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RESEARCH ARTICLE

Strongly rational sets for normal-form games

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Abstract We introduce the concept of minimal strong curb sets which is a set-theoretic coarsening of the notion of strong Nash equilibrium. Strong curb sets are product sets of pure strategies such that each player's set of recommended strategies contains all actions she may rationally select in every coalition she might belong to, for any belief each coalition member may have that is consistent with the recommendations to the other players. Minimal strong curb sets are shown to exist and are compared with other well-known solution concepts. We provide a dynamic learning process leading the players to play strategies from a minimal strong curb set only.

Keywords Set-valued solution concept · Strong Nash equilibrium · Coalition · Strong curb set · Learning

JEL Classification C70 · C72

1 Introduction

Basu and Weibull (1991) have introduced a set-theoretic coarsening of the notion of strict Nash equilibrium: minimal curb (closed under rational behavior) sets. This set-

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valued solution concept combines a standard rationality condition, stating that the set of recommended strategies of each player contains all her best responses to whatever belief she may have that is consistent with the recommendations to the other players, with players' aim at simplicity, which encourages them to maintain a set of strategies as small as possible.

We propose the concept of minimal strong curb sets which is a set-theoretic coarsening of the notion of strict strong Nash equilibrium.[1] We require the sets to be immune not only against individual deviations, but also against group deviations. Strong curb sets are product sets of pure strategies such that each player's set of recommended strategies contains all actions she may rationally select in every coalition she might belong to, for any belief each coalition member may have that is consistent with the recommendations to the other players. A strong curb set is minimal if it does not properly contain another strong curb set. Think of the set of recommendations to a player in a minimal strong curb set as a well-packed bag for a sports weekend: you may want to be prepared for different kinds of sports since you may like playing tennis with player 2 or playing golf with playing 3 or playing bridge with players 2, 3 and 4 or going alone for a jog. When a coalition forms, we assume that its members may talk to each other and commit to take a joint action. That is, we do not restrict the set of possible deviations by requiring that the deviation should be self-enforcing. By allowing for every possible deviations, our concept may be weak at determining actual play, but predicts with confidence: if players have beliefs with support in a minimal strong curb set (say because they realize this is an equilibrium or because they are able to observe history of play to gain information on likely choices of others), then we can predict with confidence that players will actually choose an action in the set.

In Sect. 2 we recall notations and definitions. In Sect. 3, we formally define the concept of minimal strong curb sets and we show that minimal strong curb sets exist in general. In Sect. 4 we compare minimal strong curb sets with strong Nash equilibria, coalition-proof Nash equilibria and (bayesian) coalitionally rationalizable strategy profiles. In Sect. 5, we provide a dynamic learning process where at each period players choose a best-response, possibly in groups, against beliefs formed on the basis of strategies used in the recent past. We show that if agents' memory is long enough, play settles down in a minimal strong curb set. In Sect. 6 we conclude.

2 Preliminaries

Strict set inclusion is denoted by $\subsetneq$ and weak set inclusion is denoted by $\subseteq$. A normal-form game is a tuple $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$, where $N = \{1, 2, \ldots, n\}$ is a finite set of players, each player $i \in N$ has a nonempty, finite set of pure strategies (or actions) A_i and a von Neumann–Morgenstern utility function $u_i : A \to \mathbb{R}$, where $A = \times_{j\in N} A_j$. The set of all games is denoted by Γ. For every $X = \times_{i\in N} X_i \subseteq A$, let $X_{-i} = \times_{j\in N\setminus\{i\}} X_j$, $\forall i \in N$. The subgame obtained from G by restricting the action set of each player $i \in N$ to a subset $X_i \subseteq A_i$ is denoted—with a minor abuse of notation from restricting the domain of the utility functions u_i to $\times_{j\in N} X_j$—

[1] Many games of interest lack strong Nash equilibria Aumann (1959).

by $G_X = \langle N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$. The set of mixed strategies of player $i \in N$ with support in $X_i \subseteq A_i$ is denoted by $\Delta(X_i)$. Payoffs are extended to mixed strategies in the usual way. Beliefs are profiles of mixed strategies. The profile of strategies where player $i \in N$ plays $a_i \in A_i$ and the remaining agents play according to the mixed strategy profile $\alpha_{-i} = (\alpha_j)_{j\in N\setminus\{i\}} \in \times_{j\in N\setminus\{i\}}\Delta(A_j)$ is denoted (a_i, α_{-i}). For $i \in N$ and $\alpha_{-i} \in \times_{j\in N/\{i\}}\Delta(A_j)$, $\mathrm{BR}^i(\alpha_{-i}) = \{a_i \in A_i \mid u_i(a_i, \alpha_{-i}) \geq u_i(a_i', \alpha_{-i})$ for each $a_i' \in A_i\}$ is the set of pure best responses of player i against her belief α_{-i}.

