ON THE BOUNDARIES, ASYMPTOTIC LAW AND BERNOULLI-DOOB REPRESENTATION OF HOMOGENEOUS BOUNDED MARTINGALES

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LIDAM Discussion Paper LFIN
2026 / 03

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On the boundaries, asymptotic law and Bernoulli-Doob representation of homogeneous bounded martingales

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Abstract

We study homogeneous diffusion martingales evolving in a bounded state space $D = [a, b]$, where a and b are zeros of the diffusion coefficient. We call a process of the form $Z_t = \mathbb{E}[B \mid \mathcal{F}_t]$, with B a Bernoulli random variable, a *Bernoulli-Doob martingale*. Our main results establish a complete equivalence: every such diffusion martingale is a Bernoulli-Doob martingale (Theorem 2) and, conversely, every continuous time-homogeneous Markov Bernoulli-Doob martingale on a Brownian filtration arises from such a diffusion (Theorem 3). The intuitive reason is that a bounded martingale has constant expectation while accumulating variance, so it converges to the maximum-variance distribution with given mean and range, namely the Bernoulli. We further show that this Bernoulli limit is *truly asymptotic*: for any fixed finite horizon T, the probability of not yet having reached the boundary is strictly positive (Theorem 4), even when the individual boundaries are accessible. We clarify the relationship between Feller's boundary classification, the pathwise SDE framework, and the martingale constraint, showing that the martingale property forces absorption at any attainable boundary. The theory is illustrated with the Φ-martingale, the Jacobi martingale, and applications to credit-risk modelling.

Keywords: Bounded stochastic processes, martingales, asymptotic law, Bernoulli-Doob martingale, boundary classification, Feller theory

1. Introduction

Conic martingales, i.e., martingales evolving between real bounds [8], appear in many applications, but may exhibit very different behaviors. This is particularly striking when looking at some common processes in quantitative finance.

- When confronted to non-negative interest rates, the discounted price of a default-free zero-coupon bond (ZCB) paying one unit of currency at expiry T,

$$\tilde{P}_t(T) := P_t(T)/\beta_t \,,$$

is a non-negative martingale confined to $[0,1]$, i.e., a $[0,1]$-martingale, under the domestic pricing measure $\mathbb{P}$. In the case of a non-explosive and strictly positive short-rate model, the terminal value cannot reach the bounds because

$$\mathbb{P}\Big(0 < \tilde{P}_T(T) < 1\Big) = \mathbb{P}\Big(0 < e^{-\int_0^T r_s\,ds} < 1\Big) = 1\,, \qquad \forall\, T > 0\,.$$

In fact, $\tilde{P}(T)$ is typically confined in $(0,1)$ in most non-negative interest rate models.

- A defaultable ZCB with zero recovery behaves differently: although its discounted price remains a $[0,1]$-martingale, it hits the lower bound zero before maturity if the default time ξ of the issuer takes place before T, which is typically modeled as an event with positive probability for all $T > 0$.

- The survival probability process $S^{\mathbb{F}}(T)$ defined as $S^{\mathbb{F}}_t(T) := \mathbb{P}(\xi > T \mid \mathcal{F}_t)$, is the conditional expectation of the bounded random variable $\mathbb{1}_{\{\xi > T\}}$, hence is a $(\mathbb{P},\mathbb{F})$-martingale in $[0,1]$ which, just like the defaultable zero-recovery ZCB, can hit 0 before T. However, $S^{\mathbb{F}}_t(T)$ belongs to $\{0,1\}$ almost surely for $t \geq T$ if ξ is an $\mathbb{F}$-stopping time.

These three examples show that bounded martingales may behave quite differently near their bounds, depending on the context. Despite their practical importance, the general probabilistic properties of $[a,b]$-martingales have received limited attention. Notably, among previously studied examples, the Φ-martingale [8] admits a complete analysis: it is a $[0,1]$-martingale whose limit law is the Bernoulli distribution with parameter equal to its initial value, and whose boundaries are never reached in finite time. A natural question is thus whether the Bernoulli convergence is specific to the Φ-martingale or is a universal feature of homogeneous diffusion martingales with constant bounds.

Conic martingales with uniform marginals at all times are studied in [3, Sections 2 & 3], suggesting that Bernoulli is not always the limit law of bounded martingales. However, these examples are different as the related diffusion processes evolve between time-dependent bounds of form $[-b(t), b(t)]$ with $b : t \to b(t)$ an increasing function satisfying $b(0) = 0$. Examples of diffusions evolving between constant boundaries and displaying uniform marginals are provided in [3, Sections 4], but the related processes are mean-reverting, hence, are not martingales. Therefore, our question above remains open.

This paper offers a complete treatment of this question. We consider homogeneous diffusions Z with initial value z, solving

$$dZ_t = \sigma(Z_t)\,dW_t\,, \qquad Z_0 = z \in [a,b]\,, \tag{1}$$

where σ is locally Hölder-1/2 on (a,b), strictly positive on (a,b), and satisfies $\sigma(a) = \sigma(b) = 0$. In addition, we establish a connection between such diffusions and what we call *Bernoulli-Doob martingales*, which are processes $Z = (Z_t)_{t\geq 0}$ of the form $Z_t = \mathbb{E}[B \mid \mathcal{F}_t]$, where B is a Bernoulli random variable and $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. Those processes arise naturally whenever a binary event is revealed progressively through time.[1] Moreover, they are bounded in $[0, 1]$ by construction, have constant expectation equal to $\mathbb{P}(B = 1)$, and converge almost surely to a $\{0, 1\}$-valued limit.

We make the following contributions to the literature.

1. **Existence and uniqueness** (Theorem 1). Under mild regularity on σ, the SDE (1) admits a unique strong absorbed solution on $D = [a, b]$, which is a bounded uniformly integrable martingale.
2. **Bernoulli-Doob representation** (Theorems 2 and 3). Every homogeneous diffusion martingale satisfying the above conditions admits a Bernoulli-Doob representation, i.e., $Z_t = \mathbb{E}[B \mid \mathcal{F}_t]$ for $B \sim \mathrm{Ber}(z)$. Conversely, every continuous time-homogeneous Markov Bernoulli-Doob martingale on a Brownian filtration arises from such a diffusion.
3. **Truly asymptotic Bernoulli limit** (Theorem 4). For any fixed $T < \infty$, $\mathbb{P}(\tau > T) > 0$, where τ is the first hitting time of $\{a, b\}$. This shows that the Bernoulli distribution cannot be attained at any finite time, even when the individual boundaries are accessible.
4. **Boundary classification** (Section 3). We reconcile Feller's semigroup-based classification with the pathwise SDE framework and the martingale constraint, showing that the martingale property forces absorption at any attainable boundary.

The paper is organised as follows. Section 2 introduces the setup and the existence and uniqueness result. Section 3 develops the boundary theory. Section 4 establishes Bernoulli convergence and the Bernoulli-Doob representation. Section 5 provides variance analysis and hitting time results for several tractable examples. Section 6 discusses applications in financial modelling which are deeply connected with our tractable examples. Appendix A recalls Feller's classification; Appendix B verifies the four examples; Appendix C solves the Jacobi moment recursion.

2. Setup, Existence, and Uniqueness

We consider the driftless SDE (1) on the compact interval $D = [a, b]$ under the following standing assumptions, in force throughout the paper.

Assumption 1 (Diffusion coefficient)**.** *Unless stated otherwise, we consider the following assumptions about the diffusion coefficient in eq.* (1)*:*

(A1) $\sigma(a) = \sigma(b) = 0$*;*
(A2) $\sigma(x) > 0$ *for all* $x \in (a, b)$*;*
(A3) σ *is locally Hölder-*$1/2$ *continuous on* (a, b)*.*

Remark 1 (Bounded local martingales are true martingales)**.** *Any solution to* (1) *is a continuous local martingale. Since it is bounded in* $[a, b]$*, it is automatically a uniformly integrable true martingale. We use this fact repeatedly below.*

[1] This is exactly the case of our survival probability example, where $S_t^{\mathbb{F}}(T) = \mathbb{E}[B|\mathcal{F}_t]$ with $B = \mathbb{1}_{\{\xi > T\}}$.

