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Optimal Growth Models and the Lagrange Multiplier*

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Abstract

We provide sufficient conditions on the objective functional and the constraint functions under which the Lagrangean can be represented by a ℓ^1 sequence of multipliers in infinite horizon discrete time optimal growth models.

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1 Introduction

The classical optimal growth models have played a central role in the theory of economic dynamics. The model is described by the presence of a planner maximizing the sum of discounted utilities of consumption subject to a convex one-sector production set. This simple and elegant model proves to be useful in explaining many aspects of capital accumulation.¹

The study of this type of optimal capital accumulation problems can be considerably enhanced by means of Lagrange multipliers techniques.² Lagrange multipliers have been widely used in solving finite dimensional constrained optimization problems as they also facilitate the analysis on the properties of the solution [see Rockafellar (1976)]. However, in infinite horizon optimal growth models, these multipliers will typically belong to an infinite dimensional decision space. Therefore, the questions whether the Lagrange multipliers exist and whether they can be represented by a summable sequence arise.³

In this paper we extend the Lagrangean to infinite dimensional spaces and provide sufficient conditions for Lagrangean to be represented by a summable sequence of multipliers in infinite discrete time horizon optimal growth models. We take the approach of placing restrictions directly on the asymptotic behaviors of the objective functional and the constraint functions. In this sense, our work is closely related to Dechert (1982).

Considering the set of problems in the form of: $\inf \{F(x) \mid \phi(x) \leq 0\}$, where $F : \ell^\infty \rightarrow \mathbb{R}$ and $\phi : \ell^\infty \rightarrow \ell^\infty$ are convex functions and $x \in \ell^\infty$, Dechert (1982) extends the Lagrangean to infinite dimensional spaces. However, a limitation to the direct use of his results in infinite horizon optimal growth models stems from the fact that he assumes that the objective functional and the constraint functions are real valued on ℓ^∞ . That avoids the analysis of the cases where the usual Inada type conditions are assumed for the utility (in this case there is no extension of utility value to $\mathbb{R}$) and the production functions.

The paper is organized as follows. In section 2, we first present the problem in general form and give the extension of the Kuhn-Tucker theorem where the multipliers are represented in $(\ell^\infty)'$, the dual space of ℓ^∞ . Second, we present the sufficient conditions for having an ℓ^1 representation of the multipliers and state our main result. Finally, section 3 is devoted to the applications to a one-sector optimal growth model and an incentive-constrained optimal growth model.

¹See Aliprantis, Border and Burkinshaw (1990, 1997) for some recent studies and Stokey et al. (1989) for more details.

²Dynamic programming is another technique used in analyzing such models; see for instance Alvarez and Stokey (1998), Le Van and Morhaim (2002). See also Weitzman (1973) for the close connection between duality theory for infinite horizon convex models and dynamic programming.

³The existence of separating vectors and whether they can be represented by a sequence of real numbers in infinite dimensional spaces have been widely studied; see for instance Bewley (1972), Majumdar (1972), Colonius (1983), Rustichini (1998) and McKenzie (1986).

2 Model

Consider the following intertemporal optimization problem (P):

$$\begin{aligned} \min \quad & F(x) \\ \text{s.t.} \quad & \phi(x) \leq 0 \\ & x \in \ell^\infty \end{aligned}$$

where $F : \ell^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$ and $(\phi_t)_{t=0,\dots,\infty} : \ell^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$ are convex functions. Let $\phi(x)$ denote for $(\phi_t(x))_{t=0,\dots,\infty}$. We define $C = \text{dom}(F) = \{x \in \ell^\infty \mid F(x) < +\infty\}$, $\Gamma = \text{dom}(\phi) = \{x \in \ell^\infty \mid \phi_t(x) < +\infty, \forall t\}$ and assume that $C \neq \emptyset$.

By means of an application of the separation theorem on Banach spaces, we give the following extension of the Kuhn-Tucker theorem in which the multipliers are represented in $(\ell^\infty)'$:

Theorem 1 *Let x^* be a solution of (P). Assume that $\forall x \in \Gamma$ we have $\phi(x) \in \ell^\infty$. Suppose $\exists x^0 \in \ell^\infty$ such that*

$$x^0 \in C, \sup_t \phi_t(x^0) < 0 \text{ (Slater condition)}.$$

Then $\exists \Lambda^ \in (\ell^\infty)'$ such that*

$$\begin{aligned} (i) \quad & \Lambda^* \geq 0, \\ (ii) \quad & \Lambda^* \phi(x^*) = 0, \\ (iii) \quad & F(x) + \Lambda^* \phi(x) \geq F(x^*), \forall x \in (\Gamma \cap C). \end{aligned}$$

Conversely, if $\exists x^ \in (\Gamma \cap C)$, $\Lambda^* \in (\ell^\infty)'$ such that (i) – (iii) hold, then x^* is a solution.*

Proof. Define the set Z as:

$$Z = \{ (\rho, z) \in \mathbb{R} \times \ell^\infty : \exists x \in \ell^\infty, \rho > F(x) - F(x^*), z_t > \phi_t(x), \forall t \geq 0 \}.$$

Z is non-empty (e.g. take $\rho = F(x_0) - F(x^*) + \varepsilon$ and $z_t = 0, \forall t \geq 0$, where $\varepsilon > 0$ and x_0 satisfies the Slater condition). It is obvious that Z is convex and it satisfies $Z \cap (\mathbb{R}_- \times \ell^\infty) = \emptyset$. Since $\text{int}(\ell^\infty) \neq \emptyset$, we can apply the separation theorem: $\exists (c, \Lambda) \in \mathbb{R} \times (\ell^\infty)' \setminus \{0\}$ such that:

$$\forall (\rho, z) \in Z, \forall (\rho', z') \in (\mathbb{R}_- \times \ell^\infty) : c\rho' + \Lambda z' \leq c\rho + \Lambda z. \quad (1)$$

In particular, this relation implies that:

$$c\rho + \Lambda z \geq 0, \forall (\rho, z) \in Z. \quad (2)$$

Note that $(\rho, z) \in Z$ if $\rho > 0$ and $z_t > 0, \forall t$ (take $x = x^*$ in the definition of Z).

We first want to prove that $c \geq 0$. On the contrary, if $c < 0$ then keeping z fixed and letting $\rho \rightarrow +\infty$ in (2), we obtain a contradiction. Hence $c \geq 0$.

Now we claim that $\Lambda \in (\ell^\infty)_+^1$, i.e. $\Lambda z \geq 0, \forall z \in \ell_+^\infty$. For that, take $z \in \ell_+^\infty$. Let $\theta = (1, 1, 1, \dots, 1, \dots)$ and $\rho = 1$. Then, $(\rho, \theta + \mu z) \in Z$ for any $\mu > 0$. From relation (2), we notice that $c\rho + \Lambda(\theta + \mu z) \geq 0, \forall \mu > 0$ or equivalently $\mu^{-1}(c\rho + \Lambda\theta) + \Lambda z \geq 0, \forall \mu > 0$. Letting $\mu \rightarrow +\infty$, we obtain that $\Lambda z \geq 0, \forall z \in \ell_+^\infty$. Hence, $\Lambda \in (\ell^\infty)_+^1$.

