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The Three-step method in a dynamic setting

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Abstract

A crucial issue in a dynamic framework, is how risk valuations at different times are interrelated. In this regard, the notion of time consistency was widely introduced and discussed in the literature. A time-consistent dynamic valuation is a pricing method according to which a product that will be, in almost all states of nature, cheaper than another one at a future date should already be cheaper today. This paper aims to construct a time-consistent, dynamic version of the Three-step method introduced in [Deelstra et al., 2020] for hybrid life Pure Endowment products, employing a backward iteration scheme. The backward scheme is illustrated in a dual-iteration approach using a Pure Endowment product without profit sharing. Furthermore, we explore the continuous-time limit of the backward scheme, incorporating profit-sharing into the Pure Endowment to investigate a hybrid life payoff. Our analysis demonstrates that the presence of the diversifiable component undermines the time-consistency of the dynamic Three-step method. Consequently, the time-consistent price of the actuarial part shows a notable increase. To address this, and in accordance with [Devolder and Lebègue, 2016], we present a reduced time-consistent variant by decreasing the safety loads in each iterative step of the backward scheme.

Keywords:

Premium principles, time-consistent, fair dynamic valuation, backward iteration scheme, hybrid life payoff.

1 Introduction

Valuation methods serve as quantitative instruments for assessing the liabilities of an insurance company and must adhere to regulatory frameworks. In particular, the Solvency 2 regulation requires that fairness be preserved in both the financial and actuarial markets. This entails ensuring adherence to the market-consistent property and the actuarial-consistent property. In a complete financial market without arbitrage opportunities, the market-consistent property guarantees the valuation of purely financial payoffs by employing the risk-neutral measure. Market-consistent valuations have been largely studied and we refer to [Malamud et al., 2008] for an overview. In continuation of this theoretical concept, [Pelsser and Stadje, 2014] has proposed a valuation technique referred to as the two-step method. This method systematically converts individual conditional valuations into market-consistent valuations, thereby enabling the generation of a diverse set of market-consistent valuations. Moreover, it was shown in [Dhaene et al., 2017] that market consistent valuations can be characterised by the hedge-based valuations, which under the assumption of market completeness corresponds with the two-step valuations.

Nonetheless, there has been recent critique in the literature regarding market-consistency in the evaluation of long-term insurance liabilities, as discussed, for example, by [Le Courtois et al., 2020]. Based on pooling and diversification arguments, the actuarial-consistent property asserts that applying insurance premium principles is essential in the valuation of purely actuarial products; see also [Barigou et al., 2023].

The life insurance industry is increasingly directing its attention towards hybrid products distinguished by the presence of both hedgeable financial risks and non-hedgeable actuarial risks. An illustrative category of hybrid products includes participating life insurance contracts, wherein the insurer initially extends a return guarantee and subsequently offers the policyholder a share in the event that the realized return surpasses the guaranteed return. The profit-sharing component may be earmarked for maturity or scheduled periodically, as illustrated in [Bacinello, 2001]. Additionally, the apportionment of profits could be contingent upon financial risk, actuarial risk, or a combination of both, as elaborated in [Devolder and Azizieh, 2013]. The literature extensively covers fair valuation operators designed for the assessment of hybrid payoffs. While brief mention of works such as [Chen et al., 2020] and [Chen et al., 2021] is by no means exhaustive, our study is particularly focused on exploring the Three-step method elucidated within a static framework in [Deelstra et al., 2020].

A crucial issue in the dynamic framework, is how risk valuations at different times are interrelated. This inquiry prompts the exploration of the time-consistent property. In essence, this property asserts that if, it is anticipated that a certain product will be less expensive than another at a subsequent date, in almost all states of nature, this discernment should, hold true in the current time-frame, as discussed in [Acciaio and Penner, 2011] in the context of random variables representing purely financial positions. The main contribution of this paper is twofold. First, we extend the fair Three-step valuation of [Deelstra et al., 2020] to a dynamic context, thereby providing updated fair values aligned with the available information. In common with the majority of non-linear valuation operators, the dynamic Three-step method is not time-consistent. By adapting the iterative version of time-consistency outlined in [Bion-Nadal, 2009] for hybrid life pure endowment products, against the backdrop of our target dynamic valuation, namely the Three-step method, we progressively derive a succession of time-consistent values, evolving from the initially non time-consistent version. Secondly, to alleviate the risk premium of time-consistency, we adopt the approach proposed by [Devolder and Lebègue, 2016], proven to reduce the time-consistent Solvency Capital Requirement (SCR). The approach involves reducing the safety loads employed in the backward scheme.

The remainder of the paper is organized as follows. In section 2, we present the main assumptions used throughout the paper, and an exposition of the Three-step method within in a dynamic framework. Section 3 introduces the definition of the time-consistent property for financial products, establishing its association with a backward iteration scheme. Furthermore, by centering on the Three-step method dynamic, we broaden the scope of the latter backward procedure to encompass hybrid life Pure Endowment products. Using two iterations, the backward procedure is applied to the Three-step method in the context of a Pure Endowment product without profit sharing. In a continuous-time limit framework, section 4 is dedicated to deriving the partial differential equation (PDE) for the Three-step method time-consistent dynamic. By studying a hybrid life product, namely the Pure Endowment with terminal bonus, we present a numerical resolution of the PDE. Section 5 concludes the paper.

2 Extending the Three-step method to a dynamic framework

2.1 Assumptions

We consider an arbitrage-free and complete financial market. Moreover, we assume that there are risks that are not traded on the financial market, which we label as 'actuarial' risks. These risks are assumed to be independent from the financial market. Such strong assumptions have already been imposed in some papers in the literature, such as [Deelstra et al., 2020] and [Belhouari et al., 2024]. The set of trading dates is $[0, T]$, where T is the maturity. We consider a portfolio of n insured individuals, all of age x at time 0; their remaining lifetimes are denoted by $(\tau_1, \tau_2, \dots, \tau_n)$ and these remaining lifetimes are assumed to be identical in distribution. We denote by N_t the number of survivors up to time t , formulated as $N_t = \sum_{j=1}^n 1_{\tau_j > t}$. Furthermore, we assume that for each $t \in [0, T]$, $(\tau_i)_{1 \leq i \leq N_t} \sim \tau_1$. Assuming that actuarial and financial risks are independent, we consider the following probability space

$$(\Omega^a \times \Omega^f, \mathcal{F}^a \vee \mathcal{F}^f, \mathbb{P}^a \times \mathbb{P}^f).$$

Ω^a , Ω^f denote the actuarial and financial space, respectively. Therefore, the product space $\Omega^a \times \Omega^f$ reflects all the possible financial and mortality events. $\mathbb{P}^a$, $\mathbb{P}^f$ denote the real world actuarial and financial probability, respectively. Referring for instance to [Artzner et al., 2020], we introduce the risk-neutral financial probability $\mathbb{Q}^f$. $\mathcal{F}^f$ is the financial filtration while $\mathcal{F}^a$ is the actuarial filtration composed of the filtration $(\mathfrak{S}_t)_t$ generated by systematic mortality risk factors and the filtration $\bigvee_{i=1}^n \mathfrak{S}^i$ encompassing information on unsystematic mortality risk. At any given time $t < T$, for each individual $1 \leq i \leq n$, we define $\mathfrak{S}_t^i = \sigma(1_{\tau_i \leq u} : u \leq t)$. Thus, it follows that $\mathcal{F}^a = \mathfrak{S} \vee (\bigvee_{i=1}^n \mathfrak{S}^i)$. Notably, we define $\mathcal{F} \triangleq \mathcal{F}^a \vee \mathcal{F}^f$. Furthermore, we will assume that the lifetimes of the surviving insured at each instant t display conditional independence, given the systematic mortality risk information at maturity.¹

For the actuarial instruments, stochastic mortality rates $(\lambda_t)_{0 \leq t \leq T}$, adapted to the filtration $\mathfrak{S}$ will be used. Thus, the probability of an individual at time $t \leq u$ to survive until maturity $u \leq T$ is²

$${}_{u-t}p_{x+t} \triangleq \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_s ds}]. \quad (1)$$

¹In a more precise mathematical formulation, this means that for any $t \in [0, T]$, $\mathbb{P}^a(\tau_1 > t, \tau_2 > t, \dots, \tau_{N_t} > t | \mathfrak{S}_T) = \prod_{i=1}^{N_t} \mathbb{P}^a(\tau_i > t | \mathfrak{S}_T)$.