Basu and Weibull (1991) have introduced the concept of curb sets. A curb set is a product set $X = \times_{i\in N}X_i$ where (a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies; (b) for each $i \in N$ and each belief α_{-i} of player i with support in X_{-i}, the set X_i contains all best responses of player i against her belief: $\forall i \in N$, $\forall \alpha_{-i} \in \times_{j\in N\setminus\{i\}}\Delta(X_j)$, $\mathrm{BR}^i(\alpha_{-i}) \subseteq X_i$. Curb sets are product sets of pure strategies such that each player's set of recommended strategies contains all best-replies to whatever belief she may have that is consistent with the recommendations to the other players. Since the full strategy space is a curb set, particular attention is devoted to minimal curb sets. A curb set X is minimal if no curb set is a proper subset of X. Every game G possesses at least one minimal curb set (Basu and Weibull 1991).

A strong Nash equilibrium is a strategy profile such that no coalition has a joint deviation that benefits all of them (Aumann 1959). Coalitions are nonempty subsets of players (J such that $J \subseteq N$ and $J \neq \emptyset$). For every $X \subseteq A$ and $J \subseteq N$, let $X_{-J} = \times_{j\in N\setminus J}X_j$. The profile of strategies where players belonging to coalition J play according to the strategy profile $a_J \in \times_{i\in J}A_i$ and the remaining players play according to the mixed strategy profile $\alpha_{-J} = (\alpha_j)_{j\in N\setminus J} \in \times_{j\in N\setminus J}\Delta(A_j)$ is denoted (a_J,α_{-J}). For every $J \subseteq N$, $i \in J$, and $\alpha_{-i} = (\alpha_j)_{j\in N\setminus\{i\}} \in \times_{j\in N\setminus\{i\}}\Delta(A_j)$, we denote by α_{-i}^{-J} the marginal distribution of α_{-i} over A_{-J}. The strategy profile a^* $\in \times_{i\in N}A_i$ is a strong Nash equilibrium if and only if, for each $a \neq a^*$, there exists some player i such that $a_i \neq a_i^*$ and $u_i(a) \leq u_i(a^*)$. A strong Nash equilibrium is strict if the last inequality holds strictly.

3 Strong curb sets

While the concept of curb sets is a set-theoretic coarsening of the notion of strict Nash equilibrium, we now introduce the concept of strong curb sets which is a set-theoretic coarsening of the notion of strict strong Nash equilibrium. We require the set to be immune not only against individual deviations, but also against coalitional deviations.

Definition 1 For each profile of beliefs $\alpha = (\alpha_{-i})_{i\in N}$ with $\alpha_{-i} \in \times_{j\in N\setminus\{i\}}\Delta(A_j)$, the set of coalitional best-responses of coalition $J \subseteq N$ is

$$\begin{aligned}\mathrm{CBR}^J(\alpha) = \{b_J \in \times_{i\in J}A_i \mid \ &\text{(i) } \forall i \in J, u_i(a_i, \alpha_{-i}) \leq u_i(b_J, \alpha_{-i}^{-J}), \forall a_i \in A_i \\ &\text{and (ii) } \nexists\, b_J' \in \times_{i\in J}A_i \text{ such that } \forall i \in J, u_i(b_J, \alpha_{-i}^{-J}) < u_i(b_J', \alpha_{-i}^{-J})\}.\end{aligned}$$

Given a profile of beliefs α, a strategy profile b_J for coalition J is a coalitional best-response if (i) each member $i \in J$ prefers to join coalition J and playing b_J rather than playing her individual best-response against her belief α_{-i}; (ii) there is no other profile $b'_J \neq b_J$ such that all members of J strictly prefer b'_J to b_J. Conditions (i) and (ii) capture some rudimentary form of coalitional rationality. First, a sensible concept of coalitional rationality should prescribe coordination on strategy profiles so that all coalition members have incentives to join the group. Second, it should be conceivable that members of coalition J will never coordinate their play on strategy profiles that are Pareto dominated. Of course, $\text{CBR}^{\{i\}}(\alpha)$ coincides with $\text{BR}^i(\alpha_{-i})$ $\forall i \in N$.

Example 1 Consider the normal-form games G_1 and G_2.