Theorem 1 (Existence and Uniqueness of the Absorbed Strong Solution)**.** *Let $D = [a, b]$ be a compact interval and consider* (1) *under Assumption 1. Define*

$$\tau_x := \inf\{t > 0 : Z_t = x\} \quad \textit{and let} \quad \tau := \tau_a \wedge \tau_b$$

denote the first hitting time to a boundary. Then:

(a) *For every $z \in (a, b)$ there exists a unique strong solution of* (1) *on* $[0, \tau)$.
(b) *The stopped process $Z_t^\tau := Z_{t\wedge\tau}$ is a strong solution of* (1) *on all of* $[0, \infty)$.
(c) *If $z \in \{a, b\}$, the unique solution to* (1) *in D is the constant process $Z_t \equiv z$.*
(d) *Z^τ is the unique strong solution within the class of absorbed solutions on D.*
(e) *Z^τ is a bounded uniformly integrable martingale.*

Proof. **Interior dynamics (proving (a)).** Let $z \in (a, b)$. For every compact subinterval $K_\varepsilon = [a + \varepsilon, b - \varepsilon] \subset (a, b)$, assumption (A3) implies that σ satisfies the Yamada–Watanabe criterion on K_ε. Hence pathwise uniqueness holds on each such compact set. Standard localization arguments yield a unique strong solution up to the first exit time from (a, b), i.e., up to τ.

Boundary initialization (proving (c)). Suppose $z = a$. Since $\sigma(a) = 0$, the constant process $Z_t \equiv a$ satisfies the SDE. If Z is any other solution taking values in D, then $Z_t \geq a$ a.s. Every local martingale bounded from below is a supermartingale (by Remark 1 applied to $Z - a$), so $\mathbb{E}[Z_t] \leq Z_0 = a$. Since $Z_t - a \geq 0$ a.s., we conclude $Z_t = a$ a.s. for all $t \geq 0$. The argument for $z = b$ is identical.

Absorbed extension (proving (b)). If $\tau = \infty$ there is nothing to prove. Otherwise, for $t \geq \tau$, $Z_t^\tau = Z_\tau \in \{a, b\}$, so $\sigma(Z_t^\tau) = 0$ by (A1), and $dZ_t^\tau = 0 = \sigma(Z_t^\tau)\, dW_t$. Hence Z^τ satisfies the SDE globally.

Uniqueness among absorbed solutions (proving (d)). Let $\widehat{Z}$ be another absorbed solution. Pathwise uniqueness on (a, b) gives $\widehat{Z}_t = Z_t^\tau$ for $t < \tau$. At time τ both processes reach the same boundary point and remain constant thereafter, so $\widehat{Z}_t = Z_t^\tau$ for all $t \geq 0$ a.s..

Martingale property (proving (e)). The stopped process Z^τ is a bounded local martingale. By Remark 1, it is uniformly integrable and therefore a true martingale. □

Remark 2 (Pathwise admissibility versus diffusion startability)**.** *When we say a boundary point is not startable, this is meant in the classical Feller sense: startability as a non-degenerate diffusion with a Feller semigroup. This should not be confused with pathwise admissibility of the SDE. For all examples in this paper, if $\sigma(a) = 0$, then the constant process $Z_t \equiv a$ is a perfectly valid strong solution even if Feller classifies a as non-startable (see Theorem 1(c)). Such solutions are absorbing and degenerate; they are not diffusions if started at the boundary, and are therefore excluded from Feller's diffusion classification, even though they are entirely acceptable for our purposes and are included in our classification tables.*

3. Boundary Behaviour of Bounded Martingales

In the previous section, we looked at solutions to (1) living in $D = [a, b]$, in the particular case where σ obeys Assumption 1. Let us now take a step back and analyse the SDE (1) for more general σ and D. In this broader context, uniqueness of solutions may be lost, the boundaries of the process may not coincide with the roots of σ, and the behaviour of solutions at the zeros of σ depends on the shape of the diffusion coefficient.

3.1. Roots of σ, state space, and Feller classification

We begin with a guiding example that illustrates these subtleties, then state the general classification.

Consider (1) with $a = 0$, $b = 1$, and

$$\sigma(x) = |x(1-x)|^{1/4}. \tag{2}$$

This diffusion coefficient satisfies Assumption 1. In particular, the Hölder exponent at 0 and 1 is $1/4 < 1/2$, proving $(A3)$. The maximal domain of this SDE is $A = \mathbb{R}$, since σ is defined everywhere. If we took $D = A = \mathbb{R}$, the roots 0 and 1 would be interior equilibrium points. Indeed, since $\sigma(x) = |x(1-x)|^{1/4}$ is Hölder-1/2 on both sides of 0 and 1, the Yamada–Watanabe theorem guarantees a unique strong solution on all of $\mathbb{R}$, and the local time at 0 and 1 is zero; the process can therefore cross these points (see Remark 3 below). In order to force solutions to be confined between the zeros, one must impose $D = [0, 1]$. We proceed in four steps.

1. *Identify the zeros.* The points $x = 0$ and $x = 1$ are roots of σ. By Theorem 1(c), if $z = 0$ the unique solution in D is $Z_t \equiv 0$; similarly for $z = 1$.
2. *Define the state space $D = [0, 1]$.* We restrict to processes confined between these equilibrium points. The roots transition from interior points of A to topological boundaries of D. Imposing this state space restricts the set of admissible solutions.
3. *Apply Feller's classification on D.* Because 0 and 1 are now topological boundaries, Feller's theory applies. Evaluating the relevant integrals (Appendix B), both endpoints are classified as regular.
4. *Impose the martingale constraint.* As established by Proposition 1 below, the martingale property forces absorption at any attainable boundary. Of all the boundary behaviours Feller allows for regular boundaries (reflection, absorption, stickiness), only absorption is compatible with the bounded martingale requirement.

A key principle emerges: *a root of σ is not intrinsically a boundary.* If the state space is not restricted to the interval delimited by the roots, solutions can cross the zeros of σ, which then act as interior points rather than boundaries. Its classification depends on the definition of the state space. For instance, $\sigma(x) = \sqrt{|x|}$ with $D = \mathbb{R}$ has $x = 0$ as an interior point; with $D = [0, \infty)$ and $\sigma(x) = \sqrt{x}$, $x = 0$ is a topological boundary to which Feller's theory applies. In this paper, when σ has two roots $a < b$ and $z \in [a, b]$, we always take $D = [a, b]$.

Remark 3 (Non-uniqueness, local time, and boundary conditions)**.** *The regularity of σ at a root x^* determines whether the solution is unique beyond the first hitting time of x^*, and whether x^* acts as a boundary or an interior point.*

Hölder-1/2 on both sides: interior point. If σ is Hölder-1/2 on both sides of x^*, the Yamada–Watanabe theorem guarantees a unique strong solution on all of $\mathbb{R}$, and x^* is an interior point that can be crossed. For example, $\sigma(x) = \sqrt{|x|}$ on $\mathbb{R}$ gives a unique solution for which $x = 0$ is interior.

Hölder-1/2 on one side only: boundary. If σ is defined and Hölder-1/2 only on one side of x^* (e.g., $\sigma(x) = \sqrt{x}$ on $[0, \infty)$), then x^* is a topological boundary. Feller's theory classifies whether it is attainable.