²At time t , we denote the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_t]$ and the conditional variance $\mathbb{V}[\cdot | \mathcal{F}_t]$ as $\mathbb{E}_{\mathcal{F}_t}[\cdot]$ and $\mathbb{V}_{\mathcal{F}_t}[\cdot]$, respectively; these are only almost surely unique.

In addition to the actuarial risk, we include the financial risk. In the realm of financial instruments, our focus will be on a sole stochastic investment fund denoted as $(S_t)_{0 \leq t \leq T}$, coupled with stochastic interest rates captured by $(r_t)_{0 \leq t \leq T}$.³ The determination of the value for a zero-coupon bond with maturity $u \leq T$ at a time $t \leq u$ can be accomplished through the following equation

$$P(t, u) \triangleq \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_s ds}]. \quad (2)$$

2.2 Axiomatization

The examination of axioms will take place using a general class of hybrid product described by the following general set of random variables (rv.s)

$$\forall t \in [0, T], \quad \overline{\mathcal{H}}_t \triangleq \{ \overline{H}_t \text{ rv} / \mathcal{F}_t - \text{mesurable \& payable at } t \}. \quad (3)$$

Furthermore, we introduce two more refined subsets, building exclusively knowing the financial filtration or the actuarial filtration. The precise definitions of these subsets are outlined below

$$\forall t \in [0, T], \quad \overline{\mathcal{H}}_t^f \triangleq \{ \overline{h}_t \text{ rv} / \mathcal{F}_t^f - \text{mesurable \& payable at } t \}. \quad (4)$$

$$\forall t \in [0, T], \quad \overline{\mathcal{H}}_t^a \triangleq \{ \overline{g}_t \text{ rv} / \mathcal{F}_t^a - \text{mesurable \& payable at } t \}. \quad (5)$$

According to [Barigou et al., 2019], we follow some axioms that a valuation operator must verify in the following; see also [Belhouari et al., 2024].

Definition 1 (t^u -Valuation operator).⁴

Let $t \in [0, T]$, $u \in [0, T]$ such as $t \leq u$. Let $\pi_t^u : \overline{\mathcal{H}}_u \rightarrow \overline{\mathcal{H}}_t$, a function which represents the value at time t of the hybrid payoff $\overline{H}_u \in \overline{\mathcal{H}}_u$.

We say that π_t^u is a t^u -valuation operator if it verifies the following axioms

-**Normalization** $\pi_t^u(0) = 0$.

-**Translation invariance** $\forall x_t, \mathcal{F}_t$ -mesurable and payable at the instant u , $\forall \overline{H}_u \in \overline{\mathcal{H}}_u$

$$\pi_t^u(\overline{H}_u + x_t) = \pi_t^u(\overline{H}_u) + P(t, u)x_t.$$

Definition 2 (Dynamic valuation).

The sequence $(\pi_t^u)_{0 \leq t \leq u \leq T}$, where each π_t^u is a t^u -valuation operator, is referred to as a dynamic valuation.

In a complete and arbitrage-free financial market, discounted purely financial payoffs exhibit martingale behavior under the risk-neutral measure $\mathbb{Q}^f$. This naturally guides us toward formulating the following t^u -financial valuation operator $\pi_t^{u,f}$

$$\pi_t^{u,f}(\overline{H}_u | \mathcal{F}_t) \triangleq \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_s ds} \overline{H}_u], \quad (6)$$

with $\overline{H}_u \in \overline{\mathcal{H}}_u$ and $0 \leq t \leq u \leq T$.

Definition 3 (Market-consistent property).

³The interest rate process $(r_t)_{0 \leq t \leq T}$ is a financial risk correlated with the stochastic investment fund $(S_t)_{0 \leq t \leq T}$.

⁴To avoid heavy notation, we will use $\pi_t[\overline{H}_T]$ to represent the t^T -valuation operator, indicating the value at time t of the hybrid payoff $\overline{H}_T \in \overline{\mathcal{H}}_T$.

Let $0 \leq t \leq u \leq T$. A t^u -valuation operator π_t^u is market-consistent with respect to the t^u -financial operator $\pi_t^{u,f}$ defined in (6) if and only if

$$\forall \bar{h}_u \in \overline{\mathcal{H}_u^f}, \forall \bar{H}_u \in \overline{\mathcal{H}_u} : \pi_t^u(\bar{H}_u + \bar{h}_u | \mathcal{F}_t) = \pi_t^u(\bar{H}_u | \mathcal{F}_t) + \pi_t^{u,f}(\bar{h}_u | \mathcal{F}_t).$$

On the actuarial front, we assume that establishing actuarial consistency will draw upon two actuarial principles, namely, the standard deviation premium principle and the variance premium principle. Given the stochastic nature of interest rates in our study, we will adopt the framework proposed by [Belhouari et al., 2024], albeit with minor modifications.

The standard deviation premium principle is

$$\pi_t^{u,SD}[\bar{H}_u | \mathcal{F}_t] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_s ds} \bar{H}_u] + \frac{\alpha_t}{2} \sqrt{\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f}[\bar{H}_u]} \times \sqrt{u-t} \times P(t, u), \quad (7)$$

for $0 \leq t \leq u \leq T$, $\bar{H}_u \in \overline{\mathcal{H}_u}$, and $\alpha_t > 0$ is the portion of the load used at time t . The standard deviation term is multiplied by the factor $\sqrt{u-t}$; this small change is necessary to solve the dimensionality problem that arise from the different time-scales produced by the expectation and the standard deviation, see [Pelsser and Ghalehjooghi, 2016]. The variance premium principle is

$$\pi_t^{u,V}[\bar{H}_u | \mathcal{F}_t] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_s ds} \bar{H}_u] + \frac{\alpha_t}{2k} \times \mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f}[\bar{H}_u] \times P(t, u), \quad (8)$$

for $0 \leq t \leq u \leq T$, $\bar{H}_u \in \overline{\mathcal{H}_u}$, and $\alpha_t > 0$ indicates the portion of the load used at time t . Expressed in monetary units, the expectation differs from the variance. Hence, the parameter $\frac{\alpha_t}{2}$ necessitates multiplication by $\frac{1}{k}^5$ to address the issue.

Definition 4 (Actuarial-consistent property).