We now prove that $c > 0$. Assume on the contrary that $c = 0$. In this case, we know that $\Lambda \neq 0$. Note that the closure of Z is:

$$\bar{Z} = \{ (z_0, z) \in \mathbb{R} \times \ell^\infty : \exists x \in \ell^\infty, z_0 \geq F(x) - F(x^*), z_t \geq \phi_t(x), \forall t \}.$$

Observe that we still have,

$$c\rho + \Lambda z \geq 0, \forall (\rho, z) \in \bar{Z}. \quad (3)$$

As $(F(x^0) - F(x^*), \phi(x^0)) \in \bar{Z}$, from (3), we obtain a contradiction:

$$0 > \Lambda \phi(x^0) \geq 0,$$

due to the fact that $\sup_t \phi_t(x^0) < 0$.

Let $\Lambda^* = c^{-1}\Lambda$ in order to obtain the claim (i) of the theorem. Now, we prove the claim (ii) of the theorem. Since $\Lambda^* \geq 0, (0, \phi(x^*)) \in \bar{Z}$ and $\phi(x^*) \in \ell_-^\infty$, we have on the one hand $\Lambda^* \phi(x^*) \leq 0$ and $\Lambda^* \phi(x^*) \geq 0$ on the other. Hence, $\Lambda^* \phi(x^*) = 0$.

Finally, take $z_0 = F(x) - F(x^*)$ and $z = \phi(x), \forall x \in (\Gamma \cap C)$. Then, from inequality (3), we obtain that:

$$\begin{aligned} F(x) - F(x^*) + \Lambda^* \phi(x) &\geq 0 \Rightarrow \\ F(x) + \Lambda^* \phi(x) &\geq F(x^*) + \Lambda^* \phi(x^*), \forall x \in (\Gamma \cap C). \end{aligned}$$

As for the converse, let $x \in \ell^\infty$ satisfy $\phi(x) \leq 0$. Then, $x \in \Gamma$. If $x \notin C$ then $F(x) = +\infty$ and of course we have $F(x) \geq F(x^*)$. Now assume that $x \in C$. Relations (i), (ii) and (iii) imply that

$$F(x) \geq F(x) + \Lambda^* \phi(x) \geq F(x^*) + \Lambda^* \phi(x^*) = F(x^*).$$

Thus, we have proved that x^* is an optimal solution. ■

Corollary 1 *Let the assumptions of Theorem 1 be satisfied. If x^* (solution to the optimization problem P) satisfies that $x^* \in \text{int}(\Gamma \cap C)$, then we have*

$$\forall x \in \ell^\infty, F(x) + \Lambda^* \phi(x) \geq F(x^*) + \Lambda^* \phi(x^*)$$

where $\Lambda^* \in (\ell^\infty)_+^1$ and $\Lambda^* \phi(x^*) = 0$.

Proof. We know that $\exists \Lambda^* \in (\ell^\infty)_+^I$ such that:

$$\forall x \in (\Gamma \cap C), F(x) + \Lambda^* \phi(x) \geq F(x^*) + \Lambda^* \phi(x^*) \text{ and } \Lambda^* \phi(x^*) = 0.$$

Now assume that $x \in \ell^\infty$. As $x^* \in \text{int}(\Gamma \cap C)$, we have $(\mu x^* + (1 - \mu)x) \in \text{int}(\Gamma \cap C)$ where $\mu \in [0, 1]$ and μ approaches to 1. Then, we can write that :

$$F(\mu x^* + (1 - \mu)x) + \Lambda^* \phi(\mu x^* + (1 - \mu)x) \geq F(x^*) + \Lambda^* \phi(x^*).$$

As F and ϕ are convex and $\Lambda^* \geq 0$, we obtain that:

$$\mu F(x^*) + (1 - \mu) F(x) + \mu \Lambda^* \phi(x^*) + (1 - \mu) \Lambda^* \phi(x) \geq F(x^*) + \Lambda^* \phi(x^*) \Rightarrow$$

$$F(x) + \Lambda^* \phi(x) \geq F(x^*) + \Lambda^* \phi(x^*), \forall x \in \ell^\infty. \blacksquare$$

2.1 An ℓ^1 representation of the multipliers

In this section we provide an ℓ^1 representation of the multipliers. As F and ϕ are convex and a Slater condition on ϕ is satisfied, by means of a standard separation argument on Banach spaces we have shown the existence of a $\Lambda^* \in (\ell^\infty)_+^I$. However, we know that $(\ell^1)^I = \ell^\infty$ but $(\ell^\infty)^I \neq \ell^1$.

Denoting the set of bounded linear functionals generated by the purely finitely additive measures on integers with ℓ^s [see Rudin (1973) for a broad analysis of those functionals], we can identify the dual space of ℓ^∞ with a direct sum of two subspaces: $(\ell^\infty)^I = \ell^1 \oplus \ell^s$. Then, for each $\Lambda^* \in (\ell^\infty)_+^I$, we adopt the notation $\Lambda^* = \Lambda^1 + \Lambda^s$ where $\Lambda^1 \in \ell^1$ and $\Lambda^s \in \ell^s$. Our aim is to avoid the ℓ^s part of $\Lambda^* \in (\ell^\infty)_+^I$ so that each constraint $\phi_t(x)$ will have a multiplier λ_t associated with it. To that end, we will take the approach of presenting sufficient conditions so that $\Lambda^s = 0$.

Definition 1 Let $x \in \ell^\infty$, $y \in \ell^\infty$ and $T \in \mathbb{N}$. A sequence $x^T(x, y)$ is defined by:

$$x^T(x, y) = \begin{cases} x_t, & t \leq T, \\ y_t, & t > T. \end{cases}$$

We make the following assumptions which will prove to be useful in our analysis:

Assumption 1 If $x \in C$, $y \in \ell^\infty$ satisfy $\forall T \geq T_0, x^T(x, y) \in C$ then

$$F(x^T(x, y)) \rightarrow F(x) \text{ when } T \rightarrow \infty.$$

Assumption 2 If $x \in \Gamma$, $y \in \Gamma$ and $x^T(x, y) \in \Gamma, \forall T \geq T_0$ then

$$(a) \phi_t(x^T(x, y)) \rightarrow \phi_t(x), \text{ as } T \rightarrow \infty,$$

$$(b) \exists M \text{ s.t. } \forall T \geq T_0, \|\phi(x^T(x, y))\| \leq M,$$

$$(c) \forall N \geq T_0, \lim_{t \rightarrow \infty} [\phi_t(x^N(x, y)) - \phi_t(y)] = 0.$$

