On the representability of actions of non-associative algebras

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Joint work with Jose Brox (*Universidad de Valladolid*), Xabier García Martínez (*Universidade de Vigo*), Tim Van der Linden and Corentin Vienne (*UCLouvain*).

- Let B, X be groups. An action of B on X is a map

$$\xi\colon B \times X \to X\colon (b,x) \mapsto b \cdot x$$

such that $(bb') \cdot x = b \cdot (b' \cdot x)$, $1_B \cdot x = x$ and $b \cdot (xx') = (b \cdot x)(b \cdot x')$.

- The *semi-direct product* $B \ltimes X$ w.r.t. ξ is $B \times X$ with multiplication

$$(b,x)(b',x') = (bb', x(b \cdot x')),$$

where $1_{B \ltimes X} = (1_B, 1_X)$ and $(b,x)^{-1} = (b^{-1}, b^{-1} \cdot x^{-1})$.

- We can associate with ξ the split extension

$$\begin{tikzcd} 1 \arrow[r] & X \arrow[r, "i_2"] & B \ltimes X \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "i_1"] \arrow[r] & 1. \end{tikzcd}$$

- Conversely, given

$$1 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 1$$

then we can define an action

$$B \times X \to X \colon (b, x) \mapsto \beta(b)k(x)\beta(b)^{-1}.$$

- $\mathsf{Act}(-, X) \cong \mathsf{SplExt}(-, X)$.
- In **Grp**, actions are represented by automorphisms, i.e.

$$\mathsf{Act}(-, X) \cong \mathsf{Hom}_{\mathbf{Grp}}(-, \mathsf{Aut}(X)).$$

The semi-abelian context

- Let $\mathcal{C}$ be a semi-abelian category (i.e. $\mathcal{C}$ is pointed, Barr-exact, Bourn-protomodular with finite coproducts).
- Let B, X be objects of $\mathcal{C}$. A *split extension* of B by X is a diagram

$$0 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$

  in $\mathcal{C}$ such that $\alpha \circ \beta = 1_B$ and (X, k) is a kernel of α.
- For any object X in $\mathcal{C}$, we consider the functor

$$\mathrm{SplExt}(-, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

  where $\mathrm{SplExt}(B, X)$ is the set of isomorphism classes of split extensions of B by X.
- We have a natural isomorphism $\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X)$.

Action representable categories

Definition (F. Borceux, G. Janelidze and G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ is representable. This means that there exists an object $[X]$ of $\mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]).$$

- Prototype examples: **Grp**, **Lie** and any abelian category.
- In **Grp**, the actor of X is $\mathrm{Aut}(X)$.
- In **Lie**, the actor of X is $\mathrm{Der}(X)$.
- In any abelian category, the actor of X is 0.

- The notion of action representable category has proven to be quite restrictive.

Theorem (X. García Martínez, M. Tsishyn, T. Van der Linden and C. Vienne, 2021)

Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2$ and let $\mathcal{V}$ be a variety of non-associative algebras over $\mathbb{F}$. If $\mathcal{V}$ is action representable, then $\mathcal{V} = \mathbf{AbAlg}$ or $\mathcal{V} = \mathbf{Lie}$.

Weakly action representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every object X in $\mathcal{C}$, the functor $\mathsf{SplExt}(-, X)$ admits a weak representation. This means that there exist an object T of $\mathcal{C}$ and a monomorphism of functors

$$\tau \colon \mathsf{SplExt}(-, X) \rightarrowtail \mathsf{Hom}_{\mathcal{C}}(-, T).$$

An object T as above is called *weak actor* of X, the pair (T, τ) is called *weak representation* of $\mathsf{SplExt}(-, X)$ and $(\varphi \colon B \to T) \in \mathsf{Im}(\tau_B)$ is called *acting morphism*.

Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is a natural isomorphism.