Let $0 \leq t \leq u \leq T$. A t^u -valuation operator π_t^u is actuarial-consistent with respect to the t^u -actuarial operator $\pi_t^{u,a}$ if and only if $\forall \bar{g}_u \in \overline{\mathcal{H}_u^a}$,

$$\pi_t^u(\bar{g}_u | \mathcal{F}_t) = \pi_t^{u,a}(\bar{g}_u | \mathcal{F}_t),$$

with $\pi_t^{u,a}$ is given by (7) or (8).

Definition 5 (t^T -Fair valuation).

Let $0 \leq t \leq u \leq T$. A t^u -fair valuation is a t^u -valuation operator that is both market-consistent and actuarial-consistent.

Definition 6 (Market-consistent dynamic valuation).

A market-consistent dynamic valuation is a dynamic valuation $(\pi_t^u)_{0 \leq t \leq u \leq T}$, where for each $0 \leq t \leq u \leq T$, π_t^u is a t^u -market-consistent valuation operator.

Definition 7 (Actuarial-consistent dynamic valuation).

An actuarial-consistent dynamic valuation is a dynamic valuation $(\pi_t^u)_{0 \leq t \leq u \leq T}$, where for each $0 \leq t \leq u \leq T$, π_t^u is an t^u -actuarial-consistent valuation operator.

Definition 8 (Fair dynamic valuation).

A fair dynamic valuation is a dynamic valuation $(\pi_t^u)_{0 \leq t \leq u \leq T}$, where for each $0 \leq t \leq u \leq T$, π_t^u is a t^u -fair valuation.

⁵ $k \in \mathbb{R}_{>0}$ is a quantitative benchmark denoted in the currency monetary unit.

2.3 Three-step decomposition in a dynamic setting

Within the scope of our study, we investigate the Three-step method introduced in [Deelstra et al., 2020]. To this end, we extend the natural dynamic approach of the three-step decomposition. We consider the aggregate hybrid payoff for the insured entities, outlined as

$${}_tH_u^c = h(S_u) \sum_{j=1}^{N_t} g(\tau_j), \quad (9)$$

where $0 \leq t \leq u \leq T$, $g :]0, \infty[\rightarrow [0, \infty[$ and $h :]0, \infty[\rightarrow [0, \infty[$ are two $\mathcal{F}_u^a$ -measurable functions and $\mathcal{F}_u^f$ -measurable, respectively.⁶

We decompose $\frac{{}_tH_u^c}{N_t} \times 1_{N_t > 0}$, with respect to the full spectrum of information, as follows⁷

$${}_tH_u = \frac{{}_tH_u^c}{N_t} \times 1_{N_t > 0} = {}_tH_u^1 + {}_tH_u^2 + {}_tH_u^3, \quad (10)$$

where

$${}_tH_u^1 \triangleq \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_u^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tH_u] = \frac{\sum_{j=1}^{N_t} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)] h(S_u)}{N_t} \times 1_{N_t > 0}. \quad (11)$$

$${}_tH_u^2 \triangleq {}_tH_u - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u \vee \mathcal{F}_u^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tH_u] = \frac{\sum_{j=1}^{N_t} (g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)]) h(S_u)}{N_t} \times 1_{N_t > 0}. \quad (12)$$

$$\begin{aligned} {}_tH_u^3 &\triangleq \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u \vee \mathcal{F}_u^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tH_u] - \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_u^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tH_u] \\ &= \frac{1}{N_t} \sum_{j=1}^{N_t} \left(\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)] - \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)] \right) h(S_u) 1_{N_t > 0}. \end{aligned} \quad (13)$$

The term $\frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)]}{N_t} \times 1_{N_t > 0}$ exhibits $\mathcal{F}_t$ -measurability, then ${}_tH_u^1$ is a financial claim; consequently $\pi_t^{u,f}$ serves as the suitable operator for its assessment. The component (12) is diversifiable since this assertion hold

$$\begin{aligned} {}_tH_u^2 \Big|_{\mathcal{F}_t^f \vee \mathfrak{S}_u} &= \frac{1}{N_t} \sum_{j=1}^{N_t} \left(g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)] \right) h(S_u) \Big|_{\mathcal{F}_t^f \vee \mathfrak{S}_u} \\ &\xrightarrow{N_t \rightarrow \infty} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\left(g(\tau_1) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_1)] \right) h(S_u) \mid \mathcal{F}_t^f \vee \mathfrak{S}_u \right] = 0 \quad \text{a.s.} \end{aligned}$$

Hence, we assess it by applying actuarial premium principles. Furthermore, both the standard deviation premium principle (7) and the variance principle (8) conform to pooling arguments in relation to ${}_tH_u^2$. Indeed, they approach a zero value in an asymptotic sense⁸ Concerning the final part ${}_tH_u^3$, it represents the systematic risk which is unaffected to both hedging and diversification.

⁶It is postulated that for each $0 \leq t \leq u \leq T$, for all $j = 1, \dots, N_t$, the quantities $\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_s ds} h(S_u) g(\tau_j)]$, $\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_t^u r_s ds} h(S_u) g(\tau_j)]$, $\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_s ds} h(S_u) g(\tau_j)]$ and $\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_t^u r_s ds} h(S_u) g(\tau_j)]$ are finite.

⁷Inherently, the subscript t appended to ${}_tH_u^1$, ${}_tH_u^2$ and ${}_tH_u^3$ also determines the evaluation instance.

⁸In this formulation $\frac{\pi_t^{u,V}({}_tH_u^2 - c | \mathcal{F}_t)}{N_t} \xrightarrow{N_t \rightarrow +\infty} 0$, the variance principle cannot be diversified and differs from our valuation methodology, i.e. $\pi_t^{u,V}({}_tH_u^2 | \mathcal{F}_t) \xrightarrow{N_t \rightarrow +\infty} 0$.

To compute its valuation, we propose an Esscher transform following [Deelstra et al., 2020], detailed as follows

$$\pi_t^{u,n}(\overline{H_u}|\mathcal{F}_t) \triangleq \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} [e^{-\int_t^u r_s ds} \times \overline{H_u}], \quad (14)$$

$\overline{H_u} \in \overline{\mathcal{H}_u}$, $0 \leq t \leq u \leq T$ and

$$\frac{d\mathbb{Z}_\theta^a}{d\mathbb{P}^a} \Big|_{\mathcal{F}_t} \triangleq \frac{e^{\int_t^u \theta_{t,u}^{(1)} \lambda_s ds}}{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{\int_t^u \theta_{t,u}^{(1)} \lambda_s ds}]}, \quad (15)$$

The explicit formula for $\theta_{t,u}^{(1)}$ depends on the specified dynamics of stochastic mortality rates and will therefore be provided later; see Subsection 3.2 for further details on this point.

Definition 9 (The dynamic Three-step method).

Let $0 \leq t \leq u \leq T$. The dynamic Three-step method is defined by

$$\pi_t^{u,3}({}_t H_u|\mathcal{F}_t) \triangleq \pi_t^{u,f}({}_t H_u^1|\mathcal{F}_t) + \pi_t^{u,a}({}_t H_u^2|\mathcal{F}_t) + \pi_t^{u,n}({}_t H_u^3|\mathcal{F}_t),$$

where $\pi_t^{u,f}$ is given by (6), $\pi_t^{u,a}$ is given by (7) or (8) and $\pi_t^{u,n}$ is given by (14).

It is easy to check that the dynamic Three-step method is fair according to our axiomatization.

