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Smooth Wave Front Set and Deformations of Tempered Distributions Algebras

Une dissertation présentée pour l'obtention du titre de Docteur en Sciences par

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"Where in the Schrödinger equation do you put the joy of being alive?"

- Eugene Wigner

Remerciements

"Often I am reluctant to call myself a mathematician [...] I know enough to know it is not enough. I do not sleep and wake up with a math problem in mind... I certainly like to teach, and the more I teach the more I understand. Something I am really interested in is repetition. Artists, musicians and dancers rehearse and repeat. In French the word "répéter" means to repeat and also to rehearse. Research also means a lot of repetition, you repeat and you repeat again and at some point something new comes out of this repetition. The feeling of stagnation, that you are in a vicious circle, is preparation for creation..."

- Betül Tanbay, *Women in Mathematics throughout Europe. A gallery of portraits.*

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Introduction

"What we observe is not nature itself, but nature exposed to our method of questioning."

- Werner Heisenberg

"The goal of mathematics is the symbolic comprehension of the infinite with human, that is finite, means."

- Hermann Weyl

At the beginning of the twentieth century, advances in technology enabled physicists to observe quantum phenomena through experiments. These experiments revealed how strange the quantum world is, often defying our classical physical intuition (see, for example, the presentation of the two-slit experiment in [FLS10]). In the 1920s, these observations led to the development of quantum mechanics, a theory that is fundamentally distinct from classical Newtonian mechanics and is characterized by counterintuitive and often paradoxical principles. One of the most famous of these principles is the *Heisenberg uncertainty principle*, which states that it is impossible to simultaneously know the precise position and velocity of a particle in a quantum system. The more precisely we know a particle's position, the more uncertain our measurement of its velocity becomes, and vice versa. This principle is closely linked to another fundamental aspect of quantum mechanics: its probabilistic nature, in contrast to the deterministic nature of classical mechanics. Whereas in classical mechanics, we can precisely determine the evolution of a phenomenon from its initial conditions, in quantum mechanics, we cannot predict the exact evolution of a physical system but only the probability distribution of the various possible outcomes.

Despite experiments compelling us to accept the strange nature of the quantum world, it remains unconvincing that there should be no tangible link between classical and quantum laws. As a result, almost simultaneously with the advent of quantum mechanics, physicists and mathematicians—led by Dirac [Dir25] and Schrödinger [Sch82]—began to seek a bridge between quantum and classical theories. This effort gave rise to a new field of research at the intersection of theoretical physics and mathematics: *quantization theory*. To introduce the fundamental ideas of quantization theory, we first need to provide a brief description of classical and quantum mechanical systems.

From Classical to Quantum Mechanics

Classical Mechanics The connection between classical and quantum systems is elegantly facilitated by a geometrical formulation of classical mechanics introduced by Hamilton. Hamiltonian mechanics describes a classical physical system as a pair $(M := T^*N, H)$, where N is a smooth manifold and H is a function in $C^\infty(T^*N)$, the space of smooth functions on the cotangent bundle of N. Let us delve a bit deeper into this formalism.

- The manifold N represents the space of all possible positions of a physical object: the whole space for a free particle, the circle for a pendulum, the straight line for the harmonic oscillator, etc. It is called the *configuration space* of the system.
- The cotangent bundle $M = T^*N$ is known as the *phase space*. Its elements are pairs (q, p), where $q \in N$ represents the position of the physical object, and $p \in T^*_qN$ is its momentum. It is well known that cotangent bundles are canonically endowed with a symplectic structure. In particular, they admit a canonical symplectic form ω and an associated Poisson bracket $\{\cdot, \cdot\}$ (for a brief overview, see appendix B; for a comprehensive pedagogical treatment, see [Mei98]). The symplectic structure is, however, not essential for writing the kinematics of the system, and sometimes the base space

N is omitted directly considering the more general setting of a Poisson manifold $(M, \{\cdot,\cdot\})$.

- The algebra $C^\infty(M)$ (with pointwise product) is the *algebra of observables* of the system. It contains the physical quantities one might wish to measure: position, momentum, energy, etc.
- The particular observable H is the *Hamiltonian of the system*. It is often interpreted as the total energy of the system.

Hamiltonian mechanics postulates that the time evolution of an observable f in a classical system (M, H) is governed by the *equation of motion*

$$\frac{d}{dt}f = \{f, H\}.$$

The simplest example of a physical system is the case of a free particle in $\mathbb{R}^n$, for which $N = \mathbb{R}^n = \{q\}$, $T^*N = \mathbb{R}^{2n} = \{(q,p)\}$, $\omega = dp_i \wedge dq^i$, and

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.$$

If this system is subjected to a force F derived from a potential V, and if we define the Hamiltonian as the total energy of the system:

$$H(q,p) = \frac{p^2}{2m} + V(q), \tag{1}$$

then the Hamiltonian equations of motion for p and q are equivalent to the famous equation of Newtonian mechanics:

$$\frac{d}{dt}q = \{q, H\} \quad \text{and} \quad \frac{d}{dt}p = \{p, H\} \quad \Longleftrightarrow \quad \vec{F} = m\vec{a}.$$

Quantum mechanics As mentioned earlier, quantum mechanics operates under principles distinct from those of classical mechanics, necessitating a different formalism. Notably, the Heisenberg uncertainty principle implies that quantum observables do not commute. The simplest non-commutative objects we encounter in mathematics are matrices, or more generally, linear operators on some Hilbert space. In this context (and also because it matches experimental results), a quantum system is defined as a pair $(\mathcal{H}, \hat{H})$, where $\mathcal{H}$ is a Hilbert space and $\hat{H}$ is a fixed Hermitian operator on $\mathcal{H}$.

- Quantum observables are Hermitian operators on $\mathcal{H}$. The possible values of these observables are given by their spectrum and are thus generally discrete.
- The Hilbert space $\mathcal{H}$ is the space of states of the quantum system. A state is a unit vector $\psi \in \mathcal{H}$ that associates a probability distribution to the possible values of each observable.
- The fixed operator $\hat{H}$ is the quantum Hamiltonian of the system.

Explicitly, since a quantum observable A is a Hermitian operator, it admits an associated basis of eigenstates $A\psi_i = \lambda_i \psi_i$. The possible values for A are given by its eigenvalues $\lambda_i \in \mathbb{R}$. If the system is in an eigenstate ψ_i, then A will take the value λ_i with probability 1. For a general state

$$\psi = \sum \alpha_i \psi_i, \quad \text{with} \quad \sum |\alpha_i|^2 = 1,$$

each value λ_i will occur with probability $|\alpha_i|^2$. Quantum mechanics postulates that the time evolution of a quantum observable A is given by

$$\frac{d}{dt} A = \frac{i}{\hbar}[A, \hat{H}],$$

where $\hbar$ is a physical constant called the *Planck constant* and $[\cdot,\cdot]$ is the commutator of operators.

Quantization By quantization, we mean a process through which one can build a quantum system from the data of a classical one

$$\Big(C^\infty(M), H, \{\cdot,\cdot\}\Big) \rightsquigarrow \Big(\mathscr{L}(\mathcal{H}), \hat{H}, [\cdot,\cdot]\Big),$$

where we denote by $\mathscr{L}(\mathcal{H})$ the space of linear operators on $\mathcal{H}$. In general, these processes are realized through a *quantization map* Q that transforms classical observables into quantum ones, i.e., smooth functions on M into linear operators on $\mathcal{H}$:

$$Q \colon C^\infty(M) \to \mathscr{L}(\mathcal{H}).$$

It is usually required that Q satisfies some correspondence principles between classical and quantum kinematics, the ideal requirement being the so-called Dirac condition:

$$\{f_1, f_2\} = f_3 \quad \Longrightarrow \quad [Q(f_1), Q(f_2)] = -i\hbar Q(f_3). \tag{2}$$

Let's now see how one can quantize the classical system describing a free particle in $\mathbb{R}^n$. In this simple case, Schrödinger proposed to use the Hilbert space $L^2(\mathbb{R}^n)$ of square-integrable functions on $\mathbb{R}^n$, and to associate to the position q and momentum p the operators

$$\hat{q}^i : \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\ , \quad \varphi \mapsto \left[q \mapsto q^i \varphi(q)\right] \tag{3}$$

$$\hat{p}_i : \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\ , \quad \varphi \mapsto \left[q \mapsto -i\hbar \frac{\partial}{\partial_{q^i}} \varphi(q)\right]. \tag{4}$$

Note that these operators are, in fact, not defined on $L^2(\mathbb{R}^n)$, but rather on the dense subspace $\mathcal{S}(\mathbb{R}^n)$, called the Schwartz space, which consists of smooth functions whose derivatives decay faster than any polynomial (see Chapter I). Moreover, this satisfies the Dirac condition, which yields a mathematical formulation of the Heisenberg uncertainty principle since the commutator of $\hat{q}^i$ and $\hat{p}_i$ is non-zero:

$$[\hat{q}^i, \hat{p}_i] = i\hbar.$$

Furthermore, (3) and (4) are sufficient to quantize the classical Hamiltonian (1) when V is a polynomial in q:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}).$$

This enables us to write the quantum equation of motion

$$\frac{d}{dt}\hat{q} = \frac{i}{\hbar}[\hat{q}, \hat{H}],$$

which yields satisfactory experimental results.

However, this does not explain how to quantize observables that are not pure powers of q or p. For example, what should be the operator quantizing $q^i p_j$? One might define $\widehat{q^i p_j} = \hat{q}^i \hat{p}_j$, or the more symmetric choice $\widehat{q^i p_j} = \frac{1}{2}(\hat{q}^i \hat{p}_j + \hat{p}_i \hat{q}^i)$. Different orderings are possible, and each will provide a different extension of the Schrödinger quantization.

For $a \in \mathcal{S}(V^2)$, $\phi \in L^2(\mathbb{R}^n)$, and $q_0 \in \mathbb{R}^n$, the formula

$$\begin{aligned} &Q_W : \mathcal{S}(\mathbb{R}^{2n}) \to \mathscr{L}(L^2(\mathbb{R}^n)) \\ &\left(Q_W(a)\phi\right)(q_0) = (2\pi\hbar)^{-n} \int e^{\frac{i}{\hbar}(q_0 - q)p} a\left(\frac{q_0 + q}{2}, p\right) \phi(q) \mathrm{d}q\mathrm{d}p, \end{aligned} \tag{5}$$

defines the famous *Weyl quantization* [Wey27], which corresponds to the symmetric ordering $\widehat{q^i p_j} = \frac{1}{2}(\hat{q}^i \hat{p}_j + \hat{p}_i \hat{q}^i)$. This yields our first example of quantization:

$$\Big(\mathcal{S}(\mathbb{R}^{2n}), H, \{\cdot,\cdot\}\Big) \xrightarrow{Q_W} \Big(\mathscr{L}(L^2(\mathbb{R}^n)), \hat{H}, [\cdot,\cdot]\Big).$$

For a more comprehensive historical survey of quantization theory and a pedagogical presentation of some of its mathematical approaches, we recommend [Tod12]. As for us, we will now focus on the particular quantization approach studied in this work.

Deformation quantization

This work is motivated by a particular approach to quantization introduced in 1978 by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [BFF⁺78a, BFF⁺78b]. They suggested that quantization can be understood as *"a deformation of the structure of the algebra of classical observables, rather than as a radical change in the nature of the observables"*. This deformed algebra structure on the space of classical observables, often denoted $\star$, can be thought of as inherited from some operator composition:

$$\operatorname{Op}(a \star b) := \operatorname{Op}(a) \circ \operatorname{Op}(b). \tag{6}$$

The fact that the nature of the observables does not change drastically when the system is quantized enables us to imagine quantization as a *continuous process*. Indeed, in deformation quantization, the transition from quantum to classical mechanics is viewed as a continuous change of scale. In practice, this is realized by introducing a formal parameter $\hbar$[1] in the product $\star_\hbar$ in such a way that, as $\hbar \to 0$, we recover the classical pointwise product of functions:

$$\lim_{\hbar \to 0} a \star_\hbar b = ab.$$

This approach gave rise to the notion of a star-product on a Poisson manifold.

[1]From now on, $\hbar$ will denote a formal parameter rather than the physical Planck constant.

Definition. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold, and $C^\infty(M)[[\hbar]]$ be the space of formal power series in $\hbar$ with coefficients in $C^\infty(M)$. A *deformation quantization* on $(M, \{\cdot, \cdot\})$ is a map

$$\star_\hbar \colon C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$$

such that:

1. $\lim_{\hbar \to 0} a \star_\hbar b = ab$ and $a \star_\hbar b = ab + \sum_{k=1}^\infty \hbar^k C_k(a, b)$;
2. The antisymmetrization of C_1 is exactly the Poisson bracket:
$$[C_1] \coloneqq C_1(f_1, f_2) - C_1(f_2, f_1) = \{f_1, f_2\}. \tag{7}$$
3. For every k, the coefficient $C_k \colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a bi-differential operator.

The mapping $\star_\hbar$ is called a *formal star-product* on $(M, \{\cdot, \cdot\})$.

Note that (7) is a deformed version of Dirac's condition (2), since it implies
$$\lim_{\hbar \to 0} \frac{1}{\hbar} [a, b]_{\star_\hbar} = \{a, b\},$$
where
$$[a, b]_{\star_\hbar} = a \star_\hbar b - b \star_\hbar a.$$

Weyl quantization provides an easy example of deformation quantization. As suggested above, star-products can often be obtained through operator composition. In the case of Weyl quantization, a simple computation (the details of which are presented in the proof of Proposition 1.2.13) yields

$$\begin{aligned} &Q_W(a \star_\hbar^W b) \coloneqq Q_W(a) \circ Q_W(b) \\ &a \star_\hbar^W b\,(x_0) = (2\pi\hbar)^{-n} \int e^{\frac{2i}{\hbar}(\omega(x_0, x) + \omega(x, y) + \omega(y, x_0))} a(x) b\,(y) \mathrm{d}x \mathrm{d}y, \end{aligned} \tag{8}$$

where $a, b \in C^\infty(\mathbb{R}^{2n})$ and ω is the canonical symplectic form on $\mathbb{R}^{2n}$:

$$\omega\left((x, \xi), (y, \eta)\right) = (x, \xi) \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} y \\ \eta \end{pmatrix}.$$

The Taylor expansion of (8) around $\hbar = 0$ yields

$$\begin{aligned} a \star_\hbar^W b\,(x_0) &= ab + \frac{\hbar}{2i}\{a,b\} \\ &\quad + \sum_{k=2}^{\infty} \frac{1}{k!}\left(\frac{\hbar}{2i}\right)^k \sum_{\substack{i_1,\dots,i_k=1\\ j_1,\dots,j_k=1}}^{2} \omega^{i_1 j_1}\dots\omega^{i_k j_k}\partial_{i_1\dots i_k} f_1 \partial_{j_1\dots j_k} f_2 \end{aligned} \tag{9}$$

where ω^{ij} are the components of the inverse matrix of ω. It is easy to check that (9) defines a star-product on $C^\infty(\mathbb{R}^{2n})$. In particular, we have

$$\lim_{\hbar\to 0} a \star_\hbar^W b = ab \quad \text{and} \quad \lim_{\hbar\to 0} \frac{1}{i\hbar}[a,b]_{\star_\hbar^W} = \{a,b\}.$$

Besides formulating the link between classical and quantum mechanics in a particularly satisfying manner, deformation quantization is also a quite comprehensive theory. In 1994, Fedosov proposed a geometric construction to associate a canonical star-product with any symplectic connection on a symplectic manifold [Fed94]. Furthermore, in 2003, Kontsevich proved his own formality conjecture, showing that any finite-dimensional Poisson manifold admits a formal star-product [Kon03], providing a definitive answer to the natural question of whether formal star-products exist on a given Poisson manifold.

Twists and Universal Deformation Formulas

Quite parallel to these developments, Drinfel'd introduced the idea of using symmetries to construct formal deformations of associative algebras [Dri86a]. The succinct presentation that follows is inspired by [ESW17]. For a complete overview and further details on the objects mentioned below, we refer to [ES02].

Whenever a Lie algebra $\mathfrak{g}$ acts on an associative unital algebra $\mathcal{A}$ (over a ring K) by derivations, the choice of a *formal Drinfel'd twist* $F \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))\,[[\hbar]]$ allows one to deform any such algebra $\mathcal{A}$ by means of a *universal deformation formula*:

$$\begin{aligned} \star_F &: \mathcal{A}[[\hbar]] \otimes \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]] \\ a \star_F b &\coloneqq \mu_{\mathcal{A}}\left(F(a\otimes b)\right). \end{aligned}$$

Here, $\mu_{\mathcal{A}}\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the initial algebra multiplication, and F acts on $\mathcal{A} \otimes \mathcal{A}$ since the action of $\mathfrak{g}$ extends to its enveloping algebra $\mathcal{U}(\mathfrak{g})$ and then to $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$. Finally, all operations are extended $K[[\hbar]]$-multilinearly to formal power series. In particular, we can consider the case $\mathcal{A} = C^\infty(G)$, where G is the connected and simply connected Lie group associated with $\mathfrak{g}$. In that case, since it is well known that $\mathcal{U}(\mathfrak{g})$ is isomorphic to the algebra of left-invariant differential operators on G, the twist F is a formal power series of left-invariant bidifferential operators.

To define the notion of a Drinfel'd twist, we need to describe more precisely the canonical structure of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. In addition to its usual product, the universal enveloping algebra is equipped with a coproduct, given by the unital algebra homomorphism

$$\Delta\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}), \quad X \in \mathfrak{g} \mapsto 1 \otimes X + X \otimes 1.$$

We also introduce the homomorphism $\epsilon : \mathcal{U}(\mathfrak{g}) \to K$, defined by $\epsilon(x) = 0$ for $x \in \mathfrak{g} \setminus \{0\}$ and $\epsilon(1) = 1$. The data $(\mathcal{U}(\mathfrak{g}), \cdot, \Delta, 1, \epsilon)$ endows $\mathcal{U}(\mathfrak{g})$ with a bialgebra structure. The action of $\mathcal{U}(\mathfrak{g})$ satisfies the generalized Leibniz rule: for $X \in \mathcal{U}(\mathfrak{g})$,

$$X(\mu_{\mathcal{A}}(a \otimes b)) = \mu_{\mathcal{A}}(\Delta(X)(a \otimes b)).$$

One calls $\mathcal{A}$ a $\mathcal{U}(\mathfrak{g})$-*module algebra*.

Definition. A Drinfel'd twist based on $\mathcal{U}(\mathfrak{g})$ is an invertible element $F \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ satisfying

1. $((\mathtt{Id} \otimes \Delta)(F))(1 \otimes F) = ((\Delta \otimes \mathtt{Id})(F))(F \otimes 1)$, where we use the natural multiplication on $(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ (associativity condition);
2. $(\epsilon \otimes \mathtt{Id})F = 1 = (\mathtt{Id} \otimes \epsilon)F$;
3. $F = 1 \otimes 1 + \mathcal{O}(\hbar)$.

The notion of a Drinfel'd twist can be defined similarly on a general bialgebra, and the data of a Drinfel'd twist based on any bialgebra acting on an associative algebra $\mathcal{A}$, produces an associative deformation of $\mathcal{A}$ [GZ98, Thm. 1.3]. In particular, we have the following result for the action of a Lie algebra on an associative unital algebra.

Theorem. *Suppose $\mathfrak{g}$ is a Lie algebra acting by derivation on an associative unital algebra $\mathcal{A}$ and $F \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ is a Drinfel'd twist, then the formula*

$$a \star_F b := \mu_{\mathcal{A}} \left(F(a \otimes b) \right) \tag{10}$$

defines an associative unital algebra structure on $\mathcal{A}$. We call the algebra $(\mathcal{A}, \star_F)$ a deformation of $\mathcal{A}$ by the twist F, and (10) *the universal deformation formula associated with F.*

These deformed algebra structures are linked to star-products by a theorem due to Drinfel'd [Dri86b]. A Poisson-Lie group is a Lie group G equipped with a Poisson bracket for which the group multiplication is a Poisson map, where the manifold $G \times G$ has been given the structure of a product Poisson manifold. A star-product

$$a \star_\hbar b = ab + \sum_{k=1}^{\infty} \hbar^k C_k(a, b)$$

in some formal parameter $\hbar$, on the Poisson manifold $(G, \{\cdot, \cdot\})$, is called *left-invariant* if every bi-differential operator C_k is left-invariant for the action of G (see [MV92] for more details).

Theorem (Drinfel'd)**.** *Let G be a Poisson-Lie group. There is a one-to-one correspondence between formal Drinfel'd twists on $\mathcal{U}(\mathfrak{g})$ and left-invariant formal star-products on $C^\infty(G)$.*

In the case of an abelian Lie algebra $\mathfrak{g} \simeq \mathbb{R}^n$, any bivector $\pi = \pi^{ij} \partial_i \wedge \partial_j \in \mathfrak{g} \otimes \mathfrak{g}$ yields a Drinfel'd twist through the formal exponential

$$\exp(\hbar\pi) = \sum_{n=0}^{\infty} \frac{\hbar^n \pi^n}{n!}$$

seen as an element of $(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$. The corresponding left-invariant star-product for the action of $\mathbb{R}^n$ reads

$$\begin{aligned} a \star_\pi b &= \mu_{\mathcal{A}} \big(\exp(\hbar\pi)(a \otimes b) \big) \\ &= ab + \hbar \sum_{ij} \pi^{ij} (\partial_i a)(\partial_j b) + \frac{\hbar^2}{2} \sum_{ij} \pi^{ij} \pi^{k\ell} (\partial_i \partial_k a)(\partial_j \partial \ell b) + \cdots \end{aligned} \tag{11}$$

In particular, if $\mathfrak{g} \simeq \mathbb{R}^{2n}$ and $\pi^{ij} = \frac{i}{2} \omega^{ij}$, with ω the canonical symplectic form on $\mathbb{R}^{2n}$, equation (11) yields the Weyl star-product (9).

Beyond Formality

As discussed in the preceding paragraphs, deformation quantization is an elegant theory that has produced highly general and insightful mathematical results. However, it carries a significant limitation: its formal nature. This limitation not only prevents actual physical measurements but also poses challenges from a mathematical perspective, as applications in non-commutative geometry and quantum field theory require convergence to be meaningful. This implies that Kontsevich's and Drinfel'd results are not the end of the story and that another natural question arises: When can we construct *non-formal star-products*, i.e., non-formal associative deformed products on the largest possible algebra of observables $\mathcal{A}$

$$\star_\hbar \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A},$$

which can be approximated by a formal star-product in a certain sense?

For example, one can investigate certain integral formulas obtained by compositions of integral operators depending on a positive parameter $\hbar$ and show that the resulting formal Taylor expansion in $\hbar$ of these integral expressions yields a formal star-product. This is exactly how we derived the Weyl star-product (9) earlier. The integral (8) converges to a Schwartz function whenever a and b are Schwartz functions themselves, but it will not converge in general for $a, b \in C^\infty$! We say that (8) defines a non-formal star-product on the Schwartz space but not on the space of smooth functions. This highlights the analytical difficulties encountered in constructing non-formal star-products even in very simple frameworks.

Another approach is to follow the steps of Rieffel [Rie93], who proposed a definition of *strict deformation quantizations*, which can roughly be seen as a non-formal version of Drinfel'd universal deformation formula. However, Rieffel's construction is limited to highly restrictive contexts, and creating new examples presents significant analytical challenges. These difficulties are mainly due to the notion of *non-formal Drinfel'd twist.* We denote $\mathcal{D}(G)$ the space of smooth compactly supported functions on G, and $\mathcal{D}'(G)$ the space of continuous linear functionals on $\mathcal{D}(G)$. As explained in [BdGS13], at the non-formal level, the notion of a Drinfel'd twist based on $\mathcal{U}(\mathfrak{g})$ corresponds to a universal deformation formula[2] for the action α of the Lie group G on an associative algebra $\mathcal{A}$ of a specified topological type, such as Fréchet or C^*-algebras. This roughly consists of the data of a two-point kernel $K_\hbar \in \mathcal{D}'(G\times G)$ satisfying specific properties that guarantee a meaningful interpretation of the integral expression

$$a \star_\hbar b := \int K_\hbar(x,y)\mu_{\mathcal{A}}(\alpha_x a \otimes \alpha_y b)\mathrm{d}x\mathrm{d}y, \tag{12}$$

with $a, b \in \mathcal{A}$. Once well-defined, one also requires the associativity of the product $\star_\hbar$, as well as the limit condition $\lim_{\hbar\to 0} a \star_\hbar b = \mu_{\mathcal{A}}(a \otimes b)$ in some precise topological context. These requirements, together with the convergence of (12), raise important analytical difficulties. In what follows, we delve further into the construction of such non-formal universal deformation formulas. Unsurprisingly, they will require more structure than their formal counterparts, beginning with the data of a specific topology on $\mathcal{A}$ and a particular Lie group action.

Definition. An action of a Lie group G on a locally convex topological vector space $(E, \|\cdot\|_{n\in I})$

$$\alpha\colon G\times E \to E, \quad (g,v)\mapsto \alpha_g v,$$

is called

1. *sub-isometric* if, for all $n \in I$, $\|\alpha_g v\|_n \leq C_n \|v\|_n$;
2. *p-differentiable* or C^p if for all $v \in E$, the map $\alpha(v)\colon G \to E$ is C^p;
3. *strongly continuous* if α is C^0.

The following theorem, due to Gårding [Gar47], guarantees that the smooth vectors of a strongly continuous action form a dense subspace.

Theorem. *If α is strongly continuous, then the space of smooth vectors of E,*

$$E^\infty = \{v \in E : \alpha(v) \in C^\infty(G,E)\},$$

is dense in E.

[2]Equation (12) is called a universal deformation formula for the action α of the Lie group G because it yields a deformation of any algebra $\mathcal{A}$ on which G acts.

Let $\mathcal{A}$ be a topological associative algebra, whose topology $\mathcal{T}$ is defined by some family of seminorms indexed by a set I, $\|\cdot\|_{n\in I}$, and endowed with a sub-isometric and strongly continuous action α of a Lie group G. We also require that G acts on $\mathcal{A}$ by algebra automorphisms. A direct consequence of Gårding's theorem is that $\mathcal{A}^\infty$ is a $\mathcal{T}$-dense sub-algebra of $\mathcal{A}$. On the other hand, $\mathcal{A}^\infty$ is naturally equipped with another locally convex topology. To build the seminorms of this new topology, one needs to consider the induced action of the Lie algebra $\mathfrak{g}$ of G on $\mathcal{A}^\infty$.

$$\begin{aligned}&\mathfrak{g}\times\mathcal{A}^\infty\to\mathcal{A}^\infty,\\ &(X,a)\mapsto \mathrm{d}\alpha(X)(a):=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\alpha_{\exp(-tX)}(a),\end{aligned}$$

where exp denotes the Lie exponential. Moreover, $\mathrm{d}\alpha(X)$ is a derivation of $\mathcal{A}^\infty$ and the action of $\mathfrak{g}$ extends by iteration to its universal enveloping algebra

$$\mathcal{U}(\mathfrak{g})\times\mathcal{A}^\infty\to\mathcal{A}^\infty,$$

turning $\mathcal{A}^\infty$ into a $\mathcal{U}(\mathfrak{g})$-module algebra. The seminorms of the new locally convex topology on $\mathcal{A}^\infty$ are then given by

$$\|a\|_{(n,X)\in\{I,\mathcal{U}(\mathfrak{g})\}}=\|X(a)\|_{n\in I}.$$

We denote $\mathcal{B}^0(G,\mathcal{A})$ as the space of functions

$$\mathcal{B}^0(G,\mathcal{A}):=\left\{F\in C^\infty(G,\mathcal{A}):\forall A\in\mathcal{U}(\mathfrak{g}),\|F\|_{n,X,A}<\infty\right\},$$

where the seminorms $\|F\|_{n,X,A}$ are defined by

$$\|F\|_{n,X,A}:=\sup_G\left\{\|A(F)\|_{n,X}\right\},$$

and yield a locally convex topology on $\mathcal{B}^0$. Observe that for all $v\in\mathcal{A}^\infty$, we have $\alpha(v)\in\mathcal{B}^0(G,\mathcal{A})$. The action of G on $\mathcal{A}^\infty$ yields the inclusion:

$$\mathcal{A}^\infty\times\mathcal{A}\hookrightarrow C^\infty(G^2,\mathcal{A}\otimes\mathcal{A}),\quad (a,b)\mapsto\alpha(a)\otimes\alpha(b),$$

and we have the canonical isomorphism

$$C^\infty(G^2,\mathcal{A}^\infty\otimes\mathcal{A}^\infty)\simeq C^\infty(G^2)\,\hat{\otimes}\,(\mathcal{A}^\infty\otimes\mathcal{A}^\infty).\tag{13}$$

To replace the formal notion of Drinfel'd twist presented in the preceding paragraph, we look for an element $K_\hbar\in\mathcal{D}'(G^2)$ such that

1. $K_\hbar$ can be continuously extended to $\mathcal{B}^0(G^2)$ by some density argument;

2. the deformed product
$$\tilde{\mu}_{\mathcal{A}}(a \otimes b) := \mu_{\mathcal{A}} \Big(K_\hbar \otimes \mathtt{Id}\Big(\alpha(a) \otimes \alpha(b)\Big)\Big) \tag{14}$$
defines an associative algebra structure on $\mathcal{A}^\infty$ that is continuous with respect to its locally convex topology and satisfies
$$\lim_{\hbar \to 0} \tilde{\mu}_{\mathcal{A}}(a \otimes b) = \mu_{\mathcal{A}}(a \otimes b).$$
Here, we used (13) to define $K_\hbar \otimes \mathrm{Id}$ as a mapping
$$K_\hbar \otimes \mathrm{Id} : \mathcal{D}(G^2, \mathcal{A} \otimes \mathcal{A}) \to \mathcal{A} \otimes \mathcal{A}.$$

Such a $K_\hbar$ is called a *non-formal Drinfel'd twist* and defines a *universal deformation formula.* The following theorem, presented by Rieffel in [Rie93], gives a first family of examples.

Theorem. *Let (V, ω) be a symplectic vector space. Then the Weyl cocycle*
$$K_\hbar^W : V \times V \to \mathbb{C}, \quad (v, w) \mapsto e^{\frac{i}{\hbar}\omega(v,w)}$$
defines a non-formal Drinfel'd twist.

In particular, if we go back once more to the Weyl case with $V = \mathbb{R}^{2n}$, formula (12) then reads
$$a \star_\hbar^W b(x_0) = (2\pi\hbar)^{-n} \int e^{\frac{2i}{\hbar}\omega(x,y)} a(x + x_0) b(y + x_0)\, dx\, dy,$$
and
$$K_\hbar^W(x,y) = (2\pi\hbar)^{-n} e^{\frac{2i}{\hbar}\omega(x,y)} \in C^\infty(V^2) \subset \mathcal{D}'(V^2)$$
is the non-formal Drinfel'd twist associated with the Weyl star-product. We also introduce the application
$$\Omega_W : \phi \in \mathcal{D}(V^2) \mapsto \int K_W(z) \phi(z + x)\, dz,$$
which we call the *twist* associated with the Weyl star-product.

In the context of Rieffel's result, let's delve further into how the extension of $K_\hbar^W$ from $\mathcal{D}(V^2)$ to $\mathcal{B}^0(V^2)$ is generally achieved. The main challenge arises because the inclusion $\mathcal{D}(V^2) \subset \mathcal{B}^0(V^2)$ is not dense in the $\mathcal{B}^0$-topology. To address this, we introduce the spaces

$$\mathcal{B}^N(G,\mathcal{A}) \coloneqq \left\{ F \in C^\infty(G,\mathcal{A}) : \text{for all } A \in \mathcal{U}(\mathfrak{g}), \langle x \rangle^N \|F\|_{n,X,A} < \infty \right\},$$

where $N \in \mathbb{Z}$ and $\langle x \rangle = (1+x^2)^{\frac{1}{2}}$. Equipped with the natural seminorms, these spaces are all locally convex spaces. Clearly, $\mathcal{B}^N \subset \mathcal{B}^M$ for $N > M$, and we write $\mathcal{B}^\infty = \cup_N \mathcal{B}^N$. The space $\mathcal{D}$ is now dense in $\mathcal{B}^0$ with respect to the $\mathcal{B}^N$-topology for any $N > 0$. More generally, $\mathcal{D}$ is dense in $\mathcal{B}^M$ in the $\mathcal{B}^N$-topology provided $M < N$. Consequently, it suffices to show that $K_\hbar^W$ is continuous on $\mathcal{D}(V^2)$ in the $\mathcal{B}^N$-topology for some $N > 0$, and then apply a continuous extension result (see Theorem A.39 in Appendix A). Moreover, if $K_\hbar^W$ is continuous on $\mathcal{D}(V^2)$ in the $\mathcal{B}^N$-topology for all $N > 0$, then the deformed product (14) extends to an associative product on $\mathcal{B}^\infty$. To establish the continuity of $K_\hbar^W$ in the $\mathcal{B}^N$-topology, oscillatory integral techniques are required, relying on the fact that $e^{i\omega}$ oscillates smoothly in every direction of V^2.

Very few other examples of non-formal Drinfel'd twists exist in the literature, and they all occur in fairly restricted contexts: see [BG15] for actions of Kählerian Lie groups on Fréchet algebras, and [LW16] for abelian actions on sequentially complete locally convex algebras.

For further introductory reading on deformation quantization, we recommend Sternheimer's overview, published twenty years after his involvement in the creation of the theory [Ste98], the 28 questions on strict deformation quantization proposed by Rieffel at the end of the last century [Rie98], or, for more recent developments, the surveys by Waldmann [Wal16, Wal19].

For our part, it is now time to address the precise goal of this thesis.

Deformations of tempered distributions algebras

The τ-ordering quantization family We focus on the following family of quantizations indexed by the real parameter τ and defined, for $a \in \mathcal{S}(V^2)$, $u \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ by the integral formula

$$\left(Q_\tau(a)\right)(u)(x) = \int e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) \mathrm{d}y \mathrm{d}\xi. \tag{15}$$

For $\tau = 1/2$, (15) coincides with the Weyl quantization introduced above, and for $\tau = 0$ and $\tau = 1$, it yields two other famous quantizations called the *Kohn-Nirenberg* and *anti-Kohn-Nirenberg quantizations*, which correspond respectively to the normal and anti-normal ordering extensions of Schrödinger quantization. In fact, when $\tau \in [0, 1]$, this family of quantizations arises naturally as we change the ordering in the quantization that extends Schrödinger's quantization of the position (3) and momentum (4) operators. Therefore, it is known in the literature as the *τ-ordering quantization* family.

