times \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{2\pi i (x-y)}.$$

Using the identity $g_+ - \frac{1}{2}V + \ell/2 = -\phi_+$ on (a,b) and substituting S into the equation, by (3.121), we obtain

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} -\frac{e^{n\phi_+(y)}}{|y^2-t|^{\alpha/2}} & \frac{|y^2-t|^{\alpha/2}}{e^{n\phi_+(y)}} \end{pmatrix} S_+^{-1}(y) \\ S_+(x) \begin{pmatrix} e^{-n\phi_+(x)}|x^2-t|^{\alpha/2} \\ e^{n\phi_+(x)}|x^2-t|^{-\alpha/2} \end{pmatrix}. \quad (3.142)$$

Now, we use the fact that $S = RP$ for z close to the origin. Substituting this and the definition (3.131) of P, we obtain for $x, y \in U_0$:

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\Psi_+^{(1)}\right)^{-1} (\lambda(y); s_{n,t}) E_+^{-1}(y) \\ \times R_+^{-1}(y) R_+(x) E_+(x) \Psi_+^{(1)}(\lambda(x); s_{n,t}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Up to this point, no approximations have been made. Now, we use the fact that $R = I + \mathcal{O}(n^{-1})$ in the double scaling limit and the analyticity of E and R, which imply that

$$E_+^{-1}(y) R_+^{-1}(y) R_+(x) E_+(x) = I + \mathcal{O}(x-y),$$

as $x \to y$ for n sufficiently large. We find, for fixed u, v, that

$$\frac{1}{\psi_V(0)n} K_{n,t}\left(x_n = \frac{u}{n\psi_V(0)}, y_n = \frac{v}{n\psi_V(0)}\right) = \frac{1}{2\pi i(u-v)} \\ \times \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\Psi_+^{(1)}\right)^{-1} (\lambda(y_n); s_{n,t}) \Psi_+^{(1)}(\lambda(x_n); s_{n,t}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mathcal{O}(n^{-1}),$$

as $n \to \infty$, $t \to 0$. By (3.47) and (3.85) we have

$$\begin{pmatrix} \Phi_1(\lambda;\tau) \\ \Phi_2(\lambda;\tau) \end{pmatrix} = e^{-\frac{s}{4}\sigma_3} \Psi^{(1)}(\lambda; s = \sqrt{-\tau}) \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad (3.143)$$

and it follows from (3.130) that

$$\frac{1}{n\psi_V(0)} K_{n,t}\left(x_n = \frac{u}{n\psi_V(0)}, y_n = \frac{v}{n\psi_V(0)}\right) \\ = \mathbb{K}_\alpha^{\mathrm{PV}}(\lambda(x_n), \lambda(y_n); -s_{n,t}^2) + \mathcal{O}(n^{-1}), \quad (3.144)$$

as $n \to \infty$, $t \to 0$; with $\mathbb{K}_\alpha^{\mathrm{PV}}$ given in (3.28). In the case where $\tau_{n,t} \to \tau < 0$,

$$\lambda(x_n) = \pi u + \mathcal{O}(n^{-1}), \quad -s_{n,t}^2 = \tau_{n,t} + \mathcal{O}(n^{-1}) \quad (3.145)$$

which proves (3.27) by continuity of the kernel. If $\tau_{n,t} \to -\infty$, the error terms in (3.145) become of order $\mathcal{O}(n|t|^{3/2})$. Nevertheless we can apply (3.29) (which holds whenever $u-v$ has a limit as $\tau \to -\infty$) in (3.144) to obtain

$$\lim_{s\to\infty} \mathbb{K}_\alpha^{\mathrm{PV}}(\lambda(x_n), \lambda(y_n); -s_{n,t}^2) = \mathbb{K}^{\sin}(\lambda(x_n), \lambda(y_n)) + \mathcal{O}(s_{n,t}^{-1}),$$

as $n \to \infty, s_{n,t} \to \infty$. This yields (3.25), since $\lambda(x_n) - \lambda(y_n) = \pi(u - v) + \mathcal{O}(n^{-1})$ as $n \to \infty$, uniformly in $t < t_0$ for some sufficiently small $t_0 > 0$. Finally, if $\tau_{n,t} \to 0$, we substitute (3.30) into (3.144) and obtain (3.26). This completes the proof of Theorem 3.1.2.