3 Time-consistent dynamic valuations

3.1 Time consistency properties for financial products

Once a dynamic valuation is defined, the inquiry naturally extends to the temporal behavior of valuations across distinct time instances. The challenge lies in maintaining the hierarchical sequence of prices for products evaluated through dynamic valuation, thereby mitigating the emergence of arbitrage opportunities. We will first establish a mathematical formulation of this fundamental property for purely financial products, then broaden the scope to encompass hybrid life Pure Endowment products, with a particular focus on the dynamic Three-step method. Based on [Acciaio and Penner, 2011], we state the following definition of time-consistency.

Definition 10 (Time-consistent property for financial products).

A dynamic valuation $(\pi_t^u)_{0 \leq t \leq u \leq T}$ is time-consistent if and only if, $\forall 0 \leq t \leq u, v \leq T, \forall 0 < s \leq u - t, 0 < s \leq v - t$ ⁹, $\forall \overline{h_u} \in \overline{\mathcal{H}_u}^f, \overline{g_v} \in \overline{\mathcal{H}_v}^f$,

$$\pi_{t+s}^u[\overline{h_u}] \leq \pi_{t+s}^v[\overline{g_v}] \Rightarrow \pi_t^u[\overline{h_u}] \leq \pi_t^v[\overline{g_v}]. \quad (16)$$

For instance, the mathematical expectation is a time-consistent dynamic because of the tower rule property. Notably, $(\pi_t^{u,f})_{0 \leq t \leq u \leq T}$ is time-consistent. Alternatively, an iteratively backward scheme, closely associated to time-consistency, is frequently used in the literature.

Definition 11 (Iterated property for financial products).

A dynamic valuation $(\pi_t^u)_{0 \leq t \leq u \leq T}$ is iterated if and only if, $\forall 0 \leq t \leq u \leq T, \forall 0 < s \leq u - t, \forall \overline{h_u} \in \overline{\mathcal{H}_u}^f$,

$$\pi_t^{t+s}[\pi_{t+s}^u[\overline{h_u}]] = \pi_t^u[\overline{h_u}]. \quad (17)$$

⁹The exclusive occurrence of either $s = u - t$ or $s = v - t$ is permissible, prohibiting their simultaneous fulfillment.

Remark 1.

Each time-consistent dynamic valuation satisfies the equation (17). Moreover, under the monotonicity¹⁰ hypothesis, Definitions 11 and 10 are equivalent, see details in the Appendix.

Through Definition 11, we can progressively devise a forward-looking procedure in a discrete configuration, enabling when it is not the case initially, the establishment of time-consistency for dynamic valuations that exhibit monotonicity.

Corollary 1.

Let $\Delta t^* \in]0, T[$. Let $(\pi_t^u)_{0 \leq t \leq u \leq T}$ be a dynamic valuation that adheres to monotonicity but not necessarily time-consistency, then the resulting sequence $(\tilde{\pi}_t)_t$ ¹¹ constructed according to the outlined procedure satisfies time-consistency.

1) First step.

$$\tilde{\pi}_{T-\Delta t^*}[\overline{h_T}] = \pi_{T-\Delta t^*}[\overline{h_T}].$$

2) Second step. For $s = \Delta t^*, 2\Delta t^*, 3\Delta t^*, \dots$ and while $\tilde{s} \triangleq s + \Delta t^* \leq T$

$$\tilde{\pi}_{T-\tilde{s}}[\overline{h_T}] = \pi_{T-\tilde{s}}^{T-s}[\tilde{\pi}_{T-s}[\overline{h_T}]],$$

where $\overline{h_T} \in \overline{\mathcal{H}_T^f}$.

Several dynamic valuation operators simultaneously lack monotonicity and time consistency. Consequently, the application of the procedure explicated in corollary 1 is not universal. Nonetheless, counter examples arise more frequently in extreme contexts, offering limited practical value. Therefore, we will use corollary 1 as a tool to consider time-consistent valuations. Moreover, it is pertinent to acknowledge that some contributions in the literature regard the formulation of time-consistency presented in definition 11 as the primary definition; see for instance [Bion-Nadal, 2009].

3.2 The Three-step method time-consistent dynamic for hybrid life Pure Endowment products

We investigate a hybrid life insurance product, where the longevity risk corresponds to what is commonly referred to in the literature as a Pure Endowment product. The benefit consists of generating a payment only in case of survival of the insured at a fixed maturity. No payout is made in the event of premature death. For the financial payoff, we consider a broad framework involving any measurable function h of the financial investment fund. In terms of quantitative standpoint, the collective payoff for a maturity $u \leq T$ is

$${}_t H_u^c = \left(\sum_{j=1}^{N_t} 1_{\tau_j > u} \right) \times h(S_u). \quad (18)$$

¹⁰ A dynamic valuation $(\pi_t^u)_{0 \leq t \leq u \leq T}$ satisfies the condition of monotonicity if $\forall 0 \leq t \leq u \leq T, \forall \overline{h_u}, \overline{g_u} \in \overline{\mathcal{H}_u^f}$,

$$\overline{h_u} \leq \overline{g_u} \Rightarrow \pi_t^u[\overline{h_u}] \leq \pi_t^u[\overline{g_u}].$$

¹¹ The newly generated time-consistent dynamic $(\tilde{\pi}_t)_t$ is, by construction, assumed to be a sequence of t^T -valuation operators.

The distinctiveness of the Three-step decomposition described in (11), (12), and (13) lies in its use of a homogeneous payoff, which depends on the number of survivors. Under mortality risk, the number of survivors decreases over time from the initial instant to maturity. This necessitates ensuring price equilibrium across different time points, taking into account the number of survivors at each point in time. When dealing with hybrid life products, this adjustment should be integrated into the iterative time consistency property specified in definition 11.

Definition 12 (Time-consistency for hybrid life Pure Endowment products).

Consider the collective payoff given by (18). Let ${}_tH_u^1$, ${}_tH_u^2$, and ${}_tH_u^3$ be its three-step decomposition, as defined respectively in (10), (11), and (12). Then, the dynamic Three-step method is time-consistent if and only if the equation below holds

$$\pi_t^{t+s,3} \left[\frac{N_{t+s}}{N_t} \times \left(\pi_{t+s}^{u,f} [{}_{t+s}H_u^1] + \pi_{t+s}^{u,a} [{}_{t+s}H_u^2] + \pi_{t+s}^{u,n} [{}_{t+s}H_u^3] \right) \right] = \pi_t^{u,f} [{}_tH_u^1] + \pi_t^{u,a} [{}_tH_u^2] + \pi_t^{u,n} [{}_tH_u^3], \quad (19)$$

for all $0 \leq t \leq u \leq T$, $0 < s \leq u - t$.