Below Hölder-1/2: non-uniqueness and local time. When σ fails to be Hölder-1/2 at x^* (e.g., $\sigma(x) = |x|^{1/4}$), pathwise uniqueness is not guaranteed after the first hitting time of x^*. Multiple extensions of the solution exist, parameterised by the local time coefficient θ in

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + \theta\, L_t^{x^*}(X)\,, \tag{3}$$

where $L_t^{x^*}(X)$ is the local time of X at x^*, defined for any continuous semimartingale by

$$L_t^{x^*}(X) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{|X_s - x^*| < \varepsilon}\; d\langle X\rangle_s = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{|X_s - x^*| < \varepsilon}\, \sigma^2(X_s)\, ds\,.$$

The four canonical choices are:

- *$\theta = 0$: absorption at x^* (the unique solution to the SDE itself, since the local time term vanishes);*
- *$\theta > 0$: reflection at x^*;*
- *$\theta < 0$: x^* is an interior crossing point;*
- *$\theta = \infty$: sticky boundary (the process spends positive Lebesgue time at x^*).*

The SDE (1) without additional boundary conditions corresponds to $\theta = 0$, selecting the absorbed solution. The generator perspective is complementary: the operator $\mathcal{L}f(x) = \frac{\sigma^2(x)}{2} f''(x)$ is not defined at roots of σ, and specifying a boundary condition at x^* corresponds to extending the domain of $\mathcal{L}$. Imposing $f(x^*) = 0$ gives absorption; imposing $f'(x^{*+}) = 0$ gives reflection above x^*; imposing $f'(x^{*+}) = f'(x^{*-})$ gives an interior point; imposing $\mathcal{L}f(x^*) = kf''(x^*)$ gives a sticky boundary. The SDE is thus the canonical representation of the generator with absorption at x^*.

As a concrete illustration, the equation $dX_t = \operatorname{sgn}(X_t)\, dW_t$ has no unique solution: one solution behaves as a standard Brownian motion (crossing zero), while others are absorbed at zero. All solutions share the same quadratic variation $\langle X\rangle_t = t$, but differ in their behaviour at zero. Imposing the absorbing boundary condition uniquely selects the martingale solution. For our bounded martingales, the martingale property independently forces $\theta = 0$, as established by Proposition 1.

3.2. The Martingale Trilemma

The next proposition shows that the boundaries of a $[0,1]$-martingale are necessarily absorbing.

Proposition 1 (Martingale Trilemma)**.** *Let Z be a continuous process on $[0,1]$ with $\sigma(0)=0$. The following three conditions cannot hold simultaneously:*

(a) *Startability: $\mathbb{P}(Z_t>0)>0$ for some $t>0$ when $Z_0=0$.*
(b) *Bounded domain: $Z_t \geq 0$ for all $t\geq 0$.*
(c) *Martingale property: $\mathbb{E}[Z_t]=Z_0=0$ for all $t\geq 0$.*

Consequently, within the class of processes satisfying (b) *and* (c)*, the only admissible boundary behaviour at* 0 *is absorption: $Z_t=0$ for all $t\geq\tau_0$ on $\{\tau_0<\infty\}$.*

Proof. Suppose (a) and (b) hold. Then $Z_t>0$ with positive probability for some $t>0$, and $Z_t\geq 0$ a.s., so $\mathbb{E}[Z_t]>0=Z_0$, violating (c). Suppose (a) and (c) hold. Then $\mathbb{E}[Z_t]=0$ with $Z_t>0$ on a set of positive probability, which forces $Z_t<0$ on a set of positive probability, violating (b). Hence (a) must be abandoned: once Z reaches 0, it must remain there. □

Remark 4. *The Trilemma contrasts with the CIR process $d\lambda_t=\kappa(\theta-\lambda_t)\,dt+\eta\sqrt{\lambda_t}\,dW_t$, where the drift $\kappa\theta\,dt>0$ at $\lambda_t=0$ acts as a deterministic engine returning the process to the interior. Our driftless martingales have no such engine: the noise shuts off at the boundary and the process stops.*

Imposing absorption does not violate the martingale property. The absorbed process $Z_{t\wedge\tau_0}$ is the stopped process; by Doob's Optional Stopping Theorem, if Z_t is a true martingale in the interior then $Z_{t\wedge\tau_0}$ is a true martingale for all $t\geq 0$. The expectation is preserved by the law of total expectation:

$$\mathbb{E}[Z_{t\wedge\tau_0}]=\underbrace{\mathbb{E}[Z_t\mid\tau_0\leq t]}_{=0}\,\mathbb{P}(\tau_0\leq t)+\mathbb{E}[Z_t\mid\tau_0>t]\,\mathbb{P}(\tau_0>t)=z\,.$$

Paths absorbed at 0 *contribute* 0*; the surviving paths drift upward to compensate.*

3.3. Four boundary classification cases

We begin with an example that cements our earlier discussion on the importance of choosing the relevant state space from the start.

Example 1. *The case $\sigma(x)=\sqrt{x(1-x)}$ with $D=[0,1]$ gives a strong pathwise unique solution with* 0 *and* 1 *as absorbing (exit) boundaries. However, $\sigma(x)=\sqrt{|x(1-x)|}$ with $D=\mathbb{R}$ also gives a strong pathwise unique solution, but* 0 *and* 1 *are interior points: the process can cross them. The difference is that $\sqrt{|x(1-x)|}$ is Hölder-1/2 on both sides of* 0 *and* 1 *(the local time at these points is zero), so the process is not absorbed. The shape of σ near its roots, together with the choice of state space, jointly determine whether the roots act as boundaries.*

Case $\sigma(x)$	$x = 0$	$x = 1$
$(x(1-x))^{1/4}$	Regular: attainable & startable Absorbing	Regular: attainable & startable Absorbing
$x(1-x)$	Natural: inaccessible, not startable †	Natural: inaccessible, not startable †
$\sqrt{x(1-x)}$	Exit: attainable, not startable Absorbing	Exit: attainable, not startable Absorbing
$x^{1/4}(1-x)$	Regular: attainable & startable Absorbing	Natural: inaccessible, not startable †

Table 1: Combined Feller and martingale boundary classification for four diffusion coefficients with $D = [0, 1]$, $\sigma(0) = \sigma(1) = 0$. The Feller type (Regular/Exit/Natural) is determined by the integrability of σ^{-2}; see Appendix A. Imposing the martingale property implies that all attainable boundaries are necessarily absorbing (Proposition 1). If $z \in \{0, 1\}$, the unique solution is $Z_t \equiv z$ for all t (Theorem 1(c)). Entries marked † denote the case where $z \in \{0, 1\}$ gives a degenerate constant solution that is pathwise valid but falls outside Feller's diffusion classification; see Remark 2.

Table 1 presents four examples with their combined Feller and martingale classifications. Detailed calculations are in Appendix B.

Remark 5 (Regular boundaries under the martingale constraint)**.** *Table 1 includes Regular boundaries, which might appear to contradict the statement that attainable boundaries are always exit type for bounded martingales. The resolution is as follows. Feller's classification is a property of the generator and does not depend on the martingale constraint. A Regular boundary is attainable and startable in the Feller sense. However, Proposition 1 shows that, once the martingale property is imposed, the process cannot restart from a boundary without violating either the domain constraint or the zero-drift condition. Thus, Regular boundaries are realized as absorbing under the martingale constraint, behaving effectively like Exit boundaries in terms of pathwise dynamics. The Feller type describes the analytic properties of the generator; the realized pathwise behaviour is determined by the combined Feller classification and the martingale constraint.*

We now discuss each case.

Case 1 ($\sigma(x) = (x(1-x))^{1/4}$, both boundaries regular). This case was used earlier to illustrate the importance of clarifying the state space. We keep it here for completeness. Both boundaries are attainable from the interior and may serve as initial values in the Feller sense. Since $\sigma(0) = \sigma(1) = 0$, the Martingale Trilemma forces absorption upon hitting.

Case 2 ($\sigma(x) = x(1-x)$, both boundaries natural). Neither boundary is attainable from the interior. An alternative proof uses the fact that Z has the same dynamics as the logistic transform of X_t, where $dX_t = \frac{\eta^2}{2}\tanh(X_t/2)\,dt + \eta\,dW_t$ (see Example 3.5 in [8]). Therefore $\{Z_t \in \{0,1\}\} \Leftrightarrow \{X_t \in \{-\infty, +\infty\}\}$, which has probability zero for all $t \geq 0$.

