

WHAT IS AN IDEAL A ZERO-CLASS OF?

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ABSTRACT. We characterise, in pointed regular categories, the ideals as the zero-classes of surjective relations. Moreover, we study a variation of the *Smith is Huq* condition: two surjective left split relations commute as soon as their zero-classes commute.

1. INTRODUCTION

The description of congruences, and of some other relations, in terms of their zero-classes is a very classical topic in universal algebra. It led to the study of different notions of subalgebras in pointed varieties; let us mention here the ones of ideal [9, 13, 25] and clot [1].

Later these notions have been considered in a categorical context [11, 12, 15]. Clots were characterised as zero-classes of internal reflexive relations, and ideals were characterised as regular images of clots. However, a characterisation of ideals as zero-classes of suitable relations was still missing, both in universal and categorical algebra.

The aim of the present paper is to fill this gap. We prove that, in every pointed regular category, the ideals are the zero-classes of what we call *surjective relations*. Such is any relation from an object X to an object Y where the projection on Y is a regular epimorphism. In fact, we can always choose a *left split* surjective relation to represent a given ideal, which means that moreover the projection on X is a split epimorphism. We also show that, in general, it is not possible to describe ideals by means of endorelations on an object X . The table at the end of the introduction gives an overview of the description of all the notions mentioned above in terms of zero-classes.

A related issue is to consider a variation of the so-called *Smith is Huq* condition, which says that two equivalence relations on the same object commute in the Smith–Pedicchio sense [24, 23] if and only if their zero-classes commute in the Huq sense [10]. Our condition is then the following: two semi-split surjective relations commute if and only if their zero-classes (their associated ideals) commute. This provides a conceptual interpretation of the *admissibility* condition introduced in [16] and further explored in [8, 21]. We consider some equivalent and some stronger conditions, and we compare them with the standard *Smith is Huq* condition.

The paper is organised as follows. In Section 2 we recall the notions of ideal and clot, both from the universal and the categorical-algebraic points of view, and

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we prove some stability properties of ideals. In Section 3 we prove that ideals are exactly zero-classes of surjective relations (or, equivalently, of semi-split surjective relations) and we consider some concrete examples. In Section 4 we study the above-mentioned variations of the *Smith is Huq* condition.

any (left split) relation	surjective (left split) relation	reflexive relation	equivalence relation	effective equivalence relation
monomorphism	ideal	clot	normal monomorphism	kernel

TABLE 1. Several types of monomorphisms in pointed regular categories

2. IDEALS AND CLOTS

The notion of *ideal* was introduced in [9] in the context of groups with multiple operators (also called Ω -groups), and then extended in [13]—and further studied in [25] and in subsequent papers—to varieties of algebras with a constant 0. We recall here the definition in the case of **pointed** varieties: those with a unique constant 0.

Definition 2.1. A term $t(x_1, \dots, x_m, y_1, \dots, y_n)$ in a pointed variety $\mathcal{V}$ is said to be an **ideal term** in $y_1, \dots, y_n$ if $t(x_1, \dots, x_m, 0, \dots, 0) = 0$ is an identity in $\mathcal{V}$. A subalgebra I of an algebra A in $\mathcal{V}$ is an **ideal** of A if $t(x_1, \dots, x_m, i_1, \dots, i_n)$ belongs to I for all $x_1, \dots, x_m \in A$, all $i_1, \dots, i_n \in I$ and every ideal term t .

Later, as an alternative, in the paper [1] the concept of *clot* was introduced:

Definition 2.2. A subalgebra K of A in $\mathcal{V}$ is called a **clot** in A if

$$t(a_1, \dots, a_m, 0, \dots, 0) = 0$$

and $k_1, \dots, k_n \in K$ imply $t(a_1, \dots, a_m, k_1, \dots, k_n) \in K$ for all $a_1, \dots, a_m, k_1, \dots, k_n$ in A and every $(m+n)$ -ary term function t of A .