We cannot proceed by applying the iterative time-consistency formula to each three-step component in isolation. This comes from the fact that a unique initial decomposition at maturity u cannot be preserved as time progresses. For instance, while the risk ${}_{t+s}H_u^1$ is hedgeable at time $t + s$, the risk $\left(\pi_{t+s}^{u,f} [{}_{t+s}H_u^1] \times \frac{N_{t+s}}{N_t} \right)$ ceases to be hedgeable at time t . Consequently, applying the financial operator $\pi_t^{t+s,f}$ to $\left(\pi_{t+s}^{u,f} [{}_{t+s}H_u^1] \times \frac{N_{t+s}}{N_t} \right)$ would be inappropriate. Likewise, there is no assurance that the risk $\left(\pi_{t+s}^{u,a} [{}_{t+s}H_u^2] \times \frac{N_{t+s}}{N_t} \right)$ will remain diversifiable at time t , excluding its revaluation through a t^{t+s} -actuarial operator. Viewed from time t ,

$${}_tLY_{t+s} \triangleq \frac{N_{t+s}}{N_t} \times \left(\pi_{t+s}^{u,f} [{}_{t+s}H_u^1] + \pi_{t+s}^{u,a} [{}_{t+s}H_u^2] + \pi_{t+s}^{u,n} [{}_{t+s}H_u^3] \right) \quad (20)$$

is a payoff that matures at $t + s$. Hence, once again, it must be decomposed into a hedgeable component, a diversifiable component, and a systematic component to apply the Three-step method and determine the valuation at time t .

As an introductory step, we aim to assess the time-consistency of the Three-step approach, disregarding the diversifiable risk. This results in the three-step decomposition being simplified to a 'Two-step' model, comprising a hedgeable part, and a systematic part, as detailed below

$$\begin{cases} {}_tH_u^h \triangleq \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_u^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tH_u] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_s ds}] \times h(S_u) \times 1_{N_t > 0}, \\ {}_tH_u^{sy} \triangleq {}_tH_u - \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_u^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tH_u] = \left(\frac{N_u}{N_t} - \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_s ds}] \right) \times h(S_u) \times 1_{N_t > 0}. \end{cases}$$

Therefore, $\pi_t^{u,3} [{}_tH_u]$ introduced in definition 9, is reduced to

$$\pi_t^{u,2} [{}_tH_u] \triangleq \pi_t^{u,f} [{}_tH_u^h] + \pi_t^{u,n} [{}_tH_u^{sy}]. \quad (21)$$

Lemma 1.

The dynamic $(\pi_t^{u,2})_{0 \leq t \leq u \leq T}$ is time-consistent; consequently, its time-consistent valuation is given, by the one-period valuation, as follows

$$\tilde{\pi}_t^2 [{}_tH_T | \mathcal{F}_t] = \pi_t^2 [{}_tH_T | \mathcal{F}_t] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^T r_v dv} h(S_T)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^T \lambda_v dv}],$$

with $t \in [0, T]$.

Proof.

See the Appendix. $\square$

Next, we include the diversifiable risk into the previous decomposition, finding the full three-step decomposition as provided below

$$\begin{cases} {}_tH_u^1 = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_s ds}] \times h(S_u) \times 1_{N_t > 0}, \\ {}_tH_u^2 = \left(\frac{N_u}{N_t} - e^{-\int_t^u \lambda_s ds} \right) \times h(S_u) \times 1_{N_t > 0}, \\ {}_tH_u^3 = (e^{-\int_t^u \lambda_s ds} - \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_s ds}]) \times h(S_u) \times 1_{N_t > 0}. \end{cases}$$

Lemma 2.

The dynamic Three-step method $(\pi_t^{u,3})_{0 \leq t \leq u \leq T}$ introduced in definition 9, is not time-consistent.

Proof.

See the Appendix. $\square$

Since the dynamic Three-step method is not time-consistent, we strive to iteratively construct a sequence of time-consistent valuations by applying corollary 1 in line with definition 12.

Corollary 2.

Let $\Delta t^* \in]0, T[$. Let $(\pi_t^{u,3})_{0 \leq t \leq u \leq T}$ be the dynamic Three-step method defined in 9, then the resulting sequence $(\tilde{\pi}_t^3)_t$ constructed according to the outlined procedure is time-consistent

1) First step.

$$\tilde{\pi}_{T-\Delta t^*}^3 [T-\Delta t^* H_T] = \pi_{T-\Delta t^*}^3 [T-\Delta t^* H_T].$$

2) Second step. For $s = \Delta t^*, 2\Delta t^*, 3\Delta t^*, \dots$ and while $\tilde{s} \triangleq s + \Delta t^* \leq T$

$$\tilde{\pi}_{T-\tilde{s}}^3 [T-\tilde{s} H_T] = \pi_{T-\tilde{s}}^{T-s,3} \left[\frac{N_{T-s}}{N_{T-\tilde{s}}} \times \tilde{\pi}_{T-s}^3 [T-s H_T] \right], \quad (22)$$

for ${}_tH_u = \frac{\left(\sum_{j=1}^{N_t} 1_{\tau_j > u} \right) \times h(S_u)}{N_t}$ and where $0 \leq t \leq u \leq T$.

To derive explicit time-consistent valuations based on corollary 2, it is essential to specify the dynamics of $(\lambda_t)_{0 \leq t \leq T}$, $(S_t)_{0 \leq t \leq T}$ and $(r_t)_{0 \leq t \leq T}$. Under $\mathbb{P}^a$, the stochastic mortality rate λ_t follows a stochastic solution of the following equation commonly recognized as Ornstein–Uhlenbeck intensity process (see [Luciano and Vigna, 2008])

$$d\lambda_t = \mu_m \lambda_t dt + \sigma_m dW_m(t),$$

where $(W_m(t))_{t \geq 0}$ is a Brownian motion and $\mu_m > 0, \sigma_m > 0$.

Thus, for $0 \leq t \leq u \leq T$, ${}_{u-t}p_{x+t} = e^{-\lambda_t l_{t,u} + \frac{1}{2} s_{t,u}^2}$, where $l_{t,u} = \frac{e^{\mu_m(u-t)} - 1}{\mu_m}$, and $s_{t,u}^2 = \frac{\sigma_m^2}{2\mu_m^3} (1 - e^{\mu_m(u-t)})^2 + \frac{\sigma_m^2}{\mu_m^2} ((u-t) - \frac{e^{\mu_m(u-t)} - 1}{\mu_m})$. The parameter $\theta_{t,u}^{(1)}$ present in (15) is adjusted in accordance with the Cost of Capital approach detailed in [Belhouari et al., 2024], producing $\theta_{t,u}^{(1)} = \frac{-0.16}{s_{t,u}}$.

The dynamics of the couple stochastic investment fund and stochastic interest rates will be characterized using the following Black-Scholes-Vasicek model, described under $\mathbb{P}^f$ by

$$\begin{cases} dr_t = (b - ar_t)dt + \sigma_r d\tilde{W}_r(t), \\ dS_t = \mu_S S_t dt + S_t \sigma_S (\rho d\tilde{W}_r(t) + \sqrt{1 - \rho^2} d\tilde{W}_S(t)). \end{cases}$$

where $(\tilde{W}_S(t))_{t \geq 0}$ is a Brownian motion independent from $(\tilde{W}_r(t))_{t \geq 0}$, a is the constant speed of mean reversion, b represents a strictly positive scalar such that the ratio $\frac{b}{a}$ is the long-term mean value and σ_r is the constant volatility of interest rates. μ_S is the instantaneous return of $(S_t)_{t \geq 0}$, σ_S is the volatility of the financial investment fund, and ρ defines the correlation between the financial investment fund and interest rates.