For each quantization Q_τ, a non-formal star-product is induced on the Schwartz space, defined through operator composition via an integral formula similar to that of Weyl quantization (8):

$$\begin{aligned} &Q_\tau(a) \circ Q_\tau(b) = Q_\tau(a \star_\tau b) \\ &a \star_\tau b\,(x_0) = \int K_\tau(x, y) a(x + x_0) b\,(y + x_0) \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{16}$$

As we did for the Weyl case in the preceding paragraph, we associate to each kernel K_τ an application

$$\Omega_\tau : \mathcal{S}(V^2) \to \mathcal{S}(V^2)\ , \quad \phi \mapsto \int K_\tau(z)\phi(z + x)\, dz$$

that we call the *twist* corresponding to the star-product $\star_\tau$. In Section 1.2, we will see that the τ-ordering quantization family also arises naturally in the context of Shubin calculus.

Since the Schwartz space does not contain most of the observables of physical interest, it is natural to ask whether the star-products

$$\star_\tau : \mathcal{S}(V) \times \mathcal{S}(V) \to \mathcal{S}(V)$$

can be extended *in a non-formal way* to some larger algebras. The usual approach to this problem is to construct extensions of the star-products to some subset of the space $\mathcal{S}'(V)$ of continuous linear functionals on $\mathcal{S}(V)$, known as *tempered distributions*. Rieffel's theorem, presented above, is in line with this philosophy since it yields an extension of the Weyl star-product to $\mathcal{B}^\infty(V)$ which is a subspace of $\mathcal{S}'(V)$. Unfortunately, Rieffel's result cannot be used in the Kohn-Nirenberg case. Indeed, the Kohn-Nirenberg cocycle, defined on V^2, for $z_i = (q_i, p_i)$, by

$$K_0(z_1, z_2) = e^{-q_2 p_1} \delta(q_1)\delta(p_2),$$

does not oscillate smoothly in all of V^2 as the Weyl cocycle does. Consequently, the oscillatory integration techniques used in the proof of Rieffel's theorem do not apply in this context.

The smooth wave front set This thesis is motivated by the intrusion in these matters of an object originating from the field of microlocal analysis. It is well-known that microlocal analysis provides a criterion to define the pointwise product of distributions. Indeed, the obstruction to multiply any two distributions is coded by a microlocal object introduced by Hörmander: the *wave front set*. Briefly, providing a careful condition on the wave front set of two distributions, one is able to define their pointwise product as the continuous extension of the usual pointwise product of functions [Hö90]. In 2019, it was observed by Bahns and Schulz [BS19] that a variant of the wave front set, called the *smooth wave front set*, also introduced by Hörmander in [Hö89], could be used to build algebras of tempered distributions under the pointwise product but also under the *twisted convolution product*, i.e., the product defined, for any skew-symmetric matrix θ, by

$$u \star_\theta v(x) := \mathcal{F}^{-1}_{\xi \to x} \int_{\mathbb{R}^{2m}} \mathcal{F}(\xi - \eta)\mathcal{F}(g)(\eta) e^{-\frac{i}{2} x^T \theta y} \mathrm{d}y, \tag{17}$$

where $\mathcal{F}$ denotes the Fourier transform. This is a famous family of star-products containing the Weyl star-product since for

$$\theta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

formula (17) is exactly the Weyl star-product (8). However, no other element of the τ-ordering family can be obtained as a twisted convolution product. It is well known that formula (17) defines an associative algebra structure on the Schwartz space:

$$\star_\theta : \mathcal{S} \times \mathcal{S} \to \mathcal{S}.$$

Without delving into technical details for the moment, the smooth wave front set of a tempered distribution is a cone in $T^*\mathbb{R}^n$, i.e., a subset stable under dilatation, which simultaneously encodes the singular behavior at finite points of a tempered distribution and its asymptotic growth at infinity. For a given cone $\Gamma \subset T^*\mathbb{R}^n$, one can consider the subspace $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ of tempered distributions whose smooth wave front set is contained in Γ. Bahns and Schulz identified a sufficient conditions on the cone Γ under which the twisted convolution product extends from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ in a non-formal way.

The work of Bahns and Schulz is *not* a direct consequence of Rieffel's theorem. Indeed, the action of V by translations on $\mathcal{S}'_\Gamma(V)$ is not strongly continuous with respect to the topological structure used on $\mathcal{S}'_\Gamma(V)$ and some of the algebras $\mathcal{S}'_\Gamma(V)$ obtained by Bahns and Schulz strictly contain the space $\mathcal{B}^\infty$.

Extensions of the twisted convolution product were already proposed in the literature using other techniques, such as those in [Mai86], [GBV88], [Sol12, Sol15]. The smooth wave front set approach has the advantage of providing a simple sufficient criterion for the extension of the twisted convolution product, while highlighting, in an intuitive way, the analytic obstructions arising in the process. Moreover, the smooth wave front set approach can be adapted to the τ-ordering quantization family and, in particular, to the Kohn-Nirenberg framework.

Structure of the thesis

Chapter I To set the groundwork, we define the Schwartz space and the space of tempered distributions, presenting some of their key properties. We then introduce Shubin's calculus for pseudo-differential operators on $\mathbb{R}^n$. Finally, we show that the family of star-products $\star_\tau$ arises naturally through the composition of such operators (see Theorem 1.2.11 and Proposition 1.2.13).

Chapter II This chapter is dedicated to the notion of the *smooth wave front set*, a central focus of our research.

In Section 2.1, we present three definitions, including a new formulation based on unitary representations of the Heisenberg group (see Definition 2.1.29). Additionally, we introduce new examples that we find enlightening and provide explicit proofs for Proposition 2.1.8 and Corollary 2.1.9, which were previously unproven in the literature.

In Section 2.2, we introduce a criterion developed by Bahns and Schulz [BS19, Thm. 3.1], based on results by Hörmander [Hö89], to extend the pointwise product to an algebra $\mathcal{S}'_\Gamma$ of tempered distributions. We recall that this algebra consists of tempered distributions whose wave front sets are contained within the cone Γ. However, it appears that several topological arguments were omitted in their work, and we aim to clarify the topological properties of the algebra $\mathcal{S}'_\Gamma$.

First, we introduce the Γ-topology, a locally convex topology on $\mathcal{S}'_\Gamma$ (see Lemma 2.2.9 and Definition 2.2.10) based on a notion of convergence originally introduced by Hörmander in [Hö89] (Definition 2.2.1). We then prove in Proposition 2.2.11 and Proposition 2.2.12 that this topology is Hausdorff, sequentially complete, and admits the Schwartz space $\mathcal{S}$ as a dense subset. In Theorem 2.2.14, we further demonstrate that, under Hörmander's existing condition (see [Hö89, Prop. 2.9]), the pullback by an injective linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ defines a continuous map

$$A^* : \mathcal{S}'_\Gamma(\mathbb{R}^m) \to \mathcal{S}'_{A^*\Gamma}(\mathbb{R}^n)$$

in the Γ-topology. Finally, we present Theorem 2.2.16, which recovers Bahns and Schulz's criterion for constructing algebras of tempered distributions under the pointwise product, while clarifying the topological properties of these algebras.

Chapter III In the third chapter of this work, we extend the result of Bahns and Schulz to the family of star-products $\star_\tau$, with particular emphasis on the Kohn-Nirenberg star-product. We tackle this problem by focusing on the twists Ω_τ associated with each star-product.

In Section 3.1, we observe that the stability of the cone Γ under the twist is the key condition for the non-formal extension of $\star_\tau$ to a space $\mathcal{S}'_\Gamma$ of tempered distributions. Then, in Theorem 3.1.6, we derive a criterion on the cone Γ ensuring that Ω_τ defines a continuous map in the Γ-topology, which stabilizes Γ:

$$\Omega_\tau \colon \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_\Gamma(\mathbb{R}^n).$$

In Section 3.2, we build on this result to state Theorem 3.2.2, which provides, for every value of τ, including the Kohn-Nirenberg case, a criterion to extend $\star_\tau$ to a non-formal star-product on $\mathcal{S}'_\Gamma(\mathbb{R}^n)$. In particular, in

Example 3.2.3, we establish that $\mathcal{S}'_{V\times\{0\}}(V)$ forms an associative algebra under each product $\star_\tau$.

In Section 3.3, we prove that the spaces $\mathcal{B}^N(V)$, central in Rieffel's construction of a non-formal universal deformation formula for the Weyl cocycle, are contained within the algebra $\mathcal{S}'_{V\times\{0\}}(V)$ and that the twists Ω_τ stabilize the spaces $\mathcal{B}^N(V)$ (see Proposition 3.3.1 and Proposition 3.3.2). Finally, in Corollary 3.3.3, we prove that the union $\mathcal{B}^\infty(V)$ of all $\mathcal{B}^N(V)$ spaces is a graded subalgebra of $\mathcal{S}'_{V\times\{0\}}(V)$ for the product $\star_\tau$:

$$\left(\mathcal{B}^\infty(V) = \bigcup_N \mathcal{B}^N(V), \star_\tau\right) \subseteq \left(\mathcal{S}'_{V\times\{0\}}(V), \star_\tau\right).$$

These results represent a significant step toward generalizing Rieffel's construction to the Kohn-Nirenberg kernel, and we hope that this work will serve as a foundation for future advancements in this direction.

The appendix contains a brief introduction to topological vector spaces and an outline of Kirillov's orbit method. A list of recurring symbols and notations used in this work is provided at the end.

Chapter I

Tempered distributions and pseudo-differential operators in $\mathbb{R}^n$

"Since Dirac introduced the famous $\delta(x)$ function, which would be zero everywhere except for $x = 0$ and would be infinite for $x = 0$ such that $\int \delta(x)dx = 1$, the formulas of symbolic calculus have become even more unacceptable to mathematicians' rigour. Stating that the Heaviside function $Y(x)$, which is 0 for $x < 0$ and 1 for $x \geq 0$, has as its derivative the Dirac delta function $\delta(x)$, whose very definition is mathematically contradictory, and then discussing the higher-order derivatives $\delta'(x)$, $\delta''(x)$... of this function that lacks actual existence, goes beyond the limits permitted to us. How can we explain the success of these methods? When such a contradictory situation arises, it is rare that it does not result in a new mathematical theory that justifies, in a modified form, the language of physicists; this is even an important source of progress in mathematics and physics."

- Laurent Schwartz, *Théorie des distributions*[1]

[1] Translated by the author from the original French text.

This chapter is a condensed presentation of the main definitions and results about tempered distributions and pseudo-differential operators that will serve our later purposes. For a more thorough study the reader might refer to [Hö90], [Mel03], [RS78], or [Shu01].

1.1 Tempered distributions

1.1.1 Definition and examples

The Schwartz space, $\mathcal{S}(\mathbb{R}^n)$, is the space of smooth functions $\phi\colon \mathbb{R}^n \to \mathbb{C}$ such that for all $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^\alpha \phi(x) \right| < \infty,$$

where $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, with $\partial_i := \partial/\partial x_i$. An element ϕ of $\mathcal{S}(\mathbb{R}^n)$ is called a Schwartz function. It is clear that ϕ, along with all its derivatives, decays to zero faster than any polynomial at infinity. We say that ϕ and its derivatives are rapidly decreasing.

Clearly, every compactly supported smooth function is a Schwartz function:

$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n).$$

However, the reverse inclusion does not hold, as the Schwartz space contains functions such as $e^{-|x|^2}$, which are obviously not compactly supported. It is naturally endowed with a topological vector space (TVS) structure given by the locally convex topology defined by the family of seminorms

$$\|\phi\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^\alpha \phi(x) \right|. \tag{I.1}$$

This topology is called the *Schwartz topology.* The theoretical background about TVS and related topological notions is presented in appendix A.

Proposition 1.1.1. *The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ endowed with the Schwartz topology is a* Fréchet space*, i.e., a separated, complete, metrizable TVS.*

Proof. The fact that the Schwartz topology is Hausdorff is a direct consequence of Proposition A.23, since 0 is the only point in $\mathcal{S}(\mathbb{R}^n)$ for which every seminorm vanishes. Furthermore, this topology is generated by a countable family of seminorms and is thus metrizable (see [Die90a, Prop. 12.14.5]). For example, one can show that (I.1) is equivalent to the family of seminorms

$$\|\phi\|_k := \max_{|\alpha|+|\beta|\leq k} \sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^\alpha \phi(x) \right|. \tag{I.2}$$

We can then equip $\mathcal{S}(\mathbb{R}^n)$ with the metric

$$\mathrm{d}(\phi_1,\, \phi_2) = \sum_{k\geq 0} \frac{\|\phi_1 - \phi_2\|_k}{1 + \|\phi_1 - \phi_2\|_k}.$$

It is easy to see that the topology induced by this metric is the locally convex topology defined by (I.2). For the completeness of $\mathcal{S}(\mathbb{R}^n)$, see the proof of [RS78, Thm. V.9]. □

Proposition 1.1.2. *The space $C_c^\infty(\mathbb{R}^n)$ of smooth compactly supported functions is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. See [Hö90, Lem. 7.1.8]. □

The dual $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$, i.e., the space of all linear continuous functionals on $\mathcal{S}(\mathbb{R}^n)$, is called the space of *tempered distributions*. For a linear map $u\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, the continuity condition reads

$$\exists\, \alpha, \beta \text{ such that } |u(\phi)| \leq C_{\alpha,\beta} \|\phi\|_{\alpha,\beta} \quad \text{for all} \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

For now, we equip $\mathcal{S}'(\mathbb{R}^n)$ with the weak topology, i.e., the weakest topology making every map

$$p_\phi\colon \mathcal{S}'(\mathbb{R}^n) \to \mathbb{C}, \quad u \mapsto u(\phi)$$

continuous. The weak topology is equivalently defined by the seminorms

$$|u|_\phi = |u(\phi)|, \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

Remark 1.1.3. The class $\mathcal{S}'(\mathbb{R}^n)$ is not the most common class of distributions. It is more frequent to encounter the space $\mathcal{D}'(\mathbb{R}^n)$ of continuous linear functionals on the space $\mathcal{D}(\mathbb{R}^n) := C_c^\infty(\mathbb{R}^n)$ of compactly supported smooth functions. The main difference between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ is that tempered distributions behave slightly better at infinity.

Explicitly, their growth is at most polynomial (see Theorem 1.1.11 below), while the growth of a distribution in $\mathcal{D}'(\mathbb{R}^n)$ might very well be exponential. This is because we extended the space of test functions; the fact that a Schwartz function might not have compact support prevents tempered distributions from exploding at infinity. It is a general idea that larger spaces have smaller duals and vice versa. Here we have the inclusions

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n).$$

Example 1.1.4. There is a continuous injection of $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$:

$$\iota\colon \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \quad \phi \mapsto \left[\psi \mapsto \int \phi(x)\psi(x)\,dx\right].$$

Similarly, any polynomially bounded continuous function can be used to define a tempered distribution. For every continuous function u such that $u(x) \leq C\langle x\rangle^N$ for some $N \in \mathbb{R}$, the mapping

$$F_u\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}, \quad F_u(\phi) \coloneqq \int u(x)\phi(x)\,dx$$

defines a tempered distribution. We will see in Theorem 1.1.11 that all distributions are of this form, modulo the appearance of weak derivatives (see Example 1.1.10). It is a classical abuse of notation to use u to refer both to the distribution F_u and the actual function u. Sometimes, we write $F_u(\phi) = (u, \phi)$. One must be careful not to confuse the pairing $(\,\cdot\,,\,\cdot\,)$ of a distribution with a test function with the inner product $\langle\,\cdot\,,\cdot\,\rangle$ of L^2, which is the integral of the product of the complex conjugate of the first argument with the second:

$$(\psi, \phi) \neq \langle\psi, \phi\rangle \coloneqq \int \overline{\psi}(x)\phi(x)\,dx.$$

Proposition 1.1.5. *The continuous injection $\iota\colon \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is dense.*

Example 1.1.6. The delta distribution δ, defined by

$$\delta(\phi) = \phi(0),$$

is a tempered distribution. We will sometimes write, slightly abusively,

$$\delta(\phi) = \int \delta(x)\phi(x)\,dx.$$

The delta distribution can be translated, which yields a distribution δ_{x_0}

$$\delta_{x_0}(\phi) = \int \delta(x)\phi(x + x_0)\,dx = \phi(x_0).$$

Example 1.1.7. The exponential function, even though it is a smooth function, does not define a tempered distribution through integral pairing since the integral

$$\int e^x \phi(x)\, dx$$

does not converge for all $\phi \in \mathcal{S}(\mathbb{R})$. Indeed, for $\phi(x) = e^{-\sqrt{1+x^2}} \in \mathcal{S}(\mathbb{R})$, this integral diverges.

1.1.2 Continuous extension and transpose map

Given a continuous map $T\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, Proposition 1.1.5 yields the continuous map $\iota \circ T\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$, there is at most one continuous extension

$$\overline{T}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

of $\iota \circ T$ (see Corollary A.38). With slight abuses of language and notation, we call $\overline{T}$ *the continuous extension of* T, and we write $\overline{T}|_{\mathcal{S}(\mathbb{R}^n)} = T$ instead of

$$\left(\overline{T}(\iota \circ \phi)\right)(\psi) = (\iota \circ T(\phi))\,(\psi), \quad \forall\, \phi,\, \psi \in \mathcal{S}(\mathbb{R}^n),$$

and, when no confusion arises, the continuous extension $\overline{T}$ of T will also be denoted T.

Definition 1.1.8. Given a continuous map $S\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, the *transpose* of S is the continuous map defined by

$$\begin{aligned} &S'\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \quad u \mapsto S'u, \\ &(S'u)(\phi) := u(S(\phi)), \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

Remark 1.1.9. The transpose is trivially a continuous map for the weak topology on $\mathcal{S}'(\mathbb{R}^n)$. Therefore, it provides an easy way to construct continuous maps from $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, requiring only a continuous map from $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. The transpose, however, is generally not the continuous extension of the original map, though it can be used to construct it. Explicitly, if we want to find the continuous extension of a continuous map $T\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, it is sufficient to find a map $S\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ such that $S'|_{\mathcal{S}(\mathbb{R}^n)} = T$. Then $\overline{T} = S'$ is the continuous extension of T.

Example 1.1.10 (Operations on tempered distributions)**.** Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Putting into practice what we discussed in Remark 1.1.9, we describe below four basic operations on tempered distributions as continuous extensions of their well-known versions in $\mathcal{S}(\mathbb{R}^n)$: multiplication by a polynomially bounded smooth function, weak derivative, linear coordinate changes, and phase-space shifts.

1. Let $\mathcal{O}(\mathbb{R}^n)$ be the set of smooth functions on $\mathbb{R}^n$ which, together with all their derivatives, are polynomially bounded. From Example 1.1.4, the inclusion $\mathcal{O}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is clear. Moreover, if $f \in \mathcal{O}(\mathbb{R}^n)$, then pointwise multiplication by f defines a continuous mapping
$$\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad \phi \mapsto f\phi.$$
Hence, one can define the continuous mapping
$$\begin{aligned}&\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \quad u \mapsto fu,\\ &(fu)(\phi) = u(f\phi), \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).\end{aligned}$$

2. For $\alpha \in \mathbb{N}^n$, we write $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Note that ∂^α defines a continuous mapping
$$\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad \phi \mapsto \partial^\alpha\phi.$$
Again, this induces a continuous mapping
$$\begin{aligned}&\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \quad u \mapsto \partial^\alpha u,\\ &\partial^\alpha u(\phi) = (-1)^{|\alpha|}u(\partial^\alpha\phi), \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).\end{aligned}$$
The operator ∂^α on $\mathcal{S}'(\mathbb{R}^n)$ is called the *weak derivative.*

3. Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. The pullback $A^*\phi \coloneqq \phi \circ A$ defines a continuous mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. Hence, the pullback of a tempered distribution by a linear isomorphism A is defined as
$$A^*u(\phi) = |\det(A)|u((A^{-1})^*\phi).$$

4. Let $T_x\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $M_\xi\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be the translation and modulation operators, respectively:
$$T_x\phi(y) \coloneqq \phi(y-x) \quad \text{and} \quad M_\xi\phi(y) \coloneqq e^{iy\xi}\phi(y).$$

They both continuously extend to $\mathcal{S}'(\mathbb{R}^n)$ as follows: for $u \in \mathcal{S}'(\mathbb{R}^n)$ and for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$T_x u(\phi) \coloneqq u(T_{-x}\phi) \quad \text{and} \quad M_\xi u(\phi) \coloneqq u(M_\xi \phi).$$

Hence, the phase-space shift

$$\Pi(x,\xi)\phi(y) \coloneqq M_\xi T_x \phi(y)$$

also continuously extends to $\mathcal{S}'(\mathbb{R}^n)$: for $u \in \mathcal{S}'(\mathbb{R}^n)$ and for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$(\Pi(x,\xi)u)(\phi) \coloneqq u\left(\Pi(-x,\xi)\phi\right).$$

As suggested in Example 1.1.4, every tempered distribution is of the form $\partial^\beta u$ where u is some polynomially bounded continuous function and ∂^β denotes the weak derivative of order $\beta \in \mathbb{N}^n$. This remarkable fact is known as *the regularity theorem* for tempered distributions.

Theorem 1.1.11 (Regularity theorem for tempered distributions)**.** *For any $T \in \mathcal{S}'(\mathbb{R}^n)$, there exists a polynomially bounded continuous function u and $\beta \in \mathbb{N}^n$ such that $T = \partial^\beta u$, that is,*

$$T(\phi) = \int_{\mathbb{R}^n} (-1)^{|\beta|} u(x)(\partial^\beta \phi)(x)\, \mathrm{d}x.$$

This theorem justifies the frequent abuse of notation:

$$u(\phi) = (u,\phi) = \int u(x)\phi(x)\, \mathrm{d}x.$$

The proofs of Theorem 1.1.11 and Proposition 1.1.5 rely on the realization of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ as sequence spaces. This elegant construction can be found in [RS78, appendix to V.3]. It also implies the stronger version of Proposition 1.1.5 stated below.

Proposition 1.1.12. *If we endow $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ with the topologies defined in the previous section, and $L^2(\mathbb{R}^n)$ with the topology induced by its usual Hilbert space structure, then the following inclusions are dense:*

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \quad \textit{and} \quad \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

1.1.3 Fourier transform

The Fourier transform of a function f in $L^1(\mathbb{R}^n)$ is a linear map usually defined as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,\mathrm{d}x.$$

Example 1.1.13. The Fourier transform of the Gaussian function is itself:

$$\text{if } \ f(x) = e^{-\frac{|x|^2}{2}} \quad \text{then} \quad \hat{f}(\xi) = e^{-\frac{|\xi|^2}{2}}.$$

To prove this, it suffices to consider the one-dimensional case since, in $\mathbb{R}^n$, $\hat{f}$ is the product of n identical integrals. Observe that both f and $\hat{f}$ are solutions to the Cauchy problem

$$u' = -xu, \quad u(0) = 1.$$

To see this, one can apply the Fourier transform to both sides of the equation and then use (I.3). By uniqueness of the solution, we conclude that $f = \hat{f}$.

An important property of the classes $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ is that they are compatible with the Fourier transform in the sense that it defines a continuous isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, which continuously extends to an isomorphism from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. This relies on the following straightforward yet essential observation: for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} \widehat{D_j\phi}(\xi) &= \xi_j \hat{\phi}(\xi), \\ \widehat{x_j\phi}(\xi) &= -D_j \hat{\phi}(\xi). \end{aligned} \tag{I.3}$$

Notation 1.1.14. We have used the notation $D_j := -i\partial/\partial x_j \neq \partial_j := \partial/\partial x_j$ above in order to avoid the appearance of a factor of i in (I.3). The multi-index notation is used in the same way:

$$D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad \text{for all } \alpha \in \mathbb{N}^n.$$

This notation is standard in harmonic analysis and pseudodifferential operator theory. It avoids the need to track powers of i while performing integration by parts.

Equation (I.3) directly implies that $\mathcal{F}$ is a continuous map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ (for details, see [Hö90, Lem. 7.1.3]). Moreover, $\mathcal{F}$ has a continuous inverse and is therefore a continuous isomorphism of $\mathcal{S}(\mathbb{R}^n)$.

Theorem 1.1.15 (Fourier inversion theorem)**.** *The Fourier transform is a continuous isomorphism of $\mathcal{S}(\mathbb{R}^n)$. The inverse Fourier transform is given by*

$$\check{\psi}(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} \psi(\xi)\,\mathrm{d}\xi.$$

A nice proof of this theorem, based on Example 1.1.13, is presented in [Duo01, Thm. 1.13]. Observe also that

$$\int \hat{\phi}(x)\psi(x)\,\mathrm{d}x = \int \phi(x)\hat{\psi}(x)\,\mathrm{d}x, \quad \forall\, \phi,\, \psi \in \mathcal{S}(\mathbb{R}^n). \tag{I.4}$$

Hence, one can use Remark 1.1.9 to define the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ as its unique continuous extension to $\mathcal{S}'(\mathbb{R}^n)$:

$$\hat{u}(\phi) = u(\hat{\phi}). \tag{I.5}$$

The following result is a consequence of (I.4), Theorem 1.1.15, and Proposition 1.1.12.

Corollary 1.1.16. 1. *The Fourier transform extends continuously to an isomorphism of $L^2(\mathbb{R}^n)$ satisfying*

$$\|\hat{f}\|_2 = \|f\|_2 \coloneqq \left(\int_{\mathbb{R}^n} |f(x)|^2 \right)^{1/2}.$$

2. *The Fourier transform extends continuously to an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$.*

The following proposition shows that (I.3) remains valid in $\mathcal{S}'(\mathbb{R}^n)$. This is unsurprising since $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$ and every operation in (I.3) is continuous on $\mathcal{S}'(\mathbb{R}^n)$. However, we prove it by explicit computations to put into practice what we have discussed so far.

Proposition 1.1.17. *For all $u \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform satisfies*

$$\begin{aligned} \widehat{D_j u} &= \xi_j \hat{u}, \\ \widehat{x_j u} &= -D_j \hat{u}. \end{aligned}$$

Proof. One computes:

$$\begin{aligned}
\widehat{D_j u}(\phi) &= [D_j u](\hat{\phi}) && \text{by (I.5)}\\
&= -u(D_j \hat{\phi}) && \text{by Example 1.1.10, point 2}\\
&= u(\widehat{x_j \phi}) && \text{by (I.3)}\\
&= \hat{u}(\xi_j \phi) && \text{by (I.5)}\\
&= [\xi_j \hat{u}](\phi) && \text{by Example 1.1.10, point 1.}
\end{aligned}$$

Note that we change the variable name from x_j to ξ_j to stick to the notation in the proposition. The second equality is left to the reader. □

Example 1.1.18. Let 1 denote the tempered distribution defined by

$$1(\phi) := \int \phi(x)\,\mathrm{d}x.$$

Then the Fourier transform of the delta distribution is

$$\hat{\delta}(\phi) = \delta(\hat{\phi}) = \hat{\phi}(0) = \frac{1}{(2\pi)^{n/2}} \int \phi(x)\,\mathrm{d}x = \frac{1}{(2\pi)^{n/2}} 1(\phi).$$

Hence, Theorem 1.1.15 implies

$$\frac{1}{(2\pi)^{n/2}} \check{1}(\phi) = (2\pi)^{-n} \int e^{ix\cdot\xi} \phi(\xi)\,\mathrm{d}x\,\mathrm{d}\xi = \int \delta(\xi)\phi(\xi)\,\mathrm{d}x = \delta(\phi).$$

In practice, when a variable x in a pure phase is *free* under an integral (i.e., it appears nowhere else in the integrand function), we can replace the phase by $c\delta(x)$ for some constant $c \in \mathbb{R}$. This will be extremely useful in future computations. Moreover, by Proposition 1.1.17,

$$\widehat{D^\alpha \delta}(\xi) = \frac{1}{(2\pi)^{n/2}} \xi^{|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n.$$

We conclude this section by giving an integral expression for partial differential operators using the Fourier transform. This will be convenient for understanding the definition of pseudo-differential operators in Section 1.2. First, we need the notion of the support of a tempered distribution.

Definition 1.1.19. Let $u \in \mathcal{S}'(\mathbb{R}^n)$. A point x is not in the support of u, denoted $\operatorname{supp}(u)$, if there exists a neighborhood U of x such that for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}(\phi) \subseteq U$, we have $u(\phi) = 0$.

Lemma 1.1.20. *Let* $u \in \mathcal{S}'(\mathbb{R}^n)$. *Then* $\mathrm{supp}(u) \subseteq \{0\}$ *if and only if*

$$u = \sum_{\textit{finite}} c_\alpha D^\alpha \delta.$$

Proof. Let u be a finite linear combination of delta distributions as in the statement. For all $x \neq 0$, there exists an open ball $B_{x,\epsilon}$ that does not contain 0. For all $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathrm{supp}(\phi) \subset B_{x,\epsilon}$, we have

$$u(\phi) = \sum_{\text{finite}} (-1)^\alpha c_\alpha \delta(D^\alpha \phi) = \sum_{\text{finite}} (-1)^\alpha c_\alpha D^\alpha \phi(0) = 0.$$

Hence, $\mathrm{supp}(u) \subseteq \{0\}$.

The reciprocal follows from [RS78], problem 29, p. 177. □

Corollary 1.1.21. *The Fourier transform yields an isomorphism*

$$\{u \in \mathcal{S}'(\mathbb{R}^n) : \mathrm{supp}(u) \subseteq \{0\}\} \longleftrightarrow \mathbb{C}[\xi] = \{\textit{polynomials in } \xi\}.$$

One way to look at this isomorphism is to consider partial differential operators with constant coefficients

$$P(D)\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad P(D) = \sum c_\alpha D^\alpha.$$

Proposition 1.1.17 then implies, for all $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$\widehat{P(D)u}(\xi) = P(\xi)\hat{u}(\xi), \quad \text{where} \quad P(\xi) = \sum c_\alpha \xi^{|\alpha|}. \tag{I.6}$$

The polynomial P is called the (full) characteristic polynomial of $P(D)$. It determines $P(D)$ uniquely. Applying the inverse Fourier transform to (I.6) yields an integral expression for partial differential operators with constant coefficients:

$$P(D)u(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} P(\xi) u(y) \,\mathrm{d}y \,\mathrm{d}\xi. \tag{I.7}$$

1.1.4 Tensor product of tempered distributions

We are accustomed to encountering the tensor product of functions:

$$f \otimes g(x,y) = f(x)g(y).$$

It is less obvious what the tensor product of two tempered distributions should be. We present below a theorem of Hörmander that both provides a satisfying description of the tensor product of tempered distributions and yields a Fubini-type result.

Lemma 1.1.22. *If ϕ is a function in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and u is a distribution in $\mathcal{S}'(\mathbb{R}^n)$, then*

$$x \mapsto u(\phi(\,\cdot\,, x))$$

is a Schwartz function in $\mathcal{S}(\mathbb{R}^m)$.

Proof. See [Hö90, Thm. 2.1.3]. □

Theorem 1.1.23. *If $u_1 \in \mathcal{S}'(\mathbb{R}^n)$ and $u_2 \in \mathcal{S}'(\mathbb{R}^m)$, then there exists a unique $u \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ such that*

$$u(\phi_1 \otimes \phi_2) = u_1(\phi_1)u_2(\phi_2), \quad \phi_1 \in \mathcal{S}(\mathbb{R}^n),\ \phi_2 \in \mathcal{S}(\mathbb{R}^m).$$

The distribution u is denoted $u_1 \otimes u_2$ or $u_2 \otimes u_1$ and is called the tensor product of u_1 and u_2. Moreover, for all $\phi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, we have

$$u(\phi) = u_1(u_2(\phi(x, \cdot))) = u_1(u_2(\phi(\,\cdot\,, y))).$$

Proof. A proof for distributions in $\mathcal{D}'$ can be found in [Hö90, Thm. 5.1.1]. To adapt it to the $\mathcal{S}'$ framework, observe that $\mathcal{S}'$ is a linear subspace of $\mathcal{D}'$ and use the density of C_c^∞ in $\mathcal{S}$. □

The following result is a direct consequence of the above theorem.

Corollary 1.1.24. *Let u and v be tempered distributions. We have*

$$\operatorname{supp}(u \otimes v) = \operatorname{supp}(u) \times \operatorname{supp}(v).$$

Definition 1.1.25. Let X, Y, and Z be topological spaces. A mapping

$$f\colon X \times Y \to Z$$

is *jointly continuous at* (x_0, y_0) if for each neighbourhood W of $f(x_0, y_0)$, there exist a neighbourhood U of x_0 and a neighbourhood V of y_0 such that $f(U \times V) \subseteq W$. We say that f is jointly continuous if it is jointly continuous at every point of $X \times Y$.

Proposition 1.1.26. *The tensor product of tempered distributions is a jointly continuous bilinear map for the weak topology:*

$$\otimes\colon \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m).$$

Proof. See [Tre67, Chap. 41, Cor. 2]. □

1.1.5 The Schwartz kernel theorem

In the early 1950s, Laurent Schwartz discovered that each separately continuous bilinear functional on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m)$ uniquely corresponds to a jointly continuous functional on $\mathcal{S}(\mathbb{R}^{n+m})$ [Sch54b]. This result became known as the *Schwartz Kernel Theorem*. A few years later, Alexandre Grothendieck introduced the concept of *nuclear spaces*, which encompasses both $\mathcal{S}$ and $\mathcal{S}'$, and proved a generalization of Schwartz's theorem within that framework [Gro54]. While we will make use of the nuclearity of $\mathcal{S}$ and $\mathcal{S}'$ later on, this work does not aim to study nuclear spaces in general. Thus, the results in this section are specifically stated for $\mathcal{S}$ and $\mathcal{S}'$. For a broader study of nuclear spaces and a more general formulation of these theorems, we refer the reader to [Tre67] and [Sch82].

We begin with a version of the Schwartz Kernel Theorem that is close to Schwartz's original formulation.

Theorem 1.1.27 (Schwartz kernel theorem, first version)**.** *Let $B(\phi, \psi)$ be a separately continuous bilinear functional on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m)$. Then there exists a unique tempered distribution $T \in \mathcal{S}'(\mathbb{R}^{n+m})$ such that*

$$B(\phi, \psi) = T(\phi \otimes \psi),$$

where

$$\phi \otimes \psi(x_1, \ldots, x_{n+m}) = \phi(x_1, \ldots, x_n)\psi(x_{n+1}, \ldots, x_{n+m}).$$

Proof. See [RS78, Thm. V.12]. □

In particular, for all continuous linear operators

$$A\colon \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^n),$$

the theorem claims that there exists a unique $K_A \in \mathcal{S}'(\mathbb{R}^{n+m})$ such that

$$A(\phi)(\psi) = K_A(\phi \otimes \psi) = \int K_A(x,y)\phi(x)\psi(y)\,\mathrm{d}x\,\mathrm{d}y,$$

where we have made the usual abuse of notation inspired by Theorem 1.1.11. We call K_A the *kernel* of A.