3.5.7 Proof of Theorem 3.1.1 for $t < 0$

We start from the differential identity (3.108). Following the transformations $Y \mapsto T \mapsto S \mapsto R$ given in (3.117), (3.121) and (3.139), and recalling the definition (3.131) of the local parametrix, we find the following form for $Y^{-1}Y'$. Write $j = 1$ for $\operatorname{Im} z > 0$ and $j = 2$ for $\operatorname{Im} z < 0$. Then

$$\begin{aligned}\left(Y^{-1}Y'\right)_{2,2}(z) = \left(B(z) + W^{-1}(z)W'(z) + e^{-ng(z)\sigma_3}\left(e^{ng(z)\sigma_3}\right)'\right)_{2,2} \\ + \left(((\Psi^{(1)})^{-1}(\Psi^{(1)})'_z)(\lambda(z))\right)_{j,j}, \quad (3.146)\end{aligned}$$

where $'$ in general means the derivative with respect to the main argument, $'_z$ means the derivative with respect to z , and where

$$B(z) = (\Psi^{(1)}(\lambda))^{-1}(RE)^{-1}(z)(RE)'(z)\Psi^{(1)}(\lambda).$$

We will show that $B(z)$ is bounded uniformly for $s \in (0, \infty)$ and n large, but we will first calculate the other terms in the above expression. We have

$$\begin{aligned}&\left((W^{-1}W')(z) + e^{-ng(z)\sigma_3}\left(e^{ng(z)\sigma_3}\right)'\right)_{2,2} \\ &\qquad = -\frac{n}{2}V'(z) + \frac{\alpha/2}{z-z_0} + \frac{\alpha/2}{z+z_0},\end{aligned}$$

where we used

$$\phi'_\pm(x) + g'_\pm(x) = \frac{V'}{2}. \qquad (3.147)$$

By condition (d) of the RH problem for Ψ^- and (3.85), it follows that

$$((\Psi^{(1)})^{-1}(\Psi^{(1)})'_z)_{1,1}\,(\lambda(z)) = -\frac{2i\lambda'(z_0)}{s_{n,t}}(H^{-1}H')_{1,1}(1) - \frac{\alpha/2}{z-z_0} + \mathcal{O}(1) \quad \text{as } z \to z_0, \tag{3.148}$$

and

$$((\Psi^{(1)})^{-1}(\Psi^{(1)})'_z)_{2,2}\,(\lambda(z)) = -\frac{2i\lambda'(-z_0)}{s_{n,t}}(G^{-1}G')_{2,2}(0) - \frac{\alpha/2}{z+z_0} + \mathcal{O}(1) \quad \text{as } z \to -z_0, \tag{3.149}$$

where both $\mathcal{O}(1)$ terms are uniform for large n and sufficiently small $|t|$. Now, we use the following lemma, which we will prove below.

Lemma 3.5.2. (1) The following identity holds:

$$(G^{-1}G')_{2,2}(0) = -(H^{-1}H')_{1,1}(1) = \frac{s}{2} + \frac{\sigma_\alpha^-(s)}{\alpha} - \frac{\alpha}{2}.$$

(2) $B(z)$ is bounded uniformly as $n \to \infty$ for $t < t_0$ for some sufficiently small t_0.

Equations (3.146)-(3.149) and Lemma 3.5.2 can be substituted into (3.108) to obtain

$$\frac{d}{dt}\log \widehat{Z}_n(t,\alpha,V) = -\frac{\alpha}{2z_0}\left(\frac{4i\lambda'(z_0)}{s_{n,t}}\left(\frac{s_{n,t}}{2} + \frac{\sigma_\alpha^-(s_{n,t})}{\alpha} - \frac{\alpha}{2}\right) + \frac{\alpha}{2z_0} + \frac{n}{2}\left(V'(-z_0) - V'(z_0)\right) + B(z_0) - B(-z_0) + \mathcal{O}(1)\right) \tag{3.150}$$

as $n \to \infty$ and for sufficiently small $|t|$. After a straightforward calculation in which one uses (3.11) and (3.130), Theorem 3.1.1 follows upon integration.