G_1:

	L	R
U	4, 5	0, 0
D	0, 0	3, 2

G_2:

	L	R
U	2, 0	0, 0
D	0, 0	0, 2

Take the normal-form game G_1 and let $J = \{1, 2\}$. Condition (i) makes that (U, R) and (D, L) are never coalitional best-responses for J whatever α. Condition (ii) makes that (D, R) is not a coalitional best-response for J whatever α. However, the strategy profile (U, L) satisfies both conditions whatever α. Thus, $\text{CBR}^{\{1,2\}}(\alpha) = \{(U, L)\}$. Notice that the set of coalitional best-responses, $\text{CBR}^J(\alpha)$, may be empty if $|J| \geq 2$. Take the normal-form game G_2 and consider the beliefs $\alpha = (\alpha_{-1}, \alpha_{-2})$ with $\alpha_{-1}(L) = 1$ and $\alpha_{-2}(D) = 1$. Then, $\text{BR}^1(\alpha_{-1}) = \{U\}$ and $\text{BR}^2(\alpha_{-2}) = \{R\}$ and the expected payoffs are $u_1(U, \alpha_{-1}) = 2$ and $u_2(R, \alpha_{-2}) = 2$. Thus, we have that $\text{CBR}^{\{1,2\}}(\alpha) = \emptyset$.

A set X is a strong curb set if the belief that only strategies in X are played implies that players and coalitions have no incentives to use other strategies than those belonging to X.

Definition 2 A strong curb set is a product set $X = \times_{i\in N} X_i$ where

(a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies;
(b) for each $J \subseteq N$ and each vector of beliefs $\alpha = (\alpha_{-1}, \ldots, \alpha_{-N})$ of the players with each belief α_{-i} having support in X_{-i}, the product set $X_J = \times_{j\in J} X_j$ contains all coalitional best-responses of coalition J against the beliefs of its members:

$$\forall J \subseteq N, \forall \alpha = (\alpha_{-1}, \ldots, \alpha_{-n}) \text{ with } \alpha_{-i} \in \times_{l \in N\setminus\{i\}} \Delta(X_l), \quad i \in N,$$
$$\text{CBR}^J(\alpha) \subseteq \times_{j\in J} X_j.$$

Strong curb sets are product sets of pure strategies such that each player's set of recommended strategies contains all actions she may rationally select in every coalition she might belong to, for any belief each coalition member may have that is consistent with the recommendations to the other players.[2] Each coalition member is allowed

[2] Players choose pure strategies and hold mixed beliefs. The notion of strong curb set can be easily extended to mixed strategies simply by accommodating the definition of CBR.

to hold a different belief concerning the play of others in the set of recommended strategies to assess the profitability of the deviation. Thus, the coalition members may disagree on where the deviation leads to. In addition, players do not update their belief by trying to understand why some coalitional action is a best-response for the other players of the coalition. Rather, a deviation is deterred if, for each profile of beliefs of the coalition members, at least one player of the group is strictly better off by blocking the deviation. Notice that each strong curb set is a curb set and hence contains the support of at least one Nash equilibrium in mixed strategies.

A strong curb set X is minimal if no strong curb set is a proper subset of X.[3] Establishing existence of minimal strong curb sets in finite games is simple. The entire pure-strategy space A is a strong curb set. Hence the collection of strong curb sets is nonempty, finite (since A is finite) and partially ordered by set inclusion. Thus, a minimal strong curb set exists.[4]

Proposition 1 *Every normal-form game G has a minimal strong curb set.*

Proposition 2 *If X is a minimal strong curb set of $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$, then X is a minimal strong curb set of the subgame $G_X = \langle N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$.*

Proof Let X be a minimal strong curb set of G and $Y \subseteq X$. It implies that Y is not a strong curb set of G. Thus, there exists $J \subseteq N$, $\widetilde{a}_J \in \times_{j\in J} A_j \backslash Y_j$ and $\alpha = (\alpha_{-i})_{i\in N}$ with $\alpha_{-i} \in \times_{j\in N\backslash\{i\}} \Delta(Y_j)$ such that $u_j(\widetilde{a}_J, \alpha_{-j}^{-J}) \geq u_j(b_j, \alpha_{-j})$ for all $b_j \in A_j$, for all $j \in J$. Having $\widetilde{a}_J \in \times_{j\in J} A_j \backslash X_j$ would contradict that X is a strong curb set of G. Thus, $\widetilde{a}_J \in \times_{j\in J} X_j \backslash Y_j$, implying that Y is not a minimal strong curb set of G_X. □

Proposition 2 implies that it is not possible to refine the notion by iteratively eliminating strategies that do not belong to a particular minimal strong curb set.