It was shown in [1] that clots are exactly 0-classes of semi-congruences, that is, of those reflexive relations which are compatible with all the operations in the variety. Thus, for any algebra A in any variety $\mathcal{V}$ there is an inclusion

$$N(A) \subseteq \text{Cl}(A) \subseteq \text{I}(A),$$

where $N(A)$ is the set of normal subalgebras of A (that are the 0-classes of the congruences on A), $\text{Cl}(A)$ is the set of clots of A and $\text{I}(A)$ is the set of ideals.

All these notions were then studied in a categorical context (see [11, 12, 15]). Before recalling the categorical counterparts of the definitions above, we need to introduce some terminology. The context that we consider is the one of pointed regular categories.

Definition 2.3. Given a span

$$\begin{array}{ccc} & R & \\ d \swarrow & & \searrow c \\ X & & Y \end{array} \quad (\mathbf{A})$$

a **zero-class** of it is the arrow $i: I \rightarrow Y$ in the pullback

$$\begin{array}{ccc} I & \xrightarrow{l} & R \\ i \downarrow & \lrcorner & \downarrow \langle d, c \rangle \\ Y & \xrightarrow{\langle 0, 1_Y \rangle} & X \times Y. \end{array} \quad (\mathbf{B})$$

Definition 2.4. A **normalisation** of $(\mathbf{A})$ is the composite $ck: K \rightarrow X$, where $k: K \rightarrow R$ is a kernel of d .

Observe that, for our purposes, (d, c) and (c, d) are different spans. If the span (d, c) is a relation, which means that d and c are jointly monomorphic, then its zero-class is a monomorphism, since pullbacks preserve monomorphisms. Similarly, the normalisation of a relation is a monomorphism, too. Of course the zero-class and the normalisation of a span are unique up to isomorphism, so (with abuse of terminology) we may talk about “the” zero-class and “the” normalisation. In fact, the two procedures give the same result:

Proposition 2.5. *For any span (d, c) its zero-class coincides with its normalisation.*

Proof. It is easily seen that the morphism l in the diagram $(\mathbf{B})$ is a kernel of d . As a consequence, $i = cl$. On the other hand, any square such as $(\mathbf{B})$ in which $l = \ker(d)$ and $i = cl$ is a pullback. $\square$

The second part of the proof also follows from the observation that the zero-class and the normalisation of a span are unique up to isomorphism.

Definition 2.6. A **normal subobject** of an object A is the zero-class of an equivalence relation on A .

We observe that this notion is a generalisation of the notion of **kernel** of a morphism: indeed, kernels are exactly zero-classes of effective equivalence relations. It is also easy to see that, in the pointed case, the definition above is equivalent to the one introduced by Bourn in [4]: see [15] and Example 3.2.4, Proposition 3.2.12 in [2].

Definition 2.7. A **clot** of A is the zero-class of a reflexive relation on A .

The original categorical definition of clot, given in [11], was different: roughly speaking, a clot of an object A was defined as a subobject which is invariant under the *conjugation action* on A . However, the two definitions are equivalent, as already observed in [11].

The following categorical definition of ideal was proposed in [12]. It was observed in [11] that, in the varietal case, it coincides with Definition 2.1 above.

Definition 2.8. A monomorphism $i: I \rightarrow Y$ is an **ideal** if there exists a commutative square

$$\begin{array}{ccc} K & \xrightarrow{q} & I \\ k \downarrow & & \downarrow i \\ X & \xrightarrow{p} & Y \end{array} \quad (\mathbf{C})$$

in which p and q are regular epimorphisms and k is a clot. In other words, an ideal is the **regular image** of a clot.

The following fact was already observed in [12, Corollary 3.1]:

Proposition 2.9. *Every ideal is the regular image of a kernel along a regular epimorphism.*

Proof. Proposition 2.5 tells us that the morphism k in Diagram (C) is of the form cl for some kernel l and some split epimorphism c . The claim now follows, since a composite of two regular epimorphisms in a regular category is still a regular epimorphism. $\square$

The first aim of this paper is to characterise the ideals as the zero-classes of suitable relations. Before doing that, we prove some stability properties of clots and ideals.