Case 3 ($\sigma(x) = \sqrt{x(1-x)}$, Jacobi martingale, both boundaries exit). Both boundaries are attainable with probability one; the martingale requirement forces absorption upon hitting.

Case 4 ($\sigma(x) = x^{1/4}(1-x)$, mixed). The left boundary 0 is regular (attainable, absorbed by the martingale constraint). The right boundary 1 is natural (inaccessible).

3.4. Absorbing boundaries

Lemma 1. *Any boundary point reached by a martingale Z bounded in $[0,1]$ is absorbing. More precisely, let $\tau_b := \inf\{t \geq 0 : Z_t = b\}$, $b \in \{0,1\}$. Then, on $\{\tau_b < \infty\}$, $Z_t = b$ for all $t \geq \tau_b$. Moreover, if Z is a time-homogeneous diffusion with coefficient $\sigma(Z_t)$, then $\sigma(b) = 0$.*

Proof. Focus on $b = 1$; the case $b = 0$ is analogous. For all $t \geq \tau_1$, $Z_t \leq 1$ a.s. By the Optional Stopping Theorem, $\mathbb{E}[Z_t \mid \mathcal{F}_{\tau_1}] = Z_{\tau_1} = 1$. Hence Z_t is a bounded random variable whose conditional expectation given $\mathcal{F}_{\tau_1}$ equals its upper bound 1, forcing $Z_t = 1$ a.s. on $\{\tau_1 < \infty\}$ for all $t \geq \tau_1$. If $\sigma(1) \neq 0$, then $d\langle Z\rangle_t = \sigma^2(1)\,dt > 0$, contradicting the fact that Z_t is stuck at 1. □

4. Asymptotic Law and Bernoulli-Doob Representation

In the previous section, we discussed the possibility for a process Z solving (1) to reach the boundaries. In this section, we investigate the asymptotic law of Z. We find that the limit is a Bernoulli, independently of the boundary classification. Moreover, we establish the equivalence between such processes and Bernoulli-Doob martingales.

4.1. Variance-based proof of Bernoulli convergence

Remark 6 (Fokker-Planck heuristic)**.** *Before the rigorous proof, we offer an informal but illuminating argument via the Fokker-Planck equation. This requires the additional assumption $\sigma \in C^2$ and a formal interchange of limit and differentiation that we do not justify here; the rigorous proof follows in Proposition 2 and Corollary 1 below.*

Suppose Z admits a stationary density in the sense that $\lim_{t\to\infty} p(t,x) = p^\star(x)$ for some generalised density $p^\star$. The Fokker-Planck equation reads

$$\frac{\partial p(t,x)}{\partial t} = \frac{\partial^2[\sigma^2(x)p(t,x)]}{\partial x^2}\,.$$

Setting the left-hand side to zero gives $\sigma^2(x)p^\star(x) = k_1 x + k_2$ for constants k_1, k_2. Since $p^\star \geq 0$ and $\sigma^2(a) = \sigma^2(b) = 0$, we need $k_1 a + k_2 = 0$ and $k_1 b + k_2 = 0$, which forces $k_1 = k_2 = 0$. Hence $p^\star(x) = 0$ for all $x \in (a,b)$, so $p^\star$ is supported entirely on $\{a,b\}$:

$$p^\star(x) = k\,\delta(x-a) + (1-k)\,\delta(x-b)\,.$$

The martingale condition $\int x\,p^\star(x)\,dx = z$ gives $k = 1-(z-a)/(b-a)$, which is precisely the generalised Bernoulli $\mathcal{B}(a,b,z)$. The intuition is transparent: the only affine function that is non-negative and vanishes at both a and b is the zero function, so the stationary density must be supported entirely on the boundary.

Proposition 2 (Variance characterization of Bernoulli convergence)**.** *Under assumptions* (A1)–(A3)*, let* $v(t) := \mathbb{V}[Z_t]$.

(i) (Max-variance law.) *The generalised Bernoulli* $\mathcal{B}(a, b, z)$ *is the unique maximum-variance distribution on* $[a, b]$ *with mean* z*:*

$$\mathcal{B}(a, b, z) = \mathit{argmax}_{\mathcal{L} \in \mathcal{P}(a,b,z)} \mathbb{V}[\mathcal{L}] \, ,$$

where $\mathcal{P}(a, b, z)$ *is the set of distributions on* $[a, b]$ *with mean* z*.*

(ii) (Variance strictly increasing.) $v(t)$ *is strictly increasing as long as* $\mathbb{P}(Z_t \in \{a, b\}) < 1$.

Proof. **(i)** We prove for $a = 0$, $b = 1$; the general case follows by shifting and scaling. Let X have law $\mathcal{L}_X \in \mathcal{P}(0, 1, z)$ and $Y \sim \mathcal{B}(0, 1, z)$, so $\mathbb{E}[X] = \mathbb{E}[Y] = z$. Then

$$\mathbb{V}[X] = \int_0^1 x^2 \, \mu(dx) - z^2 \leq \int_0^1 x \, \mu(dx) - z^2 = z(1 - z) = \mathbb{V}[Y] \, ,$$

since $x^2 \leq x$ on $[0, 1]$, with equality if and only if $x \in \{0, 1\}$. Equality holds if and only if μ assigns all mass to $\{0, 1\}$ with mass z at 1, i.e., if and only if $\mathcal{L}_X = \mathcal{B}(0, 1, z)$.

(ii) By Itô's isometry,

$$v(t) = \int_0^t \mathbb{E}[\sigma^2(Z_s)] \, ds \, .$$

The integrand is strictly positive unless $\mathbb{P}(\sigma^2(Z_s) = 0) = 1$, i.e., unless $Z_s \sim \mathcal{B}(a, b, z)$ (in which case paths are frozen at the bounds for all $s' \geq s$). Hence v increases strictly until the distribution becomes Bernoulli. □

Corollary 1 (Almost sure Bernoulli convergence)**.** *Under Assumption 1,* $Z_t \to \{a, b\}$ *a.s. as* $t \to \infty$*, and the limit law is* $\mathcal{B}(a, b, z)$*.*

Proof. Since $v(t)$ is strictly increasing and bounded above by $(b - z)(z - a)$, we have $\mathbb{E}[\sigma^2(Z_t)] \to 0$ as $t \to \infty$. This is equivalent to $\mathbb{P}(Z_t \in \{a, b\}) \to 1$. Since Z is a bounded martingale, it converges a.s. to an integrable limit Z_∞ (see e.g. [5], Th. 5.2.1). By the above, $Z_\infty \in \{a, b\}$ a.s., and $\mathbb{E}[Z_\infty] = z$ gives the Bernoulli weights. □

4.2. Bernoulli-Doob representation theorems

Theorem 2 (Bounded homogeneous diffusions are Bernoulli-Doob martingales)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *satisfy the usual conditions with* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ *and* $\mathcal{F} = \mathcal{F}_\infty$*. Let* W *be a Brownian motion and* $Z = (Z_t)_{t \geq 0}$ *take values in* $[0, 1]$ *and solve*

$$dZ_t = \sigma(Z_t) \, dW_t \, , \qquad Z_0 = z \in [0, 1] \, ,$$

where $\sigma : [0, 1] \to [0, \infty)$ *is continuous, strictly positive on* $(0, 1)$*, and satisfies* $\sigma(0) = \sigma(1) = 0$*. Then* Z *is a continuous Bernoulli-Doob martingale:*

$$Z_t = \mathbb{E}[B \mid \mathcal{F}_t] \, , \qquad B \sim \mathrm{Ber}(z) \, , \quad t \geq 0 \, .$$