Example 1.1.28. 1. The Fourier transform can be seen as a continuous linear operator

$$\begin{aligned}
&\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\\
&[\mathcal{F}(\phi)](\psi) = \phi(\mathcal{F}(\psi)) = \frac{1}{(2\pi)^{n/2}} \int e^{-ixy}\phi(y)\psi(x)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Its kernel is simply

$$K_{\mathcal{F}}(x,y) = \frac{1}{(2\pi)^{n/2}} e^{-ixy}.$$

2. Using the formula (I.7), differential operators can be written as

$$\begin{aligned}
&P(D)\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\\
&[P(D)(\phi)](\psi) = (2\pi)^{-n} \int e^{i(x-y)\xi} P(\xi)\phi(y)\psi(x)\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}x.
\end{aligned}$$

Hence the kernel of a pseudo-differential operator $P(D)$ is

$$K_P(x,y) = (2\pi)^{-n} \int e^{i(x-y)\xi} P(\xi)\,\mathrm{d}\xi.$$

Definition 1.1.29. Let E and F be locally convex topological vector spaces. We call the π-topology (or projective topology) on $E \otimes F$ the strongest locally convex topology on this vector space for which the canonical bilinear mapping

$$E \times F \to E \otimes F, \quad (u,v) \mapsto u \otimes v$$

is continuous. We shall denote by $E \widehat{\otimes}_\pi F$ the completion of $E \otimes F$ with respect to the π-topology.

The following isomorphisms are direct consequences of the nuclearity of $\mathcal{S}$ and $\mathcal{S}'$. They can be regarded as variants of the Schwartz kernel theorem.

Theorem 1.1.30 (Schwartz kernel theorem, second version)**.** *We have the following topological isomorphisms:*

$$\mathcal{S}(\mathbb{R}^n)\widehat{\otimes}_\pi\mathcal{S}(\mathbb{R}^m) \simeq \mathcal{S}(\mathbb{R}^{n+m}), \tag{I.8}$$

$$\mathcal{S}'(\mathbb{R}^n)\widehat{\otimes}_\pi\mathcal{S}'(\mathbb{R}^m) \simeq \mathcal{S}'(\mathbb{R}^{n+m}). \tag{I.9}$$

Proof. See [Tre67, Thm. 51.6]. □

1.2 Pseudo-differential operators

Pseudo-differential operators should be understood as a generalisation of differential operators described by formula (I.7). One aims to replace the class of polynomials $P(\xi)$ by a larger class of functions that depend on both x and y:

$$Au(x) = (2\pi)^{-n}\int e^{i(x-y)\xi}a(x,y,\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi. \tag{I.10}$$

The function a is called the *amplitude* of the pseudo-differential operator, and its properties are carefully chosen to assign a precise meaning to formula (I.10), which is a priori meaningless. This is interpreted as an *oscillatory integral.* The family of linear operators defined in this way are called *pseudo-differential operators.*

We start with a presentation of the main properties of *symbol spaces.* Then we show that formula (I.10) yields a map

$$\{\text{symbols}\} \to \{\text{linear operators on } \mathcal{S} \text{ or } \mathcal{S}'\}.$$

Such a map is called a *quantization map*, following the ideas presented in the introduction of this work. We will see that this map yields the τ-ordering quantization family (15). We will then examine the composition law of pseudo-differential operators and show that the family of star-products $\star_\tau$, introduced in the introduction (see (16)), naturally arises through the symbol of the operator resulting from this composition. Our main reference for this section is [Shu01].

Notation 1.2.1. In what follows, we write:

- $\langle x\rangle \coloneqq (1+\|x\|^2)^{\frac{1}{2}}$, the Japanese bracket ;

- $\partial_i := \partial/\partial x_i$ and, for $\alpha \in \mathbb{N}^n$, $\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$;
- for $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$;
- $\Delta_x := \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, the Laplacian operator ;
- $f \lesssim g$ if there exists $C > 0$ such that $f \leq Cg$.

1.2.1 Symbol spaces

Symbols are polynomially bounded smooth functions whose derivatives are also polynomially bounded. In particular, they extend the notion of polynomials. Depending on the context, different classes of symbols can be defined. In this work, we use the so-called *Shubin symbols*.

Definition 1.2.2. Let m, ρ be real numbers with $0 < \rho \leq 1$. The space of *Shubin symbols* of order m and type ρ, denoted $G^m_\rho(\mathbb{R}^n)$, is the space of smooth functions $a \in C^\infty(\mathbb{R}^n)$ satisfying

$$|\partial_z^\alpha a(z)| \leq C_\alpha \langle z\rangle^{m-\rho|\alpha|}, \quad \forall \alpha \in \mathbb{N}^n.$$

The space G^m_ρ is a Fréchet space with respect to the seminorms

$$\sup_z |\partial_z^\alpha a(z)|\,\langle z\rangle^{-m+\rho|\alpha|}, \quad \alpha \in \mathbb{N}^n.$$

The maps

$$\begin{aligned} &G^m_\rho \times G^n_\rho \to G^{m+n}_\rho, \quad (a,b) \mapsto ab, \\ &G^m_\rho \to G^{m-\rho|\alpha|}_\rho, \quad a \mapsto \partial^\alpha a \end{aligned}$$

are continuous, and we have

$$\bigcap_{m\in\mathbb{R}} G^m_\rho = \mathcal{S}, \quad G^m_\rho \subset \mathcal{S}' \quad \forall m \in \mathbb{R}. \tag{I.11}$$

When $\rho = 1$, we simply write G^m, and the union of all symbol spaces with $\rho = 1$ is denoted

$$G^\infty := \bigcup_{m\in\mathbb{R}} G^m.$$

The most important property of symbols is that they are characterised by an *asymptotic expansion*.

Definition 1.2.3. Let $a_j \in G_\rho^{m_j}$, $j = 1, 2, \ldots$, with $m_j \to -\infty$, and $a \in C^\infty(\mathbb{R}^n)$. We say that a_j is an asymptotic expansion of a, and we write $a \sim \sum_{j=1}^{\infty} a_j$, if for all $r \geq 2$,

$$a - \sum_{j=1}^{r-1} a_j \in G_\rho^{m_r}, \quad \text{where}\, m_r = \max_{j \geq r} m_j.$$

Proposition 1.2.4. *Let $a_j \in G_\rho^{m_j}$, $j = 1, 2, \ldots$, with $m_j \to -\infty$, and $a \in C^\infty(\mathbb{R}^n)$. Then there exists a function a such that $a \sim \sum_{j=1}^{\infty} a_j$. Moreover, if another function a' has the same property, then $a - a' \in \mathcal{S}$.*

Proof. See [Shu01, Prop. 23.1]. □

1.2.2 Pseudo-differential operators in $\mathbb{R}^n$

Definition 1.2.5. We denote $A_\rho^m(\mathbb{R}^{3n})$ the set of functions $a \in C^\infty(\mathbb{R}^{3n})$ that satisfy for some $m' \in \mathbb{R}$

$$\left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \xi)\right| \leq C_{\alpha,\beta,\gamma} \langle z\rangle^{m-\rho|\alpha+\beta+\gamma|} \langle x - y\rangle^{m'+\rho|\alpha+\beta+\gamma|}, \tag{I.12}$$

where $z = (x, y, \xi)$. We call $A_\rho^m(\mathbb{R}^{3n})$ the class of amplitudes of order m and type ρ.

The following property is a consequence of basic oscillatory integral theory. The trick is mainly based on appropriate use of integration by parts. For a general presentation on this topic, the reader might refer to [Die90b, Chap. 23.17] or [Shu01, Chap. 1].

Proposition 1.2.6. *For any amplitude, the integral expression* (I.10) *defines a continuous linear operator from $\mathcal{S}$ to $\mathcal{S}$.*

Proof. Let m, ρ be any real numbers, a any amplitude in $A_\rho^m(\mathbb{R}^{3n})$, and $\phi \in \mathcal{S}$. A classical idea when dealing with oscillatory integrals is to apply integration by parts, taking advantage of the following observation:

$$\begin{aligned} e^{i(x-y)\xi} &= \langle \xi\rangle^{-2M} (1 - \Delta_y)^M e^{i(x-y)\xi}, \\ e^{i(x-y)\xi} &= \langle x - y\rangle^{-2N} (1 - \Delta_\xi)^N e^{i(x-y)\xi}. \end{aligned}$$

Since ϕ is rapidly decreasing, integration by parts in y yields

$$A\phi(x) = \int e^{i(x-y)\xi} \langle \xi \rangle^{-2M} (1-\Delta_y)^M \left(a(x,y,\xi)\phi(y)\right) \mathrm{d}y \, \mathrm{d}\xi.$$

For M sufficiently large, the integrand goes to zero as $\xi \to \infty$. Hence, integration by parts in ξ yields

$$\int e^{i(x-y)\xi} \langle x-y \rangle^{-2N} (1-\Delta_\xi)^N \left(\langle \xi \rangle^{-2M} (1-\Delta_y)^M \left(a(x,y,\xi)\phi(y)\right)\right) \mathrm{d}y \, \mathrm{d}\xi.$$

Then, applying the inequality

$$\langle x-y \rangle^{-1} \lesssim \langle x \rangle^{-1} \langle y \rangle,$$

one obtains

$$A\phi(x) \lesssim \langle x \rangle^{-2N} \int e^{i(x-y)\xi} \langle y \rangle^{2N} \\ (1-\Delta_\xi)^N \left(\langle \xi \rangle^{-2M} (1-\Delta_y)^M \left(a(x,y,\xi)\phi(y)\right)\right) \mathrm{d}y \, \mathrm{d}\xi.$$

By the Leibniz rule, estimate (I.12), and the inequality

$$\langle x-y \rangle / \langle z \rangle \leq 1, \quad \text{for } z = (x,y,\xi),$$

the right-hand side can be written as a finite sum in which all terms are bounded (up to a constant depending only on M and N) by

$$\langle x \rangle^{-2N+m'+m} \int \langle y \rangle^{2N+m'+m} b_M(\xi) \langle \xi \rangle^{-2M+m} \tilde{\phi}(y) \, \mathrm{d}y \, \mathrm{d}\xi,$$

where $\tilde{\phi}$ is some derivative of ϕ, and $b_M(\xi)$ is a bounded function given by some derivative of $\langle \xi \rangle^{-2M}$. Since ϕ is Schwartz, the integral above converges for $M > \frac{m+m'}{2}$, and for all $k \in \mathbb{R}$,

$$\sup \langle x \rangle^k |A\phi(x)| \leq C_{k,m,m'} \|\phi\|_{0,N+m+m'}.$$

A similar estimate holds for $\partial_x^\alpha(A(u(x)))$. □

Observe that the transpose operator (in the sense of Definition 1.1.8) is

$$A'u(y) = (2\pi)^{-n} \int e^{i(x-y)\xi} a(x,y,\xi) u(y) \, \mathrm{d}x \, \mathrm{d}\xi.$$

By Remark 1.1.9, A admits the following continuous extension on $\mathcal{S}'$:

$$\begin{aligned} &A : \mathcal{S}' \to \mathcal{S}', \quad u \mapsto Au, \\ &Au(\phi) = u(A'(\phi)), \quad \forall \phi \in \mathcal{S}. \end{aligned}$$

Definition 1.2.7. We denote Ψ_ρ^m the class of operators on $\mathcal{S}'$ defined by the integral expression (I.10) with amplitude $a \in A_\rho^m(\mathbb{R}^{3n})$. Such operators are called *pseudo-differential operators* of order m and type ρ on $\mathbb{R}^n$.

The following theorem characterizes pseudo-differential operators in terms of Shubin symbols in $G_\rho^m(\mathbb{R}^{2n})$.

Theorem 1.2.8. *Let $A \in \Psi_\rho^m$. Then for any $\tau \in \mathbb{R}$, the operator A may be uniquely written as*

$$Au(x) = \int e^{i(x-y)\xi} b_\tau((1-\tau)x + \tau y, \xi) u(y)\, \mathrm{d}y\, \mathrm{d}\xi \tag{I.13}$$

*where $b_\tau \in G_\rho^m(\mathbb{R}^{2n})$ is called the τ-*symbol *of A. Moreover, b_τ admits the asymptotic expansion*

$$b_\tau(x,\xi) \sim \sum_{\beta,\gamma} \frac{1}{\beta!\gamma!} \tau^{|\beta|}(1-\tau)^{|\gamma|} \partial_x^{\alpha+\beta}(-D_x)^\beta D_y^\gamma a(x,y,\xi)\big|_{y=x}.$$

Proof. See [Shu01, Thm. 23.2]. □

Remark 1.2.9. Formula (I.13) yields a family of maps

$$\begin{aligned} &Q_\tau \colon a \in G_\rho^m(\mathbb{R}^{2n}) \mapsto Q_\tau(a) \in \mathscr{L}(\mathcal{S}'(\mathbb{R}^n)), \\ &(Q_\tau(a))(u)(x) = \int e^{i(x-y)\xi} a\left((1-\tau)x + \tau y, \xi\right) u(y)\, \mathrm{d}y\, \mathrm{d}\xi. \end{aligned}$$

Since $L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is a Hilbert space and $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$ is the phase space of the flat configuration space $\mathbb{R}^n$, we call these *quantization maps* in the spirit of what we presented in the introduction of this work.

For $\tau = 0$ and $\tau = \frac{1}{2}$, these are respectively the *Kohn-Nirenberg* and the *Weyl quantizations*. We write

$$Q_0(a) = a^{KN} \quad \text{and} \quad Q_{1/2}(a) = a^W.$$

Remark 1.2.10. It is more common to define pseudo-differential operators using Hörmander symbols [Hö90, Def. 7.8.1] since this approach allows for localization and can be performed on manifolds (see [Hö94, Chap. XVIII], [Die90b], or [Shu01, § 3]). However, pseudo-differential operators in $\mathbb{R}^n$, as defined with Shubin symbols above, enable the study of certain aspects of the behavior of functions as $|x| \to \infty$. One could say that the approach through Hörmander symbols is local, while the approach through Shubin symbols is global. In this work, we will focus on the global approach.

1.2.3 Composition formula

Theorem 1.2.11. *Let $A' \in \Psi_\rho^{m_1}$ and $A'' \in \Psi_\rho^{m_2}$. Then $A' \circ A'' \in \Psi_\rho^{m_1+m_2}$, and if $b'_{\tau_1}(x,\xi)$ and $b''_{\tau_2}(x,\xi)$ are respectively the τ_1-symbol and τ_2-symbol of A' and A'', then the τ-symbol $b_\tau(x,\xi)$ of $A' \circ A''$ has the asymptotic expansion*

$$b_\tau(x,\xi) \sim \sum_{\alpha,\beta,\gamma,\delta} C_{\alpha\beta\gamma\delta} \left(\partial_\xi^\alpha D_x^\beta b'_{\tau_1}(x,\xi)\right) \left(\partial_\xi^\gamma D_x^\delta b''_{\tau_2}(x,\xi)\right),$$

where $C_{\alpha\beta\gamma\delta}$ are constants depending on τ, τ_1, and τ_2 such that $C_{0000} = 1$, and the sum runs over the sets of multi-indices such that $\alpha+\gamma = \beta+\delta$.

Proof. See [Shu01, Thm. 23.6]. □

In particular, for $\tau = \tau_1 = \tau_2$ and using the quantization map Q_τ introduced in Remark 1.2.9, we will denote $a \star_\tau b$ as the symbol of the operator obtained from the composition of $Q_\tau(a)$ and $Q_\tau(b)$:

$$Q_\tau(a \star_\tau b) := Q_\tau(a) \circ Q_\tau(b).$$

Theorem 1.2.11 yields the following asymptotic expansion for this symbol:

$$a \star_\tau b \sim \sum_{\alpha,\beta,\gamma,\delta} C_{\alpha\beta\gamma\delta} \left(\partial_\xi^\alpha D_x^\beta b'_\tau(x,\xi)\right) \left(\partial_\xi^\gamma D_x^\delta b''_\tau(x,\xi)\right). \tag{I.14}$$

The first two points of the next result are direct consequences of formula (I.14). The third one follows from basic calculus.

Corollary 1.2.12. *Let $a \in G^m$ and $b \in G^n$.*

1. *For any value of τ, the product $\star_\tau$ is a bilinear continuous map*

$$\star_\tau \colon G^m \times G^n \to G^{m+n}.$$

2. If $\operatorname{supp}(a) \cap \operatorname{supp}(b)$ *is compact, then* $a \star_\tau b \in \mathcal{S}$.

3. If $a \in \mathcal{S}$ *and* $u \in \mathcal{S}'$, *then* $Q_\tau(a)u \in \mathcal{S}$.

Proposition 1.2.13. *Let* $V = \mathbb{R}^{2n}$ *and* $z_i = (q_i, p_i) \in V$. *If* a *and* $b \in \mathcal{S}(V)$, *then* $a \star_\tau b$ *admits the following integral expression*

$$a \star_\tau b(z_0) = \int K_\tau(z_1, z_2) a(z_1 + z_0) b(z_2 + z_0) \,\mathrm{d}z_1 \,\mathrm{d}z_2, \tag{I.15}$$

where

$$K_0(z_1, z_2) = e^{-q_2 p_1} \delta(q_1)\delta(p_2), \tag{I.16}$$
$$K_1(z_1, z_2) = e^{q_1 p_2} \delta(q_2)\delta(p_1), \tag{I.17}$$

and for $\tau \neq 0$ *and* $\tau \neq 1$,

$$K_\tau(z_1, z_2) = (\tau(\tau-1))^{-n} e^{\frac{i}{\tau} q_1 p_2} e^{\frac{i}{\tau-1} q_2 p_1} = (\tau(\tau-1))^{-n} e^{i z_1 J_\tau z_2},$$

with

$$J_\tau := \begin{pmatrix} 0 & \tau^{-1} I_n \\ (\tau-1)^{-1} I_n & 0 \end{pmatrix}.$$

Moreover, $(\mathcal{S}(V), \star_\tau)$ *is an associative algebra, and the product* $\star_\tau$ *is jointly continuous.*

Proof. Let $[Q_\tau(a)]$ be the Schwartz kernel of the operator $Q_\tau(a)$:

$$[Q_\tau(a)](x, y) = \int e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) \,\mathrm{d}\xi.$$

We compute

$$[Q_\tau(a)](x, y) = \mathcal{F}_2(a)(x + \tau(y - x), y - x) = (A_\tau^* \circ (1 \otimes \mathcal{F}_2)a)(x, y),$$

where $\mathcal{F}_2$ denotes the Fourier transform in the second variable, and

$$A_\tau = \begin{pmatrix} (1-\tau) I_n & \tau I_n \\ -I_n & I_n \end{pmatrix}.$$

Note that the matrix A_τ is invertible for all values of τ. We define the mapping

$$k_\tau \colon \mathcal{S}'(V) \to \mathcal{S}'(V), \quad a \mapsto [Q_\tau(a)],$$

and its inverse

$$k_\tau^{-1} \colon \mathcal{S}'(V) \to \mathcal{S}(V), \quad u \mapsto (1 \otimes \mathcal{F}_2^{-1}) \circ (A_\tau^{-1})^*(u).$$

Then, we compute

$$\begin{aligned}
a \star_\tau b(z_0) &= k_\tau^{-1}[Q_\tau(a) \circ Q_\tau(b)]\big|_{z_0} \\
&= (1 \otimes \mathcal{F}_2^{-1}) \circ (A_\tau^{-1})^*[Q_\tau(a) \circ Q_\tau(b)]\big|_{z_0} \\
&= \int e^{iqp_0} (A_\tau^{-1})^*[Q_\tau(a) \circ Q_\tau(b)](q_0, q)\, \mathrm{d}q \\
&= \int e^{iqp_0} [Q_\tau(a) \circ Q_\tau(b)](q_0 - \tau q, q - \tau q + q_0)\, \mathrm{d}q \\
&= \int e^{iqp_0} [Q_\tau(a)](q_0 - \tau q, q')[Q_\tau(b)](q', q - \tau q + q_0)\, \mathrm{d}q'\, \mathrm{d}q.
\end{aligned}$$

Suppose that $\tau = 0$ (the case $\tau = 1$ is identical and left to the reader). Writing $q' = q_2$ and using Example 1.1.18 and simple changes of variables, the above equality becomes

$$\begin{aligned}
a \star_0 b(z_0) &= \int e^{iqp_0} [Q_0(a)](q_0, q_2)[Q_0(b)](q', q + q_0)\, \mathrm{d}q_2\, \mathrm{d}q \\
&= \int e^{iqp_0} e^{i(q_0 - q_2)\xi} a(q_0, p_1) e^{i(q_2 - q - q_0)\eta} b(q_2, \eta)\, \mathrm{d}p_1\, \mathrm{d}\eta\, \mathrm{d}q_2\, \mathrm{d}q \\
&= \int e^{iq_2(p_0 - p_1)} a(q_0, p_1) e^{iq_0(p_1 - p_0)} b(q_2, p_0)\, \mathrm{d}p_1\, \mathrm{d}q_2 \\
&= \int e^{-iq_2 p_1} \delta(q_1)\delta(p_2) a(q_1 + q_0, p_1 + p_0) \\
&\qquad\qquad b(q_2 + q_0, p_2 + p_0)\, \mathrm{d}q_1\, \mathrm{d}q_2\, \mathrm{d}p_1\, \mathrm{d}p_2 \\
&= \int K_0(z_1, z_2) a(z_1 + z_0) b(z_2 + z_0)\, \mathrm{d}z_1\, \mathrm{d}z_2.
\end{aligned}$$

When τ is different from 0 and 1, the computations become slightly more involved, but the strategy remains similar: write the full expressions for $[Q_\tau(a)]$ and $[Q_\tau(b)]$, and apply the obvious changes of variables to yield the desired expression

$$a \star_\tau b(z_0) = (\tau(\tau - 1))^{-n} \int e^{iz_1 J_\tau z_2} a(z_1 + z_0) b(z_2 + z_0)\, \mathrm{d}z_1\, \mathrm{d}z_2.$$

The facts that $a \star_\tau b$ is a Schwartz function and that $\star_\tau$ is associative are easily checked. The continuity of $\star_\tau$ is a direct consequence of observation (I.11) and the first point of Corollary 1.2.12.

□

Corollary 1.2.14. *For every value of τ, the product $\star_\tau$, given by the integral formula* (I.15)*, is a non-formal star-product on $\mathcal{S}(V)$*

$$\star_\tau : \mathcal{S}(V) \times \mathcal{S}(V) \to \mathcal{S}(V).$$

As highlighted in Proposition 1.2.13, associated with the family of star-products $\star_\tau$, we have a family of applications

$$\begin{aligned} &\Omega_\tau \colon \mathcal{S}(V^2) \to \mathcal{S}'(V^2), \\ &\Omega_\tau(\phi)(\overline{x}) \coloneqq \int K_\tau(\overline{z})\phi(\overline{z}+\overline{x})\,\mathrm{d}\overline{z}. \end{aligned} \tag{I.18}$$

We call Ω_τ the *twist* associated with the product $\star_\tau$. The cases $\tau = 0$ and $\tau = \frac{1}{2}$ correspond, respectively, to the Kohn-Nirenberg and Weyl products. We will sometimes denote

$$\star_{\frac{1}{2}} = \star_W, \quad \Omega_{\frac{1}{2}} = \Omega_W \quad \text{and} \quad \star_0 = \star_{KN}, \quad \Omega_0 = \Omega_{KN}.$$

The next lemma could be proved now by simple calculus, but it is a direct consequence of an observation we make in Chapter III (see Corollary 3.1.5).

Lemma 1.2.15. *For all values of τ, Ω_τ is an automorphism of $\mathcal{S}(V^2)$.*

By Remark 1.1.9, Ω_τ continuously extends to $\mathcal{S}'(V^2)$ as follows:

$$\begin{aligned} &\Omega_\tau \colon \mathcal{S}'(V^2) \to \mathcal{S}'(V^2), \\ &(\Omega_\tau u)(\phi) \coloneqq u(\Omega'_\tau(\phi)), \quad \text{for all } \phi \in \mathcal{S}(V^2), \end{aligned} \tag{I.19}$$

where

$$\Omega'_\tau(\phi)(\overline{x}) \coloneqq \int K_\tau(\overline{z})\phi(\overline{x}-\overline{z})\,\mathrm{d}\overline{z}.$$

We have seen that the algebras defined in Proposition 1.2.13 form a family of *non-formal deformation quantizations* of the pointwise algebra of Schwartz functions in the sense presented in the introduction. In Chapter III, we will extend these star-products to subspaces of $\mathcal{S}'$. This will be achieved through an analytical description of tempered distributions in terms of a microlocal object presented in the following chapter: the smooth wave front set.

Chapter II

The smooth wave front set

> *"We call it the wave front set by analogy with the classical Huyghens construction of a propagating wave. In this construction one assumes that the location and oriented tangent plane of a wave is known at one instant of time and concludes that at a later time it has been translated in the normal direction with the speed of light. The data are thus precisely rays in the cotangent bundle."*

- Lars Hörmander, [Hö90]

The classical wave front set, introduced by Hörmander in [Hö90, Chap. 8], encodes the relationship between the smoothness of a distribution and the growth of its Fourier transform. Explicitly, a compactly supported distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is smooth if and only if its Fourier transform decreases faster than any polynomial in every direction:

$$u \in C_c^\infty(\mathbb{R}^n) \iff \mathcal{F}(u)(\xi) \leq C_N \langle \xi \rangle^{-N}, \quad N = 1, 2, \dots \tag{II.1}$$

One could say that the singularities of a distribution are triggered by certain high-frequency directions in its Fourier decomposition. If we interpret this in terms of sounds, singular functions would be associated with very high-pitched sounds, while compactly supported smooth functions would correspond to more audible frequencies. The idea of the classical wave front set is to group the singularities of a distribution and the high-frequency directions causing them into a single object. Heuristically, the wave front set of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is the subset of $T^*\mathbb{R}^n$ given by

$$\mathrm{WF}(u) = \text{singularities of } u \times \text{high-frequency directions causing them.}$$

In this work, we will not use the classical wave front set, but rather a derived notion, initially introduced by Hörmander in [Hö89] to study the propagation of singularities of pseudo-differential operators in $\mathbb{R}^n$: the *smooth wave front set*. It is a global object in the sense that it encodes information about the behavior at infinity of the studied distribution. Unlike the classical wave front set, it provides no information about the location of singularities but only about their existence. Due to this, it is invariant under translations, which is an important property in the context of deformation quantization. Its definition was originally given in terms of Weyl operators in [Hö89], but equivalent definitions exist in terms of the short-time Fourier transform (see [RW12]) or, as we will show in this chapter, in terms of matrix coefficients of unitary representations of the Heisenberg group.

2.1 Definitions and Examples

2.1.1 Weyl Operators

For any Shubin symbol $a \in G^m$, the Weyl quantization (see Remark 1.2.9) yields a map

$$a^W : u \in \mathcal{S}'(\mathbb{R}^n) \mapsto a^W(u) \in \mathcal{S}'(\mathbb{R}^n).$$

Sometimes, $a^W(u)$ turns out to be a Schwartz function. In particular, if we take $a \in \mathcal{S}(\mathbb{R}^{2n})$, then a^W maps any tempered distribution to a Schwartz function. When $a^W u \in \mathcal{S}(\mathbb{R}^n)$, we say that the symbol a regularizes the distribution u, and we denote by $G^m(u)$ the set of symbols of order m that regularize u. We also introduce the space

$$G(u) := \bigcup_m G^m(u).$$

The definition of the smooth wave front set of a given tempered distribution u is constructed as a necessary condition on symbols to regularize u. Before stating the definition, we need to introduce the notion of conic subsets, as well as those of elliptic and characteristic directions of a symbol.

Definition 2.1.1. A subset Γ of $\mathbb{R}^n$ is called a *cone* (or conic subset) if, for every $x \in \Gamma$ and all real numbers $t > 0$, tx is also in Γ.

Note that a cone is entirely determined by its intersection with the unit sphere $S^{n-1} \subset \mathbb{R}^n$. We will say that a cone Γ is open (closed) when $\Gamma \cap S^{n-1}$ is open (closed) in S^{n-1}.

Definition 2.1.2. Given a Shubin symbol $a \in G^m$, a point in the phase space $z_0 \in (T^*\mathbb{R}^n) \setminus \{(0,0)\}$ is called *non-characteristic* or *elliptic* for a if there exist $C, \epsilon > 0$ and an open conic set $\Gamma \subseteq (T^*\mathbb{R}^n) \setminus \{(0,0)\}$ such that $z_0 \in \Gamma$ and

$$|a(z)| \geq \epsilon \langle z \rangle^m, \quad z \in \Gamma, \quad |z| \geq C.$$

The set of all elliptic directions of a symbol a will be denoted $\text{ell}(a)$.

Definition 2.1.3. The *characteristic set* of a, denoted $\text{char}(a)$, is the complement of $\text{ell}(a)$ in $T^*(\mathbb{R}^n)$. If $z \in \text{char}(a)$, we say that z is *characteristic* (i.e., not non-characteristic) for a.

Definition 2.1.4. If $u \in \mathcal{S}'(\mathbb{R}^n)$, then its *smooth wave front set* $\text{WF}(u)$ is the set of all directions $z \in T^*(\mathbb{R}^n) \setminus \{(0,0)\}$ such that, for any m, $a \in G^m$ and $a^W(u) \in \mathcal{S}$, it implies $z \in \text{char}(a)$:

$$\text{WF}(u) := \bigcap_{a \in G(u)} \text{char}(a).$$

We can formulate an equivalent definition negatively: a vector $z \neq 0$ does *not* belong to the wave front set of u if and only if there exists $a \in G^m$ such that $a^W u \in \mathcal{S}$ and $z \notin \text{char}(a)$. In particular, we have the following corollary, which will be useful.

Corollary 2.1.5. *A vector $z \neq 0$ does* not *belong to the wave front set of u if z is non-characteristic for some element $a \in G(u)$.*

Note that both $\operatorname{char}(a)$ and $\operatorname{WF}(u)$ are conic subsets of $T^*\mathbb{R}^n \setminus \{(0,0)\}$; $\operatorname{char}(a)$ is an open cone, while $\operatorname{WF}(u)$ is a closed cone.

The following proposition tells us that if $z_0 \neq 0$ and $z_0 \notin \operatorname{WF}(u)$, we can construct a very convenient symbol b such that $b^W u \in \mathcal{S}$ and $z_0 \notin \operatorname{char}(b)$.

Proposition 2.1.6. *If $z_0 \neq 0$ and $z_0 \notin \operatorname{WF}(u)$, then there exist $b \in G^0$ and an open conic set $\Gamma'' \subseteq T^*\mathbb{R}^d \setminus \{(0,0)\}$ containing z_0 such that $0 \leq b \leq 1$, $b(z) = 1$ for $z \in \Gamma''$ and $|z| > 1$, and $b^W u \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. See [RW12, Prop. 1.7]. □

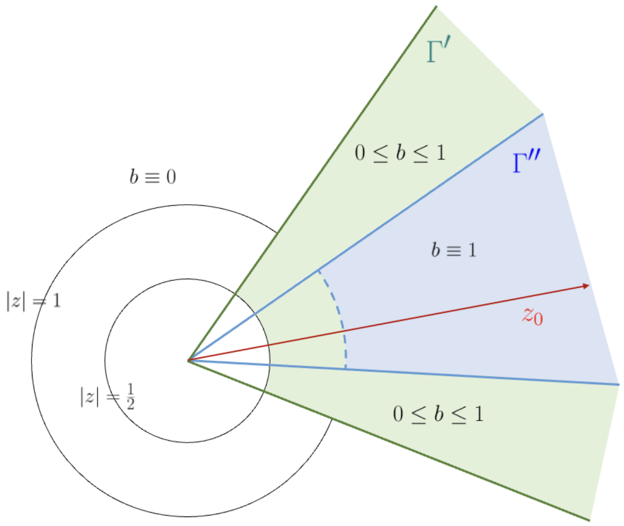


Figure II.1: Visual representation of a symbol b around a directionz_0 not contained in WF as given by Proposition 2.1.6

Definition 2.1.7. Let u be a distribution in $\mathcal{D}'(\mathbb{R}^n)$. The conic support of u, $\operatorname{conesupp}(u)$, is the set of all $x \in \mathbb{R}^n$ such that any conic open set $\Gamma_x \subset \mathbb{R}^n \setminus \{0\}$ containing x satisfies

$$\overline{\operatorname{supp}(a) \cap \Gamma_x} \quad \text{is not compact in } \mathbb{R}^n.$$

The following proposition and corollary are stated, without proof, by Hörmander in [Hö89]. An exhaustive demonstration of both results is presented below.

Proposition 2.1.8. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $a \in G^m$, then*

$$\mathrm{WF}(a^W u) \subseteq \mathrm{WF}(u) \cap \mathrm{conesupp}(a).$$

Proof. We start by showing the inclusion $\mathrm{WF}(a^W u) \subseteq \mathrm{conesupp}(a)$. Suppose $z_0 \notin \mathrm{conesupp}(a)$. Then there exists an open cone Γ' containing z_0 such that $\Gamma' \cap \mathrm{conesupp}(a) = \emptyset$ and $\Gamma' \cap \mathrm{supp}(a)$ is compact. Consider a smaller cone $\Gamma'' \subseteq \Gamma'$ still containing z_0, and let $b \in G^0$ be such that $b \equiv 1$ in $\Gamma'' \cap \{|z| > 1\}$ and $b \equiv 0$ outside Γ. Then, $\mathrm{supp}(a) \cap \mathrm{supp}(b)$ is compact, and by Corollary 1.2.12, $b \star_W a \in \mathcal{S}$. Hence,

$$b^W(a^W u) = (b \star_W a)^W(u) \in \mathcal{S},$$

which implies $z_0 \notin \mathrm{WF}(a^W u)$.