4 Relationships with other solution concepts

Minimal strong curb sets always exist and allow us to make reasonable predictions in games in which a strong Nash equilibrium does not exist.

Example 2 Consider the normal-form game G_3.

	L	C	R
U	4, 4	0, 5	0, 0
M	0, 5	2, 2	0, 0
D	0, 0	0, 0	a, 1

[3] The product set of actions chosen in every strict strong Nash equilibrium is a minimal strong curb set. Conversely, for every minimal strong curb set composed of one action per player, the strategy profile in which each player selects this action is a strict strong Nash equilibrium.

[4] In the appendix we show that the existence result holds for every game $G \in \mathcal{G}$, where $\mathcal{G}$ is the class of normal-form games $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ such that for each player $i \in N = \{1, 2, \ldots, n\}$, A_i is a compact subset of a metric space and $u_i : A \to \mathbb{R}$ is a continuous von Neumann–Morgenstern utility function.

For $a \geq 0$, this game has two Nash equilibria: the pure strategy Nash equilibrium (D, R) and the mixed one $(\sigma_U, \sigma_M, \sigma_D) = (1/4, 3/4, 0)$ and $(\sigma_L, \sigma_C, \sigma_R) = (1/3, 2/3, 0)$. For $0 \leq a < 4$ none of these Nash equilibria are immune to the joint deviation (U, L), implying that the game G_3 has no strong Nash equilibrium. The only minimal strong curb set of G_3 is $\{\{U, M\} \times \{L, C\}\}$ when $a < 4$. Indeed, for beliefs with support in the set, each player's individual best-responses lie in the set. In addition, player 2 blocks the joint deviation to (D, R) since she can be sure of having a payoff greater than 1 by playing in the set.
Even if a strong Nash equilibrium exists and is unique, it may not be the only valid prediction of the game. Consider again the game G_3 for $a > 4$. The strategy profile (D, R) is the unique strict strong Nash equilibrium. The set composed of those actions is thus a minimal strong curb set. But, $\{U, M\} \times \{L, C\}$ is another minimal strong curb set since for beliefs with support in the set, player 2 has no incentives to choose an action outside the set, even jointly with player 1. Player 1 would be willing to jointly deviate to (D, R), but without the consent of player 2 , she prefers not to play D.

Coalition-proof Nash equilibria always exist for two player games (Bernheim et al. 1987). They consist of Nash equilibria that are not Pareto dominated by another Nash equilibrium. In the game G_3, the two Nash equilibria are coalition-proof when $a > 4/3$ while only the mixed Nash equilibrium is coalition-proof when $a \leq 4/3$. For $4/3 < a < 4$, (D, R) is a coalition-proof Nash equilibrium, but the only minimal strong curb set is $\{U, M\} \times \{L, C\}$. Thus, the support of a coalition-proof Nash equilibrium is not necessarily contained in a minimal strong curb set. The following example reveals that the predictions obtained under the minimal strong curb set may be more plausible than those obtained under the notion of coalition-proof Nash equilibrium.

Example 3 (Ambrus 2006) Consider the normal-form game G_4.

	L	C	R
U	2, 1, 0	0, 0, 0	−9, −9, −9
M	2, 0, 1	1, 0, 2	−9, −9, −9
D	−9, −9, −9	−9, −9, −9	−9, −9, −9

l

	L	C	R
U	1, 2, 0	0, 2, 1	−9, −9, −9
M	0, 0, 0	0, 1, 2	−9, −9, −9
D	−9, −9, −9	−9, −9, −9	−9, −9, −9

c

	L	C	R
U	−9, −9, −9	−9, −9, −9	−9, −9, −9
M	−9, −9, −9	−9, −9, −9	−9, −9, −9
D	−9, −9, −9	−9, −9, −9	−8, −8, −8

r

The unique coalition-proof Nash equilibrium of G_4 is (D, R, r), while the unique minimal strong curb set is $\{\{U, M\} \times \{L, C\} \times \{l, c\}\}$.

Contrary to curb sets, strong curb sets may include strategies that are strictly dominated or even not rationalizable. In the Prisoner's Dilemma, the action *cooperate* is strictly dominated but belongs to the unique minimal strong curb set. Ambrus (2006) and Luo and Yang (2009) have proposed the concepts of coalitional rationalizability and bayesian coalitional rationalizability, respectively.[5] There is no relationship between minimal strong curb sets and (bayesian) coalitionally rationalizable sets. (Bayesian) coalitional rationalizability may have more cutting power than minimal strong curb sets, as in the Prisoner's Dilemma. The converse may also be true. Minimal strong curb sets may have more cutting power than (bayesian) coalitional rationalizability as in Example 2 where for $4/3 \leq a \leq 4$, every strategy is (bayesian) coalitionally rationalizable while only a subset of those belong to the unique minimal strong curb set.