Proposition 2.10. *Clots are stable under pullbacks, and an intersection of two clots is still a clot.*

Proof. Suppose that $k: K \rightarrow X$ is the zero-class of a reflexive relation (R, d, c) on X and consider a morphism $f: Y \rightarrow X$. Pull back k along f , and $\langle d, c \rangle: R \rightarrow X \times X$ along $f \times f: Y \times Y \rightarrow X \times X$, to obtain the commutative cube

$$\begin{array}{ccccc}
 K' & \xrightarrow{\quad} & K & & \\
 \downarrow k' & \searrow & \downarrow k & \searrow & \\
 & R' & \xrightarrow{\quad} & R & \\
 \downarrow \langle d', c' \rangle & & \downarrow & & \downarrow \langle d, c \rangle \\
 Y & \xrightarrow{\quad f \quad} & X & & \\
 \downarrow \langle 0, 1_Y \rangle & & \downarrow \langle 0, 1_X \rangle & & \\
 Y \times Y & \xrightarrow{\quad f \times f \quad} & X \times X & &
 \end{array}$$

Note that (R', d', c') is a reflexive relation on Y . Since the front, back, and right hand side faces of this cube are all pullbacks, also its left hand side face is a pullback. This means that k' is the zero-class of R' , so that the pullback k' of k is a clot.

Now consider two clots $k: K \rightarrow X$ and $l: L \rightarrow X$ on an object X , the respective zero-classes of the reflexive relations (R, d, c) and (S, d', c') on X . Consider the cube

$$\begin{array}{ccccc}
 K \cap L & \xrightarrow{\quad} & R \cap S & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & K & \xrightarrow{\quad} & R & \\
 \downarrow k & & \downarrow & & \downarrow \langle d, c \rangle \\
 L & \xrightarrow{\quad} & S & & \\
 \downarrow l & & \downarrow \langle d', c' \rangle & & \\
 X & \xrightarrow{\quad \langle 0, 1_X \rangle \quad} & X \times X & &
 \end{array}$$

in which all faces are pullbacks. We see that the monomorphism $K \cap L \rightarrow X$ is the zero-class of the reflexive relation $R \cap S \rightarrow X \times X$ on X . In other words, the intersection of the clots k and l is still a clot. $\square$

Proposition 2.11. *Ideals are stable under:*

- (a) *regular images;*
- (b) *pullbacks;*

(c) *compositions with product inclusions.*

Furthermore,

- (d) *an intersection of an ideal and a clot is an ideal;*
- (e) *an intersection of two ideals is an ideal.*

Proof. (a) is immediate from the definition. (b) holds because, given a morphism $f: Y' \rightarrow Y$ and an ideal i which is a regular image of a clot k as in (C), we may consider the commutative cube

$$\begin{array}{ccccc}
 K' & \xrightarrow{\quad q' \quad} & I' & & \\
 \downarrow k' & \searrow & \downarrow i' & & \\
 & X' & \xrightarrow{\quad p' \quad} & Y' & \\
 \downarrow f' & \downarrow & \downarrow & \downarrow f & \\
 K & \xrightarrow{\quad q \quad} & I & & \\
 \downarrow k & \searrow & \downarrow i & & \\
 & X & \xrightarrow{\quad p \quad} & Y &
 \end{array}$$

in which the front, left and right squares are pullbacks by construction. It follows that the back square is also a pullback, and the dotted arrows p' and q' are regular epimorphisms. Furthermore, the monomorphism k' is a clot by Proposition 2.10. As a consequence, the pullback i' of i along f is an ideal, as a regular image of the clot k' .

For the proof of (c), recall that kernels compose with product inclusions: if $k: K \rightarrow X$ is the kernel of $f: X \rightarrow X'$, then $\langle 1_X, 0 \rangle k = \langle k, 0 \rangle: K \rightarrow X \times W$ is the kernel of $f \times 1_W: X \times W \rightarrow X' \times W$ for any object W . If now i is an ideal as in (C), then $\langle 1_Y, 0 \rangle i = \langle i, 0 \rangle: I \rightarrow Y \times W$ is the direct image of $\langle k, 0 \rangle$ along the regular epimorphism $f \times 1_W$.