Now, we show the inclusion $\mathrm{WF}(a^W u) \subseteq \mathrm{WF}(u)$. Suppose $z_0 \notin \mathrm{WF}(u)$. By Proposition 2.1.6, we know there exist a symbol $0 \leq \tilde{b} \leq 1$ and two open cones $\tilde{\Gamma}' \subset \tilde{\Gamma}$ such that $\tilde{b} \equiv 1$ in $\tilde{\Gamma}' \cap \{|z| > 1\}$, $\tilde{b} \equiv 0$ outside $\tilde{\Gamma}$ (see Figure II.1) and $\tilde{b}^W u \in \mathcal{S}$. Let $\tilde{a} = 1 - \tilde{b} \in G^0$. We observe

$$u = \left(\tilde{a} + \tilde{b}\right)^W u = \tilde{a}^W u + \tilde{b}^W u,$$

where $\tilde{b}^W u \in \mathcal{S}$. Hence, to prove that $z_0 \notin \mathrm{WF}(a^W u)$, it is enough to find a symbol r for which the direction z_0 is non-characteristic such that

$$r^W\left(a^W\left(\tilde{a}^W u\right)\right) = (r \star_W a \star_W \tilde{a})^W u \in \mathcal{S}.$$

Since $\tilde{a} \equiv 0$ in $\tilde{\Gamma}' \cap \{|z| > 1\}$, we can take $r \equiv 1$ in $\tilde{\Gamma}'' \cap \{|z| > 1\}$, where $\tilde{\Gamma}'' \subset \tilde{\Gamma}'$ is an open cone containing z_0, and $r \equiv 0$ outside $\tilde{\Gamma}''$. Then $\mathrm{supp}(r) \cap \mathrm{supp}(\tilde{a})$ is compact, and $r \star_W a \star_W \tilde{a} \in \mathcal{S}$. In particular, $r^W(a^W(\tilde{a}^W u)) \in \mathcal{S}$ and $z_0 \notin \mathrm{WF}(a^W u)$, which concludes the proof. □

Corollary 2.1.9. *If* $\mathrm{conesupp}(a) \cap \mathrm{WF}(u) = \emptyset$*, then* $a^W(u) \in \mathcal{S}$*. In particular,* $\mathrm{WF}(u) = \emptyset$ *if and only if* $u \in \mathcal{S}$*.*

Proof. By Proposition 2.1.8, we know that $\mathrm{conesupp}(a) \cap \mathrm{WF}(u) = \emptyset$ implies $\mathrm{WF}(a^W u) = \emptyset$. Hence, all we need to show is that $\mathrm{WF}(u) = \emptyset$ if and only if $u \in \mathcal{S}$.

First, if $u \in \mathcal{S}$, then $1^W u \in \mathcal{S}$ and since all directions are elliptic for the symbol $1 \in G^0$, we have $\mathrm{WF}(u) = \emptyset$.

Conversely, since no direction is in the wave front set, Proposition 2.1.6 allows us to construct, around any direction z_i, a symbol $0 \leq b_i \leq 1$ such that there exist two open cones $\Gamma_i' \subset \Gamma_i''$ containing z_i, with $b_i \equiv 1$ in $\Gamma_i'' \cap \left\{|z| > \frac{1}{2}\right\}$, $b_i \equiv 0$ outside Γ_i' (as depicted in Figure II.1), and $b_i^W u \in \mathcal{S}$. Since S^{2n-1} is compact, we can choose a finite number of directions z_i and symbols b_i such that the family of bump functions

$$\psi_i \colon S^{2n-1} \to \mathbb{R}, \quad x \mapsto b_i(x),$$

forms a partition of unity on the sphere S^{2n-1}. Letting

$$b_0 = 1 - \sum_i b_i,$$

we have

$$b_0 + \sum_{i=1}^{n} b_i \equiv 1 \text{ on } T^*\mathbb{R}^n.$$

Hence,

$$u = \left(b_0 + \sum_{i=1}^{n} b_i\right)^W u = b_0^W u + \sum_i b_i^W u,$$

where all terms on the right-hand side are Schwartz functions by Proposition 2.1.6. □

Definition 2.1.10. A real symplectic vector space V is a vector space endowed with an anti-symmetric and non-degenerate bilinear form

$$\omega \colon V \times V \to \mathbb{R}.$$

Two symplectic vector spaces (V, ω) and (V', ω') are linearly symplectomorphic if there exists a linear isomorphism

$$A : V \to V' \quad \text{such that} \quad \omega'(Av, Aw) = \omega(v, w).$$

The set of linear symplectomorphisms of (V, ω) onto itself forms a Lie group denoted $\mathrm{Sp}(V)$.

Note that the smooth wave front set is an object defined on the vector space $T^*\mathbb{R}^{2n} \simeq \mathbb{R}^{2n}$, which admits a natural symplectic structure given by the two-form:

$$\omega\left((x,\xi),(y,\eta)\right) = (x,\xi) \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} y \\ \eta \end{pmatrix}.$$

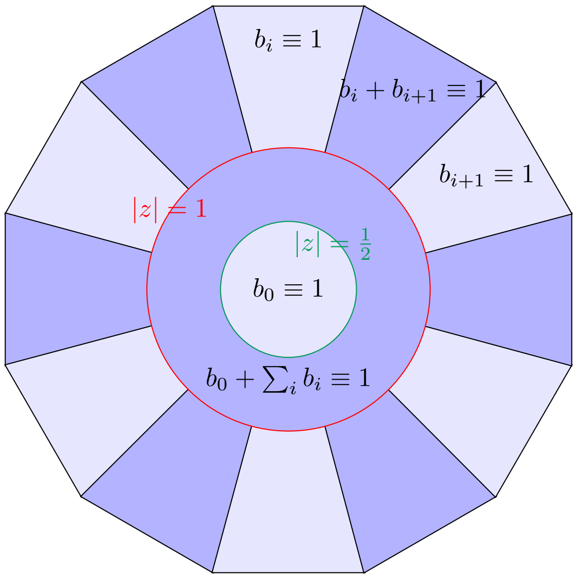


Figure II.2: Partition of unity of $T^*\mathbb{R}^n$ as built in the proof of Corollary 2.1.9. Each symbol b_i is constructed as in Figure II.1.

One of the advantages of the definition of the smooth wave front set in terms of Weyl operators is to highlight its behavior under symplectic transformations $\chi \in \mathrm{Sp}(2n)$.

Theorem 2.1.11. *For every linear symplectic transformation χ of $T^*\mathbb{R}^n$, there exists a unique (up to a phase factor) unitary operator U_χ on $L^2(\mathbb{R}^n)$, which is also an automorphism of $\mathcal{S}$ and $\mathcal{S}'$, such that*

$$U_\chi^{-1} a^W U_\chi = (a \circ \chi)^W .$$

Proof. See [Hö94, Thm. 18.5.9] and [Hö89, Prop. 2.2]. □

A direct consequence of this proposition is stated in [Sch14, Prop. 2.58].

Corollary 2.1.12. *Let $u \in \mathcal{S}'$ and χ be a linear symplectic map. Then*

$$\mathrm{WF}(U_\chi u) = \{\chi(x,\, \xi) : (x,\, \xi) \in \mathrm{WF}(u)\} .$$

In particular,

- *if $\chi\colon (x,\, \xi) \mapsto (\xi,\, -x)$, then U_χ is given by the Fourier transform and*

$$\mathrm{WF}(\mathcal{F}u) = \{(\xi,\, -x) : (x,\, \xi) \in \mathrm{WF}(u)\} ;$$

- *if* $\chi\colon (x,\,\xi) \mapsto (A^{-1}x,\, A^T\xi)$ *with* $A \in \mathrm{GL}(n,\mathbb{R})$*, then*
$$U_\chi u = \sqrt{|\det(A)|}\, u \circ A$$
and
$$\mathrm{WF}(u \circ A) = \Big\{(A^{-1}x,\, A^T\xi) : (x,\,\xi) \in \mathrm{WF}(u)\Big\}\,;$$
- *if* $\chi\colon (x,\,\xi) \mapsto (x,\, \xi + Ax)$ *with* A *a real, symmetric* $n \times n$ *matrix, then*
$$U_\chi u(x) = e^{ix^TAx} u(x)$$
and
$$\mathrm{WF}(e^{ix^TAx}u) = \{(x,\,\xi + Ax) : (x,\,\xi) \in \mathrm{WF}(u)\}\,.$$

Note that the three families of symplectic transformations considered in Corollary 2.1.12 correspond respectively to the matrices

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \left\{\begin{pmatrix} A^{-1} & 0 \\ 0 & A^T \end{pmatrix} : A \in \mathrm{GL}(n)\right\},$$
$$\left\{\begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} : A \in \mathrm{Sym}(n)\right\},$$

which form a set of generators of $\mathrm{Sp}(2n)$. Hence, any symplectic transformation $\chi \in \mathrm{Sp}(2n)$ is obtained as the product of such generators, and the unitary operator U_χ associated is given by the composition of the corresponding operators listed in Corollary 2.1.12.

Remark 2.1.13. Theorem 2.1.11 and Corollary 2.1.12 suggest an interaction between covariant and contravariant quantities in $T^*\mathbb{R}^n$. In particular, the symplectic transformation associated with the unitary operator

$$U_\chi u(x) = e^{ix^TAx} u(x)$$

yields the addition $\xi + Ax$, which makes sense if we interpret $T^*\mathbb{R}^n$ as the symplectic vector space $\mathbb{R}^{2n}$, but can be confusing if we see x and ξ respectively as a contravariant vector and a covariant vector in $T^*\mathbb{R}^n$. A way to clarify the situation is to interpret Ax as the element $(Ax)^\flat \in (\mathbb{R}^n)^*$, obtained as the image of the vector $Ax \in \mathbb{R}^n$ by the canonical isomorphism

$$V \to V^*, \quad \xi \mapsto \xi^\flat,$$

where $x^\flat$ is the unique covector in $(\mathbb{R}^n)^*$ satisfying $\langle x, v\rangle = x^\flat(v)$ for all $v \in \mathbb{R}^n$. We think the slight abuse of notation to write Ax instead of $(Ax)^\flat$ is acceptable as long as we are working with vector spaces. At this stage, we don't find it useful to introduce a peculiar notation for this. We will encounter more transformations of this form in chapter III.

2.1.2 The Short-Time Fourier Transform

It has been shown by Rodino and Wahlberg [RW12] that an equivalent definition of the wave front set can be stated in terms of the Short-Time Fourier Transform (STFT). The STFT is a central notion in time-frequency analysis, and for its detailed treatment, we refer to [Gro01].

Definition 2.1.14. Let $\psi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ be a window function, $z = (x, \xi) \in T^*\mathbb{R}^n$ and Π the phase-space shift introduced in Example 1.1.10. The Short-Time Fourier Transform (STFT) of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ is defined by the pairing

$$\mathcal{V}_\psi(u)(z) = (u, \overline{\Pi(z)\psi}).$$

Sometimes, we will use the usual abuse of notation

$$\mathcal{V}_\psi(u)(x, \xi) = \int_{\mathbb{R}^n} u(y)\overline{\psi(y-x)}e^{-i\xi\cdot y}\, dy.$$

Proposition 2.1.15. *For a given window function ψ, the STFT defines a continuous map*

$$\mathcal{V}_\psi : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{2n}),$$

and for any $u \in \mathcal{S}'(\mathbb{R}^n)$, the map

$$z \mapsto \mathcal{V}_\psi(u)(z)$$

is continuous and polynomially bounded.

Proof. See [Gro01, Thm. 11.2.3]. □

Thanks to Proposition 2.1.15, it makes sense to consider the cones in which the STFT of a tempered distribution decreases faster than any polynomial. We shall say that the directions included in such cones are not in the wave front set of u.

Definition 2.1.16. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. The pair $(x_0, \xi_0) \in T^*(\mathbb{R}^n) \setminus \{(0,0)\}$ is not in $\mathrm{WF}'(u)$ if there exists an open conic set $\Gamma_{(x_0,\xi_0)} \subseteq T^*(\mathbb{R}^n) \setminus \{(0,0)\}$ containing (x_0, ξ_0) such that

$$\sup \langle (x, \xi) \rangle^N |\mathcal{V}_\psi u(x, \xi)| < \infty \tag{II.2}$$

in $\Gamma_{(x_0,\xi_0)}$ for all $N \geq 0$.

Remark 2.1.17. It is important to note that the condition of rapid decay of the STFT in an open cone containing a given nonzero vector does not depend on the choice of the window function $\psi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ (see [RW12, Cor. 2.3]). In particular, WF' does not depend on the choice of the window function.

Theorem 2.1.18. *Definition 2.1.16 and Definition 2.1.4 are equivalent:*

$$\mathrm{WF} = \mathrm{WF}'.$$

Proof. See [RW12, Thm. 3.1 and 3.2]. □

The following lemma shows that Definition 2.1.16 can be equivalently stated using the integral estimate

$$\int \langle (x, \xi) \rangle^N |\mathcal{V}_\psi(x, \xi)| \, dx \, d\xi < \infty.$$

Lemma 2.1.19. *Let X be an open set in $\mathbb{R}^n$ and f a continuous function. Then*

$$\sup_{x \in X} \langle x \rangle^N |f(x)| < \infty \quad \textit{for} \quad N = 1, 2, \ldots \tag{II.3}$$

if and only if

$$\int_X \langle x \rangle^N |f(x)| \, dx < \infty \quad \textit{for} \quad N = 1, 2, \ldots \tag{II.4}$$

Proof. Suppose (II.3) holds. Then

$$\begin{aligned} \int_X \langle x \rangle^N |f(x)| \, dx &= \int_X \langle x \rangle^{-n-1} \langle x \rangle^{N+n+1} |f(x)| \, dx \\ &\leq C_{N,n} \int_X \langle x \rangle^{-n-1} \, dx < \infty. \end{aligned}$$

Reciprocally, if (II.4) holds, then $\lim_{x \to \infty} \langle x \rangle^N |f(x)| = 0$, which implies $|f(x)| < C_N \langle x \rangle^{-N}$ for all $N > 0$. □

The definition of the smooth wave front set in terms of the STFT allows us to easily prove some properties of WF. For example, consider the following proposition.

Proposition 2.1.20. *If $u \in \mathcal{S}'(\mathbb{R}^n)$, $v \in \mathcal{S}'(\mathbb{R}^m)$, then*

$$\mathrm{WF}(u \otimes v) \subseteq (\mathrm{WF}(u) \cup \{0\}) \times (\mathrm{WF}(v) \cup \{0\}).$$

A laborious proof of this result, based on the calculus of pseudo-differential operators, is given in [Hö89, Prop. 2.8]. Now, as observed in [Sch14], it is a direct consequence of the fact that

$$\mathcal{V}_{\psi_1 \otimes \psi_2}(u_1 \otimes u_2) = \mathcal{V}_{\psi_1}(u_1) \otimes \mathcal{V}_{\psi_2}(u_2).$$

The invariance of WF under time-frequency shifts also becomes obvious.

Proposition 2.1.21. *The smooth wave front set is invariant under time-frequency shifts. For $u \in \mathcal{S}'(\mathbb{R}^n)$ and $z = (w, \eta) \in \mathbb{R}^{2n}$,*

$$\mathrm{WF}(u) = \mathrm{WF}(\Pi(z)u).$$

Proof. One computes

$$\mathcal{V}_\psi(\Pi_z u)(x, \xi) = e^{i(\eta-\xi)\cdot w}\mathcal{V}_\psi(u)(x-w, \xi-\eta).$$

Moreover, if Γ is an open cone and $(x, \xi) \in \Gamma$, then $(tx - w, t\xi - \eta)$ is also in Γ for t sufficiently large. Hence, condition (II.2) is satisfied in Γ for u if and only if it is also satisfied for $\Pi_z u$. $\square$

2.1.3 Unitary representations of the Heisenberg group

By looking more closely at the definition of the smooth wave front set, hidden mathematical structure can be revealed. Indeed, there is a strong connection between the smooth wave front set and unitary representations of the Heisenberg group.

Representation theory of the Heisenberg group In this paragraph, we define the Heisenberg group, describe its irreducible unitary representations, and recall some notions from representation theory. For the basic concepts in representation theory, we refer to [Kir04].

Let V be a real symplectic vector space of dimension $2n$, and Z a direction in V. The Heisenberg group $\mathbb{H}_n$ is defined as the product $V \times \mathbb{R}$ endowed with the group action

$$(v,t)\cdot(w,s) = \left(v+w, t+s+\frac{1}{2}\omega(v,w)\right).$$

As such, it is a central extension of the additive group $(V,+)$:

$$0 \to \mathbb{R} \to V \times \mathbb{R} \to V \to 0.$$

It admits an obvious Lie group structure, and its Lie algebra $\mathfrak{h}_n$ is the vector space $V \oplus \mathbb{R}Z$ endowed with the Lie bracket

$$[(v,s),(w,t)] = \omega(v,w)Z.$$

Definition 2.1.22. Let F be any subspace of a symplectic vector space (V,ω). We define the ω-perpendicular space F^ω as

$$F^\omega = \{v \in V : \omega(v,w) = 0 \quad \forall w \in F\}.$$

The subspace F is called

- *isotropic* if $F \subseteq F^\omega$,
- *Lagrangian* if $F = F^\omega$,
- *symplectic* if $F \cap F^\omega = \{0\}$.

Remark 2.1.23. Any symplectic vector space (V,ω) can be decomposed into a direct sum of Lagrangian subspaces $V = Q \oplus P$. Then, any element $h \in \mathbb{H}_n$ can be written as $h = (q+p,z)$ with $q \in Q$ and $p \in P$.

Definition 2.1.24. Let V be a finite-dimensional vector space, $\mathcal{H}$ a complex Hilbert space, and G a Lie group.

1. A representation (ρ, V) of G on V is a group homomorphism

$$\rho\colon G \to \mathrm{GL}(V).$$

A representation is called *irreducible* if there is no non-trivial linear subspace $W \subset V$ such that the restriction $\rho|_W$ of ρ to W defines a representation $(\rho|_W, W)$ of G on W.

2. A unitary operator on $\mathcal{H}$ is a bounded linear operator $U\colon \mathcal{H} \to \mathcal{H}$ that satisfies $U^*U = UU^* = I_{\mathcal{H}}$, where U^* is the adjoint of U with respect to the inner product on $\mathcal{H}$, and $I_{\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}$ is the identity operator on $\mathcal{H}$. We denote by $U(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$.

3. A unitary representation of G on $\mathcal{H}$ is a group homomorphism

$$\rho\colon G \to U(\mathcal{H})$$

such that for each fixed $v \in V$, the map $g \mapsto \rho(g)v$ is continuous from G to $\mathcal{H}$. The notion of an irreducible unitary representation is identical to the finite-dimensional case.

4. For any $h_1, h_2 \in \mathcal{H}$, the function

$$g \in G \mapsto \langle \rho(g)h_1, h_2\rangle \in \mathbb{C}$$

is called a matrix coefficient of the representation ρ.

5. An irreducible unitary representation $(\rho, \mathcal{H})$ of G is called square integrable if for all non-zero vectors h_1 and h_2 in $\mathcal{H}$ the following integral converge

$$\int_G |\langle \rho(g)h_1, h_2\rangle|^2 \, \mathrm{d}g < \infty,$$

where $\mathrm{d}g$ is the Haar measure on G.

Theorem 2.1.25. *Let $V = Q \oplus P$ be a decomposition of V into Lagrangian subspaces. All irreducible unitary representations of $\mathbb{H}_n$ are equivalent to one of the following representations, where μ is any real number and v any vector in V:*

$$U_\mu\colon \mathbb{H}_n \to U(L^2(Q)),$$
$$U_\mu(q+p, z)\psi(q_0) = e^{i\mu(z+\omega(q-2q_0,\, p))}\psi(q_0 - q),$$

$$\rho_v\colon \mathbb{H}_n \to \mathbb{C}, \quad \rho_v(w, z)\psi(q_0) = e^{i\omega(v,\, w)}\psi(q_0).$$

Proof. We show this by applying the *orbit method* to the Heisenberg group (see appendix B). The explicit computations are presented in section B.3. □

Notation 2.1.26. We introduce the Dirac "bra-ket" notation for states $\psi \in L^2(Q)$:

$$|\psi\rangle \coloneqq \psi.$$

The inner product on $L^2(Q)$ then reads

$$\langle\psi|\psi'\rangle \coloneqq \int_Q \overline{\psi(q)}\psi'(q)\,\mathrm{d}q,$$

and the representation is

$$U(g)\,|\psi\rangle \coloneqq [q \mapsto U(g)\psi(q)].$$

Finally, for a given "ket" $|\psi\rangle$, its dual $\langle\psi|$ ("bra") is defined by

$$\langle\psi|\colon L^2(Q) \to \mathbb{C}, \quad |\psi'\rangle \mapsto \langle\psi|\psi'\rangle.$$

For example, if $|\psi\rangle$ is normalized, the orthogonal projection onto the line directed by $|\psi\rangle$ is given by

$$\begin{aligned}&|\psi\rangle\,\langle\psi|\colon L^2(Q) \to L^2(Q),\\ &|\psi'\rangle \mapsto |\psi\rangle\,\langle\psi|\psi'\rangle \coloneqq \langle\psi|\psi'\rangle\,|\psi\rangle\,.\end{aligned}$$

Proposition 2.1.27. *If $(\mathcal{H}, U)$ is a square-integrable irreducible unitary representation of a Lie group G, then, for all $|\psi\rangle$ different from zero,*

$$\int_G |U(g)\psi\rangle\,\langle U(g)\psi|\,\mathrm{d}g = I_{\mathcal{H}}.$$

This equality is known as the resolution of the identity*.*

Remark 2.1.28. The unitary representations of the Heisenberg group are not square-integrable on $\mathbb{H}_n = V \times \mathbb{R}$. However, if we restrict the representation U_μ from Theorem 2.1.25 to the vector space V, then for any elements h_1, h_2 of $L^2(Q)$ such that

$$\int_V |\langle U_\mu(g)h_1, h_2\rangle|^2\,\mathrm{d}g < \infty,$$

and the integral

$$\int_V |U_\mu(g)\psi\rangle\,\langle U_\mu(g)\psi|\,\mathrm{d}g$$

yields a resolution of the identity which corresponds (up to a phase factor) with the inversion formula of the STFT (see [Gro01, Cor. 3.2.3]).

A new definition for the smooth wave front set In this paragraph, we show that the matrix coefficients of the representations U_μ introduced in Theorem 2.1.25 provide a new definition of the smooth wave front set, equivalent to the two definitions given in Section 2.1.1 and Section 2.1.2. Recall that the smooth wave front set is an object defined on $V \coloneqq T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$, which has a natural structure of a symplectic vector space given by the two-form:

$$\omega\left((x,\xi),(y,\eta)\right) \coloneqq \langle \eta, x\rangle - \langle \xi, y\rangle.$$

Obviously, the sets $\{(x,0)\}$ and $\{(0,\xi)\}$ are Lagrangian subsets of (V,ω). If we restrict the unitary representation of the Heisenberg group $\mathbb{H}_n$ to $V = T^*\mathbb{R}^n$, i.e., by setting $z = 0$ and $q + p = (x,0) + (0,\xi)$ in Theorem 2.1.25, we obtain

$$U_\mu(x,\xi)\psi(q) = e^{\frac{i\mu}{2}(2q-x)\xi}\psi(q-x).$$

We propose the following definition for the smooth wave front set, formulated in terms of representation theory concepts.

Definition 2.1.29. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ be a window function. We say that $(x_0,\xi_0) \in T^*(\mathbb{R}^n) \setminus \{(0{,}0)\}$ is not in $\widetilde{\mathrm{WF}}(u)$ if there exists an open conic set $\Gamma_{(x_0,\xi_0)} \subseteq T^*(\mathbb{R}^n) \setminus \{(0,0)\}$ containing (x_0,ξ_0) such that

$$\sup \langle (x,\xi)\rangle^N \left|\langle U_\mu(x,\xi)\psi | u\rangle\right|$$

is bounded on $\Gamma_{(x_0,\,\xi_0)}$ for all $N \geq 0$.

Proposition 2.1.30. *Definition 2.1.29 is equivalent to Definition 2.1.16 and Definition 2.1.4:*

$$\widetilde{\mathrm{WF}} = \mathrm{WF} = \mathrm{WF}'.$$

Proof. Using Notation 2.1.26, we compute

$$\begin{aligned}
\langle U_\mu(x,\xi)\psi | f\rangle &= \int \overline{U_\mu(x,\xi)\psi(y)} f(y)\,\mathrm{d}y \\
&= \int e^{-i\frac{\mu}{2}(2y-x)\xi}\,\overline{\psi(y-x)} f(y)\,\mathrm{d}y \\
&= \|\psi\|_2 (2\pi)^{n/2} e^{i\frac{\mu}{2}x\xi}\,\mathcal{V}_\psi f(x,\xi).
\end{aligned}$$

Since $\left|e^{i\frac{\mu}{2}x\xi}\right| = 1$, this concludes the proof. $\square$

2.1.4 Examples

The goal of this section is to build an intuitive interpretation of the smooth wave front set of a distribution through a series of chosen examples that we find enlightening.

Example 2.1.31 (1 and δ)**.** We begin with two contrasting examples that will be crucial in building our intuition for the smooth wave front set.

1. *The wave front set of* 1*.* We compute $|\mathcal{V}_\psi 1(x,\xi)| = |\hat{\psi}(\xi)|$. Since $\hat{\psi}$ is Schwartz, $|\mathcal{V}_\psi 1(x,\xi)|$ decays rapidly in every direction (x,ξ) as long as $\xi \neq 0$. Conversely, in the cone $\mathbb{R}^n \times \{0\}$, $|\mathcal{V}_\psi 1(x,\xi)| = |\hat{\psi}(0)|$ is constant. We conclude that

$$\mathrm{WF}(1) = \mathbb{R}^n \setminus \{0\} \times \{0\}.$$

2. *The wave front set of the delta distribution δ.* We compute

$$|\mathcal{V}_\psi \delta(x,\xi)| = |\psi(-x)|.$$

For the same reason as above, this implies

$$\mathrm{WF}(\delta) = \{0\} \times \mathbb{R}^n \setminus \{0\}.$$

Note that Proposition 2.1.21 implies $\mathrm{WF}(e^{ix\xi_0}) = \mathrm{WF}(1)$ for any $\xi_0 \in \mathbb{R}^n$ and $\mathrm{WF}(\delta_{x_0}) = \mathrm{WF}(\delta)$ for any $x_0 \in \mathbb{R}^n$. We observe that singularities at finite points seem to be encoded in $\{0\} \times (\mathbb{R}^n \setminus \{0\})$, while lack of decay at infinity is detected by directions in $(\mathbb{R}^n \setminus \{0\}) \times \{0\}$.

Example 2.1.32 (The Heaviside function)**.** Based on our observations in the previous example, it is natural to wonder what happens in the case of a distribution exhibiting singularities at finite points *and* lacking decay at infinity. The Heaviside function, defined by

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

is a good example. It presents a singularity at 0 and remains constant at 1 as $x \to +\infty$. We compute

$$|\mathcal{V}_\psi H(x,\xi)| = \left| \int_{-x}^{+\infty} e^{-iy\xi} \psi(y) \, \mathrm{d}y \right|.$$

We study the behaviour of $\mathcal{V}_\psi H$ in a direction (x, ξ) by considering the function

$$t \mapsto \left| \int_{-tx}^{+\infty} e^{-iyt\xi} \psi(y) \, \mathrm{d}y \right| . \tag{II.5}$$

We consider five scenarios.

- $x = 0$, $\xi \neq 0$. Without loss of generality (see Remark 2.1.17), we assume $\psi(x) = e^{-x^2}$. Observe that

  $$|\mathcal{V}_\psi H(0, t\xi)| = \left| \frac{1}{2} \sqrt{\pi} e^{-\frac{t\xi^2}{4}} + iF\left(\frac{t\xi}{2}\right) \right| ,$$

  where F is the *Dawson function*, which behaves like $\frac{1}{2x}$ at infinity. Hence, $\mathcal{V}_\psi H$ is not rapidly decreasing in directions $x = 0$, $\xi \neq 0$.

- $x > 0$, $\xi = 0$. As t tends to ∞, the function (II.5) tends to

  $$\left| \int_{-\infty}^{+\infty} \psi(y) \, \mathrm{d}y \right| ,$$

  which is a strictly positive constant. Hence, $\mathcal{V}_\psi H$ is not rapidly decreasing in directions $x > 0$, $\xi = 0$.

- $x > 0$, $\xi \neq 0$. As t tends to ∞, the function (II.5) behaves as $|\hat{\psi}(t\xi)|$, which is rapidly decreasing since $\hat{\psi} \in \mathcal{S}$.

- $x < 0$, $\xi \neq 0$. We observe

  $$|\mathcal{V}_\psi H(tx, t\xi)| = \left| \int_{-tx}^{+\infty} e^{-ity\xi} \psi(y) \, \mathrm{d}y \right| \leq \int_{-tx}^{+\infty} |\psi(y)| \, \mathrm{d}y.$$

  Hence, the limit (II.5) tends to 0 faster than any polynomial since ψ is Schwartz.

- $x < 0$, $\xi = 0$. The function (II.5) tends to 0 faster than any polynomial since ψ is Schwartz.

Finally, we conclude that

$$\mathrm{WF}(H) = \{0\} \times \mathbb{R}^n \setminus \{0\} \cup \mathbb{R}^+ \setminus \{0\} \times \{0\}.$$

This matches the observations from the previous example: the smooth wave front set detects the singularity at 0 through the component $\{0\} \times \mathbb{R}^n \setminus \{0\}$ and the lack of decay as $x \to +\infty$ through $\mathbb{R}^+ \setminus \{0\} \times \{0\}$.

In the following lemma, we formalize some of the intuitions developed in the previous examples.

Lemma 2.1.33. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$.*

1. *$u \in \mathcal{S}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) \iff WF(u) \cap (\{0\} \times \mathbb{R}^n \setminus \{0\}) = \emptyset$;*
2. *If there exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ such that $u = v\phi$ (in particular, if u is compactly supported), then $WF(u) \cap (\mathbb{R}^n \setminus \{0\} \times \{0\}) = \emptyset$.*

Proof. Let ψ be a smooth, compactly supported function. We consider the symbols $1 \otimes \psi$ and $\psi \otimes 1$ in G^0, such that:

$$\begin{aligned}\mathrm{ell}(1 \otimes \psi) &= \mathrm{conesupp}(1 \otimes \psi) = \mathbb{R}^n \times \{0\},\\ \mathrm{ell}(\psi \otimes 1) &= \mathrm{conesupp}(\psi \otimes 1) = \{0\} \times \mathbb{R}^n,\end{aligned}$$

and

$$\begin{aligned}(\{0\} \times \mathbb{R}^n \setminus \{0\}) \cap \mathrm{conesupp}(1 \otimes \psi) = \emptyset,\\ (\mathbb{R}^n \setminus \{0\} \times \{0\}) \cap \mathrm{conesupp}(\psi \otimes 1) = \emptyset.\end{aligned}$$

1. Suppose $u \in \mathcal{S}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ (i.e., $u \in C^\infty$ and is polynomially bounded). By applying Example 1.1.18, we compute

$$\begin{aligned}(\psi \otimes 1)^W u(x) &= \int e^{i(x-y)\xi} \psi\left(\frac{x+y}{2}\right) u(y)\,\mathrm{d}y\,\mathrm{d}\xi\\ &= \int \delta(x-y) \psi\left(\frac{x+y}{2}\right) u(y)\,\mathrm{d}y\,\mathrm{d}\xi\\ &= \psi(x)u(x),\end{aligned}$$

which is clearly a Schwartz function. Since $\mathrm{ell}(\psi \otimes 1) = \{0\} \times \mathbb{R}^n$, we conclude that

$$\mathrm{WF}(u) \cap (\{0\} \times \mathbb{R}^n) = \emptyset.$$

Reciprocally, if $\mathrm{WF}(u) \cap (\{0\} \times \mathbb{R}^n \setminus \{0\}) = \emptyset$, then $(\psi \otimes 1)^W u(x) = \psi(x)u(x) \in \mathcal{S}(\mathbb{R}^n)$. This implies that u is smooth on $\mathrm{supp}(\psi)$. Since ψ is arbitrary, u is smooth on every compact subset of $\mathbb{R}^n$.

2. We compute

$$(1 \otimes \psi)^W (v\phi) = \int e^{i(x-y)\xi}\psi(\xi)v(y)\phi(y)\,\mathrm{d}y\,\mathrm{d}\xi$$
$$= \int \hat{\psi}(y-x)v(y)\phi(y)\,\mathrm{d}y,$$

which defines a Schwartz function since $\hat{\psi} \in \mathcal{S}(\mathbb{R}^n)$. Since $\mathrm{ell}(1 \otimes \psi) = \mathbb{R}^n \times \{0\}$, we conclude that

$$\mathrm{WF}(v\phi) \cap (\mathbb{R}^n \setminus \{0\} \times \{0\}) = \emptyset.$$

□

This lemma justifies the decomposition of $\mathrm{WF}(u)$ into three parts:

$$\mathrm{WF}(u) = \mathrm{WF}_x \cup \mathrm{WF}_\xi \cup \mathrm{WF}_{x\xi},$$

where $\mathrm{WF}_x \subseteq \mathbb{R}^n \setminus \{0\} \times \{0\}$, $\mathrm{WF}_\xi \subseteq \{0\} \times \mathbb{R}^n \setminus \{0\}$, and $\mathrm{WF}_{x\xi} \subseteq \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^n \setminus \{0\}$. Each of these subsets can be interpreted as follows:

- directions in WF_x represent lack of decay at infinity;
- directions in WF_ξ represent lack of smoothness at finite points;
- directions in $\mathrm{WF}_{x\xi}$ represent rays of singularities (see point 2 in Example 2.1.35), high-order oscillations (see Example 2.1.34), or superpositions of WF_x and WF_ξ.

Note that if $\mathrm{WF}_\xi(u) = \emptyset$, then u is necessarily smooth. However, if $\mathrm{WF}_x(u) = \emptyset$, we cannot conclude that u decreases faster than any polynomial at infinity, as illustrated in Example 2.1.34.

Example 2.1.34 (Quadratic phases)**.** Besides singularities at finite points in $\mathbb{R}^n \setminus \{0\} \times \{0\}$ and lack of decay at infinity in $\{0\} \times \mathbb{R}^n \setminus \{0\}$, the smooth wave front set will also detect high-order oscillations through components in the rest of $\mathbb{R}^{2n} \setminus \{0\}$. Let's consider e^{icx^2}, which exhibits quadratic oscillations. As a direct consequence of Corollary 2.1.12 and Example 2.1.31, we have

$$\mathrm{WF}(e^{icx^2}) = \{(x, cx) : x \neq 0\}.$$

We emphasize that $\mathrm{WF}_x(e^{icx^2}) = \emptyset$, but e^{icx^2} is not rapidly decreasing at infinity.