We also adopt the following conventions [see Rockafellar (1976)] in our analysis:

$$\begin{aligned} 0 (+\infty) &= (+\infty) 0 = 0, \\ \alpha (+\infty) &= (+\infty) \alpha = +\infty \text{ if } \alpha > 0. \end{aligned}$$

We have the following remark on the assumptions we have made:

Remark 1 *Assumption 1 is satisfied if F is continuous in $C = \text{dom}(F)$ for the product topology. Assumption 2-(a) is the asymptotically non-anticipatory assumption (ANA) and Assumption 2-(c) is true if the asymptotically insensitivity (AI) assumption holds [see Dechert (1982)] for ϕ in $\Gamma = \text{dom}(\phi)$. Obviously, Assumption 2-(b) is satisfied when $\Gamma = \text{dom}(\phi) = \ell^\infty$ and ϕ is continuous.*

More precisely, the restrictions on ϕ that we use lead to the following two lemmas :

Lemma 1 *Under Assumption 2-(c), we have:*

$$\Lambda^s \phi(x^N(x, y)) = \Lambda^s \phi(y), \text{ for any } N \geq T_0.$$

Proof. Follows directly from the property of ℓ^s . Indeed, for any $\Lambda^s \in \ell^s$ and $x \in \ell^\infty$ with $\lim_{t \rightarrow \infty} x(t) = 0$, $\Lambda^s x = 0$. Given N , as $\lim_{t \rightarrow \infty} [\phi_t(x^N(x, y)) - \phi_t(y)] = 0$, we have

$$\Lambda^s [\phi(x^N(x, y)) - \phi(y)] = 0,$$

completing the proof. ■

Lemma 2 *Let Assumptions (2-a) and (2-b) be satisfied. Then, $\forall \Lambda^1 \in \ell^1$,*

$$\lim_{T \rightarrow \infty} \Lambda^1 \phi(x^T(x, y)) = \Lambda^1 \phi(x).$$

Proof. We have for $T \geq T_0$,

$$\begin{aligned} |\Lambda^1 \phi(x^T(x, y)) - \Lambda^1 \phi(x)| &\leq \\ \sum_{t=0}^N |\lambda_t| |\phi_t(x^T(x, y)) - \phi_t(x)| &+ \|\phi(x^T(x, y)) - \phi(x)\| \sum_{t=N+1}^{\infty} |\lambda_t|. \end{aligned}$$

By Assumption (2-b), we know that $\exists M$ such that:

$$\forall T, \|\phi(x^T(x, y)) - \phi(x)\| \leq M' = M + \|\phi(x)\|.$$

Now, $\forall \varepsilon > 0$, $\exists N_0$ such that:

$$\sum_{t=N_0+1}^{\infty} |\lambda_t| < (2M')^{-1} \varepsilon,$$

and $\exists T^t \geq T_0$, $\forall T > T^t$,

$$|\lambda_t| |\phi_t(x^T(x, y)) - \phi_t(x)| < (2N_0)^{-1} \varepsilon.$$

Hence, $\forall T > \max_{1 \leq t \leq N_0} \{T^t\}$,

$$|\Lambda^1 \phi(x^T(x, y)) - \Lambda^1 \phi(x)| < N_0 (2N_0)^{-1} \varepsilon + M' (2M')^{-1} \varepsilon = \varepsilon,$$

completes the proof. ■

With this structure in mind we can proceed with our main result:

Theorem 2 *Let Assumptions 1-2 and the assumptions of Theorem 1 be satisfied. Let $\text{dom}(\phi_t) = \text{dom}(\phi)$, $\forall t$. Let x^* be a solution and x^0 satisfy the Slater condition. If $x^T(x^*, x^0)$ belongs to $(\Gamma \cap C)$, $\forall T \geq T_0$, then there exists $\Lambda \in \ell_+^1$ such that:*

$$\forall x \in \ell^\infty, F(x) + \Lambda \phi(x) \geq F(x^*) + \Lambda \phi(x^*)$$

and

$$\Lambda \phi(x^*) = 0.$$

Proof. By means of Theorem 1, we know that $\exists \Lambda^* \in (\ell^\infty)'$, $\Lambda^* \geq 0$ such that

$$\forall x \in (\Gamma \cap C), F(x) + \Lambda^* \phi(x) \geq F(x^*) + \Lambda^* \phi(x^*)$$

and

$$\Lambda^* \phi(x^*) = 0.$$

Write $\Lambda^* = \Lambda^1 + \Lambda^s$ where $\Lambda^1 \in \ell_+^1$ and $\Lambda^s \in \ell^s$. As $\Lambda^* \geq 0$, we have $\Lambda^1 \phi(x^*) = 0$ and $\Lambda^s \phi(x^*) = 0$. For a given $N \geq T_0$, $x^N(x^*, x^0) \in (\Gamma \cap C)$. Hence,

$$F(x^N(x^*, x^0)) + \Lambda^1 \phi(x^N(x^*, x^0)) + \Lambda^s \phi(x^N(x^*, x^0)) \geq F(x^*).$$

Note that $\Lambda^1 \phi(x^N(x^*, x^0)) \rightarrow \Lambda^1 \phi(x^*)$ when $N \rightarrow \infty$ by Lemma 2 and $\Lambda^s \phi(x^N(x^*, x^0)) = \Lambda^s \phi(x^0)$ by Lemma 1. By Assumption 1, we know that $F(x^N(x^*, x^0)) \rightarrow F(x^*)$ when $N \rightarrow \infty$. If $\Lambda^s \neq 0$, by taking the limits in the previous inequality we obtain $F(x^*) > F(x^*) + \Lambda^s \phi(x^0) \geq F(x^*)$: a contradiction. Hence, $\Lambda^s = 0$ and $\Lambda = \Lambda^1 \in \ell_+^1$. We have proved that:

$$F(x) + \Lambda \phi(x) \geq F(x^*), \forall x \in (\Gamma \cap C), \Lambda \in \ell_+^1$$

where $\Lambda \phi(x)$ can be written as:

$$\Lambda \phi(x) = \sum_{t=0}^{\infty} \lambda_t^1 \phi_t(x), \lambda_t^1 \geq 0, \forall t.$$

Let $I = \{t \mid \lambda_t^1 > 0\}$. We have

$$F(x) + \sum_{t \in I} \lambda_t^1 \phi_t(x) \geq F(x^*).$$

If $I = \emptyset$, then $F(x) \geq F(x^*)$, $\forall x \in C$. In this case, if $x \notin C$, then $F(x) = +\infty$ and the inequality still holds. Finally, what happens if $I \neq \emptyset$? If $x \notin \Gamma$ then