For the proof of (d), suppose i is an ideal as in (C) and $l: L \rightarrow Y$ is a clot. We consider the commutative cube

$$\begin{array}{ccccc}
 K \cap L' & \xrightarrow{\quad \quad} & I \cap L & & \\
 \downarrow & \searrow & \downarrow & & \\
 & L' & \xrightarrow{\quad \quad} & L & \\
 \downarrow l' & \downarrow & \downarrow & \downarrow l & \\
 K & \xrightarrow{\quad q \quad} & I & & \\
 \downarrow k & \searrow & \downarrow i & & \\
 & X & \xrightarrow{\quad p \quad} & Y &
 \end{array}$$

in which the front, left and right squares are pullbacks by construction. Then the back square is also a pullback, so that the dotted arrow is a regular epimorphism. Since the monomorphism l' , and thus also $K \cap L' \rightarrow X$, are still clots by Proposition 2.10, this proves that the intersection $I \cap L \rightarrow Y$ is an ideal.

For the proof of (e), suppose that both i and l are ideals. Repeating the above construction, through (b) and (d) we see that $K \cap L' \rightarrow X$ is an ideal, as the intersection of the clot k with the ideal l' . The result now follows from (a). $\square$

3. IDEALS AND SEMI-SPLIT SURJECTIVE RELATIONS

In order to characterise ideals as zero-classes, we shall be interested in spans where one of the legs is a regular or even a split epimorphism.

Definition 3.1. A **left split span** from X to Y is a diagram

$$\begin{array}{ccc} & R & \\ d \nearrow & & \searrow c \\ X & & Y \\ e \nwarrow & & \end{array} \quad (\mathbf{D})$$

where $de = 1_X$. A left split span (d, c, e) is called a **left split relation** when the span (d, c) is jointly monomorphic.

Proposition 3.2. For a morphism $i: I \rightarrow Y$, the following conditions are equivalent:

- (i) i is a monomorphism;
- (ii) i is the zero-class of a left split relation;
- (iii) i is the zero-class of a relation on Y .

Proof. For the equivalence between (i) and (ii) it suffices to take $X = 0$, and for the one between (i) and (iii) we consider the span $(0: I \rightarrow Y, i: I \rightarrow Y)$. In both cases the span at hand is a relation if and only if i is a monomorphism. $\square$

Definition 3.3. A **surjective span** from X to Y is a diagram

$$\begin{array}{ccc} & R & \\ d \nearrow & & \searrow c \\ X & & Y \end{array}$$

where c is a regular epimorphism. A surjective span (d, c) is called a **surjective relation** when the span (d, c) is jointly monomorphic.

Sometimes we consider both conditions together and talk about **surjective left split spans** or relations.

We are now ready to prove our main result.

Theorem 3.4. In any pointed regular category, for any morphism $i: I \rightarrow Y$, the following are equivalent:

- (i) i is an ideal;
- (ii) i is the zero-class of a surjective left split relation;
- (iii) i is the zero-class of a surjective relation.

Proof. To prove (i) $\Rightarrow$ (ii), suppose that i is an ideal as in (C) above, where k is the zero-class of a reflexive relation (R, d, c, e) . We consider the commutative cube

$$\begin{array}{ccccc} K & \xrightarrow{q} & I & & \\ \downarrow k & \searrow & \downarrow i & \searrow & \\ & R & \xrightarrow{q'} & S & \\ & \downarrow \langle d, c \rangle & \downarrow & \downarrow & \\ X & \xrightarrow{p} & Y & & \\ \downarrow \langle 0, 1_X \rangle & & \downarrow \langle 0, 1_Y \rangle & & \\ X \times X & \xrightarrow{1_X \times p} & X \times Y & & \end{array}$$

in which S is the regular image of R along $1_X \times p$ and $I \rightarrow S$ is induced by functoriality of image factorisations. We have to show that the square on the right is a pullback. Let the square on the left