Using Proposition 2.1.20, we compute

$$\begin{aligned}\mathrm{WF}(e^{icx^2} \otimes 1) &= (\{(x, cx) : x \in \mathbb{R}\} \times (\mathbb{R} \times \{0\})) \setminus \\ &= \{(x_1, x_2; cx_1, 0) : x_1, x_2 \in \mathbb{R}\} \setminus .\end{aligned}$$

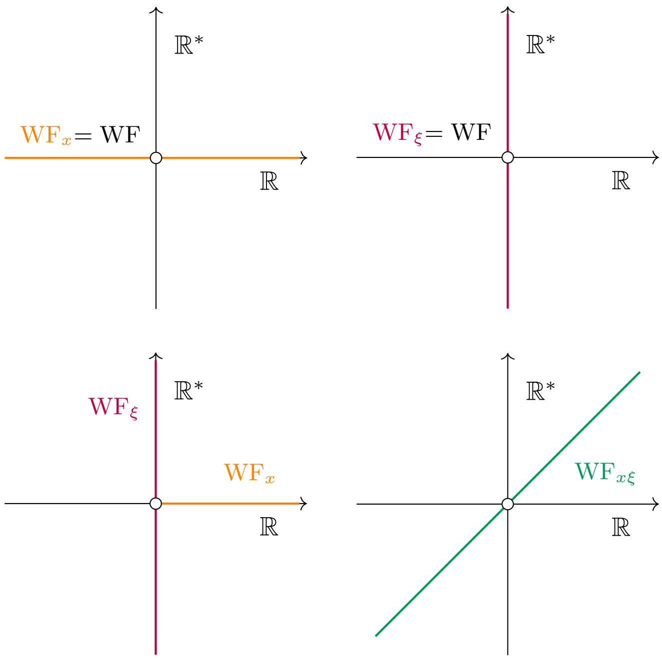


Figure II.3: In one dimension, representing the wavefront set of a tempered distribution is straightforward. Indeed, since $T^*\mathbb{R} \simeq \mathbb{R}^2$, we can draw WF in the two dimensional plan. Above, we illustrated, from left to right, the wave front sets of 1, δ, H and e^{ix^2}. The components WF_x, WF_ξ and $\mathrm{WF}_{x\xi}$ are respectively depicted in orange, red, and green.

Example 2.1.35 (Characteristic functions)**.** Characteristic functions provide visual examples that combine singularities with a lack of decay at infinity.

1. Let $\mathbb{D} := \{(x, y) : x^2 + y^2 \leq 1\}$ be the closed disc and $\chi_{\mathbb{D}}$ its characteristic function. Then we have

$$\mathrm{WF}(\chi_{\mathbb{D}}) = \{0\} \times \mathbb{R}^2 \setminus \{0\},$$

which is represented in Figure II.4c. The red arrows should be understood as elements of $(\mathbb{R}^2)^*$.

2. Let $\mathbb{H} = \{(x, y) : y > 0\}$ be the half-plane and $\chi_{\mathbb{H}}$ its characteristic function. In this case, the singularities of $\chi_{\mathbb{H}}$ exclusively arise in the y-direction. Consequently, these singularities will appear in $\mathrm{WF}(\chi_{\mathbb{H}})$ not as the whole of $\{0\} \times (\mathbb{R}^2 \backslash \{0\})$, but rather as $\{0\} \times (\{0\} \times \mathbb{R}\backslash\{0\})$. Proposition 2.1.20 enables us to verify our intuition. Since $\chi_{\mathbb{H}} = 1 \otimes H$, we have

$$\begin{aligned}\mathrm{WF}(\chi_{\mathbb{H}}) &= (\mathrm{WF}(1) \cup \{0\}) \times (\mathrm{WF}(H) \cup \{0\}) \\ &= \{(x_1, 0\,;\, 0{,}\xi_2) : \xi_2 \neq 0\} \cup \{(x_1, x_2\,;\, 0, 0) : x_2 > 0\},\end{aligned}$$

as depicted in Figure II.4b.

3. Let $\chi_{\mathbb{R}^2\backslash\mathbb{D}}$ be the characteristic function of the set $\mathbb{R}^2 \setminus \mathbb{D}$. The singularity at finite points yields a component $\{0\} \times \mathbb{R}^n \backslash \{0\}$, while lack of decay in every direction implies a component $\mathbb{R}^n \backslash \{0\} \times \{0\}$. Thus, we have

$$\mathrm{WF}(\chi_{\mathbb{R}^2\backslash\mathbb{D}}) = (\{0\} \times \mathbb{R}^n \backslash \{0\}) \cup (\mathbb{R}^n \backslash \{0\} \times \{0\})\,.$$

2.2 Topological Algebras of Distributions

It is well known that the pointwise product of functions cannot be nicely extended to distributions. This is known as the *Schwartz impossibility theorem* [Sch54a]. However, the pointwise product can be suitably extended to certain subspaces of $\mathcal{S}'(\mathbb{R}^n)$ by means of the smooth wave front set.

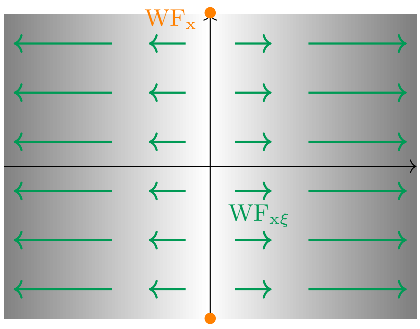
For any closed cone $\Gamma \subset T^*\mathbb{R}^n \setminus \{0\}$, we introduce the space $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ of tempered distributions whose wave front set is contained in Γ:

$$\mathcal{S}'_\Gamma(\mathbb{R}^n) \coloneqq \big\{u \in \mathcal{S}'(\mathbb{R}^n) : \mathrm{WF}(u) \subseteq \Gamma\big\}\,.$$

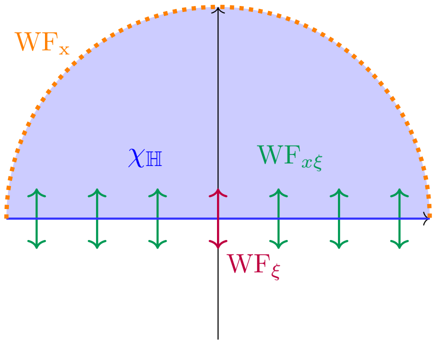
Note that $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ is a linear subspace of $\mathcal{S}'(\mathbb{R}^n)$. Our claim is the following: for a suitable choice of Γ and an appropriate topology, the usual pointwise product of functions on $\mathcal{S}(\mathbb{R}^n)$ extends continuously to a mapping

$$\mathcal{S}'_\Gamma(\mathbb{R}^n) \times \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_\Gamma(\mathbb{R}^n)$$

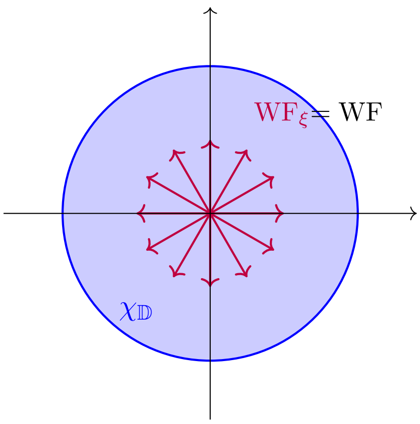
endowing $S'_\Gamma(\mathbb{R}^n)$ with a structure of topological algebra.



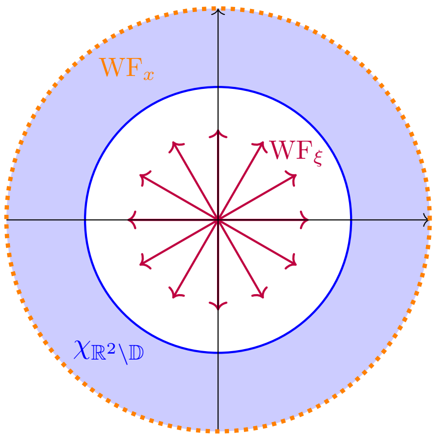
(a) Representation of $\mathrm{WF}(e^{icx^2} \otimes 1)$.



(b) Representation of $\mathrm{WF}(\chi_{\mathbb{H}})$.



(c) Representation of $\mathrm{WF}(\chi_{\mathbb{D}})$.



(d) Representation of $\mathrm{WF}(\chi_{\mathbb{R}^2\setminus\mathbb{D}})$.

Figure II.4: We propose a visual representation of the smooth wave front set for several examples in $\mathbb{R}^2$ discussed earlier. Since the wave front set is a four-dimensional object, its depiction requires a certain degree of creativity and imagination. In Figure II.4a, the gray gradient illustrates the increasing oscillation rate as x grows. In Figures II.4b to II.4d, the blue regions indicate where the functions take the constant value 1, while the darker blue lines mark the singularities, where the function 'jumps' to 0. A direction $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\} \subset T^*\mathbb{R}^2$ in the wave front set is depicted by attaching an arrow, representing the vector y, to the point $x \in \mathbb{R}^2$, where y is viewed as an element of $(\mathbb{R}^2)^*$. Red arrows from the origin represent elements of WF_ξ, while green arrows depict elements of $\mathrm{WF}_{x\xi}$. The directions $(x{,}0) \in \mathrm{WF}$ are depicted as orange dots on the boundary, corresponding to directions in which the distribution does not decay rapidly.

In Section 2.2.1, we briefly review the existing results on the subject as exposed in [Hö89], [BS19], and [Sch14]. In short, these results yield a condition on Γ under which the pointwise product of two distributions $u, v \in \mathcal{S}'_\Gamma(\mathbb{R}^n)$ is well-defined, and such that $u \cdot v$ is again an element of $\mathcal{S}'_\Gamma(\mathbb{R}^n)$. They also define a notion of convergence for sequences in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ and prove the sequential continuity of the pointwise product from $\mathcal{S}'_\Gamma(\mathbb{R}^n) \times \mathcal{S}'_\Gamma(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ (i.e., the topology of the codomain is not the sequential topology of $\mathcal{S}'_\Gamma(\mathbb{R}^n)$, but rather the weak topology on $\mathcal{S}'(\mathbb{R}^n)$).

In Section 2.2.2, we present some topological refinements of these results. Specifically, we reformulate the notion of convergence in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ mentioned above as a locally convex topology, and we try to adapt the existing results in order to construct a sequentially continuous product

$$\mathcal{S}'_\Gamma(\mathbb{R}^n) \times \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_\Gamma(\mathbb{R}^n)$$

with respect to this locally convex topology on $\mathcal{S}'_\Gamma(\mathbb{R}^n)$.

2.2.1 Review of Existing Results

In [Hö89, p.120], the following notion of convergence for sequences in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ is proposed.

Definition 2.2.1. A sequence (u_n) in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ converges to 0 if

1. For every $\phi \in \mathcal{S}(\mathbb{R}^n)$, the numerical sequence $(|u_n(\phi)|)$ tends to zero (weak convergence);
2. For any $m \in \mathbb{R}$ and for every symbol $a \in G^m$ whose cone support does not intersect Γ, the sequence of Schwartz functions $(a^W u_n)$ tends to zero in the Schwartz space topology.

This notion of convergence is referred to as the *Hörmander topology* in the literature.

Note that the second condition makes sense thanks to Corollary 2.1.9. Any sequence will converge to at most one point in the Hörmander topology since the weak topology is Hausdorff on $\mathcal{S}'(\mathbb{R}^n)$ (see the corollary following Proposition 19.3 in [Tre67]). The following lemma is due to Hörmander [Hö89].

Lemma 2.2.2. *The space $\mathcal{S}(\mathbb{R}^n)$ is sequentially dense in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ in the Hörmander topology, i.e., for all $u \in \mathcal{S}'_\Gamma(\mathbb{R}^n)$, there exists a sequence $(\phi_n) \subset \mathcal{S}(\mathbb{R}^n)$ converging to u in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$.*

The following result characterizes the pullback of a tempered distribution by a linear map. It was demonstrated by Hörmander [Hö89, Prop. 2.9]. Observe, however, that in [Hö89], the contribution of the kernel of A in $\mathrm{WF}(A^*u)$ was accidentally forgotten.

Proposition 2.2.3. *Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map, and let Γ be a closed cone in $T^*\mathbb{R}^n$ such that*

$$\Gamma \cap \{(0,\xi) : A^T\xi = 0\} = \emptyset.$$

Then, the pullback A^* is uniquely defined as a sequentially continuous map

$$A^* \colon \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m)$$

which continuously extends the usual pullback on $\mathcal{S}(\mathbb{R}^n)$

$$A^* \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m), \quad A^*(\phi) = \phi \circ A.$$

Moreover, for every u such that

$$\mathrm{WF}(u) \cap \{(0,\xi) : A^T\xi = 0\} = \emptyset,$$

we have the following inclusion for the wave front set:

$$\begin{aligned} &\mathrm{WF}(A^*u) \subset A^*\mathrm{WF}(u) \\ &\quad := \{(x, A^T\xi) \in T^*\mathbb{R}^m \setminus \{(0,0)\} : (Ax,\xi) \in \mathrm{WF}(u)\} \cup \mathrm{Ker}A \times \{0\}. \end{aligned}$$

Remark 2.2.4. Note that we wrote $A^* \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m)$ because, for $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\phi \circ A$ is generally not in $\mathcal{S}(\mathbb{R}^m)$. However, if A is injective, we have $A^* \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^m)$. When we say that

$$A^* \colon \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m)$$

continuously extends the mapping

$$A^* \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m), \quad A^*(\phi) = \phi \circ A,$$

we mean explicitly that, for every $\phi \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'_\Gamma(\mathbb{R}^n)$, the distribution $A^*\phi$ should satisfy

$$(A^*\phi, \psi) = (\phi \circ A, \psi), \quad \text{for any } \psi \in \mathcal{S}(\mathbb{R}^m).$$

Remark 2.2.5. It is important to note that the continuity statement in Proposition 2.2.3 is not as strong as one might wish. Indeed, the pullback

$$A^* \colon \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_{A^*\Gamma} \subset \mathcal{S}'(\mathbb{R}^n)$$

is stated to be a sequentially continuous map when $\mathcal{S}'_{A^*\Gamma}$ is endowed with the topology induced by the weak topology on $\mathcal{S}'(\mathbb{R}^n)$, and not with the Hörmander topology of Definition 2.2.1. Explicitly, the result states that any sequence $(u_n) \subset \mathcal{S}'_\Gamma(\mathbb{R}^n)$ converging to $u \in \mathcal{S}'_\Gamma(\mathbb{R}^n)$ is mapped to a sequence A^*u_n converging to A^*u in $\mathcal{S}'(\mathbb{R}^n)$.

Example 2.2.6. We call the map diag the *diagonal map*, defined by

$$\mathrm{diag} \colon \mathbb{R}^n \to \mathbb{R}^{2n}, \quad x \mapsto (x, x).$$

Then, for any closed cone $\Gamma \subseteq T^*\mathbb{R}^{2n}$ such that

$$\Gamma \cap \{(0, 0, \xi_1, \xi_2) : \xi_1 + \xi_2 = 0\} = \emptyset, \tag{II.6}$$

the pullback by the diagonal map is a sequentially continuous map

$$\mathrm{diag}^* \colon \mathcal{S}'_\Gamma(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^n)$$

and, for any $u \in \mathcal{S}'_\Gamma(\mathbb{R}^{2n})$,

$$\begin{aligned} \mathrm{WF}(\mathrm{diag}^* u) &= \mathrm{diag}^* \mathrm{WF}(u) \\ &= \{(x, \xi_1 + \xi_2) \in T^*\mathbb{R}^n \setminus \{(0,0)\} : (x, x, \xi_1, \xi_2) \in \mathrm{WF}(u)\}. \end{aligned}$$

This example is not without significance. Indeed, if f and g are continuous functions, then their pointwise product is given by

$$fg = \mathrm{diag}^*(f \otimes g).$$

Hence, combining Proposition 2.1.20, Proposition 2.2.3, and Example 2.2.6, we can extend the pointwise product of functions to tempered distributions as $uv = \mathrm{diag}^*(u \otimes v)$, provided some conditions on the wave front sets of $u, v \in \mathcal{S}'(\mathbb{R}^n)$.

Theorem 2.2.7. *Let $u, v \in \mathcal{S}'$ be such that $(0,\xi) \in \mathrm{WF}(u)$ implies that $(0,-\xi) \notin \mathrm{WF}(v)$. Then, the pointwise product of u and v is uniquely defined as $\mathrm{diag}^*(u \otimes v)$ and is again tempered. Moreover, we have the following inclusion for the wave front set of uv:*

$$\begin{aligned} \mathrm{WF}(uv) \subset \{(x, \xi+\eta) : (x,\xi) \in \mathrm{WF}(u) \cup \{0\}, \\ (x,\eta) \in \mathrm{WF}(v) \cup \{0\}\} \setminus \{0\}. \end{aligned}$$

Bahns and Schulz have observed that to construct an algebra of tempered distributions under the pointwise product, it is enough to find a Γ such that, if u and v are in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ and satisfy the condition of Theorem 2.2.7, then the inclusion $\mathrm{WF}(uv) \subseteq \Gamma$ holds [BS19, Sch14]. They stated the following theorem, which is straightforward in view of our preceding results.

Theorem 2.2.8. *Let Γ_1 and Γ_2 be closed cones in $\mathbb{R}^n$ such that Γ_2 is stable under addition and does not contain 0. Then the set $\mathcal{S}'_{\Gamma_1\times\Gamma_2}(\mathbb{R}^n)$ is a commutative and associative algebra under the pointwise product.*

In [BS19, Thm. 3.1], Bahns and Schulz even state that, under the conditions of Theorem 2.2.8, the pointwise product yields a sequentially continuous map

$$\mathcal{S}'_{\Gamma_1\times\Gamma_2} \times \mathcal{S}'_{\Gamma_1\times\Gamma_2} \to \mathcal{S}'_{\Gamma_1\times\Gamma_2}$$

with respect to the Hörmander topology. In our understanding, however, this statement rests on Proposition 2.2.3 and Proposition 1.1.26, which only ensures sequential continuity in $\mathcal{S}'(\mathbb{R}^n)$ (see Remark 2.2.5), and no extra argument is given to take into account the topology of $\mathcal{S}'_\Gamma(\mathbb{R}^n)$. We address this issue in Section 2.2.2.

2.2.2 Topological refinements

In this subsection, we address the topological issues raised in Remark 2.2.5 and in the brief comment following Theorem 2.2.8. First, we reformulate the Hörmander topology (Definition 2.2.1) as a sequentially complete, locally convex topology on $\mathcal{S}'_\Gamma(\mathbb{R}^n)$. To do so, we begin with the following lemma.

Lemma 2.2.9. *Let* (u_n) *be a sequence in* $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ *and* u *be a tempered distribution in* $\mathcal{S}'(\mathbb{R}^n)$ *such that*

1. *for all* $\phi \in \mathcal{S}(\mathbb{R}^n)$, $|u_n(\phi) - u(\phi)| \xrightarrow{\mathbb{C}} 0$*;*
2. *for any* $m \in \mathbb{R}$ *and for every symbol* $a \in G^m$ *whose cone support does not intersect* Γ*,*

$$\sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^\alpha a^W (u_n - u)(x) \right| \xrightarrow{\mathbb{C}} 0.$$

Then u *is in* $\mathcal{S}'_\Gamma(\mathbb{R}^n)$*.*

Proof. By contradiction, suppose there exists $z \in \mathrm{WF}(u)$ such that $z \notin \Gamma$. To get a contradiction, it is sufficient to construct a symbol a such that $\mathrm{conesupp}(a) \cap \Gamma = \emptyset$ and $z \notin \mathrm{char}(a)$. Indeed, the second condition is equivalent to the convergence of the sequence $(a^W u_n)$ to $a^W u$ in the Schwartz topology. Since the Schwartz space is a Fréchet space, it follows that $a^W u \in \mathcal{S}(\mathbb{R}^n)$. But, by the negative characterization of the wave front set (see Corollary 2.1.5), this in turn implies that $z \notin \mathrm{WF}(u)$, which yields a contradiction.

Such a symbol is constructed as follows. Since we suppose $z \notin \Gamma$ and since Γ is a closed cone, one can find two open cones Γ'' and Γ' such that $z \in \Gamma'' \subset \Gamma'$ and $\Gamma' \cap \Gamma = \emptyset$. We then build a symbol $a \in G^0$ such that $a \equiv 1$ in $\Gamma'' \backslash B^{2n}$ and $\mathrm{supp}(a) \subset \Gamma'$ (similar to the one depicted in Figure II.1). This symbol satisfies $\mathrm{conesupp}(a) \cap \Gamma = \emptyset$ and, on the other hand, $z \notin \mathrm{char}(a)$ since $z \in \Gamma''$ and $a \equiv 1$ in Γ''. The symbol a satisfies the required properties, and the proof is complete. □

Motivated by the above lemma, we introduce the following locally convex topology on $\mathcal{S}'_\Gamma(\mathbb{R}^n)$.

Definition 2.2.10. We introduce the following semi-norms on $\mathcal{S}'_\Gamma(\mathbb{R}^n)$:

1. $|u|_\phi := |(u, \phi)|$, where $\phi \in \mathcal{S}(\mathbb{R}^n)$;
2. $\|u\|^\Gamma_{a,\alpha,\beta} := \|a^W u\|_{\alpha,\beta}$, where a is a symbol whose cone support does not intersect Γ, and $\|\cdot\|_{\alpha,\beta}$ is a Schwartz semi-norm defined in (I.1) (which makes sense thanks to Corollary 2.1.9).

We denote by $\mathcal{T}_\Gamma$ the locally convex topology induced by these semi-norms and by G^Γ the set of symbols that define a semi-norm on $\mathcal{S}'_\Gamma(\mathbb{R}^n)$, i.e.

$$G^\Gamma = \{a \in G^\infty : \text{conesupp}(a) \cap \Gamma = \emptyset\} .$$

We call $\mathcal{T}_\Gamma$ the Γ-*topology* on $\mathcal{S}'_\Gamma(\mathbb{R}^n)$.

The topology $\mathcal{T}_\Gamma$ is of course equivalent to Hörmander topology (see Definition 2.2.1) in the sense that $u_n \to u$ in Hörmander topology if and only if $u_n \to u$ in $\mathcal{T}_\Gamma$. However, we find it useful to reformulate this as an actual topology rather than as a convergence criterion. Since sequential density implies density, $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ as a direct consequence of Lemma 2.2.2. Moreover, the space $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ endowed with the Γ-topology is separated, since $\mathcal{T}_\Gamma$ is finer than the weak topology. The following property follows immediately.

Proposition 2.2.11. *The space $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ endowed with the Γ-topology is separated, and $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$.*

Thanks to Lemma 2.2.9, we can show that $(\mathcal{S}'_\Gamma(\mathbb{R}^n), \mathcal{T}_\Gamma)$ is sequentially complete.

Proposition 2.2.12. *The space $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ endowed with the Γ-topology is sequentially complete.*

Proof. First, note that the dual of a Fréchet space endowed with the weak topology is sequentially complete (see [Sch82, Chap. IV.6]). In particular, since $\mathcal{S}(\mathbb{R}^n)$ is Fréchet, $\mathcal{S}'(\mathbb{R}^n)$ is sequentially complete for the weak topology. Let $(u_n) \subset \mathcal{S}'_\Gamma(\mathbb{R}^n)$ be a Cauchy sequence in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ with respect to the Γ-topology. Obviously, (u_n) is also a Cauchy sequence in the weak topology, and thus converges to a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$.

Moreover, for each symbol a in G^Γ, $(a^W u_n)$ is a Cauchy sequence in the Schwartz space topology by definition of the Γ-topology. Since the Schwartz space is a Fréchet space, the sequence $(a^W u_n)$ converges to a Schwartz function $\phi_a \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, for any symbol a, a^W is a continuous operator on $\mathcal{S}'(\mathbb{R}^n)$. Hence, $(a^W u_n)$ converges to $a^W u$ in $\mathcal{S}'(\mathbb{R}^n)$. Since convergence in $\mathcal{S}(\mathbb{R}^n)$ implies convergence in $\mathcal{S}'(\mathbb{R}^n)$, the sequence $(a^W u_n)$ also converges to ϕ_a in $\mathcal{S}'(\mathbb{R}^n)$ for any symbol $a \in G^\Gamma$. Since $\mathcal{S}'(\mathbb{R}^n)$ is separated, $a^W u = \phi_a$ in $\mathcal{S}'(\mathbb{R}^n)$ for any symbol $a \in G^\Gamma$.

Eventually, we have shown that the distribution u is such that the sequence $(u_n - u)$ converges to 0 in the Γ-topology. By Lemma 2.2.9, u is in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$, which concludes the proof. $\square$

The pullback by a linear isomorphism A is well defined as a continuous map for the Γ-topology

$$A^* \colon \mathcal{S}'_\Gamma \to \mathcal{S}'_{A^*\Gamma}$$

as a direct consequence of Corollary 2.1.12. However, when A is not a linear isomorphism, the continuity in the Γ-topology is not obvious, and consequently, neither is the definition of a continuous pointwise product of tempered distributions.

The goal of this subsection is to prove a stronger version of Theorem 2.2.8, i.e., showing that, under the right conditions on Γ and for a suitable choice of topology on $\mathcal{S}'_\Gamma(\mathbb{R}^n) \times \mathcal{S}'_\Gamma(\mathbb{R}^n)$, the pointwise product is a continuous map

$$\mathcal{S}'_\Gamma(\mathbb{R}^n) \times \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_\Gamma(\mathbb{R}^n)$$

in the Γ-topology. To do so, we need to prove a refined version of Proposition 2.2.3 at least for injective linear maps, and in particular for the diagonal maps (see Example 2.2.6).

We start with some preliminary observations about the injection map

$$\iota \colon \mathbb{R}^n \to \mathbb{R}^{n+m}, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0).$$

Let $\phi_1 \in \mathcal{S}(\mathbb{R}^n)$, $\phi_2 \in \mathcal{S}(\mathbb{R}^m)$ and $\phi = \phi_1 \otimes \phi_2 \in \mathcal{S}(\mathbb{R}^{n+m})$. For $a \in G^m(\mathbb{R}^{2n})$ and $x' \in \mathbb{R}^n$, we compute

$$a^W(\iota^*\phi)(x') = a^W(\phi_1(x'))\phi_2(0).$$

Let 1_m be the unit function on $\mathbb{R}^m$ and χ be a compactly supported smooth function on $\mathbb{R}^m$ such that $\chi(0) = 1$. Let also $x = (x', x'')$ with $x' \in \mathbb{R}^n$ and $x'' \in \mathbb{R}^m$. We define the new symbol $\overline{a} \in G^m(\mathbb{R}^{2n+2m})$ by

$$\overline{a}(x, \xi) = (a \otimes (\chi \otimes 1_m))\,(x, \xi) = a(x', \xi')\chi(x'').$$

We compute

$$\begin{aligned}
\overline{a}^W\phi(x) &= \int e^{i(x-y)\xi} a\left(\frac{x'+y'}{2},\xi'\right)\chi\left(\frac{x''+y''}{2}\right)\phi(y)\,dy'\,dy''\,d\xi'\,d\xi'' \\
&= \int \delta(x''-y'')e^{i(x'-y')\xi'} a\left(\frac{x'+y'}{2},\xi'\right)\chi\left(\frac{x''+y''}{2}\right)\phi(y)\,dy'\,dy''\,d\xi' \\
&= \chi(x'')\phi_2(x'')a^W\phi_1(x').
\end{aligned}$$

With $\alpha,\beta\in\mathbb{N}^n$, let's consider the Schwartz semi-norm

$$\left|a^W(\iota^*\phi)(x')\right|_{\alpha,\beta} = \sup_{x'}|\phi_2(0)|\left|(x')^\alpha\left(\partial^\beta a^W\phi_1\right)(x')\right|.$$

We are going to show it can be controlled by a Schwartz semi-norm of $\overline{a}^W\phi(x)$. Let $\overline{\alpha},\overline{\beta}\in\mathbb{N}^{n+m}$. We compute

$$\begin{aligned}
\left|\overline{a}^W\phi(x)\right|_{\overline{\alpha},\overline{\beta}} &= \sup_x\left|x^{\overline{\alpha}}\partial^{\overline{\beta}}\left(\chi(x'')\phi_2(x'')a^W\phi_1(x')\right)\right| \\
&= \sup_x\left|x^{\overline{\alpha}}\right|\left|\partial^{\overline{\beta}}\left(\chi(x'')\phi_2(x'')a^W\phi_1(x')\right)\right| \\
&\geq \sup_x\left|x^{\overline{\alpha}}\right|\left|\chi(x'')\phi_2(x'')\right|\left|\partial^{\overline{\beta}}\left(a^W\phi_1\right)(x')\right|.
\end{aligned}$$

If we suppose $\overline{\alpha}=(\alpha,0)$, $\overline{\beta}=(\beta,0)$, we have

$$\begin{aligned}
&\sup_x\left|x^{\overline{\alpha}}\chi(x'')\phi_2(x'')\right|\left|\partial^{\overline{\beta}}\left(a^W\phi_1\right)(x')\right| \\
&\qquad\geq \sup_{x'}\left|(x')^\alpha\right|\left|\chi(0)\phi_2(0)\right|\left|\partial^\beta\left(a^W\phi_1\right)(x')\right| \\
&\qquad= \sup_{x'}|\phi_2(0)|\left|(x')^\alpha\partial^\beta\left(a^W\phi_1\right)(x')\right| \\
&\qquad= \left|a^W(\iota^*\phi)(x')\right|_{\alpha,\beta}.
\end{aligned}$$

Thanks to our computations above, we can show the following lemma.

Lemma 2.2.13. *Let $\overline{\Gamma}$ be a closed cone in $T^*\mathbb{R}^{n+m}$ such that*

$$\overline{\Gamma}\cap\left\{(0,0,0,\xi''):\xi''\in\mathbb{R}^m\right\}=\emptyset$$

and Γ, a closed cone in $T^*\mathbb{R}^n$, defined by

$$\Gamma=\iota^*\overline{\Gamma}=\left\{(x',\xi'):\exists\xi''\in\mathbb{R}^m \text{ such that } (x',0,\xi',\xi'')\in\overline{\Gamma}\right\}.$$

The pullback by the injection map

$$\iota^* \colon \mathcal{S}(\mathbb{R}^{n+m}) \to \mathcal{S}(\mathbb{R}^n)$$

extends uniquely to a sequentially continuous map for the Γ*-topology*

$$\iota^* \colon \mathcal{S}'_{\overline{\Gamma}}(\mathbb{R}^{n+m}) \to \mathcal{S}'_{\Gamma}(\mathbb{R}^n).$$

Proof. Since $\mathcal{S}(\mathbb{R}^n)$ is sequentially dense in $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ and $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ is a sequentially complete and separated topological space, it is enough to show that ι^* is continuous on $\mathcal{S}(\mathbb{R}^{n+m})$. Then we can use a continuous extension theorem to conclude (see Theorem A.39 and Remark A.40 below).

We have to verify that, for any $\phi \in \mathcal{S}(\mathbb{R}^{n+m})$, every semi-norm from Definition 2.2.10 of $\iota^*\phi$ is controlled by a semi-norm of ϕ. Explicitly, for any $\alpha, \beta \in \mathbb{N}^n$, $\psi \in \mathcal{S}(\mathbb{R}^n)$, and $a \in G^\Gamma$, we must find $\overline{\alpha}, \overline{\beta} \in \mathbb{N}^{n+m}$, $\overline{\psi} \in \mathcal{S}(\mathbb{R}^{n+m})$ and $\overline{a} \in G^{\overline{\Gamma}}$ which satisfy

$$|\iota^*\phi|_\psi \lesssim |\phi|_{\overline{\psi}} + \|\phi\|^{\overline{\Gamma}}_{\overline{a},\overline{\alpha},\overline{\beta}} \tag{II.7}$$

$$\|\iota^*\phi\|^{\Gamma}_{a,\alpha,\beta} \lesssim |\phi|_{\overline{\psi}} + \|\phi\|^{\overline{\Gamma}}_{\overline{a},\overline{\alpha},\overline{\beta}}. \tag{II.8}$$

By nuclearity of the Schwartz space (see Theorem 1.1.30) and the fact that, for any $a \in G^\infty$, the mapping $a^W : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, we can suppose

$$\phi = \phi_1 \otimes \phi_2 \in \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^m).$$

Let χ be a compactly supported smooth function such that $\chi(0) = 1$. For (II.7), simply observe that

$$\begin{aligned}
\left|\int \iota^*\phi(x')\psi(x')\,dx'\right| &= |\phi_2(0)| \left|\int \phi_1(x')\psi(x')\,dx'\right| \\
&\leq C_{\phi_2,\chi} \left|\int \phi(x)(\psi \otimes \chi)(x)\,dx\right| \\
&\leq C_{\phi_2,\chi} \left|\int \phi_2(x'')\chi(x'')\,dx''\right| \left|\int \phi_1(x')\psi(x')\,dx'\right|.
\end{aligned}$$

For (II.8), our computations above show that

$$\|\iota^*\phi\|^{\Gamma}_{a,\alpha,\beta} \leq \|\phi\|^{\overline{\Gamma}}_{\overline{a},\overline{\alpha},\overline{\beta}}$$

for $\overline{\alpha} = (\alpha, 0)$, $\overline{\beta} = (\beta, 0)$ and

$$\overline{a}(x, \xi) = (a \otimes (\chi \otimes 1_m))(x, \xi) = a(x', \xi')\chi(x'').$$

Hence, the result is proved provided that

$$\operatorname{conesupp}(a) \cap \Gamma = \emptyset \implies \operatorname{conesupp}(\overline{a}) \cap \overline{\Gamma} = \emptyset.$$

Since

$$\operatorname{conesupp}(\overline{a}) = (\operatorname{conesupp}(a) \cup 0) \times (\{0\} \times \mathbb{R}^m),$$

we must impose

$$\overline{\Gamma} \cap ((\Gamma^c \cup 0) \times (\{0\} \times \mathbb{R}^m)) = \emptyset.$$

This is equivalent to

$$\overline{\Gamma} \cap \{(0, 0, 0, \xi'') : \xi'' \in \mathbb{R}^m\} = \emptyset$$

and

$$\iota^* \overline{\Gamma} = \left\{(x', \xi') : \exists \xi'' \in \mathbb{R}^m \text{ such that } (x', 0, \xi', \xi'') \in \overline{\Gamma}\right\} \subseteq \Gamma.$$

By Hörmander's result 2.2.3, the wave front set of $\iota^* u$ for any distribution $u \in \mathcal{S}'_{\overline{\Gamma}}$ satisfies $\mathrm{WF}(\iota^* u) \subseteq \iota^* \overline{\Gamma}$, which means it is sufficient to impose the minimal constraint

$$\Gamma = \iota^* \overline{\Gamma} = \left\{(x', \xi') : \exists \xi'' \in \mathbb{R}^m \text{ such that } (x', 0, \xi', \xi'') \in \overline{\Gamma}\right\}.$$

$\square$

For any injective linear map A, we can find a linear isomorphism L such that $A = L \circ \iota$. In particular, for

$$\operatorname{diag}\colon \mathbb{R}^n \to \mathbb{R}^{2n}, \quad x \mapsto (x, x),$$

we have

$$\operatorname{diag} = \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} \circ \iota.$$

Moreover, the pullback of a tempered distribution by a linear isomorphism A is well defined as a continuous map in the Γ-topology

$$L^*\colon \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_{L^*\Gamma}(\mathbb{R}^n)$$

without any condition on Γ, as a direct consequence of Corollary 2.1.12. Hence, the pullback by any injective linear map is defined on $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ as a continuous extension of the pullback on $\mathcal{S}(\mathbb{R}^n)$, modulo the condition of Lemma 2.2.13 applied to $L^*\Gamma$, which is equivalent to asking

$$\Gamma \cap \{(0,\xi) : A^T\xi = 0\} = \emptyset.$$

We have proved the following theorem.