$\sum_{t \in I} \lambda_t^1 \phi_t(x) = +\infty$. Since $F(x) \in \mathbb{R} \cup \{+\infty\}$, we have $F(x) + \sum_{t \in I} \lambda_t^1 \phi_t(x) = +\infty$ so that the inequality $F(x) + \Lambda\phi(x) \geq F(x^*)$ holds. On the other hand, if $x \in \Gamma$ and $x \notin C$ we have also $F(x) + \sum_{t \in I} \lambda_t^1 \phi_t(x) = +\infty$ and $F(x) + \Lambda\phi(x) \geq F(x^*)$. Thus, $\forall x \in \ell^\infty$, $F(x) + \Lambda\phi(x) \geq F(x^*)$. ■

3 Applications

Example 1 One-sector optimal growth model

Production possibilities are represented by a production function f and preferences are described by a one period reward function u and a discount factor β . At period t , a single commodity is used as capital to produce output and the output is either consumed or saved as capital until the next period. This process repeats ad infinitum. The problem is formalized as follows:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} - f(k_t) &\leq 0, \quad \forall t \\ c_t &\geq 0, \quad \forall t \geq 0 \\ k_t &\geq 0, \quad \forall t \geq 1 \\ k_0 &> 0, \text{ given.} \end{aligned}$$

Definition 2 For any initial condition $k_0 > 0$, when $\mathbf{k} = (k_1, k_2, \dots, k_t, \dots)$ is such that $0 \leq k_{t+1} \leq f(k_t)$ for all t , we say it is feasible from k_0 and the class of feasible accumulation path is denoted by $\prod(k_0)$.

Definition 3 A consumption sequence $\mathbf{c} = (c_0, c_1, c_2, \dots, c_t, \dots)$ is feasible from $k_0 > 0$ if there exists a sequence $\mathbf{k} \in \prod(k_0)$ that satisfies $0 \leq c_t \leq f(k_t) - k_{t+1}$, for all t ; and the class of feasible consumption sequences from k_0 is denoted by $\sum(k_0)$.

In section 2, actually ℓ^∞ denotes the space of uniformly bounded sequences of $\mathbb{R}^n$, $n \geq 1$. Now, we will denote by ℓ^∞ the space of sequences $\{(c_t, k_t)\}$ with $\sup_t |c_t| < +\infty$ and $\sup_t |k_t| < +\infty$. We make the following assumptions:

Assumption 3 The one period reward function u is concave in $\mathbb{R}_+$ and strictly increasing. Further, $\beta \in (0, 1)$ and $u'(0) \leq +\infty$.

Assumption 4 The gross production function f is concave, differentiable in $\mathbb{R}_+$ and strictly increasing. Further, $f(0) = 0$, $1 < f'(0) \leq +\infty$ and $f'(\infty) < 1$.

Assumption 5 In order to extend u and f on $\mathbb{R}$, we have $u(c) = -\infty$ if $c < 0$ and $f(x) = -\infty$ if $x < 0$.

The assumption of $f'(\infty) < 1$ implies the existence of a maximum sustainable capital stock $\bar{k}$ for which $f(k) < k$ iff $k > \bar{k}$ and $f(\bar{k}) = \bar{k}$. Further, if $f'(0) > 1$, then for all $k_0 > 0$, there must be some $0 < k' \leq k_0$ such that $f(k') > k'$. Then, for all $k_0 > 0$, there is a feasible interior consumption-accumulation plan described by $k' > 0$ and $c' = f(k') - k'$.

Note that this problem can be written as

$$\begin{aligned} \min \quad & F(\mathbf{x}) \\ \text{s.t.} \quad & \phi(\mathbf{x}) \leq 0 \\ & \mathbf{x} \in \ell^\infty \times \ell^\infty \end{aligned}$$

where $\mathbf{x} = (\mathbf{c}, \mathbf{k})$, $F(\mathbf{x}) = -\sum_{t=0}^{\infty} \beta^t u(c_t)$ and $F : \ell^\infty \times \ell^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$. We denote $\phi_t = (\phi_t^1, \phi_t^2, \phi_t^3)$ where $\phi_t^1(\mathbf{x}) = c_t + k_{t+1} - f(k_t)$, $\phi_t^2(\mathbf{x}) = -c_t$, $\forall t \geq 0$, and $\phi_{t+1}^3(\mathbf{x}) = -k_{t+1}$, $\forall t \geq 0$. Accordingly, $C = \text{dom}(F) = \ell_+^\infty \times \ell^\infty$, $\Gamma = \text{dom}(\phi) = \ell^\infty \times \ell_+^\infty = \text{dom}(\phi_t)$, $\forall t$ and $C \cap \Gamma = \ell_+^\infty \times \ell_+^\infty$.

Now we will show that Theorem 2 is satisfied for this optimal growth problem under Assumptions 3 to 5:

1. Since $f'(0) > 1$, for any $k_0 > 0$, there exists k' such that $0 < k' + \varepsilon < f(k_0)$ and $0 < k' + \varepsilon < f(k')$ with $\varepsilon > 0$. Let $\mathbf{k}^0 = (k_0, k', k', \dots)$, $\mathbf{c}^0 = (\varepsilon, \varepsilon, \varepsilon, \dots)$ and $\mathbf{x}^0 = (\mathbf{c}^0, \mathbf{k}^0)$. Note that $\sup_t \phi_t(\mathbf{c}^0) < 0$. Thus, **Slater condition** is verified.
2. Consider any $\tilde{\mathbf{x}} \in C$, $\tilde{\mathbf{x}} \in \ell^\infty \times \ell^\infty$ such that $\forall T, \mathbf{x}^T(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) \in C$. We know that:

$$F(\mathbf{x}^T(\tilde{\mathbf{x}}, \tilde{\mathbf{x}})) = -\sum_{t=0}^T \beta^t u(\tilde{c}) - \sum_{t=T+1}^{\infty} \beta^t u(\tilde{c}).$$

As $\tilde{\mathbf{x}} \in \ell^\infty \times \ell^\infty$, $\sup_t |\tilde{c}_t| < +\infty$. In other words, $\exists a > 0, \forall t, |\tilde{c}_t| \leq a$. Note that,

$$\sum_{t=T+1}^{\infty} \beta^t u(a) = u(a) \sum_{t=T+1}^{\infty} \beta^t \rightarrow 0 \text{ when } T \rightarrow \infty, \text{ as } \beta \in (0, 1).$$

Then, we have

$$F(\mathbf{x}^T(\tilde{\mathbf{x}}, \tilde{\mathbf{x}})) \rightarrow F(\tilde{\mathbf{x}}) \text{ when } T \rightarrow \infty$$

so that **Assumption 1** is satisfied.