5 Learning to play minimal strong curb strategies

We now provide a class of dynamic learning processes where groups of players may coordinate their actions that leads the players to play strategies from a minimal strong curb set only. In line with Hurkens (1995), players observe actions played recently, form their beliefs upon these observations, and play best-responses to those beliefs.

Suppose a game $G = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ is played once every period. The players are partitioned into classes $C_1, C_2, \ldots, C_n$ such that $u_i = u_j$ and $A_i = A_j$ if $i, j \in C_k$. In each period, one player is drawn at random from each of n disjoint classes $C_1, C_2, \ldots, C_n$, to play the game G in that period. These players are partitioned into coalitions to form a coalition structure $\mathbf{J} = (J_1, J_2, \ldots, J_M)$ such that $J_k \cap J_l = \emptyset$ for $k \neq l$ and $\cup_{k=1}^{M} J_k = N$. Let $\mathbb{J}$ be the finite set of coalition structures. In period t, the history $h^t = (a^{t-K}, \ldots, a^{t-1})$ is a description of how the game has been played in the K previous periods, where $a^k \in A$ is the action profile chosen by the n players in period k. We define the state space $H = A^K$ to consist of all histories $h = (a^{-K}, \ldots, a^{-1})$ of length K. Call $\widehat{h} \in H$ a successor of $h \in H$ if $\widehat{h}$ is obtained from h by deleting the leftmost element and by adding some element $a \in A$ to the right. Let $r(\widehat{h})$ denote the rightmost element of $\widehat{h} \in H$. For $h = (a^{-K}, \ldots, a^{-1}) \in H$, let $\pi_i(h, k) = \{a_i^{-k}, \ldots, a_i^{-1}\}$ denote the set of strategies played by player i in the k last periods, for $k \leq K$. We assume that the choice of the players is time-independent. The learning process can thus be described by a stationary Markov chain on the state space $H = A^K$. Let $P : H \times H \rightarrow [0, 1]$ be a transition matrix, where $P(h, \widehat{h})$ is the probability of moving from state $h \in H$ to state $\widehat{h} \in H$ in one period and $\Sigma_{\widehat{h} \in H} P(h, \widehat{h}) = 1$ for all $h \in H$. A learning process is described by a transition matrix $P \in \mathcal{P}$, where $\mathcal{P}$ is defined as follows.

Definition 3 Let $\mathcal{P}$ be the set of transition matrices P that satisfy for all histories h, $\widehat{h} \in H$, $P(h, \widehat{h}) > 0$ if and only if (i) $\widehat{h}$ is a successor of h, (ii) there exists some $\mathbf{J} \in \mathbb{J}$

[5] See also Ambrus (2009) and Herings et al. (2004).

and $\alpha = (\alpha_{-1}, \ldots, \alpha_{-n})$ with $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(\pi_j(h, K))$ such that $r(\widehat{h}) = (a_J)_{J\in\mathbf{J}}$ with $a_J \in \mathrm{CBR}^J(\alpha)$ if $\mathrm{CBR}^J(\alpha) \neq \emptyset$ and $a_J \in \times_{i\in J}\mathrm{BR}^i(\alpha_{-i})$ otherwise.

At each period every player chooses an action. This action can be chosen individually or in group, and is chosen after having observed the recent past play. When a group of players coordinate their actions, they choose a Pareto undominated action profile such that each member of the group benefits from playing jointly. In state h, if coalition $J \subseteq N$ has a coalitional best-response $a_J \in \mathrm{CBR}^J(\alpha)$ given a profile of beliefs with support in the set of strategies played in the recent past, then the process moves with positive probability from state h to some state $\widehat{h}$ in which coalition J plays a_J. To determine the outcome of such learning processes, what matters is to identify the set of states that can be reached from a state h with positive probability and those that cannot be reached. Since the exact probability does not matter, we do not have to specify a particular process of belief formation nor a protocol of coalition formation. We only require that every such belief with support in the set of actions played recently and every partition of the players occur with positive probability.