Proof of Lemma 3.5.2. Equation (3.46) implies that

$$(H^{-1}H')_{1,1}(1) = -(G^{-1}G')_{2,2}(0).$$

We proceed to evaluate $(G^{-1}G')_{2,2}(0)$. Substituting

$$\Psi(\zeta) = G(\zeta)\zeta^{\frac{\alpha}{2}\sigma_3}, \quad \text{for } \zeta \text{ in a neighborhood of } 0,$$

into (3.49), (3.51) and evaluating the terms of order z^{-1}, we find that

$$(G\sigma_3 G^{-1})_{2,2}(0) = \frac{2v}{\alpha} - 1. \tag{3.151}$$

Writing

$$G(z) = G_0(1 + G_1 z + \mathcal{O}(z^2)), \qquad z \to 0,$$

we also have

$$(G^{-1}G')(0) = G_1, \tag{3.152}$$

and, by (3.52),

$$\frac{\partial}{\partial s} G_0 = B_0 G_0, \tag{3.153}$$

$$\frac{\partial}{\partial s} G_1 = G_0^{-1} B_1 G_0 = -\frac{1}{2} G_0^{-1} \sigma_3 G_0. \tag{3.154}$$

By (3.151), (3.152) and (3.154), we obtain

$$\frac{\partial}{\partial s}(G^{-1}G')_{2,2}(0) = -\frac{v}{\alpha} + \frac{1}{2}. \tag{3.155}$$

By analyzing at the behavior of Ψ as $s \to +\infty$ in Section 3.2.3, one finds that as $s \to +\infty$,

$$(G^{-1}G')_{2,2}(0) = -\frac{\alpha}{2} + \frac{s}{2} + o(1). \tag{3.156}$$

Integrating (3.155) using (3.62) and comparing (3.156) to (3.11), one finds that

$$(G^{-1}G')_{2,2}(0) = \frac{s}{2} + \frac{\sigma(s)}{\alpha} - \frac{\alpha}{2}.$$

It remains to show that $B(z)$ is bounded. It follows from (3.141) and Remark 3.5.1 that

$$R^{-1}(z)R'(z) = \mathcal{O}(n^{-1}) \quad \text{as } n \to \infty, t \to 0.$$

We will show that E and $E^{-1}E'$ are bounded uniformly as $n \to \infty$ and $t \to 0$.

For z in a neighborhood of z_0, we write

$$
\begin{aligned}
E(z) &= \prod_{j=1}^{4} f_j(z),\\
f_1(z) &= D^{-1}(\infty)N_0(z),\\
f_2(z) &= D(z)\sigma_1\sigma_3(z+z_0)^{-\frac{\alpha}{2}\sigma_3}\left(\lambda(z)+\frac{is_{n,t}}{4}\right)^{-\frac{\alpha}{2}\sigma_3} n^{\frac{\alpha}{2}\sigma_3},\\
f_3(z) &= (z-z_0)^{-\frac{\alpha}{2}\sigma_3}\left(\lambda-\frac{is_{n,t}}{4}\right)^{\frac{\alpha}{2}\sigma_3} n^{-\frac{\alpha}{2}\sigma_3},\\
f_4(z) &= e^{n\phi(z)\sigma_3}e^{-i\lambda(z)\sigma_3}e^{\frac{\pi i\alpha}{2}\sigma_3}.
\end{aligned}
$$

Each $f_j, j=1,\ldots,4$ is uniformly bounded at $\pm z_0$ as $n\to\infty$ and $t\to 0$. This is clear for f_1 and f_3. To see this for f_2, one can perform a contour integral to find that

$$
D(z_0) = \begin{pmatrix} \mathcal{O}\left(z_0^{-\alpha}\right) & 0 \\ 0 & \mathcal{O}\left(z_0^{\alpha}\right) \end{pmatrix}.
$$

For f_4, it follows from the definition of λ and the fact that $\phi(z)=\overline{\phi(\overline{z})}$. One can do the same for z in a neighborhood of $-z_0$. Thus

$$
B(\pm z_0) = \lim_{z\to\pm z_0} (\Psi^{(1)})^{-1}(\lambda(z);s_{n,t})\mathcal{O}(1)\Psi^{(1)}(\lambda(z);s_{n,t}),
$$

which is bounded as long as s lies in compact subsets of $(0,+\infty)$. To see that $B(\pm z_0)$ is bounded as well as $s\to\infty$, we substitute in $\widehat{\Psi}$ from (3.83) and use the fact that it converges to the identity. To see that $B(\pm z_0)$ is bounded as $s\to 0$ we refer to the small x behavior of Ψ in [19] (formulas (4.23), (4.53), (4.61), (4.64), (4.67)).