3. Let $\mathbf{x} = (\mathbf{c}, \mathbf{k})$, $k_t \geq 0, \forall t$ and $\mathbf{y} = (\mathbf{c}', \mathbf{k}')$, $k'_t \geq 0, \forall t$. We have:

$$\phi_t^1(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) = \begin{cases} c_t + k_{t+1} - f(k_t) < +\infty, & \text{if } t < T \\ c'_t + k'_{t+1} - f(k'_t) < +\infty, & \text{if } t > T \end{cases}$$

and

$$\phi_T^1(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) = c_T + k_{T+1} - f(k_T) < +\infty.$$

Then, $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in \Gamma$, $\forall T$. It is evident that if $T > t + 1$, we have:

$$\phi_t^1(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) = \phi_t^1(\mathbf{x}).$$

It is clear that the same is also true for $\phi_t^2(\mathbf{x}^T(\mathbf{x}, \mathbf{y}))$ and $\phi_t^3(\mathbf{x}^T(\mathbf{x}, \mathbf{y}))$, as well. Thus, **Assumption 2-(a)** is satisfied.

4. For any $\mathbf{x} \in \Gamma$, $\mathbf{y} \in \Gamma$, let

$$M_1 = \|\phi(\mathbf{x})\|^\infty = \sup_{t \geq 0} |\phi_t(\mathbf{x})|$$

and

$$M_2 = \|\phi(\mathbf{y})\|^\infty = \sup_{t \geq 0} |\phi_t(\mathbf{y})|.$$

Knowing that

$$\mathbf{x}^T(\mathbf{x}, \mathbf{y}) = \begin{cases} x_t, & t \leq T \\ y_t, & t > T \end{cases}$$

and simply setting $M = \max\{M_1, M_2\}$ lead us to conclude that $\exists M$ such that, $\forall T$, $\left\| \phi \left(x^T \left(\begin{smallmatrix} x \\ \sim \end{smallmatrix}, \begin{smallmatrix} y \\ \sim \end{smallmatrix} \right) \right) \right\| \leq M$. This proves that **Assumption 2-(b)** is satisfied.

5. We know that

$$\forall N, \quad \phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) = \begin{cases} \phi_t(\mathbf{x}), & t \leq N \\ \phi_t(\mathbf{y}), & t > N \end{cases}$$

Let $\hat{t} = N + 1$. Then we have $\phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) = \phi_t(\mathbf{y})$, when $t > \hat{t} > N$. Thus, **Assumption 2-(c)** is satisfied.

6. It is obvious that $x^T(\mathbf{x}^*, \mathbf{x}^0)$ belongs to $\ell_+^\infty \times \ell_+^\infty$, $\forall T$.

Summing up, we have shown that the assumptions of Theorem 2 are fulfilled for our optimal growth problem. Hence there exists $\Lambda \in \ell_+^1$ such that $\forall \mathbf{x} \in \ell^\infty$, if $\mathbf{x}^*$ is optimal, then

$$F(\mathbf{x}) + \Lambda \phi(\mathbf{x}) \geq F(x^*) + \Lambda \phi(x^*) \quad \text{and} \quad \Lambda \phi(x^*) = 0.$$

These establish the following proposition.

Proposition 1 *If $\mathbf{x}^* = (\mathbf{c}^*, \mathbf{k}^*)$ is optimal then there exists $\lambda \in \ell_+^1$ such that $\forall \mathbf{x} = (\mathbf{c}, \mathbf{k}) \in \ell^\infty \times \ell^\infty$,*

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t u(c_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* + k_{t+1}^* - f(k_t^*)) + \sum_{t=0}^{\infty} \lambda_t^2 c_t^* + \sum_{t=0}^{\infty} \lambda_t^3 k_t^* \geq \\ \sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t + k_{t+1} - f(k_t)) + \sum_{t=0}^{\infty} \lambda_t^2 c_t + \sum_{t=0}^{\infty} \lambda_t^3 k_t \end{aligned} \quad (4)$$

and

$$\lambda_t^1 (c_t^* + k_{t+1}^* - f(k_t^*)) = 0, \forall t \quad (5)$$

$$\lambda_t^2 c_t^* = 0, \forall t \quad (6)$$

$$\lambda_t^3 k_t^* = 0, \forall t. \quad (7)$$

As by-products of this proposition, we obtain that the sequence $\{\beta^t u'(c_t^*)\}$ is in ℓ^1 and Euler equation holds for this optimal growth problem. Clearly, the property $\{\beta^t u'(c_t^*)\}$ is in ℓ^1 can be obtained in different ways even without resorting to our approach. In order to obtain that prices belong to ℓ^1 , Le Van and Dana (2003), for instance, use Euler equation and the monotonicity of the optimal paths. In fact, the monotonicity property of optimal paths in the one sector growth models goes back to Dechert and Nishimura (1983) among others. Ekeland and Scheinkman (1986) obtain that $\{\beta^t u'(c_t^*)\}$ is in ℓ^1 under more general settings by considering non-concave infinite horizon problems with unbounded returns. Their analysis does not involve Euler equation and the monotonicity of the optimal paths. Instead, their analysis relies mainly on Fatou's lemma together with some technical assumptions on the utility function (see Assumption P2(c_t), page 221). In our approach, both results are obtained independent of each other. In particular, the proof of the summability of the sequence $\{\beta^t u'(c_t^*)\}$ is immediate from our theorem.

Corollary 2 *Let Assumptions 3-5 be satisfied. Assume that u is strictly concave and continuously differentiable with $u'(0) = +\infty$. Let $\mathbf{x}^* = (\mathbf{c}^*, \mathbf{k}^*)$ be the optimal solution. Then we have:*

(i) *The sequence $\{\beta^t u'(c_t^*)\}$ is in $\ell_+^1 \setminus \{0\}$.*

(ii) *Euler equation:*

$$\forall t, u'(c_t^*) = \beta u'(c_{t+1}^*) f'(k_{t+1}^*) \quad (8)$$

Proof. It is clear that by means of Inada condition ($u'(0) = +\infty$), we have $c_t^* > 0, k_t^* > 0, \forall t$. Thus, $\lambda_t^2 = \lambda_t^3 = 0, \forall t$, from equations (6) and (7).

(i) Define $\mathbf{c}, \mathbf{k}$ as follows:

$$\begin{aligned} k_t &= k_t^*, \forall t, \\ c_t &= c_t^*, \forall t \neq T, \\ c_T &= c_T^* + \varepsilon, \varepsilon \in \mathbb{R} \text{ such that } c_T^* + \varepsilon > 0. \end{aligned}$$

Inequality (4) implies that:

$$\beta^T u(c_T^*) - \lambda_T^1 c_T^* \geq \beta^T u(c_T^* + \varepsilon) - \lambda_T^1 (c_T^* + \varepsilon),$$

for any ε in a neighborhood of zero. This implies:

$$\beta^T u'(c_T^*) - \lambda_T^1 = 0.$$

Since, from Proposition 1, $\lambda \in \ell_+^1$, then $(\beta^T u'(c_T^*)) \in \ell_+^1$.