$$\begin{array}{ccc}
 & & K \cdots \cdots \rightarrow P \\
 & & \downarrow \quad \searrow \\
 P & \xrightarrow{\quad} & S \\
 \downarrow & \lrcorner & \downarrow \\
 Y & \xrightarrow{\langle 0, 1_Y \rangle} & X \times Y
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & K & \cdots \cdots & P & \\
 & \downarrow k & & \downarrow & \\
 & R & \xrightarrow{q'} & S & \\
 \langle d, c \rangle \swarrow & & & & \searrow \langle d', c' \rangle \\
 X & \xrightarrow{p} & Y & & \\
 \langle 0, 1_X \rangle \swarrow & & \searrow \langle 0, 1_Y \rangle & & \\
 X \times X & \xrightarrow{1_X \times p} & X \times Y & &
 \end{array}$$

be the pullback in question. The induced arrow $f: I \rightarrow P$ is an isomorphism. Indeed it is a monomorphism since i is. Moreover, the bottom and left squares in the cube are pullbacks, and so the dotted arrow $K \rightarrow P$ is a regular epimorphism, being a pullback of the regular epimorphism q' . Then f is a regular epimorphism, hence an isomorphism. Note that d' is split by $q'e$ and c' is a regular epimorphism because $pc = c'q'$ is.

(ii) $\Rightarrow$ (iii) is obvious. For the proof of (iii) $\Rightarrow$ (i), let $i: I \rightarrow Y$ be the zero-class **(B)** of a surjective relation (d, c) . Consider the pullback

$$\begin{array}{ccc}
 T & \cdots \cdots & R \\
 \downarrow \langle d', c' \rangle & & \downarrow \langle d, c \rangle \\
 R \times R & \xrightarrow{d \times c} & X \times Y
 \end{array}$$

of $\langle d, c \rangle$ and $d \times c$, which defines a reflexive relation (T, d', c', e') on R , where e' is $\langle \langle 1_R, 1_R \rangle, 1_R \rangle$. We prove that i is the regular image of the zero-class k of T along the regular epimorphism c as in the square on the left.

$$\begin{array}{ccc}
 K & \xrightarrow{q} & I \\
 \downarrow k & & \downarrow i \\
 R & \xrightarrow{c} & Y
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & K & \cdots \cdots & I & \\
 & \downarrow k & & \downarrow i & \\
 & T & \xrightarrow{\quad} & R & \\
 \langle d', c' \rangle \swarrow & & & & \searrow \langle d, c \rangle \\
 R & \xrightarrow{c} & Y & & \\
 \langle 0, 1_R \rangle \swarrow & & \searrow \langle 0, 1_Y \rangle & & \\
 R \times R & \xrightarrow{d \times c} & X \times Y & &
 \end{array}$$

Here it suffices to consider the cube on the right, noting that q is a regular epimorphism because all vertical squares are pullbacks and c is a regular epimorphism by assumption. $\square$

Remark 3.5. Consider a pointed variety of universal algebras $\mathcal{V}$ and let $A \in \mathcal{V}$. According to the previous theorem, a subalgebra I of A is an ideal of A if and only if there is a surjective relation R for which I is the zero-class of R . In other words, I is an ideal if and only if there exists a subalgebra R of $B \times A$, for some $B \in \mathcal{V}$, such

that the second projection is surjective and $a \in I$ if and only if $(0, a) \in R$. A direct proof of this is in fact pretty simple. That such a zero-class is an ideal is trivial from Higgins' definition of ideals by means of ideal terms (Definition 2.1). For the converse, assume that I , as an ideal of A , is the image $f(K)$ of a clot K of some $B \in \mathcal{V}$ under a surjective homomorphism $f: B \rightarrow A$. Then K is the zero-class of some reflexive subalgebra S of $B \times B$. One easily sees that the relational product $f \circ S$, where now f means “the set-theoretic graph of the mapping f ”, is a surjective relation, whose zero-class is exactly I .

As the following example shows, in general it is not possible to see every ideal as a zero-class of a surjective endorelation. We are grateful to Sandra Mantovani for suggestions concerning this example.