Theorem 2.2.14. *Let $A\colon \mathbb{R}^m \to \mathbb{R}^n$ be an injective linear map. Let Γ be a closed cone satisfying*

$$\Gamma \cap \{(0,\xi) : A^T\xi = 0\} = \emptyset.$$

Then the pullback

$$A^*\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m)$$

extends uniquely, in the Γ-topology, to a sequentially continuous map

$$A^*\colon \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_{A^*\Gamma}(\mathbb{R}^m)$$

with

$$A^*\Gamma := \{(x, A^T\xi) \in T^*\mathbb{R}^m \setminus \{(0,0)\} : (Ax,\xi) \in \Gamma\}.$$

Remark 2.2.15. The tensor product of tempered distributions, to the best of our knowledge, has never been shown to be continuous for the Hörmander topology. However, just like the pullback by a linear map, the tensor product is central in the definition of a continuous pointwise product on $\mathcal{S}'_\Gamma(\mathbb{R}^n)$. Nevertheless, Proposition 2.1.20 yields the following inclusions:

$$\mathcal{S}'_{\Gamma_1}(\mathbb{R}^n) \times \mathcal{S}'_{\Gamma_2}(\mathbb{R}^m) \hookrightarrow \mathcal{S}'_{\Gamma_1}(\mathbb{R}^n) \otimes \mathcal{S}'_{\Gamma_2}(\mathbb{R}^m) \hookrightarrow \mathcal{S}'_{\Gamma_1\otimes\Gamma_2}(\mathbb{R}^{n+m}),$$

$$(u,v) \mapsto u\otimes v \mapsto u \otimes v,$$

where $\Gamma_1 \otimes \Gamma_2 = (\Gamma_1 \cup \{0\}) \times (\Gamma_2 \cup \{0\})$. Hence, we can endow the product $\mathcal{S}'_{\Gamma_1}(\mathbb{R}^n) \times \mathcal{S}'_{\Gamma_2}(\mathbb{R}^m)$ with the subspace topology induced by $\mathcal{S}'_{\Gamma_1\otimes\Gamma_2}(\mathbb{R}^{n+m})$.

The following refinement of Theorem 2.2.8 follows immediately from Theorem 2.2.14 and Remark 2.2.15.

Theorem 2.2.16. *Let $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_2 a closed cone in $\mathbb{R}^n$ such that Γ_2 is stable under addition and does not contain 0. Then, the space $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ is a commutative and associative algebra under the pointwise product $uv = \operatorname{diag}^*(u \otimes v)$. Moreover, if $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ and $\mathcal{S}'_{\Gamma\otimes\Gamma}(\mathbb{R}^{2n})$ are endowed with the Γ-topology, and $\mathcal{S}'_\Gamma(\mathbb{R}^n) \times \mathcal{S}'_\Gamma(\mathbb{R}^n)$ is endowed with the subspace topology induced by $\mathcal{S}'_{\Gamma\otimes\Gamma}(\mathbb{R}^{2n})$, the pointwise product is a sequentially continuous map*

$$\mathcal{S}'_\Gamma(\mathbb{R}^n) \times \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_\Gamma(\mathbb{R}^n).$$

Compared to Theorem 2.2.8, we see that in order to equip the codomain with the Γ-topology while preserving continuity, we had to adjust the topology of $\mathcal{S}'_\Gamma(\mathbb{R}^n) \times \mathcal{S}'_\Gamma(\mathbb{R}^n)$. This space is no longer viewed as the topological product of $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ with itself, but as a subspace of $(\mathcal{S}'_{\Gamma\otimes\Gamma}, \mathcal{T}_{\Gamma\otimes\Gamma})$, endowed with the induced topology. When we will state further than some products are sequentially continuous, it should be understood in that way. In the next chapter, we will deform such products.

Chapter III

Deformation of tempered distributions algebras

> *"We attempt here to establish a mathematically rigorous and physically manageable formulation for quantum mechanics in phase space. To obtain the right mathematical context, we must, for example, specify those pairs of functions whose twisted product may be formed; and a suitable function space should include as many observables of physical interest as possible."*
>
> - José M. Gracia-Bondía et Joseph C. Várilly, [GBV88]

Let V be a $2n$-dimensional symplectic vector space. In Section 1.2.3, we introduced the twists

$$\begin{aligned}
&\Omega_\tau \colon \mathcal{S}'(V^2) \to \mathcal{S}'(V^2),\\
&\Omega_\tau(\phi)(\overline{x}) := \int K_\tau(\overline{z})\phi(\overline{z}+\overline{x})\, d\overline{z},
\end{aligned}$$

and their associated family of star-products

$$a \star_\tau b(z_0) = \int K_\tau(z_1, z_2) a(z_1 + z_0) b(z_2 + z_0)\, dz_1\, dz_2.$$

In Section 3.1, we give a sufficient condition on a cone $\overline{\Gamma} \subset T^*V^2$ for the twist Ω_τ to define an automorphism

$$\Omega_\tau \colon \mathcal{S}'_{\overline{\Gamma}}(V^2) \to \mathcal{S}'_{\overline{\Gamma}}(V^2)$$

in the $\overline{\Gamma}$-topology. This is achieved by imposing the stability of $\overline{\Gamma}$ under a certain symplectic transformation of T^*V^2. Additionally, we observe that this result generalizes to a larger family of operators Ω_A.

In Section 3.2, we make the following central observation: for all $\phi, \psi \in \mathcal{S}(V)$, the family of products $\star_\tau$ can be equivalently defined by

$$\phi \star_\tau \psi = \mathrm{diag}^*(\Omega_\tau(\phi \otimes \psi)).$$

Thus, to construct a cone Γ such that $(\mathcal{S}(V),\ \star_\tau)$ forms an associative algebra, it is sufficient to combine the results on the twist Ω_τ presented in Section 3.1 with Theorem 2.2.14 and deduce the conditions required for Γ. Again, this strategy can be applied to a broader class of products $\star_A$, which includes the twisted convolution product (17) studied in [BS19].

In Section 3.3, we apply our result to the spaces $\mathcal{B}^N(V)$ presented in the introduction.

3.1 Continuity of the twist

In this section, we show that the twist Ω_τ is an automorphism of $\mathcal{S}(V^2)$ and $\mathcal{S}'(V^2)$ associated with a symplectic transformation via Theorem 2.1.11 and Corollary 2.1.12. From there, the condition for its continuity in the Γ-topology will be easy to state.

Notation 3.1.1. In what follows, we write:

- V is a $2n$-dimensional symplectic vector space and $T^*V \simeq V \times V^*$ is its cotangent bundle;
- $x = (q, p)$ is an element of V, ξ is an element of V^*, and $(x, \xi) \in T^*V$;
- $\overline{x} = (x_1, x_2) = ((q_1, p_1), (q_2, p_2))$ is an element of V^2, and $\overline{g} = (\overline{x}, \overline{\xi})$ is an element of T^*V^2.

First, we express the Weyl quantization map in terms of unitary representations of the Heisenberg group. Let $\mathbb{H} = T^*V^2 \times \mathbb{R}Z$ be the Heisenberg group associated with T^*V^2, and let $(U_\mu, L^2(V^2))$ be one of its irreducible unitary representations realized by the orbit method (see Theorem 2.1.25 and Appendix B.3). We also introduce the application $\Sigma_W \psi(q) \coloneqq \psi(-q)$.

We define the mapping

$$\begin{aligned} &\tilde{\sigma} : \mathbb{H} \to U(L^2(V^2)) \\ &\tilde{\sigma}(h) \coloneqq U_\mu(h) \Sigma_W U_\mu(h^{-1}) \end{aligned} \tag{III.1}$$

which can be restricted to $T^*V^2 \subset \mathbb{H}$, yielding a mapping

$$\begin{aligned} &\sigma : T^*V^2 \to U(L^2(V^2)) \\ &\sigma(\overline{g}) \coloneqq U_\mu(\overline{g}) \Sigma_W U_\mu(\overline{g}^{-1}). \end{aligned} \tag{III.2}$$

Explicitly, we compute

$$\sigma(\overline{x}, \overline{\xi}) \psi(\overline{z}) = e^{2i\mu(\overline{z} - \overline{x})\overline{\xi}} \psi(2\overline{x} - \overline{z}).$$

Then, it is easy to check that for any symbol a, the Weyl operator a^W is given by

$$\left(a^W \psi\right)(q_0) = \int a(g) \sigma(g) \psi(q_0)\, dg. \tag{III.3}$$

This link between quantization operators on a given space and the representation theory of a related group is not a mere coincidence. As a symmetric space, $\mathbb{R}^{2n}$ is the most basic example of a theory developed by Bieliavsky and Gayral to quantize a particular class of symmetric spaces, see [BG15, Chap. 6].

Motivated by (III.3), we prove the following technical lemma.

Lemma 3.1.2. *With Ω_τ and σ defined respectively as in formulas* (I.18) *and* (III.1)*, one has*

$$\Omega_\tau^{-1} \sigma(\overline{g}) \Omega_\tau = \left(\left(\mathbb{B}_\tau^{-1} \right)^* \sigma \right)(\overline{g}),$$

where $\mathbb{B}_\tau$ is the symplectic transformation of T^*V^2 defined by

$$\mathbb{B}_\tau \coloneqq \begin{pmatrix} I_{4n} & -\beta_\tau \\ 0 & I_{4n} \end{pmatrix},$$

with

$$\beta_\tau \coloneqq \begin{pmatrix} 0 & b_\tau \\ b_\tau^T & 0 \end{pmatrix}, \quad b_\tau \coloneqq \begin{pmatrix} 0 & \tau I_n \\ (\tau - 1) I_n & 0 \end{pmatrix}.$$

Proof. Let $\phi \in \mathcal{S}(V^2)$. Note that

$$\Omega_\tau^{-1}(\phi)(\overline{z}_0) = (\tau(\tau-1))^n \int \overline{\Omega}_\tau(\overline{z})\phi(\overline{z}_0 - \overline{z})\, d\overline{z},$$

where $\overline{\Omega}_\tau$ denotes the complex conjugate.

First, let τ be different from 0 and 1. We compute

$$\begin{aligned}
\left[\Omega_\tau^{-1}\sigma(\overline{g})\Omega_\tau\right](\phi)(x_0) &= \int \overline{\Omega}_\tau(\overline{z})\left[\sigma(\overline{g})\Omega_\tau\right](\phi)(\overline{x}_0 - \overline{z})\, d\overline{z} \\
&= \int \overline{\Omega}_\tau(\overline{z}) e^{2i(\overline{x}-\overline{x}_0+\overline{z})\overline{\xi}}\left[\Omega_\tau\right](\phi)(2\overline{x} - \overline{x}_0 + \overline{z})\, d\overline{z} \\
&= \int \overline{\Omega}_\tau(\overline{z}) e^{2i(\overline{x}-\overline{x}_0+\overline{z})\overline{\xi}}\Omega_\tau(\overline{w})\phi(2\overline{x} + \overline{w} - \overline{x}_0 + \overline{z})\, d\overline{z}\, d\overline{w} \\
&= \int \overline{\Omega}_\tau(\overline{z}) e^{2i(\overline{x}-\overline{x}_0+\overline{z})\overline{\xi}}\Omega_\tau(\overline{w} - \overline{z})\phi(2\overline{x} + \overline{w} - \overline{x}_0)\, d\overline{z}\, d\overline{w}.
\end{aligned}$$

The variable $\overline{z}$ now only appears in the phase; using Example 1.1.18, some delta distributions will appear in this expression. Setting

$$\overline{z} = ((q_1,p_1),(q_2,p_2)), \quad \overline{w} = ((y_1,\eta_1),(y_2,\eta_2)), \quad \overline{\xi} = (\xi_1^1,\xi_1^2,\xi_2^1,\xi_2^2),$$

we develop the phase:

$$\begin{aligned}
\Omega_\tau(\overline{z})\Omega_\tau(\overline{w}-\overline{z})e^{2i\overline{z}\overline{\xi}} &= \Omega_\tau(\overline{w})e^{iq_1\left(2\xi_1^1 - \frac{1}{\tau}\eta_2\right)} \\
&\qquad e^{ip_1\left(2\xi_1^2 - \frac{1}{\tau-1}\eta_1\right)}e^{iq_2\left(2\xi_2^1 - \frac{1}{\tau-1}y_2\right)}e^{ip_2\left(2\xi_2^2 - \frac{1}{\tau}\eta_2\right)}.
\end{aligned}$$

Then, after a change of variable, we get

$$\begin{aligned}
\left[\Omega_\tau^{-1}\sigma(\overline{g})\Omega_\tau\right](\phi)(x_0) &= \int \Omega(\overline{w})e^{2i(\overline{x}-\overline{x}_0)\overline{\xi}}\phi(2\overline{x}+\overline{w}-\overline{x}_0)\delta\left(2\tau\xi_1^1 - \eta_2\right) \\
&\quad \delta\left(2(\tau-1)\xi_1^2 - y_2\right)\delta\left(2(\tau-1)\xi_2^1 - \eta_1\right)\delta\left(2\tau\xi_2^2 - y_1\right)\, d\overline{w} \\
&= e^{2i\left(\overline{x}+\beta_\tau(\overline{\xi})-\overline{x}_0\right)\overline{\xi}}\phi\left(2\left(\overline{x}+\beta_\tau(\overline{\xi})\right) - \overline{x}_0\right) \\
&= \sigma\left(\overline{x}+\beta_\tau(\overline{\xi}),\overline{\xi}\right) = \sigma\left(\mathbb{B}_\tau^{-1}(\overline{x},\overline{\xi})\right).
\end{aligned}$$

For $\tau = 0$ and $\tau = 1$, the twist is different, but the computations are essentially identical.

□

Remark 3.1.3. As a direct consequence of Lemma 3.1.2 and formula (III.3), we observe that

$$(a \circ \mathbb{B}_\tau)^W = \Omega_\tau^{-1} a^W \Omega_\tau. \tag{III.4}$$

Since β_τ is symmetric and

$$\mathbb{B}_\tau = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ \beta_\tau & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} I & -\beta_\tau \\ 0 & I \end{pmatrix}$$

is symplectic, formula (III.4) and Corollary 2.1.12 imply

$$\Omega_\tau = \mathcal{F}^{-1} \circ e^{ix^T \beta_\tau x} \circ \mathcal{F}, \tag{III.5}$$

where $\mathcal{F}$ is the usual Fourier transform. In general, for any symmetric matrix A, the operator

$$\Omega_A = \mathcal{F}^{-1} \circ e^{ix^T A x} \circ \mathcal{F}$$

is a unitary operator on L^2 and an automorphism of $\mathcal{S}$ and $\mathcal{S}'$ as a direct consequence of Theorem 2.1.11. Moreover, for any tempered distribution $u \in \mathcal{S}'$, Corollary 2.1.12 implies

$$\mathrm{WF}\left(\Omega_A(u)\right) \subseteq \left\{\mathbb{B}_A(x,\xi) : (x,\xi) \in \mathrm{WF}(u)\right\},$$

where $\mathbb{B}_A$ is the symplectic transformation of $T^*\mathbb{R}^n$ given by

$$\mathbb{B}_A = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}.$$

The symplectic transformation $\mathbb{B}_A$ implies the addition of contravariant quantities $x \in \mathbb{R}^n$ with covariant quantities $\xi \in (\mathbb{R}^n)^*$. This should be understood as explained in Remark 2.1.13.

In particular, we could consider the symmetric matrix

$$\Theta = \begin{pmatrix} 0 & \theta \\ \theta^T & 0 \end{pmatrix},$$

where θ is any skew-symmetric matrix. The family of unitary operators

$$\Omega_\theta \coloneqq \mathcal{F}^{-1} \circ e^{ix^T \Theta x} \circ \mathcal{F},$$

is associated with the family of products

$$u \star_\theta v \coloneqq \mathrm{diag}^*\left(\Omega_\theta(u \otimes v)\right),$$

which coincides with the twisted convolution product (17), presented in the introduction, and aligns with the framework studied in [BS19]. This encompasses the Weyl case for

$$\theta = \frac{1}{2}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We observe that the family of operators Ω_θ, associated with the twisted convolution product (17) and indexed by skew-symmetric matrices θ, intersects with the family of twists Ω_τ, which corresponds to the τ-ordering quantization family precisely when $\tau = \frac{1}{2}$, at the Weyl star-product.

Remark 3.1.4. As stated above, the operators Ω_τ and Ω_θ are special cases of the more general Ω_A, which does not always require the symplectic vector space V. The symplectic structure of V is only necessary when the transformation $\mathbb{B}_\tau$ is involved. Therefore, whenever possible, we present our results below for Ω_A and in $\mathbb{R}^n$, as this provides a more general framework. We will return to the symplectic vector space V when discussing Ω_τ and Ω_θ.

The next result follows immediately from Remark 3.1.3. Note that it implies Lemma 1.2.15, which we left unproved in Chapter I.

Corollary 3.1.5. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and Γ be a closed cone in $T^*\mathbb{R}^n$. The operator Ω_A is a unitary operator on $L^2(\mathbb{R}^n)$ and an automorphism of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ with respect to the Schwartz topology and the weak topology, respectively. Moreover, for $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\mathrm{WF}(\Omega_A u) = \{\mathbb{B}_A(x,\xi) : (x,\xi) \in \mathrm{WF}(u)\}.$$

We can now state a simple condition on a cone Γ under which the operator Ω_A is a continuous map

$$\Omega_A : \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_\Gamma(\mathbb{R}^n)$$

in the Γ-topology.

Theorem 3.1.6. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and Γ be a closed cone in $T^*\mathbb{R}^n$ such that $\mathbb{B}_A(\Gamma) \subset \Gamma$. Then the operator Ω_A is a continuous map in the Γ-topology*

$$\Omega_A : \mathcal{S}'_\Gamma(\mathbb{R}^n) \to \mathcal{S}'_\Gamma(\mathbb{R}^n).$$

Proof. To ensure the continuity of Ω_A in the Γ-topology, we need to prove that, for any $\alpha, \beta \in \mathbb{N}^{4n}$ and any symbol $a \in G^\Gamma$, there exist $\alpha', \beta' \in \mathbb{N}^{4n}$ and a symbol $a' \in G^\Gamma$ such that the following estimate holds:

$$\|\Omega_A(u)\|^\Gamma_{a,\alpha,\beta} \lesssim \|u\|^\Gamma_{a',\alpha',\beta'}. \tag{III.6}$$

Since Ω_A is an automorphism of $\mathcal{S}(V)$ and

$$a^W(\Omega_A(u)) = \left(\Omega_A \Omega_A^{-1} a^W \Omega_A\right)(u) = \Omega_A\left((a \circ \mathbb{B}_A)^W u\right),$$

the estimate III.6 holds for $a' = a \circ \mathbb{B}_A$ as soon as

$$\mathrm{cs}(a) \cap \Gamma = \emptyset \implies \mathrm{cs}(a \circ \mathbb{B}_A) \cap \Gamma = \emptyset.$$

But by hypothesis, $\mathbb{B}_A \Gamma \subset \Gamma$, and the proof is complete. □

Corollary 3.1.7. *Let V be a symplectic vector space. Under the respective conditions $\mathbb{B}_\tau \overline{\Gamma} \subset \overline{\Gamma}$ and $\mathbb{B}_\theta \overline{\Gamma} \subset \overline{\Gamma}$, the twists Ω_τ and Ω_θ are continuous maps*

$$\Omega_\tau : \mathcal{S}'_{\overline{\Gamma}}(V^2) \to \mathcal{S}'_{\overline{\Gamma}}(V^2), \quad \textit{and} \quad \Omega_\theta : \mathcal{S}'_{\overline{\Gamma}}(V^2) \to \mathcal{S}'_{\overline{\Gamma}}(V^2).$$

Example 3.1.8. The stability condition $\mathbb{B}_A \overline{\Gamma} \subseteq \overline{\Gamma}$ is trivially satisfied for $\overline{\Gamma} = V^2 \times \{0\}$. In Section 3.3, we will see that this example is of particular interest.

3.2 Construction of the Star-Product

For all $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$, and for a symmetric matrix A, we define the family of products

$$\phi \star_A \psi = \mathrm{diag}^*(\Omega_A(\phi \otimes \psi)),$$

which includes the τ-ordering family of star-products $\star_\tau$ and the twisted convolution products $\star_\theta$. Combining Proposition 2.1.20, Theorem 2.2.14, and Theorem 3.1.6, we can establish a sufficient condition under which the product $\star_A$ of two tempered distributions is well-defined. Moreover, when $\star_A$ is well-defined, we provide an inclusion for the smooth wavefront set of $u \star_A v$.

Theorem 3.2.1. *Let $\overline{\Gamma}$ be a closed cone in $T^*\mathbb{R}^{2n}$ and*

$$\Gamma \coloneqq \operatorname{diag}^* \overline{\Gamma} = \Big\{ \Big(x,\, \operatorname{diag}^T(\eta)\Big) : (\operatorname{diag}(x),\, \eta) \subset \overline{\Gamma} \Big\}$$

such that

C_1 *:* $\overline{\Gamma} \cap \Big(\{0\} \times \ker(\operatorname{diag}^T)\Big) = \emptyset$*;*

C_2 *:* $\mathbb{B}_A(\overline{\Gamma}) \subset \overline{\Gamma}$.

If u and v are two tempered distributions in $\mathcal{S}'(\mathbb{R}^n)$ satisfying $\mathrm{WF}(u \otimes v) \subseteq \overline{\Gamma}$*, then the product $u \star_A v$ is well-defined as a tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$, and*

$$\mathrm{WF}(u \star_A v) \subseteq \Gamma.$$

Proof. Condition C_2 ensures that Ω_A is a continuous map from $\mathcal{S}'_{\overline{\Gamma}}(\mathbb{R}^{2n})$ to $\mathcal{S}'_{\overline{\Gamma}}(\mathbb{R}^{2n})$ by Theorem 3.1.6. In particular, it implies that $\mathrm{WF}(\Omega_\tau(u \otimes v))$ is included in $\overline{\Gamma}$.

Condition C_1 ensures that the pullback by the diagonal map is well-defined and continuous on $\mathcal{S}'_{\overline{\Gamma}}(\mathbb{R}^{2n})$. In particular, the product $u \star_A v = \operatorname{diag}^*(\Omega_\tau(u \otimes v))$ is well-defined, and $\mathrm{WF}(u \star_A v)$ is included in Γ by Theorem 2.2.14.

The theorem follows immediately from the cited results. □

To build algebras of tempered distributions under the family of products $\star_A$ based on Theorem 3.2.1, it is sufficient to add a condition on Γ to control the wavefront set of $u \star_A v$.

Theorem 3.2.2. *Let $\overline{\Gamma}$ be a closed cone in $T^*\mathbb{R}^{2n}$ and $\Gamma \coloneqq \operatorname{diag}^* \overline{\Gamma}$ such that*

C_1 *:* $\overline{\Gamma} \cap \Big(\{0\} \times \ker(\operatorname{diag}^T)\Big) = \emptyset$*;*

C_2 *:* $\mathbb{B}_A(\overline{\Gamma}) \subset \overline{\Gamma}$*;*

C_3 *:* $(\Gamma \cup \{0\}) \times (\Gamma \cup \{0\}) \subseteq \overline{\Gamma}$.

Then the product

$$\star_A : \mathcal{S}'_\Gamma(\mathbb{R}^n)\otimes\mathcal{S}'_\Gamma(\mathbb{R}^n) \subset \mathcal{S}'_{\overline{\Gamma}}(\mathbb{R}^{2n}) \to \mathcal{S}'_\Gamma(\mathbb{R}^n), \quad a\star_A b := \mathrm{diag}^*(\Omega_A(a\otimes b))$$

is well-defined as a sequentially continuous map for the Γ*-topology. If* Ω_A *is equal to* Ω_τ *or* Ω_θ*, this product endows* $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ *with an associative algebra structure.*

Proof. Condition C_3 ensures that for all $u, v \in \mathcal{S}'_\Gamma(\mathbb{R}^n)$, $\mathrm{WF}(u\otimes v)$ is included in $\overline{\Gamma}$, i.e., that $u\otimes v \in \mathcal{S}'_{\overline{\Gamma}}(\mathbb{R}^{2n})$ (see Proposition 2.1.20).

The continuity, understood in the sense of Remark 2.2.15 and Theorem 2.2.16, follows from the continuity of diag^* (Theorem 2.2.14) and Ω_A (Theorem 3.1.6). The associativity of $\star_\tau$ and $\star_\theta$ is trivial. □

Example 3.2.3. Let $\overline{\Gamma} = \mathbb{R}^{2n}\times\{0\}$. Then, all conditions of Theorem 3.2.2 are trivially satisfied, and $\mathcal{S}'_{\mathbb{R}^n\times\{0\}}(\mathbb{R}^n)$ is an associative algebra under both the twisted convolution product and the τ-ordering star-product.

Example 3.2.4. Let $\mathbb{R}^n_+ := \{(x_1,\dots,x_n)\in\mathbb{R}^n : x_1 > 0\}$. Then the cone

$$\overline{\Gamma} = \big((\mathbb{R}^n\times\mathbb{R}^n_+)\cup\{0\}\big)\times\big((\mathbb{R}^n\times\mathbb{R}^n_+)\cup\{0\}\big)\setminus\{0\}$$

satisfies every condition of Theorem 3.2.2, and $\mathrm{diag}^*(\overline{\Gamma}) = \mathbb{R}^n\times\mathbb{R}^n_+$. Hence, $\left(\mathcal{S}'_{\mathbb{R}^n\times\mathbb{R}^n_+}(\mathbb{R}^n), \star_A\right)$ is an algebra for any symmetric matrix A.

Example 3.2.5. More generally, let $\overline{\Gamma}$ be a cone in $T^*\mathbb{R}^{2n}$ of the form

$$\overline{\Gamma} = ((\Gamma_1\times\Gamma_2)\cup\{0\})\times((\Gamma_1\times\Gamma_2)\cup\{0\})\setminus\{0\},$$

where Γ_2 is a cone in $(\mathbb{R}^n)^*$ stable under addition and not containing zero, and Γ_1 is a cone in $\mathbb{R}^n$. Condition C_1 is satisfied since $\xi\in\Gamma_2$ implies $-\xi\notin\Gamma_2$, and

$$\ker(\mathrm{diag}^T) = \{(\xi,-\xi)\}.$$

Thus,

$$\ker(\mathrm{diag}^T)\cap\Gamma_2\times\Gamma_2 = \emptyset.$$

Condition C_3 is satisfied since

$$\mathrm{diag}^*(\overline{\Gamma}) = \{(x,\xi+\eta) : x\in\Gamma_1,\, \xi\in\Gamma_2, \eta\in\Gamma_2\},$$

which is clearly a subspace of $\Gamma_1 \times \Gamma_2$ because Γ_2 is stable under addition. Thus, $\operatorname{diag}^*(\overline{\Gamma}) = \Gamma_1 \times \Gamma_2$.

For condition C_2 to hold, any $x, y \in \Gamma_1$ and $\xi, \eta \in \Gamma_2$ must satisfy

$$\mathbb{B}_A((x, y), (\xi, \eta)) = ((x, y) + A(\xi, \eta), (\xi, \eta)) \in \overline{\Gamma}.$$

Note that this is always true for $\Gamma_1 = \mathbb{R}^n$.

In summary, if Γ_2 is stable under addition and does not contain zero, and if $\overline{\Gamma}$ satisfies the condition

$$\mathbb{B}_A((x, y), (\xi, \eta)) \in \overline{\Gamma},$$

then $\mathcal{S}'_{\Gamma_1 \times \Gamma_2}(\mathbb{R}^n)$ is an algebra under the product $\star_A$.

In the case of the twisted convolution product $\star_\theta$, condition C_2 becomes

$$x \in \Gamma_1,\, \xi \in \Gamma_2 \implies x + \theta\xi \in \Gamma_1.$$

Note that asking for $x - \theta^T\xi \in \Gamma_1$ would be redundant since θ is skew-symmetric. This condition corresponds to the result of Bahns and Schulz [BS19, Cor. 4.2]. In particular, all deformed algebras of tempered distributions obtained by Bahns and Schulz fall within the scope of Theorem 3.2.2. However, the algebra $\mathcal{M}$ presented in [GBV88] cannot be obtained through our result. Indeed, the algebra $\mathcal{M}$ cannot be reconstructed within the smooth wavefront set framework, as the conditions on WF are too restrictive to build examples containing Dirac deltas, which are included in $\mathcal{M}$. Nevertheless, the construction in [GBV88] is restricted to the Weyl case and is not valid within the Kohn-Nirenberg framework.

Remark 3.2.6. In the spirit of Remark 2.1.13, we introduce a slight abuse of notation to give a precise meaning to the sums of covariant and contravariant quantities that arise due to the transformation of the wavefront set imposed by the twists through $\mathbb{B}_\tau$. This time, however, we take advantage of the symplectic structure of V. We denote $b_\tau\xi$ the element $(b_\tau\xi)^\sharp \in V$, defined by the canonical isomorphism induced by the symplectic form ω on V,

$$V^* \to V, \quad \xi \mapsto \xi^\sharp,$$

where $\xi^\sharp$ is the unique vector in V such that $\xi(x) = \omega(\xi, x)$ for all $x \in V$. We systematically use this notation throughout this section.

Example 3.2.7. In the case of the $\star_\tau$ product, let $\overline{\Gamma}$ be a cone in T^*V^2 of the form

$$\overline{\Gamma} = ((\Gamma_1 \times \Gamma_2) \cup \{0\}) \times ((\Gamma_1 \times \Gamma_2) \cup \{0\}) \setminus \{0\},$$

where Γ_2 is a cone in V^* stable under addition and not containing zero, and Γ_1 is a cone in V. Condition C_2 becomes

$$x \in \Gamma_1,\, \xi \in \Gamma_2 \implies x + b_\tau \xi \in \Gamma_1 \quad \text{and} \quad x - b_\tau^T \xi \in \Gamma_1.$$

For the sake of completeness, we state the following two corollaries, which are direct consequences of the discussion in Example 3.2.5 and Example 3.2.7. The first corollary is exactly [BS19, Cor. 4.2], while the second one is, to the best of our knowledge, a novel result. In these statements, additions of elements of V with elements of V^* must be understood in the sense of Remark 2.1.13 and Remark 3.2.6.

Corollary 3.2.8. *Let θ be a skew-symmetric $n \times n$ matrix. Let Γ_2 be a conic subset of $\mathbb{R}^n \setminus \{0\}$ stable under addition, and let Γ_1 be a closed cone in $\mathbb{R}^n$ such that*

$$x \in \Gamma_1,\, \xi \in \Gamma_2 \implies x + \theta\xi \in \Gamma_1.$$

Then, $\mathcal{S}'_{\Gamma_1 \times \Gamma_2}(\mathbb{R}^n)$ is an associative algebra under pointwise multiplication and the twisted convolution product $\star_\theta$, and both products are sequentially continuous in the Γ-topology.

Corollary 3.2.9. *Let V be a symplectic vector space, and let Γ_1 and Γ_2 be closed cones in V such that Γ_2 is stable under addition and does not contain zero, and*

$$x \in \Gamma_1,\, \xi \in \Gamma_2 \implies x + b_\tau \xi \in \Gamma_1 \quad \textit{and} \quad x - b_\tau^T \xi \in \Gamma_1.$$

Then, the space $\mathcal{S}'_{\Gamma_1 \times \Gamma_2}(V)$, endowed with the Γ-topology, is an associative algebra under the pointwise product and the product $\star_\tau$, and both products are sequentially continuous in the Γ-topology.

We conclude this section with a practical result that can be useful for quickly verifying whether the product $\star_A$ of two tempered distributions is well-defined and for estimating its smooth wavefront set. This result and the corollaries that follow again involve the addition of covariant and contravariant quantities, interpreted as outlined in Remark 2.1.13 and Remark 3.2.6.

Theorem 3.2.10. *Let $u, v \in \mathcal{S}'(\mathbb{R}^n)$ and $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Let $(\overline{x}, \overline{\xi})$ represent $(x_1, x_2; \xi_1, \xi_2)$. If the linear system*

$$\overline{x} + A(\xi, -\xi) = 0$$

has no solution such that $(x_1, \xi) \in \mathrm{WF}(u)$ and $(x_2, -\xi) \in \mathrm{WF}(v)$, then the product

$$u \star_A v := \mathrm{diag}^*(\Omega_A(u \otimes v))$$

is well-defined as a tempered distribution, and its wavefront set satisfies the inclusion

$$\begin{aligned}\mathrm{WF}(u \star_A v) \subseteq \{\, (x, \xi_1 + \xi_2) : (x, x) = \overline{x} + A\overline{\xi} \textit{ with} \\ (x_1, \xi_1) \in \mathrm{WF}(u) \cup \{0\} \textit{ and } (x_2, \xi_2) \in \mathrm{WF}(v) \cup \{0\}\}\backslash\{0\}.\end{aligned}$$

Proof. Proposition 2.1.20 implies that

$$\mathrm{WF}(u \otimes v) \subseteq ((\mathrm{WF}(u) \cup \{0\}) \times (\mathrm{WF}(v) \cup \{0\})) \setminus \{0\}.$$

Moreover, Corollary 3.1.5 gives the inclusion

$$\begin{aligned}\mathrm{WF}(\Omega_A(u \otimes v)) \subseteq \{\mathbb{B}_A(\overline{x}, \overline{\xi}) : (x_1, \xi_1) \in \mathrm{WF}(u) \cup \{0\} \\ \text{and } (x_2, \xi_2) \in \mathrm{WF}(v) \cup \{0\}\}\backslash\{0\}.\end{aligned} \tag{III.7}$$

By Theorem 2.2.14, the pullback $\mathrm{diag}^*(\Omega_A(u \otimes v))$ is well-defined if

$$\mathrm{WF}(\Omega_A(u \otimes v)) \cap \{(0, 0; \xi, -\xi)\} = \emptyset.$$

Thus, the inclusion in (III.7) provides our sufficient condition. The wavefront set inclusion follows immediately from Theorem 2.2.14. □

The following two results are particular cases of Theorem 3.2.10. The first corollary, to the best of our knowledge, is a novel result. It provides a criterion under which the τ-ordering star-product studied in this work (including the (anti-)Kohn-Nirenberg case) extends to some subspace of $\mathcal{S}'$. Note that if we set β_τ equal to zero, we recover Theorem 2.2.16, and the proof remains valid.