(ii) Define

$$\begin{aligned} c_t &= c_t^*, \quad \forall t \\ k_t &= k_T^*, \quad \forall t \neq T+1 \\ k_{T+1} &= k_{T+1}^* + \varepsilon, \quad \varepsilon \in \mathbb{R} \text{ such that } k_{T+1}^* + \varepsilon > 0. \end{aligned}$$

By (4), we have:

$$-\lambda_T^1 k_{T+1}^* + \lambda_{T+1}^1 f(k_{T+1}^*) \geq -\lambda_T^1 (k_{T+1}^* + \varepsilon) + \lambda_{T+1}^1 f(k_{T+1}^* + \varepsilon),$$

for any ε in a neighborhood of zero. Therefore,

$$-\lambda_T^1 + \lambda_{T+1}^1 f'(k_{T+1}^*) = 0.$$

From (i), we know that $\lambda_T^1 = \beta^T u'(c_T^*)$. Hence, Euler equation follows. ■

Remark 2 In order to have an ℓ^1 representation of Lagrangean, Dechert (1982) introduces the hypothesis that:

$$\partial F(x^*) \subset \ell^1,$$

together with two constraint conditions: ϕ is asymptotically insensitive (AI) and non-anticipatory (ANA). Contrary to Dechert (1982), the hypothesis $\partial F(x^*) \subset \ell^1$ is not necessarily verified. Indeed, suppose that $f'(0) < \frac{1}{\beta}$. Let u be differentiable in $\mathbb{R}_+$ with $u'(0) = +\infty$. If $\partial F(x^*) \subset \ell^1$, then we will have:

$$\partial F(x^*) = \{(u'(c_0^*), \beta u'(c_1^*), \dots, \beta^t u'(c_t^*), \dots)\}.$$

Hence, F is differentiable at x^* and thus, continuous at x^* (for the topology ℓ^∞). Hence, $\forall \varepsilon > 0, \exists \eta$,

$$\sup_t |c_t^* - c_t| \leq \eta \Rightarrow \left| \sum_{t=0}^{\infty} \beta^t u(c_t^*) - \sum_{t=0}^{\infty} \beta^t u(c_t) \right| \leq \varepsilon.$$

We know that $c_t^* \rightarrow 0$. Then, $\exists T, \forall t \geq T, 0 \leq c_t^* < \eta$. So define $c_t = c_t^*$, $t \neq T$ and $c_T = c_T^* - \eta < 0$. $(c_0, \dots, c_t, \dots)$ verifies $\sup_t |c_t^* - c_t| = \eta$ and $u(c_t) = -\infty$. Thus, $\left| \beta^T u(c_T^*) - \beta^T u(c_T) \right| = +\infty$ and $\left| \sum_{t=0}^{\infty} \beta^t u(c_t^*) - \sum_{t=0}^{\infty} \beta^t u(c_t) \right| \leq \varepsilon$, does not hold.

Remark 3 *The assumption that $f'(0) > 1$ is crucial to have multipliers in ℓ^1 . Indeed, assume, on the contrary, that $f'(0) \leq 1$. Assume that Inada condition holds. We then have the Euler equation:*

$$-\lambda_T^1 + \lambda_{T+1}^1 f'(k_{T+1}^*) = 0.$$

Since f' is decreasing, we have $f'(k_{T+1}^) \leq f'(0) \leq 1$. Hence $\lambda_{T+1}^1 \geq \lambda_T^1, \forall T$. In other words, $\lambda_T^1 \geq \lambda_0^1 = u'(c_0^*) > 0, \forall T$. That proves that the multipliers λ^1 are not in ℓ^1 .*

Example 2 *The Recursive Utility and Optimal Growth*

In the previous example, one can use the support prices approach according to Weitzman (1973) and McKenzie (1986) tradition. But this approach is specific to separable utilities. We give below an example where their method fails. Actually, if we want to generalize their method to this example, we will not find support prices period by period, but the whole sequence of support prices. It turns out to prove that the optimal growth model has a summable sequence of multipliers.

Regarding production technology we keep the same assumptions as in Example 1. We know that for all k_0 , there exists $A(k_0) > 0$ such that $\mathbf{k} \in \Pi(k_0)$ implies $k_t \leq A(k_0)$ for all t . Let π denote the first coordinate projection function and $\mathcal{S}$ be the shift operator, i.e. $\pi \mathbf{x} = x_0$ and $\mathcal{S}(\mathbf{x}) = (x_1, \dots, x_t, \dots)$. Following Dana and Le Van (1990, 1991), we focus on utility functions that can be expressed as:

$$U(\mathbf{c}) = W(\pi \mathbf{c}, U(\mathcal{S} \mathbf{c})) \quad (9)$$

where $W : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an aggregator function that satisfies the following properties:

- (W1) W is continuous and satisfies $W(0, 0) = 0$;
- (W2) W is concave and nondecreasing in both arguments;
- (W3) There exists $\delta \in (0, 1)$ such that

$$|W(c, z) - W(c, z')| \leq \delta |z - z'|, \forall z, \forall z', \forall c.$$

Under assumption (W3), one can check that for any $A \geq 0$, one can define the utility of the stationary sequence $(A, A, \dots, A, \dots)$ denoted by $W(A, W(A, \dots))$ as the limit of $W(c_0, W(c_1, \dots, W(c_T, 0)))$ with $c_t = A$, for every t . We actually have $W(A, W(A, \dots)) \leq (1 - \delta)^{-1} W(A, 0)$.

Dana and Le Van (1991) established that, under (W1)-(W3), with every aggregator W is associated a unique utility function U , which satisfies (9) and meets other regularities conditions such as continuity and the concavity. As in Dana and Le Van (1990), we add:

(W4) W is C^1 . It satisfies Inada conditions:

$$\begin{aligned}\frac{\partial W}{\partial c}(0, z) &= +\infty, \forall z \geq 0, \\ \frac{\partial W}{\partial z}(c, z) &< +\infty, \forall c > 0, \forall z \geq 0.\end{aligned}$$

This optimal growth problem with recursive preferences can be written as follows:

$$\begin{aligned}\min \quad & F(\mathbf{x}) \\ \text{s.t.} \quad & \phi(\mathbf{x}) \leq 0 \\ & \mathbf{x} \in \ell_+^\infty \times \ell_+^\infty\end{aligned}$$

where $\mathbf{x} = (\mathbf{c}, \mathbf{k})$, $\phi_t(\mathbf{x}) = c_t + k_{t+1} - f(k_t)$, $\forall t \geq 0$, $k_0 > 0$ is given and $F(\mathbf{x}) = -U(\mathbf{c})$ with $F: \ell_+^\infty \times \ell_+^\infty \rightarrow \mathbb{R}_+$.

Proposition 2 *The optimal growth problem with recursive preferences under (W1)-(W4) has a solution and multipliers in ℓ^1 .*

Proof. The proof of the existence of a solution is standard and follows from the continuity of U and f . We will show that Theorem 2 applies :

i) We know that for all k_0 , there exists $A(k_0) > 0$ such that $\mathbf{k} \in \Pi(k_0)$ implies $k_t \leq A(k_0)$ for all t . Then U is well defined and bounded from above over $\Sigma(k_0)$ as a consequence of $A(k_0)$ also bounding feasible consumption. In other words, $\forall \tilde{c}$ feasible, $U(\tilde{c}) \leq W(A(k_0), W(A(k_0) \dots)) < \infty$.