For each $k \in \mathbb{N}$, $P^k : H \times H \to [0, 1]$ denotes the k-step transition probabilities of the Markov process with transition matrix $P \in \mathcal{P}$: $P^1 = P$ and $P^k = P \circ P^{k-1}$ for $k > 1$. We will write $h \rightsquigarrow \widehat{h}$ if there exists $k \in \mathbb{N}$ satisfying $P^k(h, \widehat{h}) > 0$. Now $\rightsquigarrow$ defines a weak order on H. We can define an equivalence relation on H: $h \sim \widehat{h} \Leftrightarrow h \rightsquigarrow \widehat{h}$ and $\widehat{h} \rightsquigarrow h$. Let $[h]$ denote the equivalence class that contains h and let $Q = \{[h] \mid h \in H\}$ denote the set of equivalence classes. A partial order $\preceq$ on Q is given by: $[h] \preceq [\widehat{h}] \Leftrightarrow \widehat{h} \rightsquigarrow h$. The minimal elements with respect to the order $\preceq$ are called ergodic sets. The other elements are called transient sets. If the process leaves a transient set it can never return to that set. If the process is in an ergodic set it can never leave that set. The elements belonging to ergodic and transient sets are called ergodic and transient states, respectively. In any finite Markov chain, no matter where the process starts, the probability that the process is in an ergodic state after k steps tends to 1 as k tends to infinity. Proposition 3 states that if memory is long enough (K high enough), each ergodic set Z of a learning process with transition matrix $P \in \mathcal{P}$ is composed of minimal strong curb elements only, i.e., Z is such that $Z \subseteq X^K$ for some minimal strong curb set X of G.

Proposition 3 *There exists $\underline{K} \in \mathbb{N}$ such that for all finite $K \geq \underline{K}$ and every Markov chain with transition matrix $P \in \mathcal{P}$, if $Z \subseteq H$ is an ergodic set then*

(a) *$Z \subseteq X^K$ for some minimal strong curb set X.*
(b) *$\nexists Y \subsetneq X$ such that $Z \subseteq Y^K$ for some minimal strong curb set X.*

The following lemma will be useful to prove Proposition 3.

Lemma 1 *Let $h^t = (x^{K-t}, \ldots, x^1, a^1, \ldots, a^t)$ be a particular history. If the players draw their beliefs from $\times_{i\in N}\pi_i(h^t, t)$ then*

(a) *the process moves with positive probability to an history h^{t+1} such that $\times_{i\in N}\pi_i(h^t, t) \subsetneq \times_{i\in N}\pi_i(h^{t+1}, t+1)$ if $\times_{i\in N}\pi_i(h^t, t)$ is not a strong curb set;*
(b) *the process moves with probability 1 to an history h^{t+1} such that $\times_{i\in N}\pi_i(h^t, t) = \times_{i\in N}\pi_i(h^{t+1}, t+1)$ if $\times_{i\in N}\pi_i(h^t, t)$ is a strong curb set.*

Proof Easy and therefore omitted. □

Proof of Proposition 3. Let $L = \Sigma_{i=1}^{n} |A_i| - (n-1)$. Let $M = \Sigma_{i=1}^{n} |A_i| - n$. Take $\underline{K} = L + M - 1$ and let $K \geq \underline{K}$ be finite. Let $P \in \mathcal{P}$.

(a) We will show that (i) from any history $h^1 \in H$, the process moves with positive probability in $L-1$ steps to a state $h^L \in H$ such that $\times_{i\in N}\pi_i(h^L, L)$ is a strong curb set, (ii) from state h^L, the process moves with positive probability in M steps to a state $h^{L+M} \in H$ such that $\times_{i\in N}\pi_i(h^{L+M}, M)$ is a minimal strong curb set, and (iii) if $Z \subseteq H$ is an ergodic set then $Z \subseteq X^K$ for some minimal strong curb set X.

(a.i) Let $a^1, \ldots, a^T \in A$ be such that $a^{t+1} \notin \times_{i\in N}\pi_i(h^t, t)$ for all $t = 1, \ldots, T-1$. By definition of L, we have $T \leq L$ since $\times_{i\in N}\pi_i(h^1, 1)$ contains n actions, $\times_{i\in N}\pi_i(h^{t+1}, t+1)$ contains at least one additional action than $\times_{i\in N}\pi_i(h^t, t)$ and the action space A, which is the largest strong curb set, contains $\Sigma_{i=1}^{n} |A_i|$ of them. Thus, there exists a $\tau \leq L$ such that, starting from $h^1 = (x^{K-1}, \ldots, x^1, a^1)$ and applying τ times part (a) of Lemma 1, we have $h^1 \rightsquigarrow h^\tau = (x^{K-\tau}, \ldots, x^1, a^1, \ldots, a^\tau)$ and $\times_{i\in N}\pi_i(h^\tau, \tau)$ is a strong curb set. From part (b) of Lemma 1, we have $h^\tau \rightsquigarrow h^L = (x^{K-L}, \ldots, x^1, a^1, \ldots, a^\tau, \ldots, a^L)$ such that $\times_{i\in N}\pi_i(h^L, L) = \times_{i\in N}\pi_i(h^\tau, \tau)$.