Corollary 3.2.11. *Let V be a symplectic vector space, and let $u, v \in \mathcal{S}'(V)$. If there is no solution to the linear system*

$$\begin{cases} x + b_\tau \xi = 0, \\ y - b_\tau^T \xi = 0, \end{cases}$$

such that $(x, \xi) \in \mathrm{WF}(u)$ and $(y, -\xi) \in \mathrm{WF}(v)$, then the star-product

$$u \star_\tau v := \mathrm{diag}^*(\Omega_\tau(u \otimes v))$$

is well-defined as a tempered distribution, and its wavefront set satisfies the inclusion

$$\mathrm{WF}(u \star_\tau v) \subseteq \Big\{ (x,\, \xi + \eta) : (x + b_\tau \eta, \xi) \in \mathrm{WF}(u) \cup \{0\} \text{ and } \Big(x + b_\tau^T \xi, \eta\Big) \in \mathrm{WF}(v) \cup \{0\} \Big\} \backslash \{0\}.$$

Proof. By Proposition 2.1.20,

$$\mathrm{WF}(u \otimes v) \subseteq ((\mathrm{WF}(u) \cup \{0\}) \times (\mathrm{WF}(v) \cup \{0\})) \setminus \{0\},$$

and by Corollary 3.1.5,

$$\mathrm{WF}(\Omega_\tau(u \otimes v)) \subseteq \{\mathbb{B}_\tau(x, y; \xi, \eta) : (x, \xi) \in \mathrm{WF}(u) \cup \{0\} \text{ and } (y, \eta) \in \mathrm{WF}(v) \cup \{0\}\} \backslash \{0\}.$$

Equivalently, we have

$$\mathrm{WF}(\Omega_\tau(u \otimes v)) \subseteq \Big\{ \Big(x - b_\tau \eta, y - b_\tau^T \xi\,; \xi, \eta\Big) : (x, \xi) \in \mathrm{WF}(u) \cup \{0\} \text{ and } (y, \eta) \in \mathrm{WF}(v) \cup \{0\} \Big\} \backslash \{0\}. \tag{III.8}$$

By Theorem 2.2.14, the pullback $\mathrm{diag}^*(\Omega_\tau(u \otimes v))$ is well-defined if

$$\mathrm{WF}(\Omega_\tau(u \otimes v)) \cap \{(0, 0; \xi, -\xi)\} = \emptyset.$$

Hence, the sufficient condition follows from (III.8). The wavefront set inclusion follows immediately from Theorem 2.2.14. □

The second corollary is exactly the result stated in [BS19, Thm. 3.1], which provides a similar sufficient condition to extend the twisted convolution product (17). Again, if we set $\theta = 0$, we recover Theorem 2.2.16.

Corollary 3.2.12. *Let* $A = \begin{pmatrix} 0 & \theta \\ \theta^T & 0 \end{pmatrix}$, *where* θ *is a skew-symmetric* $n \times n$ *matrix, and let* $u, v \in \mathcal{S}'(\mathbb{R}^n)$ *be such that the linear system*

$$x + \theta\xi = 0$$

has no solution (x, ξ) *with* $(x, \xi) \in \mathrm{WF}(u)$ *and* $(x, -\xi) \in \mathrm{WF}(v)$. *Then the twisted convolution product of* u *and* v *is well-defined as*

$$u \star_\theta v := \mathrm{diag}^*(\Omega_\theta(u \otimes v)),$$

and its wavefront set satisfies the inclusion

$$\mathrm{WF}(u \star_\theta v) \subseteq \{\, (x, \xi + \eta) : (x + \theta\eta, \xi) \in \mathrm{WF}(u) \cup \{0\} \\ (x - \theta\xi, \eta) \in \mathrm{WF}(v) \cup \{0\}\}.$$

Proof. Using the skew-symmetry of θ, we observe that

$$\mathrm{WF}(\Omega_\theta(u \otimes v)) \subseteq \Big\{ (x - \theta\eta, y + \theta\xi\,; \xi, \eta) : (x, \xi) \in \mathrm{WF}(u) \cup \{0\} \\ \text{and } (y, \eta) \in \mathrm{WF}(v) \cup \{0\} \Big\} \backslash \{0\}.$$

The rest of the proof is identical to that of Corollary 3.2.11. $\square$

3.3 Analysis of the $\mathcal{B}^N$-spaces, towards a Rieffel's type deformation formula

In the previous two sections, we have shown that the space $\mathcal{S}'_{V\times\{0\}}(V)$ behaves well under our construction (see Example 3.1.8 and Example 3.2.3). This space is of particular interest to us since it contains the space $\mathcal{B}^\infty(V) = \cup_N \mathcal{B}^N(V)$, outlined in the introduction, where

$$\mathcal{B}^N(V) := \Big\{ u \in C^\infty(V) : \forall \alpha \in \mathbb{N}^n,\ |\partial^\alpha u(y)| \leq C \langle y \rangle^N \Big\}.$$

Proposition 3.3.1. *Let* E *be an* n*-dimensional vector space. The space* $\mathcal{B}^\infty(E)$ *is contained in* $\mathcal{S}'_{E\times\{0\}}(E)$.

Proof. Let $u \in \mathcal{B}^N(E)$. We will show that $\mathrm{WF}(u) \subset E \times \{0\}$ for all $N > 0$. To do so, we will use the definition of the wavefront set WF in terms of the Short-Time Fourier Transform (STFT) (see Definition 2.1.16 and Lemma 2.1.19). Specifically, we need to prove that for any cone Γ in T^*E such that $\Gamma \cap (E \times \{0\}) = \emptyset$, we have, for any $M > 0$,

$$\int_\Gamma \langle (x,\xi)\rangle^M |\mathcal{V}_\psi(u)(x,\xi)|\, dx\, d\xi < \infty.$$

The condition on Γ implies that for all directions $(x,\xi) \in \Gamma$, there exist constants $C'_\Gamma, C_\Gamma > 0$ such that

$$\langle x\rangle \le C_\Gamma \langle \xi\rangle \quad \text{and} \quad \langle (x,\xi)\rangle \le C'_\Gamma \langle \xi\rangle . \tag{III.9}$$

Using these estimates, we perform integration by parts and use the fact that $u \in \mathcal{B}^N$. We obtain:

$$\begin{aligned}
&\int_\Gamma \langle (x,\xi)\rangle^M |\mathcal{V}_\psi(u)(x,\xi)|\, dx\, d\xi \\
&\quad = \int_\Gamma \langle (x,\xi)\rangle^M \left| \int e^{-iy\xi}\psi(y-x)u(y)\, dy \right| dx\, d\xi \\
&\quad = \int_\Gamma \langle (x,\xi)\rangle^M \left| \int \langle \xi\rangle^{-2K} e^{-iy\xi}(1-\Delta_y)^K(\psi(y-x)u(y))\, dy \right| dx\, d\xi \\
&\quad \lesssim \int_\Gamma C'_\Gamma \langle \xi\rangle^{M-2K} \int \left| \tilde{\psi}(y-x) \langle y\rangle^N \right| dy\, dx\, d\xi,
\end{aligned}$$

where $\tilde{\psi}$ is a Schwartz function corresponding to some derivative of ψ. Now, since $\tilde{\psi}$ is Schwartz, the following estimate holds for any P:

$$|\psi(y-x)| \lesssim \langle x-y\rangle^{-P} \lesssim \langle x\rangle^P \langle y\rangle^{-P} .$$

This estimate, combined with (III.9), gives the following bound:

$$\begin{aligned}
&\int_\Gamma \langle (x,\xi)\rangle^M |\mathcal{V}_\psi(u)(x,\xi)|\, dx\, d\xi \\
&\qquad\qquad \lesssim \int_\Gamma C'_\Gamma \langle \xi\rangle^{M-2K} \int \langle x\rangle^P \langle y\rangle^{N-P}\, dy\, dx\, d\xi \\
&\qquad\qquad \lesssim \int_\Gamma C_\Gamma C'_\Gamma \langle \xi\rangle^{M+P-K+n+1} \int \langle y\rangle^{N-P} \langle x\rangle^{-n-1}\, dy\, dx\, d\xi.
\end{aligned}$$

Since K and P can be chosen arbitrarily large, we choose $P \gg N$ and $K \gg M+P$, and the expression is bounded. Thus, we conclude that $\mathrm{WF}(u) \subset E \setminus \{0\} \times \{0\}$, and consequently, $\mathcal{B}^\infty \subset \mathcal{S}'_{E\times\{0\}}$. □

Proposition 3.3.1 implies that the space $\mathcal{B}^\infty(V)$ is contained in the associative algebra $\mathcal{S}'_{V\times\{0\}}(V)$.

In the following, we show that the twist Ω_τ can be restricted from the space $\mathcal{S}'_{V^2\times\{0\}}(V^2)$ to $\mathcal{B}^\infty(V^2)$, and that $\mathcal{B}^\infty(V)$ forms a subalgebra of $\mathcal{S}'_{V\times\{0\}}(V)$ for all values of τ.

Proposition 3.3.2. *For any value of τ and any $N > 0$, the twist Ω_τ stabilizes the space $\mathcal{B}^N(V^2)$,*

$$\Omega_\tau \colon \mathcal{B}^N(V^2) \to \mathcal{B}^N(V^2).$$

Proof. Let $\overline{z} = (z_1, z_2) = (q_1, p_1, q_2, p_2)$ be an element of V^2, and let $u \in \mathcal{B}^N(V^2)$, i.e., $|\partial^\alpha u(\overline{z})| \leq \langle \overline{z}\rangle^N$ for all multi-indices $\alpha \in \mathbb{N}^n$. We wish to show that for any multi-index $\alpha \in \mathbb{N}^n$, the following estimate holds:

$$|\partial^\alpha_{\overline{x}} \Omega_\tau(u)(\overline{x})| = \left| \int K_\tau(\overline{z})\, \partial^\alpha_{\overline{x}} u(\overline{z} + \overline{x})\, d\overline{z} \right| \lesssim \langle \overline{x} \rangle^N . \tag{III.10}$$

We proceed separately for the special cases $\tau = 0$ and $\tau = 1$, and for the other values of τ.

As shown in Proposition 1.2.13, when τ takes values other than 0 and 1, the kernel K_τ associated with the twist Ω_τ is the smooth function

$$K_\tau = C_\tau e^{i z_1 J_\tau z_2}, \quad \text{where} \quad J_\tau := \begin{pmatrix} 0 & \tau^{-1} I_n \\ (\tau - 1)^{-1} I_n & 0 \end{pmatrix}.$$

By performing integration by parts, we obtain the following expression for the left-hand side of (III.10) with $C_\tau > 0$:

$$C_\tau \left| \int K_\tau(\overline{z})\, \langle z_2 \rangle^{-2M_2} (1 - \Delta_{z_1})^{M_2} \left[\langle z_1 \rangle^{-2M_1} (1 - \Delta_{z_2})^{M_1} \{ \partial^\alpha_{\overline{x}} u(\overline{z} + \overline{x}) \} \right] d\overline{z} \right| .$$

Since derivatives of $\langle z_1 \rangle^{-2M_1}$ are of the form $b_{M_1}(z_1)\, \langle z_1 \rangle^{-2M_1}$, where $b_{M_1}(z_1)$ is a bounded function, and $|K_\tau(\overline{z})| = 1$, and since $u \in \mathcal{B}^N(V^2)$ by hypothesis, this yields the estimate

$$\begin{aligned} |\partial^\alpha_{\overline{x}} \Omega_\tau(u)(\overline{x})| &\lesssim \int \langle z_2 \rangle^{-2M_2} \langle z_1 \rangle^{-2M_1} \langle \overline{z} + \overline{x} \rangle^N \, d\overline{z} \\ &\lesssim \langle \overline{x} \rangle^N \int \langle \overline{z} \rangle^{N-M}\, d\overline{z}, \quad \text{where} \quad M = 2\min\{M_1, M_2\}. \end{aligned}$$

Since M can be made arbitrarily large, choosing $M \gg N$ provides the desired estimate.

When $\tau = 0$ or $\tau = 1$, the kernels K_0 and K_1 are distributional (see Proposition 1.2.13):

$$\begin{aligned} K_0(z_1, z_2) &= e^{-q_2 p_1} \delta(q_1)\delta(p_2), \\ K_1(z_1, z_2) &= e^{q_1 p_2} \delta(q_2)\delta(p_1). \end{aligned}$$

We consider the case $\tau = 0$ below; the argument for $\tau = 1$ is similar. Let us compute:

$$\begin{aligned} |\partial_{\overline{x}}^{\alpha} \Omega_{K0}(u)(\overline{x})| &= \left| \int e^{-iq_2 p_1} \delta(q_1)\delta(p_2)\, \partial_{\overline{x}}^{\alpha} u(\overline{z} + \overline{x})\, d\overline{z} \right| \\ &= \left| \int e^{-iq_2 p_1}\, \partial_{\overline{x}}^{\alpha} u(\overline{x} + (0, p_1, q_2, 0))\, dp_1\, dq_2 \right|. \end{aligned}$$

Successive integration by parts in p_1 and q_2 yields:

$$\left| \int e^{-iq_2 p_1} \langle p_1 \rangle^{-2M_1} (1 - \Delta_{q_2})^{M_1} \left[\langle q_2 \rangle^{-2M_2} (1 - \Delta_{p_1})^{M_2} \{ \partial_{\overline{x}}^{\alpha} u(\overline{x} + (0, p_1, q_2, 0)) \} \right] dp_1\, dq_2 \right|.$$

Since derivatives of $\langle q_2 \rangle^{-2M_2}$ are of the form $b_{M_2}(q_2) \langle q_2 \rangle^{-2M_2}$, where $b_{M_2}(q_2)$ is a bounded function, $|e^{iq_2 p_1}| = 1$, and $u \in \mathcal{B}^N(V^2)$ by hypothesis, this yields the estimate

$$\begin{aligned} |\partial_{\overline{x}}^{\alpha} \Omega_{K0}(u)(\overline{x})| &\lesssim \int \langle q_2 \rangle^{-2M_2} \langle p_1 \rangle^{-2M_1} \langle q_2 \rangle^{N} \langle p_1 \rangle^{N} \langle \overline{x} \rangle^{N}\, d\overline{z} \\ &\lesssim \langle \overline{x} \rangle^{N} \int \langle (p_1, q_2) \rangle^{N-M}\, dp_1\, dq_2, \end{aligned}$$

where $M = 2\min\{M_1, M_2\}$. Since M can be made arbitrarily large, choosing $M \gg N$ provides the desired estimate.

□

Corollary 3.3.3. *For any value of τ, the product $\star_\tau$ satisfies*

$$\star_\tau \colon \mathcal{B}^M \times \mathcal{B}^N \to \mathcal{B}^{M+N}.$$

In particular, $(\mathcal{B}^\infty(V), \star_\tau)$ is a graded subalgebra of $(\mathcal{S}'_{V\times\{0\}}, \star_\tau)$.

Proof. We aim to show that for any $u \in \mathcal{B}^M(V)$ and $v \in \mathcal{B}^N(V)$, the following estimate holds:

$$|\partial_x^{\alpha} (u \star_\tau v)(x)| = \left| \int K_\tau(\overline{z})\, \partial_x^{\alpha} (u(z_1 + x) v(z_2 + x))\, d\overline{z} \right| \lesssim \langle x \rangle^{M+N},$$

for any multi-index $\alpha \in \mathbb{N}^n$, and any value of τ.

Note that $\partial_x^\alpha(u(x)v(x))$ is a finite sum of products of derivatives of u and v, and hence is bounded by $\langle x\rangle^{M+N}$. In other words, the pointwise product gives $\mathcal{B}^\infty(V)$ a structure of graded algebra:

$$\mathcal{B}^M(V) \times \mathcal{B}^N(V) \to \mathcal{B}^{M+N}(V).$$

We proceed by considering the special cases $\tau = 0$ and $\tau = 1$ separately, and then deal with the remaining values of τ using the techniques from Proposition 3.3.2.

We consider the case $\tau = 0$ below; the argument for $\tau = 1$ is similar. Integration by parts yields:

$$\begin{aligned} |\partial_x^\alpha (u \star_{KN} v)(x)| = \bigg| \int & e^{-iq_2p_1} \langle p_1\rangle^{-2K_1} (1-\Delta_{q_2})^{K_1} \\ & \Big[\langle q_2\rangle^{-2K_2} (1-\Delta_{p_1})^{K_2} \{\partial_x^\alpha u(x+(0,p_1))v(x+(q_2,0))\}\Big]\, dp_1\, dq_2 \bigg|. \end{aligned}$$

This gives the estimate

$$\begin{aligned} |\partial_x^\alpha (u \star_{KN} v)(x)| &\lesssim \int \langle q_2\rangle^{-2K_2} \langle p_1\rangle^{-2K_1} \langle \overline{q}_1\rangle^{M} \langle q_2\rangle^{N} \langle x\rangle^{M+N}\, dp_1\, dq_2 \\ &\lesssim \langle x\rangle^{M+N} \int \langle p_1\rangle^{M-2K_1} \langle q_2\rangle^{N-2K_2}\, dp_1\, dq_2. \end{aligned}$$

Since K_1 and K_2 can be made arbitrarily large, choosing $K_1 \gg M$ and $K_2 \gg N$ provides the desired estimate.

When τ takes values other than 0 and 1, integration by parts gives, with $C_\tau > 0$,

$$\begin{aligned} |\partial_x^\alpha (u \star_\tau v)(x)| = C_\tau \bigg| \int & K_\tau(\overline{z}) \langle z_2\rangle^{-2K_2} (1-\Delta_{z_1})^{K_2} \\ & \Big[\langle z_1\rangle^{-2K_1} (1-\Delta_{z_2})^{K_1} \{\partial_x^\alpha u(z_1+x)v(z_2+x)\}\Big]\, d\overline{z} \bigg|. \end{aligned}$$

This implies the estimate

$$\begin{aligned} |\partial_x^\alpha (u \star_\tau v)(x)| &\lesssim \int \langle z_2\rangle^{-2K_2} \langle z_1\rangle^{-2K_1} \langle \overline{z}_1\rangle^{M} \langle z_2\rangle^{N} \langle x\rangle^{M+N}\, d\overline{z} \\ &\lesssim \langle x\rangle^{M+N} \int \langle \overline{z}\rangle^{M+N-K}\, d\overline{z}, \end{aligned}$$

where $K = 2\min\{K_1, K_2\}$. Since K can be made arbitrarily large, choosing $K \gg M+N$ provides the desired estimate.

$\square$

Proposition 3.3.2 and Corollary 3.3.3 imply that any star-product on a symplectic vector space V that can be extended to $\mathcal{S}'_{V\times\{0\}}$ using the smooth wave front set method automatically provides a graded algebra structure on $\mathcal{B}^\infty(V) = \bigcup_N \mathcal{B}^N$. If this star-product is defined by an integral formula associated with a twist K,

$$u \star v(x) = \int K(\overline{z})u(z_1 + x)v(z_2 + x)\, d\overline{z},$$

then K is a non-formal Drinfeld twist in the sense presented in the introduction, and satisfies Rieffel's hypothesis for defining a universal deformation formula. This is an important step in the direction of generalizing Rieffel's result to all such star-products, including $\star_\tau$, $\star_\theta$, and the Kohn-Nirenberg star-product $\star_{KN}$.

Conclusion and further prospects

To conclude, let us briefly review the key contributions of this work:

1. We have introduced a new definition of the smooth wave front set, highlighting its connection to the irreducible unitary representations of the Heisenberg group, realized through Kirillov's orbit method [Kir04].
2. We have developed new examples of the smooth wave front set and provided explicit demonstrations of previously unproven results in the literature.
3. We extended the work of Bahns and Schulz [BS19] by introducing a new approach based on Drinfel'd twists, thereby broadening their results to the entire τ-ordering quantization family, including the Kohn-Nirenberg and anti-Kohn-Nirenberg cases. Furthermore, we have supplied continuity arguments that were omitted in [BS19].
4. We extended each twist Ω_τ to the space $\mathcal{B}^\infty = \bigcup_N \mathcal{B}^N$ and constructed an algebra of tempered distributions under $\star_\tau$ that contains $\mathcal{B}^\infty$. This is a crucial step in Rieffel's construction of a non-formal universal deformation formula in the Weyl case.

Our results open the door to several exciting prospects:

First, our new definition of the smooth wave front set, framed through representation theory, provides a foundation for generalizing this concept beyond the Euclidean setting. There is potential for extending it to non-abelian groups, with strategies from Howe's work [How79] offering promising avenues for such adaptations.

Moreover, the use of Kirillov's orbit method to realize the unitary irreducible representations of the Heisenberg group establishes a link with quantization (see [Jag12]), a connection that remains to be fully explored. This approach may extend to Lie groups where Kirillov's method is effective, particularly nilpotent Lie groups. More broadly, we hope this work demonstrates the potential for using the smooth wave front set in the context of non-formal deformation quantization.

Finally, our extension of Bahns and Schulz's results to the Kohn-Nirenberg framework, alongside the construction of a twisted algebra that includes the $\mathcal{B}^N$-spaces, lays the groundwork for extending Rieffel's theorem to the Kohn-Nirenberg cocycle. We anticipate that these results will be instrumental in adapting the proof of Rieffel's theorem, leading to a non-formal universal deformation formula based on the Kohn-Nirenberg twist.

Appendix

A Topological Vector Spaces

The goals of this first appendix are:

1. To provide the basic background on topological vector spaces (TVS);
2. To prove a continuous extension theorem for TVS;
3. To state the results on nuclear spaces used in this work.

The first section provides the essential topological definitions in terms of filters, which are instrumental in understanding the topological nuances that arise in the spaces considered here. In particular, we address the distinctions between continuity, density, completeness, and their sequential counterparts: sequential continuity, sequential density, and sequential completeness.

In the second section, we present important results on topological vector spaces and locally convex topological vector spaces. We also prove the continuous extension theorem that is fundamental to the results in this work.

The primary references for this appendix are [Tre67] and the appendix of [Jag17]. For a more detailed and pedagogical treatment with complete proofs, we recommend [Ram18] and [Cla16].

A.1 Basic Filter Topology

Definition A.1. Let E be a set. A filter $\mathscr{F}$ is a family of subsets of E such that:

F_1 : $\emptyset \notin \mathscr{F}$;

F_2 : If $U, V \in \mathscr{F}$, then $U \cap V \in \mathscr{F}$;

F_3 : If $U \in \mathscr{F}$ and $U \subset V$, then $V \in \mathscr{F}$.

Definition A.2. A family $\mathscr{B}$ of subsets of E is a basis of a filter $\mathscr{F}$ on E if it satisfies the following two conditions:

B_1 : $\mathscr{B} \subset \mathscr{F}$, i.e., if $U \in \mathscr{B}$, then $U \in \mathscr{F}$;

B_2 : For every $U \in \mathscr{F}$, there exists $V \in \mathscr{B}$ such that $V \subseteq U$.

Proposition A.3. *A basis $\mathscr{B}$ determines a unique filter $\mathscr{F}$ on E.*

Proof. Suppose that $\mathscr{F}'$ and $\mathscr{F}$ are two filters admitting $\mathscr{B}$ as a basis. For any $U \in \mathscr{F}'$, there exists $V \in \mathscr{B}$ such that $U \subseteq V$. By B_2, $V \in \mathscr{F}$, and by F_3, $U \in \mathscr{F}$. We have shown that $\mathscr{F}' \subseteq \mathscr{F}$. The other inclusion is shown in a similar fashion, so $\mathscr{F} = \mathscr{F}'$. □

Proposition A.4. *A collection of non-empty subsets $\mathscr{B}$ of E is a basis for some filter $\mathscr{F}$ on E if and only if for every $U, V \in \mathscr{B}$, there exists $W \in \mathscr{B}$ such that $W \subseteq U \cap V$. In this case, $\mathscr{F}$ is uniquely determined by $\mathscr{B}$.*

Proof. Let $\mathscr{F}$ be the collection of all subsets U of E containing at least one element V from $\mathscr{B}$. To show that $\mathscr{F}$, as defined, is a filter on E, we need to verify conditions F_1, F_2, and F_3. Condition F_3 is straightforward. Conditions F_1 and F_2 are ensured, respectively, by the fact that $\mathscr{B}$ does not contain the empty set and that the intersection of two elements of $\mathscr{B}$ contains an element of $\mathscr{B}$. Moreover, conditions B_1 and B_2 follow directly from the definition of $\mathscr{F}$, and hence $\mathscr{B}$ is indeed a basis for $\mathscr{F}$. The uniqueness of $\mathscr{F}$ follows directly from the previous proposition. □

Definition A.5. Let (x_n) be a sequence of points in E. The filter defined by the basis composed of the subsets $S_n = \{x_n, x_{n+1}, \dots\}$ is called the *associated filter of the sequence* (x_n).

Definition A.6. A topology $\mathcal{T}$ on a set E is defined by providing, at each point $x \in E$, a filter $\mathscr{F}(x)$ such that:

T_1 : If $U \in \mathscr{F}(x)$, then $x \in U$;

T_2 : If $U \in \mathscr{F}(x)$, then there exists $V \in \mathscr{F}(x)$ such that for every $y \in V$, we have $U \in \mathscr{F}(y)$.

The filter $\mathscr{F}(x)$ is called the *neighbourhood filter of* x, and we say that E endowed with the topology $\mathcal{T}$ is a *topological space*.

Definition A.7. Let X be a subset of a topological space E.

1. X is *open* if, for every point $x \in X$, there exists a neighbourhood $V \in \mathscr{F}(x)$ such that $V \subseteq X$.
2. X is *closed* if its complement X^c is open.
3. The *closure* of X, denoted $\overline{X}$, is the smallest closed set containing X.
4. The *interior* of X, denoted X°, is the largest open set contained in X.

Definition A.8.
1. A topological space E is *first-countable* if, for every point $x \in E$, the filter $\mathscr{F}(x)$ has a countable basis.
2. A topological space E is *second-countable* if its topology is generated by a countable collection of open sets. Explicitly, E is second-countable if there exists a countable collection $\mathcal{U} = \{U_i\}_{i=0}^{\infty}$ such that any open subset of E can be written as a union of elements from some subfamily of $\mathcal{U}$.

Definition A.9.
1. A filter $\mathscr{F}$ on E converges to $x \in E$ if $\mathscr{F}(x) \subseteq \mathscr{F}$. We say that $\mathscr{F}$ is *finer* than $\mathscr{F}(x)$.
2. A sequence $(x_n) \subset E$ converges to $x \in E$ if, for every $U \in \mathscr{F}(x)$, there exists $n(U)$ such that for all $m \geq n(U)$, we have $x_m \in U$.

Proposition A.10. *A sequence $(x_n) \subset E$ converges to $x \in E$ if and only if its associated filter converges to x.*

In general, filters and sequences on a topological space E can converge simultaneously to several different points (consider two filters with a non-empty intersection on a space endowed with the trivial topology)! To avoid this undesirable feature, we often require the topological spaces we study to be *separated* or *Hausdorff*.

Definition A.11. A topological space E is *separated* (or *Hausdorff*) if for every pair of distinct points $x, y \in E$, there exist $U \in \mathscr{F}(x)$ and $V \in \mathscr{F}(y)$ such that $U \cap V = \emptyset$.

Proposition A.12. *In a separated topological space, the following hold:*

1. *Filters and sequences converge to at most one point;*
2. *Singletons are closed.*

Proposition A.13.
1. *If V is closed, then $y \in V$ if and only if, for every $U \in \mathscr{F}(y)$, we have $U \cap V \neq \emptyset$.*
2. *Let E be a topological space. If A is dense in E, then $\overline{A} = E$.*

Definition A.14. Let E be a topological space and $A \subseteq E$. The *sequential closure* $\mathrm{sc}(A)$ of A in E is defined as the set

$$\mathrm{sc}(A) := \{z \in E : \exists (a_n) \subset A \text{ converging to } z\}.$$

Because of our experience with metric spaces, we often equate the closure of a set with its sequential closure. However, in general topological spaces, limits do not necessarily capture the closure of a set. Topological spaces in which $\mathrm{sc}(A) = \overline{A}$ are called *Fréchet-Urysohn spaces.* A weaker condition is to require that every sequentially closed[1] subset is closed. Spaces satisfying this property are called *sequential spaces.* We will show below that first-countable spaces are Fréchet-Urysohn.

Proposition A.15. *Let E be a first-countable topological space and $A \subseteq E$. Then, the sequential closure of A coincides with the closure of A:*

$$\overline{A} = \mathrm{sc}(A).$$

[1] A subset A of a topological space is sequentially closed if, for any sequence $(x_n) \subset A$ converging to a point z, we have $z \in A$.

Proof. If $z \in \overline{A}$, then every neighbourhood of z intersects A non-trivially. Let $\mathcal{U} = \{U_i\}$ be a countable basis of $\mathscr{F}(z)$. We can construct a sequence $(a_n) \subset A$ converging to z by picking each a_i from $A \cap U_i$.

Conversely, if there exists a sequence $(a_n) \subset A$ converging to a point $z \in E$, then for every $U \in \mathscr{F}(z)$, there exists $n(U)$ such that for all $m > n(U)$, we have $a_m \in U$. In other words, every neighbourhood of z intersects A, and hence $z \in \overline{A}$. □

In particular, every metrizable space is Fréchet-Urysohn. Note that the standard definition of a topological manifold requires second-countability, and hence manifolds are another example of Fréchet-Urysohn spaces.

Definition A.16. Let A and B be two subsets of a topological space E.

1. A is *dense* in B if for any $x \in A$, every $U \in \mathscr{F}(x)$ satisfies $B \cap U \neq \emptyset$.
2. A is *sequentially dense* in B if for every $x \in B$, there exists a sequence in A that converges to x.

Note that a sequentially dense subset is always dense, but the reverse only holds in sequential spaces.

Definition A.17. Let $f\colon E \to F$ be a function between topological spaces, and let $\mathscr{F}$ be a filter on E. The image $f\mathscr{F}$ of the filter $\mathscr{F}$ by the function f is the unique filter on F whose basis is $\{f(V) : V \in \mathscr{F}\}$, where $f(V) = \{f(x) \in F : x \in V\}$.

Definition A.18. Let $f\colon E \to F$ be a function between topological spaces. We write $b = \lim_{x\to a} f(x)$ if for every $V \in \mathscr{F}(b)$, there exists $U \in \mathscr{F}(a)$ such that $f(U) \subseteq V$. We call b the *limit* of $f(x)$ as x tends to a.

Definition A.19. Let E and F be two topological spaces.

1. A function $f\colon E \to F$ is *continuous* at a point x if for every $V \in \mathscr{F}(f(x))$, the set $f^{-1}(V) = \{x \in E : f(x) \in V\}$ is an element of $\mathscr{F}(x)$. The function f is continuous on E if it is continuous at every point of E.

2. A function $f\colon E \to F$ is *sequentially continuous* if for every sequence $(x_n) \subset E$ converging to $x \in E$, the sequence $(f(x_n)) \subset F$ converges to $f(x) \in F$.

Proposition A.20. *Let $f\colon E \to F$ be a function between topological spaces. If f is continuous and the sequence $(x_n) \subset E$ converges to $x \in E$, then the sequence $(f(x_n)) \subset F$ converges to $f(x) \in F$. In other words, continuity implies sequential continuity.*

Again, the reciprocal of Proposition A.20 only holds in sequential spaces and, in particular, in first-countable spaces. However, we can still express the link between continuity and convergence in terms of filters.

Proposition A.21. *A function $f\colon E \to F$ between topological spaces is continuous if and only if for every filter $\mathscr{F}$ on E converging to x, the filter $f\mathscr{F}$ converges to $f(x)$.*

A.2 Topological Vector Spaces

A topological vector space (TVS) is a vector space E equipped with a topology compatible with its vector space structure, i.e., such that the maps

$$E \times E \to E; \quad (x, y) \mapsto x + y \quad \text{and} \quad \mathbb{C} \times E \to E; \quad (\lambda, x) \mapsto \lambda x$$

are continuous with respect to the usual topology on $\mathbb{C}$ and the product topologies. Morally, this implies that the topology is invariant under translations and dilations.

Proposition A.22. *Let E be a TVS. The filter of neighbourhoods $\mathscr{F}(x)$ of a point $x \in E$ is given by the family of subsets $V + x$ where $V \in \mathscr{F}(0)$.*

Thus, to study the topology of a TVS, it is sufficient to study the filter $\mathscr{F}(0)$. For example, the next proposition ([Tre67, Prop. 4.1]) allows us to verify that a topological space is separated by considering what happens near zero.

Proposition A.23. *A TVS E is separated if and only if for every point $x \neq 0$ in E, there exists a neighbourhood U of 0 such that $x \notin U$.*

Proposition A.24. *Let E be a TVS. For every $x \in E$, $\mathscr{F}(x)$ has a basis composed of closed sets.*

Proof. By Proposition A.22, it suffices to show the property for $\mathscr{F}(0)$. Let $U \in \mathscr{F}(0)$. The map $s : (x, y) \mapsto x - y$ is continuous on E. Thus, $s^{-1}(U)$ is a neighbourhood of $(0, 0)$ in $E \times E$, which must contain a square $W \times W$ where W is a neighbourhood of 0 in E. Therefore, W is such that $W - W \subset U$. We will show that $\overline{W} \subset U$. If $x \in \overline{W}$, then every neighbourhood of x, and in particular $x + W$, intersects W. This implies that there exist $y, z \in W$ such that $x + y = z$, or equivalently $x = z - y \in W - W$, proving $\overline{W} \subset U$ and concluding the proof. □

In this work, we have encountered a particular type of topological vector space, known as locally convex topological vector spaces (LCTVS). These spaces are termed 'locally convex' because they admit sufficiently many 'convex' neighbourhoods of the origin. While they are typically defined using this property, we have chosen to adopt the seminorms approach below, as this is how the topologies we dealt with in this work naturally arise.

Definition A.25. A seminorm on a complex vector space E is a non-negative function $p\colon E \to \mathbb{R}^+$ such that

1. for all $v, w \in E$, $p(v + w) \leq p(v) + p(w)$;
2. for all $\lambda \in \mathbb{C}$ and for all $v \in E$, $p(\lambda v) = |\lambda| p(v)$.

Definition A.26. A locally convex topological vector space is a vector space endowed with a family of seminorms and equipped with the coarsest topology making all these seminorms continuous.