□

3.6 Asymptotic analysis of T as $n\to\infty$ for $t>0$

In this section, we will analyse asymptotically the RH problem for T in the case where $t>0$, which means that the singularities $\pm\sqrt{t}$ are real and approach the origin as $t\to 0$. Many of the notations here will be the same as the ones in Section 3.5 to emphasize the parallels in the analysis. We will refer to many arguments given in the previous section, so it is recommended to read Section 3.5 before this one.

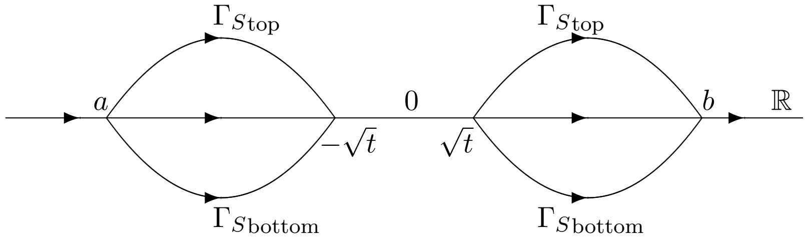

Figure 3.8: The contour Γ_S.

3.6.1 Opening of the lens for $t > 0$

We open the lens as in Figure 3.8. We define S in the same way as for $t < 0$ in (3.121), except that we use the contour pictured in Figure 3.8 instead of the one in Figure 3.7, and except for the fact that we define $(z^2 - t)^\alpha$ such that it is positive on $(a, -\sqrt{t})$ and $(\sqrt{t}, b)$:

$$(z^2 - t)^\alpha = |z^2 - t|^\alpha e^{i\alpha \arg(z-\sqrt{t})} e^{i\alpha \arg(z+\sqrt{t})},$$

with the conventions $-3\pi/2 < \arg(z - \sqrt{t}) < \pi/2$ and $-\pi/2 < \arg(z + \sqrt{t}) < 3\pi/2$.

The RH problem for S is the following.

RH problem for S

(a) S is analytic for $z \in \mathbb{C} \setminus \Gamma_S$ where $\Gamma_S = \Gamma_{S\text{top}} \cup \Gamma_{S\text{bottom}} \cup \mathbb{R}$ is the union of the lens and the real line, see Figure 3.8.

(b) S has the following jump relations on Γ_S,

$$S_+(x) = S_-(x) \begin{pmatrix} 1 & |x^2 - t|^\alpha e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix} \quad \text{for } x < a \text{ and } b < x,$$

$$S_+(x) = S_-(x) \begin{pmatrix} 0 & |x^2 - t|^\alpha \\ -\frac{1}{|x^2-t|^\alpha} & 0 \end{pmatrix} \quad \text{for } x \in (a, -\sqrt{t}) \cup (\sqrt{t}, b),$$

$$S_+(z) = S_-(x) \begin{pmatrix} 1 & 0 \\ \frac{e^{2n\phi(z)}}{(z^2-t)^\alpha} & 1 \end{pmatrix} \quad \text{for } z \in \Gamma_{S\text{top}} \cup \Gamma_{S\text{bottom}},$$

$$S_+(x) = S_-(x) \begin{pmatrix} e^{2n\phi_+(x)} & |x^2 - t|^\alpha \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} \quad \text{for } x \in [-\sqrt{t}, \sqrt{t}].$$

(c) $S(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.