Proof. By Remark 1, Z is a true martingale. By the martingale convergence theorem, there exists $Z_\infty \in L^1$ with $Z_t \to Z_\infty$ a.s. and in L^1 [10, Problem 3.19]. Moreover [10, Problem 3.20], there exists $X \in L^1$ with $Z_t = \mathbb{E}[X \mid \mathcal{F}_t]$ and $\mathbb{E}[X \mid \mathcal{F}] = Z_\infty$ a.s. The tower law gives $Z_t = \mathbb{E}[Z_\infty \mid \mathcal{F}_t]$. By Corollary 1 with $(a,b) = (0,1)$, $Z_\infty \in \{0,1\}$ a.s. Setting $B := Z_\infty$ and noting $\mathbb{E}[B] = \mathbb{E}[Z_\infty] = z$, we conclude $B \sim \mathrm{Ber}(z)$. ◻

Theorem 3 (Characterization by Bernoulli variables)**.** *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfy the usual conditions with $\mathcal{F} = \mathcal{F}_\infty$. Define $Z_t = \mathbb{E}[B \mid \mathcal{F}_t]$ where $B \sim \mathrm{Ber}(z)$. Then Z is an $\mathbb{F}$-martingale bounded in $[0,1]$ with $Z_0 = z$.*

Moreover, suppose $\mathbb{F}$ is a Brownian filtration and Z is a continuous time-homogeneous Markov process. Then Z can be written as a homogeneous diffusion

$$dZ_t = \sigma(Z_t)\, dW_t \quad \text{where} \quad Z_0 = z$$

for some $\sigma : [0,1] \to [0,\infty)$ and an $\mathbb{F}$-Brownian motion W. Furthermore, $\sigma(b) = 0$ at any boundary $b \in \{0,1\}$ reachable in finite time.

Proof. Since B is integrable and $0 \le B \le 1$, Z is an $\mathbb{F}$-martingale with $0 \le Z_t \le 1$ and $Z_0 = \mathbb{E}[B] = z$.

Assume Z is a continuous Markov process and $\mathbb{F}$ is a Brownian filtration. Then Z is a continuous square-integrable martingale adapted to $\mathbb{F}$. By the martingale representation theorem, there exist an $\mathbb{F}$-Brownian motion W' and an $\mathbb{F}$-predictable process H with

$$Z_t = Z_0 + \int_0^t H_s\, dW'_s, \qquad \langle Z \rangle_t = \int_0^t H_s^2\, ds\,.$$

By Lévy's characterization, $W_t := \int_0^t \mathrm{sign}(H_s)\, dW'_s$ is an $\mathbb{F}$-Brownian motion and $dZ_t = |H_t|\, dW_t$.

For a one-dimensional continuous time-homogeneous Markov process, the quadratic variation is a homogeneous additive functional of the process. By the theory of additive functionals [13, Ch. V], there exists a Borel function $a : [0,1] \to [0,\infty)$ such that $\langle Z \rangle_t = \int_0^t a(Z_s)\, ds$, i.e., $|H_t| = \sqrt{a(Z_t)}$ for $dt \times d\mathbb{P}$-a.e. (t,ω). Setting $\sigma(x) := \sqrt{a(x)}$ gives $dZ_t = \sigma(Z_t)\, dW_t$.

The last claim follows from Lemma 1. ◻

Remark 7. *By the same argument as in Theorem 2, the Bernoulli variable B in Theorem 3 coincides a.s. with Z_∞. Without time-homogeneity, the martingale representation need not reduce to a coefficient depending only on the current state.*

4.3. The Bernoulli limit is truly asymptotic

Theorem 4. *Consider an $[a,b]$-martingale Z solving (1) with $a < z < b$ and σ bounded on $[a,b]$.*[2] *Then*

$$\mathbb{P}(\tau > t) > 0 \qquad \forall\, t > 0\,.$$

[2] Boundedness of σ on $[a,b]$ is automatic under (A1)–(A3) since σ is continuous on the compact set $[a,b]$.

Proof. Since $a < Z_0 < b$, $\mathbb{P}(\tau > 0) = 1$. Since σ is bounded, there exists $M > 0$ with $|\sigma(x)| \leq M$ on $[a, b]$. The process $N_t := Z_{t\wedge\tau} - z$ is a continuous martingale with $N_0 = 0$ and bounded quadratic variation

$$\langle N \rangle_t = \int_0^{t\wedge\tau} \sigma(Z_s)^2 \, ds \leq M^2 t \,. \tag{4}$$

By the Dambis-Dubins-Schwarz theorem, there exists a Brownian motion $B = (B_s)_{s\geq 0}$ with $N_t = B_{\Theta_t}$, where $\Theta_t = \langle N \rangle_t$. Let $d := \min\{z - a, b - z\} > 0$ and

$$A_t := \left\{ \sup_{0\leq s\leq M^2 t} |B_s| < d \right\} .$$

From (4), $\Theta_s \in [0, M^2 t]$ for $s \leq t$. On A_t, $|B_{\Theta_s}| < d$ for all $s \leq t$, so $|N_s| < d$ and $Z_{s\wedge\tau} \in (a, b)$ for all $s \leq t$. Therefore $A_t \subseteq \{\tau > t\}$, giving $\mathbb{P}(\tau > t) \geq \mathbb{P}(A_t) > 0$ since B has strictly positive probability of remaining in $(-d, d)$ over any finite interval. □

Corollary 2. *For any $T < \infty$, $\mathbb{P}(Z_T \notin \{a, b\}) > 0$. Even when the individual boundaries are accessible (as in the Jacobi case), the event $\{\tau \leq T\}$ has strictly positive probability for all $T > 0$ but never equals 1 for any finite T. For $[0, 1]$-martingales, the Bernoulli distribution acts as a genuine asymptotic limit, not a finite-time distribution.*

5. Some Tractable Examples

We illustrate the theory with four tractable examples for $(a, b) = (0, 1)$. The first two are homogeneous diffusion cases, chosen to represent the two qualitatively different boundary regimes: natural (inaccessible) and exit (accessible). The last two highlight the importance of the time-homogeneity assumption in Theorems 2 and 4.

5.1. The Φ-martingale

The Φ-martingale [8] with constant volatility η corresponds to the process

$$Z_t := \Phi(X_t) \,, \quad X_t := e^{\frac{\eta^2}{2} t}(k + I_t) \,, \quad I_t := \eta \int_0^t e^{-\frac{\eta^2}{2} s} \, dW_s \,, \quad k = \Phi^{-1}(z) \,,$$

where Φ is the standard normal CDF and $\phi = \Phi'$. As studied independently by Carr [4], this process can be thought of as the natural analogue of Brownian motion on the compact set $[0, 1]$. It is easy to check from Itô's lemma that Z solves the time-homogeneous SDE (1) with[3]

$$\sigma(x) = \eta\phi(\Phi^{-1}(x)) \,, \quad z \in [0, 1] \,.$$

[3]The function $\phi(\Phi^{-1}(x))$ is locally Hölder-1/2, provable via Mills' formula; see e.g. [6]. Hence Z is a particular case of Theorem 1.

Both boundaries are natural (inaccessible). The variance is

$$v(t) = \Phi_2\big(\Phi^{-1}(z), \Phi^{-1}(z); \rho(t)\big) - z^2\,, \qquad \rho(t) = 1 - e^{-\eta^2 t}\,,$$

where Φ_2 is the bivariate standard normal CDF. As $t \to \infty$, $\rho(t) \to 1$ and $v(t) \to z(1-z)$, confirming Bernoulli convergence. Since Z is the image of a Gaussian process under Φ, the boundaries are never reached in finite time, i.e., $\mathbb{P}(\tau > t) = 1$ for all $t \geq 0$ when $z \in (0,1)$, consistent with the natural boundary classification.