(a.ii) Let $X \subseteq \times_{i\in N}\pi_i(h^L, L)$ be a minimal strong curb set. We have $h^L \rightsquigarrow h^{L+M} = (\ldots, a^1, \ldots, a^L, b^1, \ldots, b^M)$ such that $\times_{i\in N}\pi_i(h^{L+M}, M) = X$ since every strategy in a minimal strong curb set is an element of a coalitional best-response to some belief with support in the set. We then have $h^{L+M} \rightsquigarrow h^{L+K} = (b^1, \ldots, b^M, c^1, \ldots, c^{K-M})$ such that $\times_{i\in N}\pi_i(h^{L+K}, K) = X$ by application of part (b) of Lemma 1. Finally, for all $k \in \mathbb{N}$, for all h^{L+K+k} such that $h^{L+K} \rightsquigarrow h^{L+K+k}$, we have $\times_{i\in N}\pi_i(h^{L+K+k}, K) \subseteq X$ by Definition 3. The set X^K thus contains an ergodic set.

(a.iii) By contradiction, suppose there exists an ergodic set Z such that $Z \not\subseteq X^K$ for all minimal strong curb set X of G. Thus Z contains an ergodic state $h \in H$ such that $h \notin X^K$ for all minimal strong curb set X. Applying (a.i) and (a.ii), we have $h \rightsquigarrow h'$ such that $h' \in Y^K$ for some minimal strong curb set Y and there exists an ergodic set $W \subseteq Y^K$. We do not have $h' \rightsquigarrow h$ since $h \notin W$ by assumption, contradicting the fact that Z is an ergodic set.

(b) Follows directly from Proposition 2 and part (a) of Lemma 1. □

It is shown in Proposition 3 that if memory is long enough, the probability that players are playing a minimal strong curb strategy profile after k steps of the learning process tends to 1 as k tends to infinity. To prove this result, we show that from every state h, there exists a finite sequence of steps that occurs with positive probability leading to a state h' where the set of strategies used in the recent past span a minimal strong curb set. Once h' is reached, the players draw their belief from the minimal strong curb set and thus keep best-responding in that set. Let us make some remarks about this result. First, notice that it may be that some strategy profiles $a \in X$ are never played in any repetition of the game. In the Prisoner's Dilemma, either both players cooperate or both defect in each period. However, once the process has converged to a minimal strong curb set, the players will select infinitely often each strategy belonging to this set. In other words, play will not settle down to a proper subset of a minimal strong curb set. Second, the lower bound $\underline{K}$ we use in the proof of Proposition 3 is not tight. It may well be that $K < \underline{K}$ and each ergodic set of the game are subset of minimal strong curb histories. Third, the time k needed to move in a minimal number of steps from a strong curb history h^t to a history h^{t+k} such that the strategies chosen in the k last period span a minimal strong curb set X is exactly $k = \max(|X_1|, \ldots, |X_n|)$ if every action $a_i \in X_i$ of each player $i \in N$ is an individual best-response to some belief in the set. This time may increase if minimal strong curb actions only belong to some coalitional best-responses and the same player is involved in different coalitional moves. The maximal number of period needed to realize the transition from h^t to h^{t+k} in a minimal number of steps is $M = \Sigma_{i=1}^{n} |A_i| - n$. This is illustrated through the following example.

Example 4 Consider the normal-form game G_5

l:

	L	R
U	2, 2, 1	0, 3, 3
D	3, 0, 1	1, 1, 1

r:

	L	R
U	0, −1, 0	2, 0, 2
D	1, −1, 0	3, 0, 0

We have that the unique minimal strong curb set is $\{\{U, D\} \times \{L, R\} \times \{l, r\}\}$ and $M = \Sigma_{i=1}^{n} |A_i| - n = 3$. Suppose the process is in state h^t where $\times_{i \in N} \pi_i(h^t, M) = A$. Let k be the smallest integer such that $h^t \rightsquigarrow h^{t+k}$ with the property that $\times_{i \in N} \pi_i(h^{t+k}, k) = A$. We have $k = M = 3$ since player 2 selects her strategy L only when coalition $\{1, 2\}$ plays (U, L). Player 3 selects her strategy r only when coalition $\{1, 3\}$ plays (U, r). A third period is needed for player 1 to play D.