(d) For $\alpha < 0$, S has the following asymptotics as $z \to \pm\sqrt{t}$:

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|z \mp \sqrt{t}|^{\alpha}) \\ \mathcal{O}(1) & \mathcal{O}(|z \mp \sqrt{t}|^{\alpha}) \end{pmatrix}.$$

For $\alpha \geq 0$ the following asymptotics hold:

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \qquad \text{as } z \to \pm\sqrt{t} \text{ from outside the lense,}$$

$$S(z) = \begin{pmatrix} \mathcal{O}(|z \mp \sqrt{t}|^{-\alpha}) & \mathcal{O}(1) \\ \mathcal{O}(|z \mp \sqrt{t}|^{-\alpha}) & \mathcal{O}(1) \end{pmatrix} \qquad \text{as } z \to \pm\sqrt{t} \text{ from inside the lense.}$$

(e) $S(z)$ is bounded as $z \to a$ and as $z \to b$.

3.6.2 Global parametrix

As for $t < 0$, we now ignore small neighborhoods of $a, b, 0$ and exponentially small jumps.

The solution to the RH problem obtained in this way is constructed as for $t < 0$: N is given by (3.122)-(3.125), where the parameter t is positive in (3.125).

The function N now has singularities at $\pm\sqrt{t}$. Applying a contour deformation argument on the integral in (3.125), we obtain the following behavior,

$$N(z) = \mathcal{O}(|z \pm \sqrt{t}|^{-\frac{\alpha}{2}\sigma_3}), \qquad z \to \mp\sqrt{t}. \tag{3.157}$$

3.6.3 Local parametrices at endpoints a and b

The local parametrices at the endpoints a and b remain unchanged with respect to Section 3.5.3. They satisfy the same jumps as S near a and b, and as $n \to \infty$, they match with the global parametrix N on the boundary of fixed disks around a and b.

3.6.4 Local parametrix at the origin

Let U_0 be a sufficiently small disk around the origin which is independent of n. We will construct a function P which satisfies the following conditions.

RH problem for P

(a) $P : U_0 \setminus \Gamma_S \to \mathbb{C}^{2\times 2}$ is analytic.

(b) P has same jump relations as S:

$$P_-^{-1}(z)P_+(z) = S_-^{-1}(z)S_+(z), \quad \text{for } z \in U_0 \cap \Gamma_S. \tag{3.158}$$

(c) As $n \to \infty$ and simultaneously $t \to 0$ in such a way that $n^2 t$ tends to a non-zero constant, we have

$$P(z) = \left(I + \mathcal{O}\left(n^{-1}\right)\right) N(z),$$

uniformly for z on the boundary ∂U_0.

These RH conditions are exactly the same as in the case $t < 0$, but it has to be noted that the function S and its jump contour are different here than in the case $t < 0$. Therefore the construction of the local parametrix differs from the construction done before.

In analogy with the case $t < 0$, define

$$\lambda(z) = -in \times \begin{cases} \phi(z) - \frac{\phi_+(\sqrt{t})+\phi_+(-\sqrt{t})}{2} & \text{for } \operatorname{Im} z > 0, \\ -\phi(z) - \frac{\phi_+(\sqrt{t})+\phi_+(-\sqrt{t})}{2} & \text{for } \operatorname{Im} z < 0. \end{cases} \tag{3.159}$$

By (3.119), λ is a conformal map in U_0,

$$\lambda(\sqrt{t}) = -\lambda(-\sqrt{t}) = -\frac{in}{2}(\phi_+(\sqrt{t}) - \phi_+(-\sqrt{t})),$$

and

$$\lambda'(0) = n\pi\psi_V(0).$$

We have that $\lambda(z)$ is a conformal map which sends $\sqrt{t}$ to $|s_{n,t}|/4$ and $-\sqrt{t}$ to $-|s_{n,t}|/4$, with

$$s_{n,t} = -4i\lambda(\sqrt{t}) = -2n(\phi_+(\sqrt{t}) - \phi_+(-\sqrt{t})). \tag{3.160}$$

By (3.119), we have

$$\begin{aligned} s_{n,t} &= -2\pi in \int_{-\sqrt{t}}^{\sqrt{t}} \psi_V(s)ds \\ &= -4in\pi\sqrt{t}\psi_V(0) + \mathcal{O}(nt^{3/2}) = -i\sqrt{\tau_{n,t}} + \mathcal{O}(nt^{3/2}), \end{aligned} \tag{3.161}$$

as $t \to 0$, $n \to \infty$, similar to the case $t < 0$.