The Φ-martingale admits the Bernoulli-Doob representation with $B := Z_\infty = \mathbb{1}_{\{k+I_\infty>0\}} \sim \mathrm{Ber}(z)$, confirming Theorem 2. To see this, set $\mathcal{Z} \sim \mathcal{N}(0,1)$ independently of $\mathcal{F}_t$. Then,

$$\mathbb{E}[\mathbb{1}_{\{k+I_\infty>0\}} \mid \mathcal{F}_t] = \mathbb{P}\left(\eta^2 \sqrt{\int_t^\infty e^{-\eta^2 s}\,ds} \cdot \mathcal{Z} > -(k+I_t) \,\middle|\, \mathcal{F}_t\right) = \Phi\left(\frac{k+I_t}{\sqrt{e^{-\eta^2 t}}}\right) = \Phi(X_t) = Z_t\,.$$

5.2. The Jacobi martingale

The Jacobi martingale corresponds to (1) with $\sigma(x) = \sqrt{\eta x(1-x)}$, which is Hölder-1/2 on $[0,1]$. This process is also known as the (driftless version of the) Wright-Fisher process [9]. This is related to the process we encountered earlier but in the special case $\eta = 1$. Both boundaries are of exit type (attainable, absorbing). The second moment $g(t) := \mathbb{E}[Z_t^2]$ satisfies the ODE $\dot{g} = \eta z - \eta g$ with $g(0) = z^2$, giving

$$v(t) = z(1-z)(1-e^{-\eta t})\,,$$

which converges to $z(1-z)$ as $t \to \infty$, confirming Bernoulli convergence.

The expected first hitting time to either boundary, starting from $Z_0 = x$, satisfies $\mathcal{L}m = -1$ with $m(0) = m(1) = 0$, where $\mathcal{L}f(x) = \frac{\eta}{2}x(1-x)f''(x)$. This gives

$$\mathbb{E}_x[\tau] = -\frac{2}{\eta}\big(x \ln x + (1-x)\ln(1-x)\big)\,,$$

which is finite for all $x \in (0,1)$, so $\mathbb{P}_x(\tau < \infty) = 1$. The conditional expected hitting times are

$$\mathbb{E}_x[\tau_0] = \mathbb{E}_x[\tau \mid Z_\tau = 0] = -\frac{2}{\eta}\frac{x \ln x}{1-x}\,,$$

and $\mathbb{E}_x[\tau_1]$ is obtained by replacing x by $1-x$.

By Theorem 4, despite $\mathbb{P}_x(\tau < \infty) = 1$, we have $\mathbb{P}_x(\tau > T) > 0$ for all $T < \infty$. The Bernoulli limit is truly asymptotic.

5.3. A time-inhomogeneous case

It is easy to check using Itô's lemma that the process $Z = (Z_t)_{t\geq 0}$ defined as $Z_t = \Phi\Big(X_t/\sqrt{1+e^{\eta^2 t}}\Big)$, where X is as in Section 5.1 with $k = \Phi^{-1}(z)\sqrt{2}$, solves the SDE

$$dZ_t = \sigma(t, Z_t)\,dW_t \quad \text{where} \quad \sigma(t,x) := \frac{\phi(\Phi^{-1}(x))}{\sqrt{1+e^{\eta^2 t}}}\,, \quad Z_0 = z \in [0,1]\,.$$

Interestingly, Z can also be written as the conditional expectation of a random variable Z_∞ bounded in $[0,1]$. Indeed, setting $\mathcal{Z} \sim \mathcal{N}(0,1)$ independently of $\mathcal{F}_t$:

$$\mathbb{E}[\Phi(k + I_\infty) \mid \mathcal{F}_t] = \mathbb{E}\left[\Phi\left(k + I_t + \eta^2 \sqrt{\int_t^\infty e^{-\eta^2 s}\, ds} \cdot \mathcal{Z}\right)\right] = \Phi\left(\frac{k + I_t}{\sqrt{1 + e^{-\eta^2 t}}}\right) = Z_t \,.$$

This shows that Z is a true martingale in $[0,1]$ and can be written as the conditional expectation of $Z_\infty = \Phi(k + I_\infty)$, which is bounded in $[0,1]$. However, in contrast with the Φ-martingale, Z_∞ is not Bernoulli distributed: $Z_\infty = \Phi(k+I_\infty) \sim \Phi(k+\mathcal{Z})$ with $\mathcal{Z} \sim \mathcal{N}(0,1)$, which is a continuous distribution on $(0,1)$. This does not contradict Theorem 2 because the diffusion coefficient is not time-homogeneous: compared to the Φ-martingale, the denominator $\sqrt{1 + e^{\eta^2 t}}$ in $\sigma(t,x)$ grows without bound, reducing the volatility of the process along each trajectory and preventing it from diverging to the bounds. This explains why the asymptotic behaviour is no longer Bernoulli.

5.4. A time-changed $[0,1]$*-martingale*

Theorem 4 shows that no homogeneous diffusion martingale can hit the bounds almost surely before a fixed finite time T. However, time-inhomogeneous diffusions can achieve $\mathbb{P}(\tau \leq T) = 1$. One way to reproduce this behaviour is to use an exploding time-change of a homogeneous martingale, as we now show.

Given a homogeneous $[0,1]$-martingale Z solving (1), define the time-changed process $Y_u := Z_{t(u)}$ where

$$u(t) = T(1 - e^{-\lambda t}) \quad \Leftrightarrow \quad t(u) = -\frac{1}{\lambda} \log\left(1 - \frac{u}{T}\right)$$

for some $\lambda > 0$. As t ranges $[0,\infty)$, u ranges $[0,T)$. The dynamics of Y are

$$dY_u = \sqrt{(\lambda(T-u))^{-1}}\, \sigma(Y_u)\, d\tilde{W}_u \,, \qquad u \in [0,T) \,,$$

where $\tilde{W}$ is a standard Brownian motion under the time-changed filtration. Since $Y_T = Z_\infty \in \{a,b\}$ a.s., the process hits one of the bounds before T almost surely. This does not contradict Theorem 4 because the diffusion coefficient is time-inhomogeneous. For $u > T$, one may extend by replacing u/T with $(u \wedge T)/T$, after which Y remains constant at the attained bound.

In the special case where the latent process Z is the Φ-martingale discussed in Section 5.1, the diffusion coefficient of Y is similar to that of the process Z in Section 5.3: the only difference is the denominator, which changes from $\sqrt{1 + e^{\eta^2 t}}$ to $\sqrt{\lambda(T-t)}$. In contrast with the former, the latter vanishes as $t \to T$, causing σ to explode at $t = T$, and explains why $Y_T \in \{0,1\}$ almost surely.

6. Applications to Credit Risk Modelling

The Bernoulli-Doob representation arises naturally in credit risk modelling, where $Z_t = \mathbb{E}[B \mid \mathcal{F}_t]$ coincides with $S_t^{\mathbb{F}}(T)$ when $B = \mathbb{1}_{\{\xi > T\}}$. For conciseness, we restrict ourselves to provide three examples associated with the main families of default models.

The default time ξ of a reference entity is a stopping time under the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\{\xi < u\}, u \leq t\})$. In this case, the survival probability process $S^{\mathbb{G}}(T)$ defined as $S_t^{\mathbb{G}}(T) := \mathbb{P}(\xi > T \mid \mathcal{G}_t)$ is a $\mathbb{G}$-martingale in $[0,1]$ converging to 0 or 1 as $t \to T$, and hitting 0 before T whenever $\xi < T$. This may not be the case for $S^{\mathbb{F}}(T)$, unless ξ is an $\mathbb{F}$-stopping time. The following default models show how various information structures lead to different boundary types for the $\mathbb{F}$-martingale $Z = S^{\mathbb{F}}(T)$, which is known to play a key role in default models [1]. All these results are consistent with the Bernoulli-Doob framework exposed above. Table 2 summarises the three cases. We note $z = Z_0 = S_0^{\mathbb{F}}(T)$.