Switching to the risk-neutral probability involves the application of Girsanov's theorem, which defines the Brownian motions $(W_r(t))_t$ and $(W_S(t))_t$ in the following manner $\mathbb{P}^f$

$$\begin{cases} W_r(t) \triangleq \tilde{W}_r(t) - \nu t, \\ W_S(t) \triangleq \tilde{W}_S(t) + \int_0^t \left(\frac{\mu_S - r_u}{\sqrt{1 - \rho^2} \sigma_S} + \frac{\nu \rho}{\sqrt{1 - \rho^2}} \right) du. \end{cases}$$

Drawing upon this, the Radon-Nikodym derivative of $\mathbb{Q}^f$ with respect to $\mathbb{P}^f$ is

$$\frac{d\mathbb{Q}^f}{d\mathbb{P}^f} \Big|_{\mathcal{F}_t} = e^{\nu \tilde{W}_r(T) - \int_t^T \left(\frac{\mu_S - r_u}{\sqrt{1 - \rho^2} \sigma_S} + \frac{\nu \rho}{\sqrt{1 - \rho^2}} \right) d\tilde{W}_S(u) - \frac{1}{2} \int_t^T \left(\nu^2 + \left(\frac{\mu_S - r_u}{\sqrt{1 - \rho^2} \sigma_S} + \frac{\nu \rho}{\sqrt{1 - \rho^2}} \right)^2 \right) du}.$$

Hence, the following dynamics are obtained under $\mathbb{Q}^f$

$$\begin{cases} dr_t = (\theta - ar_t)dt + \sigma_r dW_r(t), \\ dS_t = r_t S_t dt + S_t \sigma_S (\rho dW_r(t) + \sqrt{1 - \rho^2} dW_S(t)), \end{cases}$$

with $\theta = b + \sigma_r \nu$.

3.3 The backward scheme in a dual-iteration approach

The achievement of closed forms through the recursive equation (22) with a substantial number of iterations given the stochastic risks on both the financial and mortality fronts, as assumed in our study, poses a formidable challenge, if not unattainable. To simplify the problem, we consider two periods $[0, t^*]$ and $[t^*, T]$ in a constant interest rate framework¹², and where we set $h(S_u) = 1$. In this case, we are delving into the purely actuarial, Pure Endowment product. Owing to the requirements of derivation and computation, we will use the variance premium principle, defined in (8), as the operator for evaluating the diversifiable risk component of the Three-step method. This choice enables the derivation of a closed-form solution, unlike the standard deviation premium principle. In light of the newly introduced denotations

$$\begin{aligned} {}_i\tilde{f}_j &= e^{-i\lambda_0 l_{t^*, T} e^{\mu_m t^*} + j \frac{(e^{\mu_m t^*} + 1)}{2} \frac{l_{t^*, T}^2 l_{0, t^*}^2 \sigma_m^2}{2}} + j \frac{\theta_{0, t^*}^{(1)} l_{0, t^*}^2 l_{t^*, T}^2 \sigma_m^2}{2}, \\ {}_i f_k^j &= e^{-i\lambda_0 l_{0, t^*} + j \frac{l_{t^*, T}^2 l_{0, t^*}^2 \sigma_m^2}{2} + k \frac{1}{2} s_{0, t^*}^2}, \end{aligned}$$

we hereby present the following lemma.

¹²i.e $P(t, T) = e^{-r_0(T-t)}$.

Lemma 3.

The Three-step valuation of the Pure Endowment product at the instant t^* is $\tilde{\pi}_{t^*}^3[t^* H_T | \mathcal{F}_{t^*}] = P(t^*, T) \left(e^{-\lambda_{t^*} l_{t^*, T} + \frac{1}{2} s_{t^*, T}^2} e^{-\theta_{t^*, T}^{(1)} s_{t^*, T}^2} + \frac{\alpha_{t^*}}{2k} \times \frac{e^{-\lambda_{t^*} l_{t^*, T} + \frac{1}{2} s_{t^*, T}^2} e^{-2\lambda_{t^*} l_{t^*, T} + 2s_{t^*, T}^2}}{N_{t^*}} \right)$. While the time-consistent valuation at the instant 0 is

$$\tilde{\pi}_0^3[{}_0H_T | \mathcal{F}_0] = \Gamma_0^{(h, sy)} + \Gamma_0^{(div)},$$

where

$$\begin{aligned} \Gamma_0^{(h, sy)} &= P(0, T) \left(e^{-\lambda_0 l_{0, T} - s_{0, T}^2 (\theta_{0, T}^{(1)} - \frac{1}{2})} + \frac{\alpha_{t^*}}{2k} \frac{e^{\frac{1}{2} s_{t^*, T}^2} {}_1\tilde{f}_1 - e^{2s_{t^*, T}^2} {}_2\tilde{f}_4}{n} \right), \\ \Gamma_0^{(div)} &= \frac{\alpha_0}{2k} \times P(0, t^*) P(t^*, T)^2 \times e^{-2s_{t^*, T}^2 (\theta_{t^*, T}^{(1)} - \frac{1}{2})} ({}_1f_1^2 - {}_2f_4^8), \\ \theta_{t^*, T}^{(1)} &= \frac{-0.16}{s_{t^*, T}}, \\ \theta_{0, t^*}^{(1)} &= \frac{-0.16}{s_{0, t^*}}. \end{aligned}$$

Proof.

See the Appendix. □

A notable aspect is that the aggregated non-diversifiable risks, including systematic and hedgeable components of the time-consistent formula at time 0, incorporate loading terms arising from the valuation of the diversifiable component at time t^* ; as shown in $\Gamma_0^{(h, sy)}$. This implies that, during the backward scheme, the valuation of the diversifiable component from the prior iteration impacts the valuation of the subsequent non-diversifiable components.

4 Partial Differential Equation for the Three-step method time-consistent dynamic

4.1 Derivation

In the preceding section, several simplistic measures were used to derive the time-consistent valuation across two periods. To comprehensively unveil the repercussions of time consistency on pricing, a more sophisticated numerical approach surpassing two iterations is imperative. This section focuses on studying the continuous time limit of the Three-step method when integrated into the backward scheme. In order to accomplish this, and following [Pelsser and Ghalehjooghi, 2016] we will derive partial differential equation¹³ of the Three-step method time-consistent dynamic, incorporating all the stochastic processes of mortality and finance considered in our study. Considering an infinitesimally small time increment dt that tends towards 0, we position ourselves within the $(t, t + dt)$ interval and employ

$$\tilde{\pi}_t^3 = \pi_t^{t+dt, 3} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}^3 \right] \quad (23)$$

This time, we purposefully adopt the standard deviation premium principle to assess the actuarial component of the Three-step method. Interestingly, the variance principle enables the generation of a similar PDE, reflecting also a time-consistent price.

¹³We will proceed without including the payoff in the partial differential equation.

Theorem 1.