Associative algebras and Leibniz algebras

- **G. Janelidze, 2022.** The category **Assoc** is weakly action representable. A weak actor of X is the associative algebra of *bimultipliers* of X.

$$\mathrm{Bim}(X) = \{(f{*}-, -{*}f) \in \mathrm{End}(X) \times \mathrm{End}(X)^{\mathrm{op}} \mid f{*}(xy) = (f{*}x)y,$$
$$(xy) * f = x(y * f), x(f * y) = (x * f)y, \ \forall x, y \in X\}.$$

- **A. S. Cigoli, M. M., G. Metere, 2023.** The category **Leib** is weakly action representable. A weak actor of X is the Leibniz algebra of *biderivations* of X.

$$\mathrm{Bider}(X) = \{(d, D) \in \mathrm{End}(X)^2 \mid d(xy) = d(x)y + xd(y),$$
$$D(xy) = D(x)y - D(y)x, \ xd(y) = xD(y), \ \forall x, y \in X\}.$$

W.R.A. of non-associative algebras

- Let $\mathcal{V}$ be a variety of non-associative algebras over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$;
- We suppose that $\mathcal{V}$ is *operadic* and *action accessible*.
- This is equivalent to saying that the λ/μ *rules* hold, i.e. there exist $\lambda_1, \dots \lambda_8, \mu_1, \dots \mu_8 \in \mathbb{F}$ such that

$$x(yz) = \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) +$$

$$+ \lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx),$$

$$(yz)x = \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) +$$

$$+ \mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx)$$

are identities in $\mathcal{V}$.

- We fix a set of multilinear identities

$$\Phi_{k,i}(x_1, \dots, x_k) = 0, \quad k = \deg(\Phi_{k,i}),\ i = 1, \dots, n,$$

  which determine $\mathcal{V}$.
- Our goal is to find a vector space $\mathcal{E}(X) \leq \mathrm{End}(X)^2$ together with a bilinear *partial* operation

$$\langle -, - \rangle \colon \Omega \subseteq \mathcal{E}(X) \times \mathcal{E}(X) \to \mathcal{E}(X),$$

  such that we can associate in a natural way a morphism of algebras $B \to \mathcal{E}(X)$, with every split extension of B by X in $\mathcal{V}$.

Definition

A *partial algebra* is a vector space X endowed with a *bilinear partial operation* $\Omega \subseteq X \times X \to X$.
A homomorphism of partial algebras is a linear map $f \colon X \to Y$ such that $f(xy) = f(x)f(y)$, whenever xy and $f(x)f(y)$ are defined.

Split extensions

- Let

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

  be a split extension in $\mathcal{V}$.
- We have that

$$A \cong B \ltimes X = (B \oplus X, \cdot)$$

  with

$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + x * b').$$

- The action of B on X is defined by the pair of bilinear maps

$$l \colon B \times X \to X, \quad r \colon X \times B \to X,$$

  with $b * x' = l(b, x') = \beta(b) \cdot_A k(x')$, $x * b' = r(x, b') = k(x) \cdot_A \beta(b')$.

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ as above:

Proposition

Let X and B be two algebras in $\mathcal{V}$. Given a pair of bilinear maps

$$l\colon B \times X \to X, \quad r\colon X \times B \to X,$$

we construct $(B \oplus X, \cdot)$ as before. Then $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$ if and only if

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad \forall i = 1, \dots, n,$$

where at least one of the α_1, ..., α_k is an element of of the form $(0, x)$, with $x \in X$, and the others are of the form $(b, 0)$, with $b \in B$.

- The resulting algebra is the *semi-direct product* $B \ltimes X$.
- We denote $b * x := (b, 0) \cdot (0, x)$ and $x * b := (0, x) \cdot (b, 0)$.

The partial algebra $\mathcal{E}(X)$

We define

$$\mathcal{E}(X) = \{f = (f * -, - * f) \in \mathrm{End}(X)^2 \mid \Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0\}$$

for each choice of $\alpha_j = f$ and $\alpha_t \in X$, where $t \neq j$, $fx := f * x$, $xf := x * f$, together with a bilinear map

$$\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2 \colon (f, g) \mapsto h = (h * -, - * h)$$

where

$$\begin{aligned} x * h = &\lambda_1 (x * f) * g + \lambda_2 (f * x) * g + \lambda_3 g * (x * f) + \lambda_4 g * (f * x) \\ &+ \lambda_5 (x * g) * f + \lambda_6 (g * x) * f + \lambda_7 f * (x * g) + \lambda_8 f * (g * x) \end{aligned}$$

and

$$\begin{aligned} h * x = &\mu_1 (x * f) * g + \mu_2 (f * x) * g + \mu_3 g * (x * f) + \mu_4 g * (f * x) \\ &+ \mu_5 (x * g) * f + \mu_6 (g * x) * f + \mu_7 f * (x * g) + \mu_8 f * (g * x). \end{aligned}$$