Recall the definition of $\Psi^{(2)}$ in (3.102). We now fix the lens by requiring that it is mapped to the jump contours $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$, $\widetilde{\Gamma}_3$, and $\widetilde{\Gamma}_4$ of $\Psi^{(2)}$ by λ. We search for P in the form

$$P(z) = E(z)\Psi^{(2)}(\lambda(z); s_{n,t})W(z), \tag{3.162}$$

where

$$W(z) = \begin{cases} \left((z^2-t)^{\alpha/2} e^{-n\phi(z)}\right)^{\sigma_3}\sigma_3\sigma_1 & \text{for } \operatorname{Im} z > 0,\ \operatorname{Re}\lambda(z) \notin \left[-\frac{|s_{n,t}|}{4}, \frac{|s_{n,t}|}{4}\right], \\ \left((z^2-t)^{-\alpha/2} e^{n\phi(z)}\right)^{\sigma_3} & \text{for } \operatorname{Im} z < 0,\ \operatorname{Re}\lambda(z) \notin \left[-\frac{|s_{n,t}|}{4}, \frac{|s_{n,t}|}{4}\right], \\ \left((z^2-t)^{\alpha/2} e^{\frac{\pi i\alpha}{2}} e^{-n\phi(z)}\right)^{\sigma_3}\sigma_3\sigma_1 & \text{for } \operatorname{Im} z > 0,\ \operatorname{Re}\lambda(z) \in \left[-\frac{|s_{n,t}|}{4}, \frac{|s_{n,t}|}{4}\right], \\ \left((z^2-t)^{-\alpha/2} e^{-\frac{\pi i\alpha}{2}} e^{n\phi(z)}\right)^{\sigma_3} & \text{for } \operatorname{Im} z < 0,\ \operatorname{Re}\lambda(z) \in \left[-\frac{|s_{n,t}|}{4}, \frac{|s_{n,t}|}{4}\right]. \end{cases}$$

The above construction was done in such a way that W induces the correct jump relations for P. E is an analytic function which has to be such that the matching condition (c) of the RH problem for P is valid for z on the boundary of U_0. Therefore, we recall the definition of χ in (3.103) and let

$$E(z) = N(z)W^{-1}(z)e^{-i\lambda(z)\sigma_3}\chi(\lambda(z))^{-1}.$$

Proposition 3.6.1. *$P(z)$, defined as in* (3.162)*, satisfies the RH problem for P.*

Proof. Condition (a) and condition (b) hold by construction. We proceed to prove (c). By the definition of $E(z)$, we have

$$P(z)N^{-1}(z) = E(z)\Psi^{(2)}(\lambda(z); s_{n,t})e^{-i\lambda(z)\sigma_3}\chi(\lambda(z))^{-1}E^{-1}(z).$$

For $z \in \partial U_0$, it follows from (3.104) that

$$\Psi^{(2)}\Big(\lambda(z); s_{n,t}\Big) = (I + \mathcal{O}(n^{-1}))\chi(\lambda(z))e^{i\lambda(z)\sigma_3}$$

as $n \to \infty$, where the $\mathcal{O}(n^{-1})$ is uniform for $s_{n,t}$ in compact subsets of $(0, -i\infty)$ and $\lambda(z)$ sufficiently large. Using the same calculations as in the proof of Proposition 3.5.1 we find that E is bounded on ∂U_0 and so the result follows in the same manner as in the proof of Proposition 3.5.1. ☐

3.6.5 Small norm RH problem

A small norm RH problem can be constructed in a similar way as in Section 3.5.5: define

$$R(z) = \begin{cases} S(z)P(z)^{-1} & \text{for } z \in U_0, \\ S(z)P^{(a)}(z)^{-1} & \text{for } z \in U_a, \\ S(z)P^{(b)}(z)^{-1} & \text{for } z \in U_b, \\ S(z)N(z)^{-1} & \text{for } z \in \mathbb{C} \setminus (U_0 \cup U_a \cup U_b). \end{cases}$$

Then R satisfies the following RH problem, similar to the case where $\tau < 0$.