6 Concluding comment

We have introduced the concept of minimal strong curb sets which extends the notion of curb sets by allowing for group deviations. Similarly to strong curb sets, we can define the notion of strong prep sets. Strong prep sets are product sets of pure strategies such that each player's set of recommended strategies contains at least one action she may rationally select in every coalition she might belong to, for any belief each coalition member may have that is consistent with the recommendations to the other players.[6]

[6] Prep sets (Voorneveld 2004) are product sets of pure strategies such that each player's set of recommended strategies must contain at least one best-response to whatever belief she may have that is consistent with the recommendations to the other players. Voorneveld (2005) has shown that, in generic games, persistent retracts (Kalai and Samet 1984), minimal prep sets and minimal curb sets coincide. See also Kets and Voorneveld (2008).

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Appendix

We now show that the existence of minimal strong curb sets holds in general. Let $\mathcal{G}$ be the class of normal-form games $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ where for each player $i \in N = \{1, 2, \ldots, n\}$, A_i is a compact subset of a metric space and $u_i : A \to \mathbb{R}$ is a continuous von Neumann–Morgenstern utility function. Payoffs are extended to mixed strategies in the usual way. Let $\Delta(A_i)$ be the set of Borel probability measures over A_i. If $B_i \subseteq A_i$ is a Borel set, then $\Delta(B_i)$ denotes the set of Borel probability measures with support in B_i: $\Delta(B_i) = \{\alpha_i \in \Delta(A_i) \mid \alpha_i(B_i) = 1\}$. If $G \in \mathcal{G}$, that is, payoff functions are continuous and strategy sets compact, then each set $\mathrm{BR}^i(\alpha_{-i}) \subseteq A_i$ is nonempty and compact. A strong curb set is a product set $X = \times_{i\in N} X_i$ where (a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty, compact set of pure strategies; (b) $\forall J \subseteq N, \forall \alpha = (\alpha_{-1}, \ldots, \alpha_{-n})$ with $\alpha_{-i} \in \times_{l\in N\setminus\{i\}} \Delta(X_l)$, $i \in N$, $\mathrm{CBR}^J(\alpha) \subseteq \times_{j\in J} X_j$.

Theorem 1 *Every game $G \in \mathcal{G}$ has a minimal strong curb set.*

Proof Let Q be the collection of all strong curb sets of G. A is a strong curb set of G since for every $J \subseteq N$ and $\alpha = (\alpha_{-1}, \ldots, \alpha_{-N})$ with $\alpha_{-i} \in \times_{l\in N\setminus\{i\}} \Delta(A_l)$, $i \in N$, we have $\mathrm{CBR}^J(\alpha) \subseteq \times_{j\in J} A_j$. So Q is nonempty and partially ordered via set inclusion. According to the Hausdorff Maximality Principle, Q contains a maximal nested subset R. For each $i \in N$, let $X_i = \cap_{Y\in R} Y_i$ be the intersection of player i's strategies in the nested set R. The set X_i is nonempty since the conditions of the Cantor intersection principle are satisfied, i.e., (i) the collection $\{Y_i \mid Y \in R\}$ is nested and thus satisfies the finite intersection property and (ii) each Y_i is nonempty and compact. It remains to prove that $X = \times_{i\in N} X_i$ is a minimal strong curb set. Take $\alpha = (\alpha_{-1}, \ldots, \alpha_{-N})$ with $\alpha_{-i} \in \times_{l\in N\setminus\{i\}} \Delta(X_l)$, $i \in N$. We have that $\mathrm{CBR}^J(\alpha) \cap \times_{j\in J}(A_j \setminus X_j) = \emptyset$ for $J \subseteq N$ since $\mathrm{CBR}^J(\alpha) \cap \times_{j\in J}(A_j \setminus X_j) = \mathrm{CBR}^J(\alpha) \cap (\times_{j\in J} \cup_{Y\in R} (A_j \setminus Y_j)) \subseteq \mathrm{CBR}^J(\alpha) \cap (\cup_{Y\in R} \times_{j\in J} (A_j \setminus Y_j)) = \cup_{Y\in R}(\mathrm{CBR}^J(\alpha) \cap \times_{j\in J}(A_j \setminus Y_j))$ and $\mathrm{CBR}^J(\alpha) \cap \times_{j\in J}(A_j \setminus Y_j) = \emptyset$ for all $Y \in R$ (Y is a strong curb set). This establishes that X is a strong curb set. The fact that it is minimal follows directly from the fact that R is a maximal nested subset of Q. □

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