Example 3.6. Let $\mathcal{V}$ be the variety defined by a unique constant 0 and a binary operation s satisfying just the identity $s(0, 0) = 0$. In this variety, ideal terms are all “pure”: in any term $t(x_1, \dots, x_m, y_1, \dots, y_n)$ which is an ideal term in $y_1, \dots, y_n$, necessarily $m = 0$. Therefore all subalgebras are ideals. Consider then the three element algebra $A = \{0, 1, a\}$, with $s(a, 1) = s(1, a) = s(a, a) = a$, and $s(x, y) = 0$ otherwise. $C = \{0, 1\}$ is a subalgebra, and we have that $s(a, 0) = 0$ lies in C , but $s(a, 1) = a$ does not belong to C . Hence C is an ideal, but not a clot. Suppose that there exists a surjective relation R on A such that C is its zero-class. Then there should exist $x \in A$ such that xRa . But x cannot be 0 , because $a \notin C$. $1Ra$ is impossible, too, because otherwise $s(1, 1)Rs(a, a)$, while $s(1, 1) = 0$ and $s(a, a) = a$. Similarly, aRa is impossible, otherwise $s(0, a)Rs(1, a)$, while $s(0, a) = 0$ and $s(1, a) = a$. Hence such a surjective endorelation R does not exist.

We conclude this section with the following observation. It is well known [15] that, in any pointed exact Mal'tsev category, ideals and kernels coincide. Theorem 3.4 provides us with the following quick argument.

Proposition 3.7. *In any pointed exact Mal'tsev category, ideals and kernels coincide.*

Proof. Let $i: I \rightarrow Y$ be the zero-class of a surjective left split relation (d, c, e) as in (D). Theorem 5.7 in [6] tells us that the pushout of d and c is also a pullback; as a consequence, i is the kernel of the pushout $c_*(d)$ of d along c . $\square$

4. THE *Smith is Huq* CONDITION

From now on we work in a category which is pointed, regular and weakly Mal'tsev [17]. We first recall

Definition 4.1. A finitely complete category is weakly Mal'tsev if, for any pullback of the form:

$$\begin{array}{ccc} A \times_B C & \begin{array}{c} \xleftarrow{e_2} \\ \xrightarrow{\pi_2} \end{array} & C \\ \begin{array}{c} \uparrow \pi_1 \\ \downarrow e_1 \end{array} & & \begin{array}{c} \uparrow g \\ \downarrow s \end{array} \\ A & \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} & B, \end{array} \quad (\mathbf{E})$$

where $fr = 1_B = gs$, the morphisms $e_1 = \langle 1_A, sf \rangle$ and $e_2 = \langle rg, 1_C \rangle$, induced by the universal property of the pullback, are jointly epimorphic.

We observe that any finitely complete Mal'tsev category [7] is weakly Mal'tsev. Indeed, in [3] it was proved that a finitely complete category is Mal'tsev if, for any pullback of the form (E), the morphisms $e_1 = \langle 1_A, sf \rangle$ and $e_2 = \langle rg, 1_C \rangle$ are jointly

strongly epimorphic. In particular, every Mal'tsev variety [14] is a weakly Mal'tsev category. In [18] it is shown that the variety of distributive lattices is weakly Mal'tsev. In [19] several other examples are given, amongst which the variety of commutative monoids with cancellation.

In the context of weakly Mal'tsev categories, we say that two left split spans (f, α, r) and (g, γ, s) from B to D as in

$$\begin{array}{ccccc} A & \xrightleftharpoons[r]{f} & B & \xrightleftharpoons[s]{g} & C \\ & \searrow \alpha & & \swarrow \gamma & \\ & & D & & \end{array} \quad (\mathbf{F})$$

centralise each other or **commute** when there exists a (necessarily unique) morphism $\varphi: A \times_B C \rightarrow D$, called **connector** from the pullback

$$\begin{array}{ccccc} & & C & & \\ \pi_2 \nearrow & & \downarrow s & \nearrow \gamma & \\ A \times_B C & \xrightleftharpoons[e_2]{g} & B & \xrightarrow{\beta} & D \\ \pi_1 \searrow & & \downarrow r & \nwarrow \alpha & \\ & & A & & \end{array}$$

of f and g to the object D such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$. Note that, when this happens, $\gamma s = \alpha r$; we denote this morphism by $\beta: B \rightarrow D$. In other words, the existence of β is a necessary condition for the given left split spans to centralise each other. The condition for two left split spans to commute was called *admissibility* in [16]. There it was implicit that such a condition deals with a certain type of commutativity, but it was not possible to express precisely *what* commutes. Our new interpretation makes it clear that the admissibility condition is just the commutation of left split spans.