Structural model. In standard structural models, the Bernoulli variable takes the form $B = \mathbb{1}_{\{V_T > K\}}$ where the firm-value process V and the debt level K are $\mathbb{F}$-adapted, such that ξ is an $\mathbb{F}$-stopping time. Therefore, $\{\xi > T\} \in \mathcal{F}_T$, hence, $Z_t = S_t^{\mathbb{F}}(T) = S_t^{\mathbb{G}}(T)$ for all $t \geq 0$, $Z_t \in \{0,1\}$ for all $t \geq T$, and $\mathbb{P}(\tau \leq T) = 1$. In this case, the Bernoulli law is not only reached asymptotically as $t \to \infty$, but $Z_t \sim Ber(z)$ exactly, for all $t \geq T$.

In the original Merton's model [12], where K is constant and V is an $\mathbb{F}$-GBM with parameters (μ, σ),

$$Z_t = \Phi(d_t^+) \quad \text{where} \quad d_t^+ := \frac{(\mu - \frac{\sigma^2}{2})(T-t) + \ln(V_t/K)}{\sigma\sqrt{T-t}} .$$

From Itô's lemma, the dynamics of Z read as

$$dZ_t = \mathbb{1}_{\{t<T\}} \frac{\phi(\Phi^{-1}(Z_t))}{\sqrt{T-t}} \, dW_t, \tag{5}$$

i.e., the diffusion coefficient is time-inhomogeneous on $[0,T)$. This makes clear why Z reaches the bounds almost surely by time T without contradicting Theorem 4: the diffusion coefficient is explicitly time-inhomogeneous and becomes singular as $t \uparrow T$. Likewise, there is no contradiction with Theorem 3. Although Z is a Bernoulli-Doob martingale, its quadratic variation density is

$$\frac{d\langle Z\rangle_t}{dt} = 1_{\{t<T\}} \frac{\phi(\Phi^{-1}(Z_t))^2}{T-t},$$

which depends explicitly on time and not only on the current state Z_t. Hence Z is not a one-dimensional time-homogeneous diffusion of the form $dZ_t = \sigma(Z_t) dW_t$. Interestingly, the diffusion coefficient of Z in the structural model coincides with that of the time-changed process Y considered in Section 5.4 when $\lambda = 1$ and the latent process Z therein follows the Φ-martingale dynamics discussed in Section 5.1.

Cox model. In Cox models, the Bernoulli variable reads as $B = \mathbb{1}_{\{\int_0^T \lambda_s\, ds < \mathcal{E}\}}$, where λ is $\mathbb{F}$-adapted and $\mathcal{E} \perp \mathcal{F}_\infty$, such that ξ is not an $\mathbb{F}$-stopping time: $\{\xi > T\} \notin \mathcal{F}_\infty$. Therefore, $Z = S^{\mathbb{F}}(T)$ never reaches the bounds (natural boundaries). In such models, Z is not Markovian (it depends on both λ_t and $\Lambda_t := \int_0^t \lambda_s\, ds$), so Theorem 3 does not apply and explains why the Bernoulli-Doob martingale Z may not take the form of a time-inhomogeneous diffusion.

Observe that, as in Merton's model, the limit law is not purely asymptotic, but is reached a.s. at the finite time T because $Z_t = \mathbb{P}(\mathcal{E} \geq \Lambda_T | \Lambda_T)$ for all $t \geq T$. However, in contrast with Merton, Z_t is not Bernoulli distributed after T because ξ is not an $\mathbb{F}$-stopping time. For instance, if λ is the CIR considered in Remark 4 and $\mathcal{E} \sim Exp(1)$, it holds that $Z_t = \exp\{-\Lambda_T\} \not\sim Ber(z)$ for all $t \geq T$. This does not contradict Corollary 1 because the diffusion coefficient of Z does not meet Assumption 1 on $[0, T)$. Indeed, it is not time-homogeneous:

$$dZ_t = -\mathbb{1}_{\{t<T\}}\, \eta B(T-t) Z_t \sqrt{\lambda_t} dW_t, \tag{6}$$

where $s \to B(s)$ is the usual function associated with the log-affine form

$$\mathbb{E}[\exp\{-\Lambda_T\}|\mathcal{F}_t] = A(T-t^T) \exp\{-\Lambda_{t^T} - B(T-t^T)\lambda_T\} \quad \text{where} \quad t^T := t \wedge T.$$

In fact, σ in (6) is not even of the form $\sigma(t, Z_t)$.

Φ-martingale model. In such models, the Bernoulli variable of interest is $B = \mathbb{1}_{\{\Phi^{-1}(z)+I_\infty>0\}}$ where the Brownian motion W governing the stochastic integral I is $\mathbb{F}$-adapted. We refer to [14] for a discussion about how such models can be applied to price collateralized debt obligation (CDO) options and credit valuation adjustment (CVA) under wrong-way risk.

Therefore, $\{\xi > T\} \in \mathcal{F}_\infty$ but $\{\xi > T\} \notin \mathcal{F}_t$ for all $t < \infty$. The boundaries of $Z = S^{\mathbb{F}}(T)$ are natural and Z is time-homogeneous. All assumptions of Theorem 3 are met, giving the Φ-martingale of Section 5.1.

Both the Cox and the Φ-martingale models belong to the reduced-form family, but they differ in that the Azéma supermartingale $S_t^{\mathbb{F}}(t)$ is non-increasing in the former, but features a martingale part in the latter. This specificity explains why the Φ-martingale model behaves very differently from Cox. To see this, recall that the Φ-martingale model yields

$$dZ_t = \eta\phi(\Phi^{-1}(Z_t))dW_t, \tag{7}$$

i.e., the corresponding diffusion coefficient is not killed as of $t = T$, which is a major difference with the above two families of models: the indicator $\mathbb{1}_{\{t<T\}}$ appears in (5) and (6) but not in (7). Economically speaking, this means that the agent cannot observe ξ in $\mathbb{F}$, as in Cox. However, in contrast with Cox, the likelihood of the event $\{\xi > T\}$ is continuously updated after T according to the information collected in $\mathbb{F}$. In particular, the so-called *immersion property* (also known as *H hypothesis*) holds in Cox but not in the Φ-martingale case; see [8] and [14] for details.

Model	$B \in \mathcal{F}_\infty$?	Boundaries	Z Markov?	Applicable result
Structural	Yes	Exit	No	
(Merton)	$(B \in \mathcal{F}_T)$	(accessible)	(inhomogeneous)	Section 5.4
Cox	No	Natural	No	
	$(B \notin \mathcal{F}_t, \forall t \leq \infty)$	(inaccessible)	No	—
Φ-martingale	Yes	Natural	Yes	
	$(B \notin \mathcal{F}_t, \forall t < \infty)$	(inaccessible)	Yes	Theorem 3

Table 2: Summary of the three default time models. The column "$B \in \mathcal{F}_\infty$?" indicates whether the Bernoulli variable $B = \mathbb{1}_{\{\xi > T\}}$ is measurable with respect to the market filtration at infinity. Theorem 3 applies when Z is a continuous time-homogeneous Markov process on a Brownian filtration.

7. Conclusion

This paper provides a detailed analysis of $[a, b]$-martingales, defined as one-dimensional martingales evolving between the constant bounds $a < b$. The focus is set on time-homogeneous diffusion martingale. We show that all such processes have a generalized Bernoulli $a+(b-a)B$ asymptotic distribution. Feller theory reveals that such martingales may or may not attain the bounds in finite time, depending on the integrability of the diffusion coefficient σ near the boundaries. However, even if the bounds are attainable, the Bernoulli distribution is truly asymptotic, in the sense that the probability that the hitting time is greater than any time T, $0 < T < \infty$ is strictly positive, which is not necessarily the case when the diffusion coefficient is time-inhomogeneous. Finally, we establish an (almost) equivalence between such martingales and Bernoulli-Doob martingales: any homogeneous $[0, 1]$-martingale admits a Bernoulli-Doob representation, and any one-dimensional continuous Markov martingale adapted to a Brownian filtration can be written as a time-homogeneous diffusion. Various tractable examples are provided to illuminate the theory. Finally, our theory is connected to different families of default models, making the link between shapes of the diffusion coefficient and credit risk frameworks.