RH problem for R

(a) R is analytic on $\mathbb{C} \setminus \Gamma_R$ where

$$\Gamma_R = \partial U_a \cup \partial U_b \cup \partial U_0 \cup (\Gamma_S \setminus ((a,b) \cup U_0 \cup U_a \cup U_b)). \quad (3.163)$$

(b) R has the following jump relations on Γ_R:

$$R_+(z) = R_-(z)(I + \mathcal{O}(n^{-1}))$$

as $n \to \infty$, uniformly for $z \in \Gamma_R$, in the double scaling limit where $\tau_{n,t} \to \tau > 0$.

(c) $R(z)$ has the following asymptotics as $z \to \infty$:

$$R(z) = I + \mathcal{O}(z^{-1}).$$

In the double scaling limit where $n \to \infty$ and at the same time $t \to 0$ in such a way that $n^2 t$ tends to a non-zero constant, it follows that

$$R(z) = I + \mathcal{O}(n^{-1}), \quad (3.164)$$

uniformly for z off the jump contour for R. As for $t < 0$, one can extend this result to the cases where $n \to \infty$ and $t \to 0$ in such a way that $n^2 t \to \infty$ and in such a way that $n^2 t \to 0$. We do not give the details here, as the arguments are very similar to those in [20]: see equations (5.1), (5.17), (5.18), (5.25) in [20] for $n^2 t \to \infty$ and equations (6.1), (6.28), (6.32), and (4.6) in [20] for $n^2 t \to 0$.

3.6.6 Proof of Theorem 3.1.2

In the same way as when $t < 0$, we can find an expression for the correlation kernel in terms of S when $t > 0$:

$$\begin{aligned} K_n(x,y) = \frac{1}{2\pi i(x-y)} &\begin{pmatrix} -\frac{e^{n\phi_+(y)}}{|y^2-t|^{\alpha/2}}\delta(y) & \frac{|y^2-t|^{\alpha/2}}{e^{n\phi_+(y)}} \end{pmatrix} \\ &\times S_+^{-1}(y) S_+(x) \begin{pmatrix} |x^2-t|^{\alpha/2} e^{-n\phi_+(x)} \\ e^{n\phi_+(x)}|x^2-t|^{-\alpha/2}\delta(x) \end{pmatrix} \end{aligned} \quad (3.165)$$

where

$$\delta(x) = \begin{cases} 0 & \text{for } x \in (-\sqrt{t}, \sqrt{t}), \\ 1 & \text{for } x \in (a, -\sqrt{t}) \cup (\sqrt{t}, b). \end{cases} \quad (3.166)$$

Substituting the small norm RH solution R into the formula for the correlation kernel in (3.165), one obtains:

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} -\frac{e^{n\phi_+(y)}}{|y^2-t|^{\alpha/2}}\delta(y) & \frac{|y^2-t|^{\alpha/2}}{e^{n\phi_+(y)}} \end{pmatrix}$$
$$\times W_+^{-1}(y)(\Psi_+^{(2)})^{-1}(\lambda(y);s_{n,t})E_+^{-1}(y)R_+(y)$$
$$\times R_+^{-1}(x)E(x)\Psi_+^{(2)}(\lambda(x);s_{n,t})W(x)\begin{pmatrix} e^{-n\phi_+(x)}|x^2-t|^{\alpha/2} \\ e^{n\phi_+(x)}|x^2-t|^{-\alpha/2}\delta(x) \end{pmatrix}.$$

From the asymptotic behavior (3.164) and the analyticity of R and y, we have

$$E_+^{-1}(y)R_+(y)R_+(x)E(x) = I + \mathcal{O}(x-y),$$

as $x \to y$ for n sufficiently large. Combining this with the definition for W, one finds that

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 1 & \delta(y) \end{pmatrix} (\Psi_+^{(2)})^{-1}(\lambda(y);s_{n,t})$$
$$\times (I+\mathcal{O}(x-y))\Psi_+^{(2)}(\lambda(x);s_{n,t})\begin{pmatrix} \delta(x) \\ -1 \end{pmatrix}.$$