If we take $\beta = 1_B$, we immediately recover the notion of commutativity of reflexive graphs in the Smith–Pedicchio sense:

Proposition 4.2. *Two reflexive graphs commute in the Smith–Pedicchio sense if and only if they commute in the above sense.* $\square$

We recall that the commutativity of equivalence relations was first introduced by Smith in [24] for Mal'tsev varieties, and then extended by Pedicchio [23] to Mal'tsev categories. However, weakly Mal'tsev categories are a suitable setting for the definition (because the connector, as defined above, is unique), and the commutativity can be defined, as above, just for reflexive graphs.

If, in the diagram (F), we take $B = 0$, we get the definition of commutativity of two morphisms in the Huq sense [10]: two morphisms $\alpha: A \rightarrow D$ and $\gamma: C \rightarrow D$ **commute** when there exists a (necessarily unique) morphism $\varphi: A \times C \rightarrow D$, called the **cooperator** of α and γ , such that

$$\varphi \langle 1_A, 0 \rangle = \alpha \quad \text{and} \quad \varphi \langle 0, 1_C \rangle = \gamma.$$

A pointed regular weakly Mal'tsev category satisfies the **Smith is Huq** condition [20], shortly denoted by (SH), when a pair of equivalence relations over the same object commutes as soon as their zero-classes do. (The converse is always true). We observe that the (SH) condition has the following interesting consequence. We recall that an object A is **commutative** if its identity commutes with itself (in the Huq sense); it is **abelian** if it has an internal abelian group structure.

Proposition 4.3. *If (SH) is satisfied, then every commutative object is abelian.*

Proof. The identity 1_X of an object X is the normalisation of the indiscrete relation ∇_X . If X is commutative, then 1_X commutes with itself; by the (SH) condition, the relation ∇_X commutes with itself, too. This situation is represented by the following diagram:

$$\begin{array}{ccc} & X \times X \times X & \\ \swarrow & \downarrow p & \searrow \\ X \times X & & X \times X \\ \swarrow & \downarrow & \searrow \\ & X & \end{array}$$

The connector $p: X \times X \times X \rightarrow X$ is then an internal Mal'tsev operation on X . To conclude the proof it suffices to observe that, in a pointed category, an object is endowed with an internal Mal'tsev operation if and only if it is endowed with an internal abelian group structure [2, Proposition 2.3.8]. $\square$

Our aim is to study the condition obtained by replacing equivalence relations and normal subobjects in (SH) by surjective left split relations and ideals. In order to do so, we start by introducing some terminology. We call a morphism **ideal-proper** when its regular image is an ideal; we say that a cospan is **ideal-proper** when so are the morphisms of which it consists.

In a pointed finitely complete category $\mathcal{C}$, given an object B , the category $\mathbf{Pt}_B(\mathcal{C})$ of so-called **points over** B is the category whose objects are pairs $(p: E \rightarrow B, s: B \rightarrow E)$ where $ps = 1_B$. A morphism

$$(p: E \rightarrow B, s) \rightarrow (p': E' \rightarrow B, s')$$

in $\mathbf{Pt}_B(\mathcal{C})$ is a morphism $f: E \rightarrow E'$ in $\mathcal{C}$ such that $p'f = p$ and $fs = s'$. We have, for any B , a functor (called the **kernel functor**) $\text{Ker}_B: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}$ associating with every split epimorphism its kernel. We can now formulate the main result of this section.