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Appendix A. Feller Boundary Classification

This appendix recalls Feller's boundary classification for one-dimensional diffusions in natural scale, specialised to the driftless setting of this paper. The standard references are Karlin and Taylor [11, pp. 228–250], Borodin and Salminen [2, pp. 14–15], and Rogers and Williams [13].

Proposition 3 (Feller boundary classification)**.** *Consider* $dZ_t = \sigma(Z_t)\,dW_t$ *on* $[a,b]$ *with* $Z_0 = z \in (a,b)$*,* $\sigma \in C^2([a,b])$*,* $\sigma > 0$ *on* (a,b)*,* $\sigma(a) = \sigma(b) = 0$*. Since the drift is zero, the diffusion is in natural scale* $s(x) = x$ *with speed measure density* $m(x) = 2/\sigma^2(x)$*. Fix an interior reference point* $z \in (a,b)$ *and define*

$$N_a := \int_a^z (z-x)\sigma^{-2}(x)\,dx\,, \qquad \Sigma_a := \int_a^z (x-a)\sigma^{-2}(x)\,dx\,,$$

and analogously N_b*,* Σ_b *at the right boundary* b*. The boundary type is determined by the following table.*

N_a	Σ_a	*Boundary type of* a
$< \infty$	$< \infty$	*Regular: attainable and startable*
$< \infty$	$= \infty$	*Exit: attainable, not startable*
$= \infty$	$< \infty$	*Entrance: not attainable, startable*
$= \infty$	$= \infty$	*Natural: not attainable, not startable*

Here $N_a < \infty$ *means* $\mathbb{P}_x(\tau_a < \infty) > 0$ *for all* $x \in (a,b)$*;* $N_a = \infty$ *means* $\tau_a = \infty$ *a.s.* $\Sigma_a < \infty$ *means* a *is an admissible starting point in the diffusion sense;* $\Sigma_a = \infty$ *means it is not. The same table applies to the right boundary* b*.*

Appendix B. Four Examples of Boundary Classification

We verify the four cases of Table 1 using Proposition 3. For each case we record the local asymptotics of σ^{-2} near each endpoint and determine the finiteness of (Σ, N).

Case 1: $\sigma(x) = (x(1-x))^{1/4}$ *(both endpoints regular)*

$\sigma^{-2}(x) = 1/\sqrt{x(1-x)}$. Near 0: $(z-x)\sigma^{-2}(x) = \mathcal{O}(x^{-1/2})$ and $x\sigma^{-2}(x) = \mathcal{O}(x^{1/2})$, so both Σ_0 and N_0 are finite. By symmetry the same holds at 1. Both endpoints are regular.

Closed-form primitive: using the substitution $x = \sin^2\theta$,

$$\int \frac{x-z}{\sqrt{x(1-x)}}\,dx = (1-2z)\arcsin\sqrt{x} - \sqrt{x(1-x)} + C\,.$$

Both endpoint limits are finite, confirming $(\Sigma_0, N_0) = (\text{finite}, \text{finite})$ and $(\Sigma_1, N_1) = (\text{finite}, \text{finite})$.

Case 2: $\sigma(x) = x(1-x)$ *(both endpoints natural)*

$\sigma^{-2}(x) = 1/[x^2(1-x)^2]$. Near 0: $(z-x)\sigma^{-2}(x) \sim Cx^{-2}$ and $x\sigma^{-2}(x) \sim x^{-1}$, so $\Sigma_0 = N_0 = \infty$. By symmetry both endpoints are natural.

Case 3: $\sigma(x) = \sqrt{x(1-x)}$ *(both endpoints exit)*

$\sigma^{-2}(x) = 1/[x(1-x)]$. Near 0: $(z-x)\sigma^{-2}(x) \sim zx^{-1}$ and $x\sigma^{-2}(x) \sim 1$, so $\Sigma_0 = \infty$ and $N_0 < \infty$. By symmetry both endpoints are exit. Since $\sigma(0) = \sigma(1) = 0$, the process is absorbed upon hitting.

Case 4: $\sigma(x) = x^{1/4}(1-x)$ *(left regular, right natural)*

$\sigma^{-2}(x) = x^{-1/2}(1-x)^{-2}$. Near 0: $\Sigma_0 < \infty$ and $N_0 < \infty$ (regular). Near 1: $(x-z)\sigma^{-2}(x) \sim (1-x)^{-2}$ and $(1-x)\sigma^{-2}(x) \sim (1-x)^{-1}$, so $\Sigma_1 = N_1 = \infty$ (natural).

Appendix C. Moment Recursion for the Jacobi Martingale

For the Jacobi martingale $\sigma(x) = \sqrt{\eta x(1-x)}$, the conditional moments $M_n(t) := \mathbb{E}[Z_t^n \mid Z_s = z]$ satisfy a closed-form recursion that provides an independent verification of Bernoulli convergence (Corollary 1) for this specific model, and also matches the results of [7] in the limit $\kappa \to 0$.

Let $\lambda_n := \eta n(n-1)/2$ with $\lambda_0 = \lambda_1 = 0$.

Theorem 5. *Suppose* $(M_n(t))_{n\geq 0}$ *satisfies for* $n \geq 1$ *the integral recursion*

$$M_n(t) = z^n e^{-\lambda_n(t-s)} + \lambda_n \int_s^t e^{-\lambda_n(t-u)} M_{n-1}(u)\, du\,,$$

with $M_0 \equiv 1$ *and* $M_1 \equiv z$. *Then:*

(1) ***(ODE)*** $\frac{d}{dt} M_n(t) = \lambda_n(M_{n-1}(t) - M_n(t))$, $M_n(s) = z^n$.

(2) ***(Matrix exponential)*** *With* B *the lower-bidiagonal matrix* $B_{n,n} = -\lambda_n$, $B_{n,n-1} = \lambda_n$: $M(t) = \exp((t-s)B)\, M(s)$.

(3) ***(Closed form)*** *For* $0 \leq m \leq n$:

$$M_n(t) = \sum_{m=0}^{n} z^m \sum_{j=m}^{n} e^{-\lambda_j(t-s)} \frac{\prod_{r=m+1}^{n} \lambda_r}{\prod_{\substack{r=m\\ r\neq j}}^{n} (\lambda_j - \lambda_r)}\,.$$

(4) ***(Examples)*** $M_0 = 1$, $M_1 = z$, $M_2(t) = z + z(z-1)e^{-\eta(t-s)}$,

$$M_3(t) = z + \tfrac{3}{2} z(z-1)e^{-\eta(t-s)} + \left(z^3 - \tfrac{3}{2}z(z-1) - z\right)e^{-3\eta(t-s)}\,.$$

As $t \to \infty$, $M_n(t) \to z$ *for all* $n \geq 1$, *confirming that* $Z_\infty \sim \mathrm{Ber}(z)$.

Proof. **(1)** Differentiate the integral recursion using Leibniz' rule; the integral term cancels exactly, leaving the stated ODE. Evaluating at $t = s$ gives $M_n(s) = z^n$.

(2) The ODE system $\frac{d}{dt}M(t) = BM(t)$ has unique solution $M(t) = \exp((t - s)B)M(s)$.

(3) For finite N, the $(N+1) \times (N+1)$ truncation B_N is lower-bidiagonal with explicitly computable matrix exponential entries (obtained by diagonalising each 2×2 block or by repeated variation of constants). The formula stabilises as $N \to \infty$ since each entry depends only on finitely many λ_j.

(4) For $n = 2$: $\frac{d}{dt}M_2 = \eta(z - M_2)$, $M_2(s) = z^2$, giving $M_2(t) = z + (z^2 - z)e^{-\eta(t-s)}$. For $n = 3$: substitute M_2 and integrate.

The convergence $M_n(t) \to z$ follows since $e^{-\lambda_j(t-s)} \to 0$ for all $j \geq 2$ (as $\lambda_j > 0$ for $j \geq 2$), leaving only the constant term z. □