Now, we can use (3.97) and (3.102) to express $\Psi^{(2)}$ in terms of the functions Φ_1 and Φ_2. We have the relation

$$\begin{pmatrix} \Phi_1(u;\tau) \\ \Phi_2(u;\tau) \end{pmatrix} = e^{-\frac{s}{4}\sigma_3} \begin{cases} \Psi_+^{(2)}(u;-i\sqrt{\tau})\begin{pmatrix} 0 \\ -1 \end{pmatrix}, & -|s|/4 < u < |s|/4, \\ \Psi^{(2)}(u;-i\sqrt{\tau})\begin{pmatrix} 1 \\ -1 \end{pmatrix}, & u \in \mathbb{R}\setminus[-|s|/4,|s|/4], \end{cases}$$

and after a similar calculation to the one in Section 3.5.6, using the fact that $\lambda'(0) = \pi n\psi_V(0)$, one finds that Theorem 3.1.2 holds for $\tau_{n,t} \in (0,\infty)$.

3.6.7 Proof of Theorem 3.1.1 for $t > 0$

We assume here that $\alpha > 0$. The formulas (3.146)-(3.147) which we obtained in the proof of Theorem 3.1.1 for $t < 0$, hold also in the case $t > 0$, if we replace $\Psi^{(1)}$ by $\Psi^{(2)}$. We obtain from (3.102) and condition d) in the RH problem for Ψ^+ that as $z \to \sqrt{t}$,

$$\left((\Psi^{(2)})^{-1}(\Psi^{(2)})'_z\right)_{2,2}(\lambda(z)) = -\frac{\alpha/2}{z-\sqrt{t}} + \frac{2\lambda'(z)}{|s|}\left(H^{-1}H'\right)_{2,2}(1) + \mathcal{O}(1),$$

where the error term is uniform for large n and sufficiently small t. Likewise as $z \to -\sqrt{t}$,

$$\left((\Psi^{(2)})^{-1}(\Psi^{(2)})'_z\right)_{2,2}(\lambda(z)) = -\frac{\alpha/2}{z+\sqrt{t}} + \frac{2\lambda'(z)}{|s|}\left(G^{-1}G'\right)_{2,2}(0) + \mathcal{O}(1),$$

where the error term is uniform for large n and sufficiently small t. Thus, by (3.108), we obtain

$$\begin{aligned}
&\frac{d}{dt}\log \widehat{Z}_n(t,\alpha,V) \\
&= -\frac{\alpha}{2z_0}\Bigg(\frac{2i\lambda'(z_0)}{s_{n,t}}(G^{-1}G')_{2,2}(0) - \frac{2i\lambda'(-z_0)}{s_{n,t}}(H^{-1}H')_{2,2}(1) \\
&\qquad + \frac{\alpha}{2z_0} + \frac{n}{2}\left(V'(-z_0) - V'(z_0)\right) + B(z_0) - B(-z_0) + \mathcal{O}(1)\Bigg)
\end{aligned} \tag{3.167}$$

as $n \to \infty$ and sufficiently small t. As in the case $t < 0$, we have

$$\left(G^{-1}G'\right)'_{2,2}(0) = -\left(H^{-1}H'\right)'_{2,2}(1) = \frac{\sigma'(s)}{\alpha} + \frac{1}{2},$$

and we can use the large s asymptotics given in (5.1) in [20] to find that

$$\left(G^{-1}G'\right)_{2,2}(0) = -\left(H^{-1}H'\right)_{2,2}(1) = \frac{\sigma(s)}{\alpha} + \frac{s}{2} - \frac{\alpha}{2}.$$

In the same manner as for $t < 0$, one can show that

$$B(\pm z_0) = \lim_{z\to\pm z_0}(\Psi^{(2)})^{-1}(\lambda(z);s_{n,t})\mathcal{O}(1)\Psi^{(2)}(\lambda(z);s_{n,t}),$$

which is uniformly bounded for n large and $s \in (0, -i\infty)$. To show this for $s \to -i\infty$, one uses equations (5.10) and (5.15) from [20]; while to show this for $s \to 0$, one uses equation (6.10), (6.19) and (6.23) from [20]. Substituting the formula for G and H into (3.167) with B bounded yields Theorem 3.1.1 for $t, \alpha > 0$ after integration.

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