Let $t^{(1)}$ be a strictly positive real scalar. The Three-step method time-consistent dynamic $(\tilde{\pi}_t^3)_{t \in [0, T-t^{(1)}]}$ is a solution of the following Partial Differential Equation

$$r_t \tilde{\pi}_t^3 = \Lambda_t^{(1)} + \frac{\alpha_t}{2} \times \sqrt{\Lambda_t^{(2)}}, \quad (24)$$

with

$$\begin{aligned} \Lambda_t^{(1)} &= \partial_t \tilde{\pi}_t^3 + r_t S_t \partial_S \tilde{\pi}_t^3 + (\theta - ar_t) \partial_r \tilde{\pi}_t^3 + (\overline{\theta_{t,T}^{(1)}} + \mu_m \lambda_t) \partial_\lambda \tilde{\pi}_t^3 + \frac{1}{2} \sigma_S^2 S_t^2 \partial_S^2 \tilde{\pi}_t^3 + \frac{1}{2} \sigma_m^2 \partial_\lambda^2 \tilde{\pi}_t^3 \\ &\quad + \frac{1}{2} \sigma_r^2 \partial_r^2 \tilde{\pi}_t^3 + \rho \sigma_r \sigma_S S_t \partial_r \partial_S \tilde{\pi}_t^3 + (\tilde{\pi}_t^{3*} - \tilde{\pi}_t^3) \lambda_t, \\ \Lambda_t^{(2)} &= \sigma_S^2 S_t^2 (\partial_S \tilde{\pi}_t^3)^2 + \sigma_m^2 (\partial_\lambda \tilde{\pi}_t^3)^2 + \sigma_r^2 (\partial_r \tilde{\pi}_t^3)^2 + 2\rho \sigma_S \sigma_r S_t \partial_S \partial_r \tilde{\pi}_t^3 + (\tilde{\pi}_t^3 - \tilde{\pi}_t^{3*})^2 \lambda_t, \\ \overline{\theta_{t,T}^{(1)}} &= \theta_{0,T}^{(1)} \sigma_m^2 \times \frac{e^{\mu_m(T-t)} - 1}{\mu_m}, \\ \tilde{\pi}_t^{3*}[\cdot] &= \tilde{\pi}_t^3[\cdot | N_t - 1]. \end{aligned}$$

Proof.

See the Appendix. □

Represented by $\Lambda_t^{(1)}$, the first component of the PDE captures the time-consistent price of the aggregate replicable and systematic risks. The subsequent part $\left(\frac{\alpha_t}{2} \times \sqrt{\Lambda_t^{(2)}}\right)$, relates to the time-consistent value of the diversifiable risk.

Contrary to the time-consistent case, the static continuous-time valuation framework circumvents the requirement for a partial differential equation. Indeed, a closed-form expression is readily tractable, and is given by the following formula

$$\begin{aligned} \pi_t^3[tH_T] &= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} [e^{-\int_t^T r_s ds} {}_tH_T] + \frac{\alpha_t}{2} \times \sqrt{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_T)^2]} \times \\ &\quad \sqrt{\frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_T}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_1)]]}{N_t}} \times \sqrt{T-t} \times P(t, T), \end{aligned} \quad (25)$$

for all $t \in [0, T-t^{(1)}]$, with $t^{(1)}$ denoting a strictly positive real scalar.

Let us remark that our analysis of time-consistency shall be conducted over the interval $[0, T-t^{(1)}]$. Consequently, this entails a static transition from time T to time $T-t^{(1)}$. Moreover, we shall exploit the Three-step closed-form solution in the static framework and adopt, as the terminal condition, the Three-step valuation at time $T-t^{(1)}$ of the payoff ${}_{T-t^{(1)}}H_T$ maturing at T . The rationale behind this circumvention will be explained in detail in the subsequent subsection, which outlines the numerical resolution scheme.

4.2 Numerical scheme for the resolution of the PDE using a Pure Endowment product with terminal bonus

We will concentrate our inquiry on a hybrid life product identified as the Pure Endowment with terminal bonus. Within this setting, a financial profit-sharing element is incorporated in addition

to the actuarial yield of the pure endowment product. In the event that the financial investment fund exceeds the capitalized premium P by the insurer's technical rate i , profit-sharing between the insurer and the insured will occur, given by $h(S_T) = C \times \left(1 + \beta(S_T - P(1+i)^T)^+\right)$, where β represents the profit-sharing rate, and $C > 0$ is the guaranteed minimum capital. Therefore, the defined formulation for the hybrid collective payoff is as follows $H_T^c = N_T \times C \times \left(1 + \beta(S_T - P(1+i)^T)^+\right)$. Working on a first-order basis, the premium P is calculated through $P = \left(\frac{1}{1+i}\right)^T \times_T \tilde{p}_x$, where ${}_T \tilde{p}_x$ ¹⁴ signifies the prudent survival probability, and it is entirely allocated to the financial investment fund, resulting in $S_0 = P$. Here is the unfolding of the Three-step decomposition of the Pure Endowment product with terminal bonus

$$\begin{cases} {}_{T-t(1)} H_T^1 = \mathbb{E}_{\mathcal{F}_{T-t(1)}}^{\mathbb{P}^a \times \mathbb{P}^f} \left[e^{-\int_{T-t(1)}^T \lambda_s ds} \right] \times h(S_T) \times 1_{N_{T-t(1)} > 0}, \\ {}_{T-t(1)} H_T^2 = \left(\frac{N_T}{N_{T-t(1)}} - e^{-\int_{T-t(1)}^T \lambda_s ds} \right) \times h(S_T) \times 1_{N_{T-t(1)} > 0}, \\ {}_{T-t(1)} H_T^3 = \left(e^{-\int_{T-t(1)}^T \lambda_s ds} - \mathbb{E}_{\mathcal{F}_{T-t(1)}}^{\mathbb{P}^a \times \mathbb{P}^f} \left[e^{-\int_{T-t(1)}^T \lambda_s ds} \right] \right) \times h(S_T) \times 1_{N_{T-t(1)} > 0}, \end{cases}$$

As stated in lemma 2, the dynamic Three-step method is not time-consistent because of the presence of diversifiable actuarial risks. Consequently, we will numerically address the PDE stated in Theorem 1 using the finite difference method to obtain the time-consistent price. Firstly, we discretize $(t)_t$, $(r)_t$, $(S)_t$, $(\lambda)_t$ and $(N)_t$ on the following finite grids $[0, T-t^{(1)}]$, $[r_{min}, r_{max}]$, $[0, S_{max}]$, $[0, \lambda_{max}]$, and $[0, n]$, respectively. The parameters r_{min} , r_{max} , S_{max} , and λ_{max} are configured to be attained with a probability that is very low¹⁵. We define the discretization step by $\Delta\zeta = \frac{\zeta_{max} - \zeta_{min}}{M_\zeta}$, where $\zeta \in \{t, S, r, \lambda, N\}$ and M_ζ denotes a strictly positive integer that represents the number of steps in the grid for ζ . Furthermore, we denote the time-consistent option price at time $i\Delta t$, for an underlying asset price of $j_S \Delta S$, with a mortality intensity value of $j_\lambda \Delta \lambda$, an interest rate value of $(r_{min} + j_r \Delta r)$, and a current count of surviving policyholders $j_N \Delta N$ by $\tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N}$, where $j_\zeta \in \{0, 1, \dots, M_\zeta\}$ for $\zeta \in \{S, r, \lambda, N\}$ and $i \in \{0, 1, \dots, M_t\}$.

Given that the risk factors $\tilde{M}_t^{(1)} = \frac{N_t}{N_{t-t(1)}}$ and $\tilde{M}_t^{(2)} = e^{-\int_{t-t(1)}^t \lambda_s ds}$ are not Markovian processes¹⁶, we establish the terminal condition at the instant $T - t^{(1)}$ instead of at T . The definitive formula below follows from the closed-form expression in (25), the valuation of the call option under stochastic interest rates, and the log-normal distribution of the investment fund

¹⁴The survival probability ${}_T \tilde{p}_x$ is computed according to Makeham's formula using the parameters from the Belgian Royal Decree.

¹⁵ $\lambda_{max} = 0.02$, $r_{min} = -0.1$, $r_{max} = 0.03$, $S_{max} = 7$.