Example

If $\mathcal{V} = \mathbf{Assoc}$, then $\mathcal{E}(X) = \mathrm{Bim}(X)$. If $\mathcal{V} = \mathbf{Leib}$, then $\mathcal{E}(X) \cong \mathrm{Bider}(X)$.

Remark

The bilinear map $\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$ defines a partial operation $\langle -, - \rangle \colon \Omega \to \mathcal{E}(X)$, where Ω is the preimage $\langle -, - \rangle^{-1}(\mathcal{E}(X))$ of the inclusion $\mathcal{E}(X) \hookrightarrow \mathrm{End}(X)^2$.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

Let $\mathcal{V}$ be a variety and let $U\colon \mathcal{V} \to \mathbf{PAlg}$ denote the forgetful functor.

- *Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors*

$$\tau\colon\ \mathsf{SplExt}(-, X) \rightarrowtail \mathsf{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X))$$

  where τ_B sends an element of $\mathsf{SplExt}(B, X)$ to $(b * -, - * b)$.
- *The morphism of partial algebras $U(B) \to \mathcal{E}(X)\colon\ b \mapsto (b * -, - * b)$ belongs to $\mathsf{Im}(\tau_B)$ if and only if $\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0$, as before.*
- *If $(\mathcal{E}(X), \langle -, - \rangle)$ is an object of $\mathcal{V}$, then $(\mathcal{E}(X), \tau)$ is a weak representation of $\mathsf{SplExt}(-, X)$.*

Definition

The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X. The pair $(\mathcal{E}(X), \tau)$ is called *external weak representation* of $\mathrm{SplExt}(-, X)$. When τ is a natural isomorphism, we say that $\mathcal{E}(X)$ is an *external actor* of X.

Example

We may check that, with the *obvious* choices of the λ/μ rules:

1. if $\mathcal{V} = \mathbf{AbAlg}$, then $\mathcal{E}(X) = 0$ is the actor of X;
2. if $\mathcal{V} = \mathbf{Lie}$, then $\mathcal{E}(X) \cong \mathrm{Der}(X)$ is the actor of X;
3. if $\mathcal{V} = \mathbf{CAssoc}$, then $\mathcal{E}(X) \cong \mathrm{M}(X)$ is an external actor of X;
4. if $\mathcal{V} = \mathbf{Nil}_2(\mathbf{CAlg})$, then $\mathcal{E}(X)$ is a weak actor of X;
5. if $\mathcal{V} = \mathbf{Alt}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and X is unitary, then $\mathcal{E}(X) \cong X$ is an alternative algebra and

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Alt}}(-, X)$$

What happens if $\mathcal{E}(X)$ is not an object of $\mathcal{V}$?

Remark

Let X be a commutative associative algebra. Then

$$\mathrm{SplExt}(-,X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(U(-),\mathrm{M}(X)),$$

where $\mathrm{M}(X) = \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y\}$ and $U\colon \mathbf{CAssoc} \to \mathbf{Assoc}$ denotes the forgetful functor.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

The category **CAssoc** *is weakly action representable.*

- **Key:** *Amalgamation property*.

Open problems and future directions

- Does the converse of the implication

  weakly action representable category $\Rightarrow$ *action accessible category*

  hold in the context of varieties of algebras over a field?
- To generalize the construction of the external weak actor to action accessible varieties of algebras with two not necessarily associative bilinear operations (**Pois**, **CPois**,...).
- To study the (weak) representability of actions in the context of *ideally exact categories* (**Ring**, **MVAlg**, every category of unitary algebras,...).

Moitas grazas pola vosa atención!

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