It is clear that such a topology gives E a TVS structure. Given a family of seminorms $\{p_\alpha\}_{\alpha \in I}$ on a vector space E, we can define the quasi-ball

$$\mathbb{B}^r_{\alpha_1,\dots,\alpha_n} := \{v \in E : p_{\alpha_i}(v) < r \text{ for all } 1 \leq i \leq n\} \subseteq E$$

with $r > 0$, $n \in \mathbb{N}$, and $\{\alpha_1, \dots, \alpha_n\} \subseteq I$. These quasi-balls form a basis of the neighbourhood filter $\mathscr{F}(0)$ in the locally convex topology associated. Hence, a locally convex topological vector space is first-countable if and only if its topology can be generated by a countable family of seminorms. In general, LCTVS are not sequential.

The following proposition ([Tre67, Prop. 7.7]) is of practical importance in verifying the continuity of linear maps between locally convex spaces.

Proposition A.27. *Let V and W be two locally convex spaces. A linear map $f: V \to W$ is continuous if and only if for every continuous seminorm q on W, there exists a continuous seminorm p on V such that, for all $x \in V$, $q(f(x)) \lesssim p(x)$.*

A TVS structure enables the definition of the concepts of Cauchy sequences, Cauchy filters, and uniform continuity.

Definition A.28. A sequence (x_n) in a TVS E is a *Cauchy sequence* if for every $U \in \mathscr{F}(0)$, there exists $n(U) \in \mathbb{N}$ such that for all $n, m \geq n(U)$, $x_m - x_n \in U$.

Definition A.29. A filter on a subset A of a TVS E is a *Cauchy filter* if for every $U \in \mathscr{F}(0)$, there exists $M \in \mathscr{F}$ such that $M - M \subset U$.

Proposition A.30.

1. *A filter associated with a Cauchy sequence is called a* Cauchy filter*.*
2. *For every $x \in E$, $\mathscr{F}(x)$ is a Cauchy filter.*
3. *If $\mathscr{F}$ is a Cauchy filter and $\mathscr{F} \subset \mathscr{F}'$, then $\mathscr{F}'$ is Cauchy.*
4. *Every convergent filter is Cauchy.*

Definition A.31.

1. A subspace A of a TVS E is complete if every Cauchy filter on A converges to a point $a \in A$.
2. A subspace A of a TVS E is sequentially complete if every Cauchy sequence in A converges to a point $a \in A$.

A space in which every Cauchy sequence converges is called sequentially complete. It is clear, by Proposition A.10, that completeness implies sequential completeness. Once again, the converse holds in sequential spaces.

Definition A.32. Let A be a subset of a topological space E. The topology induced by E on A is given by the neighbourhood filters $\mathscr{F}(x) \cap A$, $x \in A$, having as a basis the collection of sets $\{U \cap A : U \in \mathscr{F}(x)\}$.

Definition A.33. Let E and F be two TVS and A a subset of E. A function $f : A \to F$ is uniformly continuous if for every neighbourhood V of 0 in F, there exists a neighbourhood U of 0 in E such that for every $x_1, x_2 \in A$,

$$x_1 - x_2 \in U \implies f(x_1) - f(x_2) \in V.$$

Proposition A.34. *Let E and F be two TVS, and A a subspace of E. Every continuous linear map $f : A \to F$ is uniformly continuous. In particular, every continuous linear map from E to F is uniformly continuous.*

Proof. By continuity, for every $V \in \mathscr{F}(0_F)$, there exists $U \in \mathscr{F}(0_E)$ such that $U = f^{-1}(V)$. Thus, for every $x_1, x_2 \in A$, if $x_1 - x_2 \in U$, then $f(x_1) - f(x_2) \in V$. Since f is linear by assumption, it follows that $f(x_1) - f(x_2) \in V$. □

Proposition A.35. *Let E and F be two TVS, $f : A \subseteq E \to F$ a uniformly continuous function, and $\mathscr{F}$ a Cauchy filter on A. Then, the filter $f\mathscr{F}$ on F is a Cauchy filter.*

Proof. As a reminder, a basis of $f\mathscr{F}$ is given by $\{f(M) \subset F : M \in \mathscr{F}\}$. By uniform continuity, for every $V \in \mathscr{F}(0_F)$, there exists $U \in \mathscr{F}(0_E)$ such that for all $x_1, x_2 \in A$ with $x_1 - x_2 \in U$, we have $f(x_1) - f(x_2) \in V$. Since $\mathscr{F}$ is Cauchy, there exists $M \in \mathscr{F}$ such that for all $x_1, x_2 \in M$, we have $M - M \subseteq U$. Therefore, $f(M) - f(M) \subseteq V$, and since $f(M) \in f\mathscr{F}$, it follows that $f\mathscr{F}$ is Cauchy. □

Remark A.36. The space $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ endowed with the Γ-topology introduced in the second chapter of this work is a locally convex topological vector space whose topology is defined by an uncountable set of seminorms. Since this space is a priori not sequential, we need to be careful not to deduce too much from sequential properties. Especially since most topological results in the existing literature are formulated in terms of the Hörmander topology, which is in fact just a criterion for the convergence of sequences.

Continuous Extension Theorem

Let E and F be two TVS, A a dense subspace of E, and $f : A \to F$ a uniformly continuous function. When F is complete, it is natural to hope that for reasonable topologies, the function f might be continuously extended from A to E by

$$\overline{f} : E \to F \; ; \quad \overline{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \lim\limits_{a \to x} f(a) & \text{if } x \in E \setminus A. \end{cases}$$

In this section, we demonstrate such a result in the case where E and F are Hausdorff topological vector spaces. The reader may note that a more general formulation of the continuous extension theorem exists for uniform spaces (see, for example, [Jam87, Prop. 12.13]).

Proposition A.37. *Let f and g be two continuous functions from a topological space X into a TVS E. The set $A = \{x \in X : f(x) = g(x)\}$ is closed.*

Proof. The function $D : x \mapsto f(x) - g(x)$ is continuous, and $\{0\}$ is closed in E. Consequently, since D is continuous, the set $A = D^{-1}(\{0\})$ is closed. □

Corollary A.38. *Let f, g, X, and E be as in the previous proposition. If f and g are equal on a dense subset of X, then they are equal at every point of X.*

Proof. This is an immediate consequence of the previous proposition. Since the set $A = \{x \in X : f(x) = g(x)\}$ is dense and closed in X, we have $A = \overline{A} = X$. □

Theorem A.39. *Let E and F be two TVS, A a dense subset of E, and $f : A \to F$ a uniformly continuous function. If F is complete, then there exists a unique continuous function $\overline{f} : E \to F$ such that $\overline{f}(x) = f(x)$ for all $x \in A$.*

Proof. The uniqueness of $\overline{f}$ follows directly from Corollary A.38.

To prove its existence, let $x \in E$. Since A is dense in E, every neighbourhood $U \in \mathscr{F}(x)$ has a non-empty intersection with A. Thus, the collection $\{U \cap A : U \in \mathscr{F}(x)\}$ forms a base for a filter on A, denoted $\mathscr{F}(x) \cap A$. Since $\mathscr{F}(x)$ converges and is therefore Cauchy on E, $\mathscr{F}(x) \cap A$ is a Cauchy filter on A. By Proposition A.35, $f(\mathscr{F}(x) \cap A)$ is a Cauchy filter, and since F is a complete TVS, this filter converges to a unique limit in F, denoted $\overline{f}(x)$.

If $x \in A$, $\mathscr{F}(x) \cap A$ is the filter of neighbourhoods of x for the induced topology, and hence it converges. By the continuity of f on A, the filter $f(\mathscr{F}(x) \cap A)$ converges to $f(x)$, implying that $\overline{f}(x) = f(x)$.

Now, we show the continuity of the function $\overline{f}$. Let $x \in E$ and V be a closed neighbourhood of $\overline{f}(x)$. (By Proposition A.24, we can assume that V is closed without loss of generality.) By the definition of $\overline{f}(x)$, $V \in f(\mathscr{F}(x) \cap A)$. Consequently, there exists $U \in \mathscr{F}(x)$ such that $U \cap A \subset \overline{f}^{-1}(V)$. The goal is to show that $\overline{U \cap A} \subset \overline{f}^{-1}(V)$, and then prove that $\overline{U \cap A} = \overline{U}$, which will demonstrate the continuity of $\overline{f}$ since $\overline{U}$ is a neighbourhood of x.

Let $y \in \overline{U \cap A}$. For every $U' \in \mathscr{F}(y)$, the intersection $U' \cap U \cap A$ is non-empty. Thus, these subsets form the base of a filter $\mathcal{G}$ on E, which is obviously Cauchy as it is finer than $\mathscr{F}(y) \cap A$, and by uniform continuity, $f(\mathcal{G})$ is a Cauchy filter converging to $\overline{f}(y)$. Hence, every neighbourhood of $\overline{f}(y)$ contains a set of the form $f(U' \cap U \cap A)$ and therefore also contains points of $f(U \cap A)$, which is a subset of the neighbourhood V of $\overline{f}(x)$. Thus, every neighbourhood of $\overline{f}(y)$ intersects V. Since V is closed, this implies $\overline{f}(y) \in V$, and hence $\overline{U \cap A} \subset \overline{f}^{-1}(V)$.

Without loss of generality, we can assume that U is an open neighbourhood of x. It is clear that $\overline{U \cap A} \subset \overline{U}$. To prove $\overline{U} \subset \overline{U \cap A}$, let $y \in \overline{U}$. By the definition of closure, every open neighbourhood U' of y intersects $U \cap A$. Hence, every open neighbourhood of y has a non-trivial intersection with $U \cap A$, which implies $y \in \overline{U \cap A}$. Therefore,

$$\overline{U} = \overline{U \cap A} \subset \overline{f}^{-1}(V).$$

This shows the continuity of $\overline{f}$ and completes the proof. □

Remark A.40. If A is a sequentially dense subset of E but F is only sequentially complete, then the theorem yields a unique sequentially continuous map

$$\overline{f} : E \to F.$$

B The orbit method

The orbit method is a procedure whose ambition is to associate to every coadjoint orbit of a Lie group, a unitary representation of this Lie group. The method was first introduced in 1962 by Kirillov for nilpotent Lie groups [Kir62] but as since been extended to more general frameworks. This appendix gives a short presentation of the method and presents in an exhaustive fashion the case of the Heisenberg group for which it highlights a one-to-one correspondence between coadjoint orbits and unitary representations of the group. We start with a review of some basic notions of differential and symplectic geometry. Our main references for this appendix are Kirillov's lectures [Kir04] and the review presented in [Jag12]. We also recommend [Lee12] for a comprehensive presentation of the necessary background in differential geometry and Lie theory, and [CFM21] for the concepts in symplectic and Poisson geometry.

B.1 Preliminaries

Differential and symplectic geometry

We will call *manifold* a paracompact and Hausdorff smooth manifold. Let G be a Lie group. We denote $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}^*$ the dual of $\mathfrak{g}$, $\exp : \mathfrak{g} \to G$ the exponential map and $e \in G$ its neutral element. If M and N are two manifolds and $f\colon M \to N$ a smooth map, we also denote, for a point $x \in M$,

$$f_{*_x} : T_xM \to T_{f(x)}N$$

the differential of f at x. Any group G acts on itself by conjugation

$$C\colon G \times G \to G\ , \quad C_g g' = gg'g^{-1}.$$

Differentiating C yields an action of G on its Lie algebra $\mathfrak{g}$ called *the adjoint action.* Then, one defines *the coadjoint action* of G on $\mathfrak{g}^*$ by duality. These actions yields representations of the Lie group G, respectively on the vector spaces $\mathfrak{g}$ and $\mathfrak{g}^*$.

Definition B.1. 1. The adjoint representation of G is defined by

$$\begin{aligned}&\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}\\&(g,X)\mapsto \mathrm{Ad}_g X \coloneqq C_{g*e}X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 g\exp(tX)g^{-1}.\end{aligned}$$

2. The coadjoint representation of G is defined by

$$\begin{aligned}&\mathrm{Ad}^\flat\colon G\times\mathfrak{g}^*\to\mathfrak{g}^*\\&(g,\xi)\mapsto \mathrm{Ad}^\flat_g\,\xi \coloneqq \Big[\mathfrak{g}\to\mathbb{R}\ ;\ X\mapsto \Big\langle \xi,\mathrm{Ad}_{g^{-1}}X\Big\rangle\Big].\end{aligned}$$

Definition B.2. Let M be a manifold on which G acts smoothly

$$G\times M \xrightarrow{C^\infty} M\ ;\quad (g,x)\mapsto g\cdot x.$$

1. The fundamental vector field of the action τ associated to an element $X\in\mathfrak{g}$ is the vector field X^* on M defined by
$$X^*_x \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \exp(tX)\cdot x.$$
2. If the action of G is transitive, we say that M is a *G-homogeneous space*.
3. Let N be another manifold on which G acts smoothly. A map $F : M\to N$ is called G-equivariant if for each $g\in G$ and $x\in M$,
$$F(g\cdot x) = g\cdot F(x).$$

Definition B.3. Let M be a manifold and G a Lie group. A principal G-bundle over M is a pair (P,π) where P is a manifold endowed with a free action of G and $\pi : P\to M$ a smooth surjective map called the *projection*, satisfying the following conditions:

1. For all $x\in M$ and $p\in\pi^{-1}(x)$, $\pi^{-1}(x) = \{p\cdot g : g\in G\}$. The set $\pi^{-1}(x)$ is called the *fiber* of P above x.
2. For all $x\in M$, there exists a neighbourhood $U\subset M$ of x and a diffeomorphism
$$\phi\colon U\times G\to\pi^{-1}(U)$$
such that for all $(y,g)\in U\times G$,
$$(\pi\circ\phi)(y,g) = y \quad\text{and}\quad \phi(y,gg') = \phi(y,g)\cdot g'.$$
The diffeomorphism ϕ is called a *local trivialisation* of P.

The following two theorems ([Lee12, Thm. 21.17 and Thm. 21.18]) characterise homogeneous spaces as quotients of Lie groups by a closed subgroup.

Theorem B.4. *Let G be a Lie group and K a closed subgroup of G. The left coset space G/H admits a unique structure of smooth manifold such that the quotient map*

$$\pi\colon G \to G/H$$

is a smooth submersion. The left action of G on G/H

$$g_1 \cdot (g_2 H) = (g_1 g_2 H)$$

gives to G/H the structure of a G-homogeneous space. In other words, $\pi\colon G \to G/H$ defined a principal H-bundle over G/H.

Theorem B.5. *Let G be a Lie group, M a G-homogeneous space, and let x be any point in M. The stabiliser $K = \operatorname{stab}(x)$ of x under the action of G is a closed subgroup of G and the map $F\colon G/K \to M$ defined by*

$$F(gK) = \tau_g x$$

is an equivariant diffeomorphism.

Definition B.6.

1. A symplectic manifold is a pair (M, ω) where M is a manifold and ω a closed and non-degenerate 2-form called the symplectic form.

2. A symplectomorphism between two manifolds (M_1, ω_1) and (M_2, ω_2) is a diffeomorphism

$$F\colon M_1 \to M_2 \quad \text{s.t.} \quad F^*\omega_2 = \omega_1.$$

Let (M, ω) be a symplectic manifold. Since ω is non-degenerate, there is, for all $x \in M$, a natural isomorphism $T_xM \simeq T_x^*M$ defined by the symplectic form:

$$\begin{aligned} &v \in T_xM \mapsto v^\flat \in T_x^*M, \\ &\langle v^\flat, w\rangle := \omega_x(v, w),\, w \in T_xM. \end{aligned}$$

This isomorphism extends to the space of smooth sections of TM and T^*M (i.e., between vector fields and one-forms), respectively denoted $\Gamma^\infty(TM)$ and $\Omega(M)$:

$$X \in \Gamma^\infty(TM) \mapsto X^\flat = \omega(X,\cdot) \in \Omega(M)$$

Reciprocally, to a one-form $\xi \in \Omega(M)$, one associates a unique vector field $\xi^\sharp$ defined as the unique vector fields such that $\omega(\xi^\sharp, Y) = \xi(Y)$ for all $Y \in \Gamma^\infty(M)$. This is known as the musical isomorphism. Note that the musical signification of $\flat$ and $\sharp$ matches the physical conventions for indices. Indeed, in physics going from vector field to one-form is represented by a lowering of coordinate indices $X^i \to X_i$ and the passage from one-form to vector fields by a raise of coordinate indices $X_i \to X^i$. In musical theory, a $\flat$ lowers a note by half a tone and a $\sharp$ raises it by half a tone.

Definition B.7. A Poisson algebra is an algebra $\mathcal{A}$ equipped with a Lie structure $[\cdot,\cdot]$ such that

$$[AB, C] = A[B, C] + [A, C]B, \quad \text{for all } A, B, C \in \mathcal{A}.$$

Definition B.8. A Poisson structure on a manifold M is a map

$$\{\cdot,\cdot\} \colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

such that $(C^\infty(M), \{\cdot,\cdot\})$ is a Poisson algebra. A manifold endowed with a Poisson structure is a Poisson manifold.

Proposition B.9. *A symplectic manifold (M,ω) is automatically endowed with a Poisson structure given by*

$$\{f, g\} = \omega(X_f, X_g) \text{ , for all } f, g \in C^\infty(M).$$

Example B.10 (Cotangent bundles)**.** The cotangent bundle of any manifold M is canonically endowed with a symplectic structure. Consider the projection

$$\pi \colon T^*M \mapsto M$$

and the one-form $\theta \in \Omega(T^*M)$ defined by

$$\theta_x \colon T_x(T^*M) \to \mathbb{R} \text{ ,} \quad v \mapsto \langle x, \pi_{*_x}(v)\rangle.$$

The canonical one-form θ is called the Liouville's one-form. The exterior derivative of θ yields a 2-form $\omega := \mathrm{d}\theta \in \Omega^2(T^*M)$ which happens to be symplectic (see for example [Mei98] for details).

Example B.11. [Coadjoint orbits] If G is a Lie group we can define the coadjoint orbit of a point $\xi_0 \in \mathfrak{g}^*$, with regards to Definition B.1, as the set

$$\mathcal{O}_{\xi_0} = \mathrm{Ad}^\flat_G(\xi_0) := \left\{ \xi \in \mathfrak{g}^* : \exists g \in G \text{ s.t. } \mathrm{Ad}^\flat_g(\xi_0) = \xi \right\}.$$

By Theorem B.5, as a G-homogeneous space, $\mathcal{O}_{\xi_0} = G/K$ where

$$K := \mathrm{stab}_{\xi_0}(\mathrm{Ad}^b) = \left\{ k \in G \; \mathrm{Ad}^b_k(\xi_0) = \xi_0 \right\}.$$

It is equipped with the canonical symplectic form defined for all $x, y \in \mathfrak{g}$ and $\xi \in \mathcal{O}_{\xi_0}$ by

$$\omega_\xi(X^*_\xi, Y^*_\xi) = -\langle \xi, [x, y] \rangle$$

where X^* denotes the fundamental vector field on $\mathcal{O}_{\xi_0}$ for the coadjoint action

$$X^*_\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{Ad}^\flat_{\exp(tX)} \xi.$$

This form is known as the *Kirillov-Kostant-Souriau (KKS) symplectic form.*

B.2 Condensed overview of the method

Induced representation Let G be a Lie group, B a closed subgroup of G and (ρ, V) a representation of B on a finite dimensional vector space V. Since B is closed, Theorem B.4 ensures that G/B is endowed with a manifold structure such that the projection

$$\pi \colon G \to G/B \;, \quad g \mapsto gB$$

defines a B-principal bundle structure over G/B. Let $E_\rho := G \times_\rho V$ be the associated vector bundle through the representation (ρ, V), i.e. the set

$$E_\rho = \{[(g, v)] : g \in G, v \in V\}$$

of equivalence classes $[(G, V)]$ defined by the B action,

$$[(g, v)] = \left\{ \left(gb, \rho^{-1}(b) v \right) : b \in B \right\},$$

with the vector bundle structure defined by the projection

$$p \colon E_\rho \to G/B \;, \quad [(g, v)] \mapsto \pi(g).$$

Note that smooth sections of E_ρ are in one-to-one correspondence with B-equivariant smooth functions:

$$\Gamma^\infty(E_\rho) \simeq C^\infty(G,V)^B \coloneqq \left\{\varphi \in C^\infty(G,V) : \varphi(gb) = \rho(b)^{-1}\varphi(g)\right\}.$$

Definition B.12. The left action of G on $C^\infty(G,V)^B$, given by

$$\begin{aligned} \lambda : G \times C^\infty(G,V)^B &\to C^\infty(G,V)^B \\ (g,f) &\mapsto \lambda_g f \coloneqq \left[g' \mapsto f(g^{-1}g')\right], \end{aligned} \tag{B.1}$$

defines a representation of G on $C^\infty(G,V)^B$. This representation is sometimes called the induced representation from B to G.

Suppose that V is endowed with an inner product $\langle\cdot,\cdot\rangle_V$ and that ρ is a unitary representation of B on V for this inner product, i.e. $\langle\rho(b)v,\rho(b)w\rangle_V = \langle v,w\rangle_V$, for all $b \in B$ and $v,w \in V$. Suppose moreover that there exists a G-invariant volume form $\mathrm{d}y$ on $N \coloneqq G/B$ (a condition for the existence of such a measure is given in [BG15, Lem. 6.17]). Then the space $C_c^\infty(G,V)^B$ of smooth compactly supported B-equivariant functions on G inherits a prehilbertian structure defined by

$$\langle\varphi,\psi\rangle \coloneqq \int_N \langle\varphi(y),\psi(y)\rangle_V \ \mathrm{d}y.$$

This inner product is invariant under the action of G given by (B.1). If we denote $\mathcal{H}$ the Hilbert space defined as the completion of $C_c^\infty(G,V)_c^B$ for $\langle\cdot,\cdot\rangle$, (B.1) induced a unitary representation of G on $\mathcal{H}$ denoted $\mathrm{Ind}_B^G(\rho)$ and called the *unitary induced representation from B to G.*

Kirillov correspondence Kirillov's idea has been to look at all the representations induced by the stabilisers of coadjoint orbits of a Lie group G and compare them to the unitary dual[2] $\widehat{G}$ of G. Let $\mathcal{O}_{\xi_0}$ be the coadjoint orbit of a point $\xi_0 \in \mathfrak{g}^*$, K the stabiliser of ξ_0 under the coadjoint

[2] The set of unitary equivalence classes of irreducible unitary representations of G. It admits a topological structure. For details see [Kir04]

action and $\mathfrak{k}$ the Lie algebra of K. Note that

$$\begin{aligned}\mathfrak{k} &:= \left\{X \in \mathfrak{g} \mid \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{Ad}^\flat_{\exp(tX)}\xi_0 = 0\right\}\\ &= \left\{X \in \mathfrak{g} \mid \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left\langle \mathrm{Ad}^\flat_{\exp(tX)}\xi_0, Y\right\rangle = 0, \text{ for all } Y \in \mathfrak{g}\right\}\\ &= \left\{X \in \mathfrak{g} \mid \left\langle \xi_0, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{Ad}_{\exp(-tX)}Y\right\rangle = 0, \text{ for all } Y \in \mathfrak{g}\right\}\\ &= \{X \in \mathfrak{g} \mid \langle \xi_0, [x,y]\rangle = 0, \text{ for all } Y \in \mathfrak{g}\}\,.\end{aligned}$$

Since $\mathcal{O}_{\xi_0}$ is a symplectic manifold (see Example B.11), it has even dimension and so has $\mathfrak{g}/\mathfrak{k}$.

Definition B.13. A real polarisation associated to ξ_0 is a sub-Lie algebra $\mathfrak{b}$ of $\mathfrak{g}$ such that

1. $\mathfrak{b}$ is stable under Ad_K ;
2. for all $x, y \in \mathfrak{b}$, $\langle \xi_0, [x,y]\rangle = 0$;
3. $\dim \mathfrak{g}/\mathfrak{b} = \frac{1}{2}\dim \mathfrak{g}/\mathfrak{k}$.

A polarisation $\mathfrak{b}$ associated to ξ_0 yields a Lie algebra homomorphism

$$\xi_0 \colon \mathfrak{b} \to \mathbb{R}\,, \quad X \mapsto \langle \xi_0, X\rangle\,.$$

Suppose that the associated Lie subgroup $B := \exp(\mathfrak{b})$ is closed and write $\log(b)$ the unique element of $\mathfrak{g}$ such that $\exp(\log(b)) = b$. Moreover, suppose that ξ_0 extends to a group homomorphism between B and the unitary group $U(1)$,

$$\chi_{\xi_0} \colon B \to U(1)\,, \quad \chi(b) \mapsto e^{i\langle \xi_0,\, \log(b)\rangle}.$$

satisfying $\chi_{*_e} = \xi_0$. The corresponding representation

$$\chi_{\xi_0} \colon B \times \mathbb{C} \to \mathbb{C}\,, \quad (b,z) \to e^{i\langle \xi_0,\, \log(b)\rangle} z$$

is unitary. Under the same conditions that in the previous paragraph, one can construct the unitary induced representation from B to G, $\mathrm{Ind}_B^G(\chi_{\xi_0})$.

Definition B.14. Let $\mathcal{O}(G)$ denote the set of all coadjoint orbits of G and $\widehat{G}$ its unitary dual. The mapping

$$\mathcal{O}(G) \to \widehat{G}, \quad \mathcal{O}_{\xi_0} \mapsto \operatorname{Ind}_B^G(\chi_{\xi_0})$$

is called the *Kirillov correspondence.*

Note that all the suppositions made throughout this presentation hold for connected, simply connected nilpotent Lie groups. For other classes of groups, special attention has to be paid to each of these assumptions (see [Kir04] for a complete overview). The following theorem is a refinements of the result presented in [Kir62] due to Brown [Bro73].

Theorem B.15. *For any connected, simply connected nilpotent Lie group the Kirillov correspondence is a homeomorphism between the unitary dual of the group and the orbit space with its quotient topology.*

B.3 The Heisenberg group

In this section, we apply the orbit method to the Heisenberg group. Let (V, ω) be a symplectic vector space of dimension $2n$ and Z a direction in V. We consider the associated Heisenberg algebra $\mathfrak{h}_n = V \oplus \mathbb{R}Z$ with the Lie bracket given by $[v, v'] = \omega(v, v')Z$, the element Z being central. The corresponding simply connected Lie group $\mathbb{H}_n$ is a copy of $\mathfrak{h}_n$ equipped with the action

$$(v, \mu) \cdot (v', \mu') = \left(v + v', \mu + \mu' + \frac{1}{2}\omega(v, v')\right).$$

We introduce the musical isomorphism induced by ω

$$^\flat : V \to V^* , \quad {}^\flat v(v') = \omega(v, v')$$

which extends to the isomorphism

$$^\flat : \mathfrak{h} \to \mathfrak{h}^* , \quad {}^\flat(v, \mu)(v', \mu') = \omega(v, v') + \mu\mu'.$$

Coadjoint orbits One computes

$$\operatorname{Ad}^\flat_{(v',\mu')}({}^\flat(v, \mu)) = {}^\flat(v - \mu v', \mu).$$

Hence, the coadjoint orbits of $\mathbb{H}_n$ are of two types: if $\mu = 0$ they are reduced to a single point and we denote them $\mathcal{O}_v$, if $\mu \neq 0$ they are copies of V indexed by the value of μ and we denote them $\mathcal{O}_\mu$.

Induced representations In the spirit of the orbit method, we construct the irreducible unitary representation of $\mathbb{H}_n$ associated to each of its coadjoint orbit. For a non-trivial orbit $\mathcal{O}_\mu$, let $\xi_0 := \mu^\flat Z \in \mathcal{O}_\mu$ and $\mathfrak{p} \subset V$ be a maximal Lagrangian subspace which defines a real polarisation $\mathfrak{b} = \mathfrak{p} \oplus \mathbb{R}Z$. We denote $B := \exp(\mathfrak{b})$ the associated subgroup. Because $\mathfrak{p}$ is Lagrangian in V, the KKS form

$$\omega_{\xi_0}(x, y) := \langle \xi_0, [x, y] \rangle$$

vanishes on $[\mathfrak{b},\mathfrak{b}]$. Hence, $\xi_0 \colon \mathfrak{b} \to \mathbb{C}$ is an homomorphism of Lie algebras inducing the representation of B defined by the multiplicative character

$$\chi \colon B \to \mathbb{C}\ , \quad \chi(b) \mapsto e^{i\langle \xi_0,\, \log(b)\rangle}.$$

We define $\mathfrak{q} \subset V$ as the Lagrangian subset satisfying $\mathfrak{p} \oplus \mathfrak{q} = V$. We denote $Q := \exp(\mathfrak{q})$ the abelian subgroup associated. Hence $Q = \mathbb{H}_n/B$ and $\mathbb{H}_n$ admits a structure of B-principal bundle over $\mathbb{H}_n/B$ with the projection

$$\pi \colon \mathbb{H}_n \to Q\ , \quad ((q + p), \mu) \mapsto q.$$

The vector bundle associated to this principal bundle through the representation χ is denoted

$$E_\chi := G \times_\chi \mathbb{C}.$$

One has the linear isomorphisms

$$\Gamma^\infty(E_\chi) \simeq C^\infty(G, \mathbb{C})^B \simeq C^\infty(Q).$$

through which, the $\mathbb{H}_n$-action (B.1) on $C^\infty(Q)$ reads

$$U_\mu(qb)u(q_0) = e^{-i\mu(z+\omega(q-q_0,\,p))}u(q_0 - q).$$

The latter induces a unitary representation of $\mathbb{H}_n$ on $L^2(Q)$ which is precisely $\mathrm{Ind}_B^G(\chi)$. The trivial point-orbits yields 1-dimensional representations given by

$$\rho_v(v', \mu') = e^{i\langle {}^\flat v,\, (v', \mu')\rangle} = e^{i\omega(v, v')}.$$

By Theorem B.15, the Kirillov correspondence

$$\mathcal{O}_\mu \mapsto U_\mu \in \widehat{\mathbb{H}}_n$$
$$\mathcal{O}_v \mapsto \rho_v \in \widehat{\mathbb{H}}_n$$

is an homeomorphism between the space of coadjoint orbits $\mathcal{O}(\mathbb{H}_n)$ and the unitary dual $\widehat{\mathbb{H}}_n$ of $\mathbb{H}_n$.

List of recurring notations and symbols

Roman letters

a^{KN}	Kohn-Nirenberg operator associated with the symbol a, 39
a^W	Weyl operator associated with the symbol a, 39
$\mathbb{B}_\tau$	Symplectic transformation associated to the twist Ω_τ, 81
$\mathbb{B}_A$	Symplectic transformation associated to the operator Ω_A, 83
$\mathrm{char}(a)$	the set of characteristic points of a symbol a, 47
D_j	short for $-i\partial/\partial x_j \neq \partial_j := \partial/\partial x_j$, 28
diag	the diagonal map $x \mapsto (x,x)$, 69
$\mathrm{ell}(a)$	the set of elliptic points of a symbol a, 47
$\mathbb{H}_n$, $\mathfrak{h}_n$	the n-dimensional Heisenberg group and its Lie algebra, 56
$\mathcal{H}$	Hilbert space, 3
$\mathcal{F}(f) = \hat{f}$	Fourier transform of f, 28
$\mathscr{L}(\mathcal{H})$	space of linear operators on a Hilbert space $\mathcal{H}$, 4
Q_τ	interpolation family of quantizations, 39

$\mathcal{V}_\psi$	Short-Time Fourier Transform (STFT) with window function ψ, 53
$\mathcal{T}_\Gamma$	locally convex topology on $\mathcal{S}'_\Gamma$, 72
S^{n-1}	the unit sphere in $\mathbb{R}^n$, 47
$U(\mathcal{H})$	the space of unitary operators on a Hilbert space $\mathcal{H}$, 57
WF	the smooth wave front set, 47
$\widetilde{\mathrm{WF}}$	the smooth wave front set in terms of representations of $\mathbb{H}_n$, 59
WF$'$	The smooth wave front set in terms of the STFT, 54

Greek letters

δ, δ_{x_0}	the delta distribution at 0 and at a point x_0, 24
Δ_x	the Laplacian operator, $\Delta_x \coloneqq \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, 36
$\Pi(x,\xi)$	time-frequency shift, $\Pi(x,\xi)\phi(y) \coloneqq M_\xi T_x \phi(y)$, 27
ω	symplectic form, 114
Ω_θ	Twist associated with the twisted convolution product $\star_\theta$, 83
Ω_A	Unitary operator associated to the symmetric matrix A, 83

Functions and distributions spaces

$\mathcal{B}^N(G,E)$	smooth functions on a Lie group G valued in a locally convex topological vector space E that are polynomially bounded along with all their derivatives, 15
$\mathcal{B}^\infty(G,E)$	the union of all the $\mathcal{B}^N(G,E)$ spaces, 15
$C^\infty(M)$	smooth functions on a manifold M, algebra of observables, 3
$C_c^\infty(\mathbb{R}^n)$	the space of smooth compactly supported functions, 23

$C^\infty(G,V)^B$	B-equivariant smooth functions on G valued in V, 117
$\mathcal{D}(\mathbb{R}^n)$	the space of smooth compactly supported functions, 23
$\mathcal{D}'(\mathbb{R}^n)$	the space of continuous linear functionals on $\mathcal{D}(\mathbb{R}^n)$, 23
$G^m_\rho(\mathbb{R}^n)$	the space of Shubin symbols of order m and type ρ, 36
$G^m(u)$	the set of Shubin symbols of order m regularizing u, 46
$G(u)$	the set of Shubin symbols of any order regularizing u, 46
G^Γ	the set of symbols regularizing every distribution in $\mathcal{S}'_\Gamma$, 72
$L^2(\mathbb{R}^n)$	the space of square-integrable functions, 5
$\mathcal{O}(\mathbb{R}^n)$	polynomially bounded smooth functions with polynomially bounded derivatives, 26
$\mathcal{S}(\mathbb{R}^n)$	the Schwartz space, 22
$\mathcal{S}'(\mathbb{R}^n)$	the space of tempered distributions, 23
$\mathcal{S}'_\Gamma(\mathbb{R}^n)$	tempered distribution with WF $\subseteq \Gamma$, 65

Other Symbols

$\star_\theta$	the twisted convolution product, 17
$\star_A$	Product associated with the unitary operator Ω_A, 85
∂_i	short for $\partial/\partial x_i$, the partial derivative in the i^{th} coordinate, 22
∂^α	$\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, for $\alpha \in \mathbb{N}^n$, 22
$(\,\cdot\,,\,\cdot\,)$	pairing of a distribution with a test function, 24
$[\,\cdot\,,\cdot\,]$	commutator or Lie bracket, 4
$\langle\,\cdot\,,\cdot\,\rangle$	usual inner product on L^2 or on a finite dimensional vector space depending on the context, 24

$|\cdot\rangle$, $\langle\cdot|$, $\langle\cdot\,|\,\cdot\rangle$ the Dirac bracket notation, compatible with the inner product on the state space $L^2(Q)$, 58

$\{\cdot,\cdot\}$ Poisson bracket, 2

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