Theorem 4.4. *In any pointed, regular and weakly Mal'tsev category $\mathcal{C}$, the following are equivalent:*

- (i) *for every object B in $\mathcal{C}$, the kernel functor $\text{Ker}_B: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}$ reflects Huq-commutativity of ideal-proper cospan;*
- (ii) *a pair of surjective left split relations over the same objects commutes as soon as their zero-classes do;*
- (iii) *a pair of surjective left split spans over the same objects commutes as soon as their zero-classes do.*

Proof. The equivalence between conditions (ii) and (iii) is proved just by taking regular images. In order to prove that (i) and (ii) are equivalent, given a pair of surjective left split relations over the same object, we rewrite Diagram (F) in the shape

$$\begin{array}{ccccc} A & \xrightarrow{\langle \alpha, f \rangle} & D \times B & \xleftarrow{\langle \gamma, g \rangle} & C \\ & \searrow & \downarrow \pi_B & \swarrow & \\ & & B & & \end{array} \quad \begin{array}{c} \downarrow r \\ \downarrow s \\ \downarrow f \\ \downarrow g \end{array} \quad (\mathbf{G})$$

and consider it as a cospan $(\langle \alpha, f \rangle, \langle \gamma, g \rangle)$ in $\mathbf{Pt}_B(\mathcal{C})$. Let us prove that this cospan is ideal-proper. To do that, it suffices to notice that $\langle \alpha, f \rangle$ is the composite of the kernel $\langle 1_A, f \rangle: A \rightarrow A \times B$ with the regular epimorphism $\alpha \times 1_B: A \times B \rightarrow D \times B$.

Indeed, the outer square in the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\langle 1_A, f \rangle} & A \times B \\
 \downarrow f & \swarrow r \quad \searrow \pi_B & \downarrow f \times 1_B \\
 & B & \\
 \downarrow f & \swarrow \langle r, 1_B \rangle \quad \searrow \pi_2 & \downarrow f \times 1_B \\
 B & \xrightarrow{\langle 1_B, 1_B \rangle} & B \times B
 \end{array}$$

is a pullback in $\mathbf{Pt}_B(\mathcal{C})$. The same is true for $\langle \gamma, g \rangle$. To conclude the proof of the equivalence between (i) and (ii) it suffices then to observe that applying the kernel functor Ker_B to the cospan $(\mathbf{G})$ gives the normalisations of the two surjective split relations. $\square$

It is immediately seen that condition (i) above is equivalent to the condition that, for every morphism $p: E \rightarrow B$ in $\mathcal{C}$, the pullback functor

$$p^*: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathbf{Pt}_E(\mathcal{C})$$

—which sends every split epimorphism over B into its pullback along p —reflects Huq-commutativity of ideal-proper cospans. In the same way as for the previous theorem, it can be shown that also the following conditions are equivalent.

Proposition 4.5. *In any pointed, regular and weakly Mal'tsev category $\mathcal{C}$, the following are equivalent:*

- (iv) *for every object B in $\mathcal{C}$, the kernel functor $\text{Ker}_B: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}$ reflects Huq-commutativity of cospans;*
- (v) *a pair of left split relations over the same objects commutes as soon as their zero-classes do;*
- (vi) *a pair of left split spans over the same objects commutes as soon as their zero-classes do.* $\square$

Again, condition (iv) can be expressed equivalently in terms of all pullback functors $p^*: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathbf{Pt}_E(\mathcal{C})$. Note that the conditions (iv)–(vi) are stronger than (i)–(iii): this is easily seen by comparing conditions (i) and (iv). Indeed, condition (iv) requires that the kernel functors reflect commutativity of a wider class of cospans.

We conclude by observing that Theorem 4.4 is a generalisation of [22, Proposition 2.5] and Proposition 4.5 is a generalisation of [22, Proposition 3.1], which have been proved for pointed exact Mal'tsev categories (see also Theorem 2.1 in [5]). Indeed, in such categories ideals coincide with kernels (Proposition 3.7). In particular, in a pointed exact Mal'tsev category the conditions (i)–(iii) are equivalent to (SH), while (iv)–(vi) are equivalent to the stronger condition (SSH)—see [22].

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