Consider now $\tilde{\mathbf{x}} = (\mathbf{c}, \mathbf{k}) \in C$, $\tilde{\tilde{\mathbf{x}}} = (\mathbf{c}', \mathbf{k}') \in \ell_+^\infty \times \ell_+^\infty$ such that $\forall T$, $\mathbf{x}^T(\tilde{\mathbf{x}}, \tilde{\tilde{\mathbf{x}}}) \in C$. We have:

$$\begin{aligned}& \left| F(\mathbf{x}^T(\tilde{\mathbf{x}}, \tilde{\tilde{\mathbf{x}}})) - F(\tilde{\mathbf{x}}) \right| = \\ & \left| W(c_0, \dots, W(c_T, W(c'_{T+1}, \dots))) - W(c_0, \dots, W(c_T, W(c_{T+1}, \dots))) \right| \leq \\ & \delta^{T+1} \left| W(c'_{T+1}, W(\dots)) - W(c_{T+1}, W(\dots)) \right| \leq 2\delta^{T+1} W(A(k_0), W(A(k_0) \dots)),\end{aligned}$$

which leads to:

$$F(\mathbf{x}^T(\tilde{\mathbf{x}}, \tilde{\tilde{\mathbf{x}}})) \rightarrow F(\tilde{\mathbf{x}}) \text{ when } T \rightarrow \infty.$$

Thus, **Assumption 1** is satisfied.

ii) **Slater condition** and **Assumption 2** are easily verified from Example 1. Finally, it is obvious that $\mathbf{x}^T(\mathbf{x}^*, \mathbf{x}^0)$ belongs to $\ell_+^\infty \times \ell_+^\infty$.

All the assumptions of Theorem 2 are satisfied. ■

From Inada condition, the optimal paths satisfy $\forall t, c_t^* > 0, k_t^* > 0$. Proposition 2 implies that there exists $\lambda \in \ell_+^1$ such that $\forall \mathbf{x} = (\mathbf{c}, \mathbf{k}) \in \ell_+^\infty \times \ell_+^\infty$,

$$W(c_0^*, ..W(c_t^*, ...)) - \sum_{t=0}^{\infty} \lambda_t (c_t^* + k_{t+1}^* - f(k_t^*)) \geq \\ W(c_0, ..W(c_t, ...)) - \sum_{t=0}^{\infty} \lambda_t (c_t + k_{t+1} - f(k_t)), \quad (10)$$

and

$$\lambda_t (c_t^* + k_{t+1}^* - f(k_t^*)) = 0, \forall t.$$

Corollary 3 Assume that W is strictly concave and continuously differentiable. Let $\mathbf{x}^* = (\mathbf{c}^*, \mathbf{k}^*)$ be the optimal solution. Let U be the recursive utility defined by $U(\mathbf{c}) = W(\pi \mathbf{c}, U(S\mathbf{c}))$. Then we have:

(i) The sequence $\frac{\partial U}{\partial c_t}(c_0^*, ..., c_t^*, ...)$ is in $\ell_+^1 \setminus \{0\}$.

(ii) Euler equation:

$$\forall t, \frac{\partial W}{\partial c}(c_t^*, z_{t+1}^*) = \frac{\partial W}{\partial c}(c_{t+1}^*, z_{t+2}^*) \frac{\partial W}{\partial z}(c_t^*, z_{t+1}^*) f'(k_{t+1}^*), \quad (11)$$

where $\forall t, z_t^* = W(c_t^*, W(c_{t+1}^*, ...)) = U(c_t^*, c_{t+1}^*, ...)$.

Proof. (i) Define $\mathbf{c}, \mathbf{k}$ as follows:

$$k_t = k_t^*, \forall t, \\ c_t = c_t^*, \forall t \neq T, \\ c_T = c_T^* + \varepsilon, \varepsilon \in \mathbb{R} \text{ such that } c_T^* + \varepsilon > 0.$$

Inequality (10) implies that:

$$W(c_0^*, ..W(c_T^*, ...)) - \lambda_T c_T^* \geq W(c_0^*, ..W(c_T^* + \varepsilon, ...)) - \lambda_T (c_T^* + \varepsilon),$$

for any ε in a neighborhood of zero. Then, we have:

$$\frac{\partial U}{\partial c_T}(c_0^*, ..., c_T^*, ...) - \lambda_T = 0.$$

(ii) Define

$$c_t = c_t^*, \forall t \\ k_t = k_T^*, \forall t \neq T+1 \\ k_{T+1} = k_{T+1}^* + \varepsilon, \varepsilon \in \mathbb{R} \text{ such that } k_{T+1}^* + \varepsilon > 0.$$

By (10), we have:

$$-\lambda_T k_{T+1}^* + \lambda_{T+1} f(k_{T+1}^*) \geq -\lambda_T (k_{T+1}^* + \varepsilon) + \lambda_{T+1} f(k_{T+1}^* + \varepsilon),$$

for any ε in a neighborhood of zero. Therefore,

$$-\lambda_T^1 + \lambda_{T+1}^1 f'(k_{T+1}^*) = 0.$$

From (i), we know that $\lambda_T = \frac{\partial U}{\partial c_T}(c_0^*, \dots, c_T^*, \dots)$. It is easy to check that

$$\frac{\partial U}{\partial c_T}(c_0^*, \dots, c_T^*, \dots) = \frac{\partial W}{\partial z}(c_0^*, z_1^*) \dots \frac{\partial W}{\partial z}(c_{t-1}^*, z_t^*) \frac{\partial W}{\partial c}(c_t^*, z_{t+1}^*),$$

while

$$\frac{\partial U}{\partial c_{T+1}}(c_0^*, \dots, c_{T+1}^*, \dots) = \frac{\partial W}{\partial z}(c_0^*, z_1^*) \dots \frac{\partial W}{\partial z}(c_t^*, z_{t+1}^*) \frac{\partial W}{\partial c}(c_{t+1}^*, z_{t+2}^*).$$

Hence, Euler equation follows. ■

Example 3 *The Incentive-constrained optimal growth model [Rustichini (1998)]*

Consider the following problem ($\mathcal{G}$):

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t V(x_t, x_{t+1}), \quad 0 \leq \beta < 1, \\ \text{s.t.} \quad & (x_t, x_{t+1}) \in A \subset \mathbb{R}^{2n}, \forall t, \\ & x_0, \text{ is given,} \end{aligned}$$

where the government announces the optimal sequence of allocated capital goods (x_t) . The government is assumed to have a full commitment power. This assumption is too strong as the government has the possibility to modify its previous plan together with the specification of the continuation value that the government will receive after any deviation from the previous plan. The value of the deviation defines a constraint on the plans chosen by the government: at any date N , the value of the government's criterion, $\sum_{t=N}^{\infty} \beta^{t-N} V(x_t, x_{t+1})$ should be greater than the value of the deviation, $D(x_N)$. Then, the incentive constrained optimal growth problem can be written in the general form by adding these additional constraints to problem ($\mathcal{G}$):

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t V(x_t, x_{t+1}), \quad 0 \leq \beta < 1, \\ \text{s.t.} \quad & \sum_{t=N}^{\infty} \beta^{t-N} V(x_t, x_{t+1}) \geq D(x_N), \forall N = 0, 1, \dots, \\ & (x_t, x_{t+1}) \in A \subset \mathbb{R}^{2n}, \forall t, \\ & x_0 \in \pi_1(A), \text{ is given} \end{aligned}$$

where π_1 denotes the projection on the first component.

We make the following assumptions:

Assumption 5 The set A is convex, compact and non-empty.

Assumption 6 The function V is concave, continuous on A . The function D is continuous and convex on $\pi_1(A)$.

Assumption 7 If $(x_1^*, x_2^*, \dots, x_t^*, \dots)$ is optimal from x_0 , then there exists $\bar{x}$ such that:

- (i) $(x_0, \bar{x}) \in A, (\bar{x}, \bar{x}) \in A$,
- (ii) $D(x_0) - V(x_0, \bar{x}) - \beta(1 - \beta)^{-1} V(\bar{x}, \bar{x}) < 0, D(\bar{x}) - (1 - \beta)^{-1} V(\bar{x}, \bar{x}) < 0$,
- (iii) $\exists T_0$ such that $(x_T^*, \bar{x}) \in A$ for every $T \geq T_0$.

We then obtain the following propositions:

Proposition 3 *Under Assumptions 5-7, the Incentive-constrained problem has a solution.*

Proof. First, observe that under Assumption 7-(ii), the feasible set is non-empty. It is then straightforward to prove that V is continuous for the product topology and the feasible set is compact for this topology. Hence, there exists a solution. ■

Proposition 4 *Under Assumptions 5-7, the Incentive-constrained problem has a solution and multipliers in ℓ^1 .*

Proof. Obviously the sequence $(x_0, \bar{x}, \bar{x}, \dots, \bar{x}, \dots)$ satisfies the Slater condition by Assumption 7-(ii). Let us check whether the assumptions of Theorem 2 are satisfied:

- (i) It is obvious that $\Gamma = C = \{\mathbf{x} \in \ell^\infty : (x_t, x_{t+1}) \in A, \forall t = 0, 1, \dots\}$.
- (ii) Since A is compact and V, D are continuous, Assumption 2 (b) is satisfied. One can easily check that Assumption 1 is also satisfied.
- (iii) To check Assumption 2 (a), observe that for $\mathbf{x}, \mathbf{y} \in \Gamma$, for $T > N$, we have:

$$|\phi_N(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) - \phi_N(\mathbf{x})| \leq 2M(1 - \beta)^{-1} \beta^{T-N}$$

where $\forall i, \phi_i(\mathbf{x}) = D(x_i) - \sum_{t=i}^{\infty} \beta^{t-i} V(x_t, x_{t+1})$ and $M = \max \{V(a, b) : (a, b) \in A\}$.

Thus, when $T \rightarrow +\infty$,

$$\phi_N(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \rightarrow \phi_N(\mathbf{x}).$$

- (iv) Since, given N , for t large enough, $\phi_t(\mathbf{x}^N(\mathbf{x}, \mathbf{y})) = \phi(\mathbf{y})$, Assumption 2 (c) is satisfied.

Finally, by assumption, the sequence $\mathbf{x}^T(\mathbf{x}^*, \bar{x})$ belongs to $\Gamma = \Gamma \cap C$ for every $T \geq T_0$. All the assumptions of Theorem 2 are satisfied. ■

By Proposition 4, we obtain that the Lagrange multipliers are in ℓ^1 . We will now show how this result can help us to characterize the steady states of the incentive constrained problem.

Let $(x_1^*, x_2^*, \dots, x_t^*, \dots)$ is optimal from x_0 . By Proposition (4), $\forall (x_t, x_{t+1}) \in A$, we have:

$$\sum_{t=0}^{\infty} \beta^t V(x_t^*, x_{t+1}^*) - \sum_{N=0}^{\infty} \lambda_N \left[D(x_N^*) - \sum_{t=N}^{\infty} \beta^{t-N} V(x_t^*, x_{t+1}^*) \right] \geq \sum_{t=0}^{\infty} \beta^t V(x_t, x_{t+1}) - \sum_{N=0}^{\infty} \lambda_N \left[D(x_N) - \sum_{t=N}^{\infty} \beta^{t-N} V(x_t, x_{t+1}) \right]$$

and

$$\lambda_N \left[D(x_N^*) - \sum_{t=N}^{\infty} \beta^{t-N} V(x_t^*, x_{t+1}^*) \right] = 0, \forall N. \quad (12)$$

The first order conditions can be easily obtained as:

$$[v_t + \beta^t] V_2(x_t^*, x_{t+1}^*) + [v_{t+1} + \beta^{t+1}] V_1(x_{t+1}^*, x_{t+2}^*) - \lambda_{t+1} D'(x_{t+1}) = 0, \quad t = 0, 1, \dots \quad (13)$$

where $v_t = \sum_{N=0}^{\infty} \lambda_N \beta^{t-N}$ and V_1, V_2 denote the partial derivatives of $V(x, y)$ with respect to x and y , respectively.

Let us study the steady states $(\hat{x}_0)$ where the incentive constraints are binding. One of the necessary conditions follows from (12) as:

$$V(\hat{x}_0, \hat{x}_0) = (1 - \beta) D(\hat{x}_0). \quad (14)$$

which can have many solutions. Consider one dimensional case and assume that $(\hat{x}_0, \hat{x}_0) \in \text{int } A$. Writing (13) for $t - 1$ and t at the steady state, and noting that $v_t = \beta v_{t-1} + \lambda_t$, one can prove that [see Rustichini (1998)]:

$$\lambda_{t+1} = \frac{V_2(\hat{x}_0, \hat{x}_0) + \beta D'(\hat{x}_0)}{D'(\hat{x}_0) - V_1(\hat{x}_0, \hat{x}_0)} \lambda_t.$$

By Proposition 4, we know that $\lambda \in \ell_+^1$. Then, in addition to (14) we also have:

$$0 < \frac{V_2(\hat{x}_0, \hat{x}_0) + \beta D'(\hat{x}_0)}{D'(\hat{x}_0) - V_1(\hat{x}_0, \hat{x}_0)} < 1. \quad (15)$$

It should be noted that (14) and (15) are just necessary conditions for $\hat{x}_0$ to be a steady state of the incentive constrained problem. It is also worth mentioning that Rustichini (1998) used more assumptions (see Theorem 5.7, page 357) on the functions V and D in order to obtain (15).

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