¹⁶The derivation of the PDEs is based on Ito's lemma, whose formulation is predicated upon the foundational tenets of Markovian assumptions governing the risk factors $(S)_t$, $(r)_t$, $(\lambda)_t$, $(N)_t$.

under $\mathbb{P}^f$

$$\begin{aligned} \tilde{\pi}_{T-t^{(1)}}^3 [T-t^{(1)} H_T] &= C \times \left(P(T-t^{(1)}, T) + \beta(S_{T-t^{(1)}} \Phi(d_{T-t^{(1)}}) - P(T-t^{(1)}, T) PG \times \right. \\ &\quad \left. \Phi(d_{T-t^{(1)}} - n(T-t^{(1)}, T))) \right) \times e^{-\lambda_{T-t^{(1)}} l_{T-t^{(1)}, T} + \frac{1}{2} s_{T-t^{(1)}, T}^2} \times \\ &\quad e^{-\theta_{T-t^{(1)}, T}^{(1)} s_{T-t^{(1)}, T}^2} + C \times \frac{\alpha_{T-t^{(1)}}}{2} \sqrt{t^{(1)}} P(T-t^{(1)}, T) \left[1 + 2\beta(S_{T-t^{(1)}} \times \right. \\ &\quad \left. e^{\mu_S t^{(1)}} \Phi(\delta_{T-t^{(1)}}) - PG \Phi(\delta_{T-t^{(1)}} - \sigma_S \sqrt{t^{(1)}})) + \beta^2 (S_{T-t^{(1)}}^2 e^{2(\mu_S + \frac{1}{2} \sigma_S^2) t^{(1)}} \times \right. \\ &\quad \left. \Phi(\delta_{T-t^{(1)}} + \sigma_S \sqrt{t^{(1)}}) + (PG)^2 \Phi(\delta_{T-t^{(1)}} - \sigma_S \sqrt{t^{(1)}}) - 2PG S_{T-t^{(1)}} e^{\mu_S t^{(1)}} \times \right. \\ &\quad \left. \Phi(\delta_{T-t^{(1)}})) \right]^{\frac{1}{2}} \times 1_{N_{T-t^{(1)}} > 0} \times \\ &\quad \sqrt{\frac{1}{N_{T-t^{(1)}}} (e^{-\lambda_{T-t^{(1)}} l_{T-t^{(1)}, T} + \frac{1}{2} s_{T-t^{(1)}, T}^2} - e^{-2\lambda_{T-t^{(1)}} l_{T-t^{(1)}, T} + 2s_{T-t^{(1)}, T}^2})}, \end{aligned}$$

where

$$\begin{aligned} \delta_{T-t^{(1)}} &= \frac{(\mu_S + \frac{1}{2} \sigma_S^2) t^{(1)} + \ln(\frac{S_{T-t^{(1)}}}{PG})}{\sigma_S \sqrt{t^{(1)}}}, \\ \theta_{T-t^{(1)}, T}^{(1)} &= \frac{-0.16}{s_{T-t^{(1)}, T}}, \\ d_{T-t^{(1)}} &= \frac{\frac{1}{2} n(T-t^{(1)}, T)^2 + \ln(\frac{S_{T-t^{(1)}}}{PG}) - \ln(P(T-t^{(1)}, T))}{n(T-t^{(1)}, T)}, \\ n(T-t^{(1)}, T)^2 &= \sigma_S^2 t^{(1)} + \frac{\sigma_r^2}{a^2} t^{(1)} + \frac{\sigma_r^2}{a^3} \left(\frac{-3}{2} - \frac{1}{2} e^{-2at^{(1)}} + 2e^{-at^{(1)}} \right) + \frac{2\rho\sigma_S\sigma_r t^{(1)}}{a} \\ &\quad - \frac{2\rho\sigma_S\sigma_r (1 - e^{-at^{(1)}})}{a^2}. \end{aligned}$$

The approximations for the partial derivatives existing in Theorem 1 are detailed as follows

$$\begin{aligned} \partial_t \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N} &= \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} - \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N}}{\Delta t}, \quad i < M_t \\ \partial_S \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N} &= \begin{cases} \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} - \tilde{\pi}_{i+1,j_S-2,j_r,j_\lambda}^{j_N}}{2\Delta S} & \text{if } j_S = M_S \\ \frac{\tilde{\pi}_{i+1,j_S+2,j_r,j_\lambda}^{j_N} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}}{2\Delta S} & \text{if } j_S = 0 \\ \frac{\tilde{\pi}_{i+1,j_S+1,j_r,j_\lambda}^{j_N} - \tilde{\pi}_{i+1,j_S-1,j_r,j_\lambda}^{j_N}}{2\Delta S} & \text{else} \end{cases} \\ \partial_r \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N} &= \begin{cases} \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} - \tilde{\pi}_{i+1,j_S,j_r-2,j_\lambda}^{j_N}}{2\Delta r} & \text{if } j_r = M_r \\ \frac{\tilde{\pi}_{i+1,j_S,j_r+2,j_\lambda}^{j_N} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}}{2\Delta r} & \text{if } j_r = 0 \\ \frac{\tilde{\pi}_{i+1,j_S,j_r+1,j_\lambda}^{j_N} - \tilde{\pi}_{i+1,j_S,j_r-1,j_\lambda}^{j_N}}{2\Delta r} & \text{else} \end{cases} \end{aligned}$$

$$\begin{aligned}
\partial_\lambda \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N} &= \begin{cases} \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda-2}^{j_N}}{2\Delta\lambda} & \text{if } j_\lambda = M_\lambda \\ \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda+2}^{j_N} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}}{2\Delta\lambda} & \text{if } j_\lambda = 0 \\ \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda+1}^{j_N} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda-1}^{j_N}}{2\Delta\lambda} & \text{else} \end{cases} \\
\partial_S^2 \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N} &= \begin{cases} \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} - 2\tilde{\pi}_{i+1,j_S-1,j_r,j_\lambda}^{j_N} + \tilde{\pi}_{i+1,j_S-2,j_r,j_\lambda}^{j_N}}{\Delta S^2} & \text{if } j_S = M_S \\ \frac{\tilde{\pi}_{i+1,j_S+2,j_r,j_\lambda}^{j_N} - 2\tilde{\pi}_{i+1,j_S+1,j_r,j_\lambda}^{j_N} + \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}}{\Delta S^2}, & \text{if } j_S = 0 \\ \frac{\tilde{\pi}_{i+1,j_S+1,j_r,j_\lambda}^{j_N} - 2\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} + \tilde{\pi}_{i+1,j_S-1,j_r,j_\lambda}^{j_N}}{\Delta S^2} & \text{else} \end{cases} \\
\partial_r^2 \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N} &= \begin{cases} \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} - 2\tilde{\pi}_{i+1,j_S,j_r-1,j_\lambda}^{j_N} + \tilde{\pi}_{i+1,j_S,j_r-2,j_\lambda}^{j_N}}{\Delta r^2} & \text{if } j_r = M_r \\ \frac{\tilde{\pi}_{i+1,j_S,j_r+2,j_\lambda}^{j_N} - 2\tilde{\pi}_{i+1,j_S,j_r+1,j_\lambda}^{j_N} + \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}}{\Delta r^2} & \text{if } j_r = 0 \\ \frac{\tilde{\pi}_{i+1,j_S,j_r+1,j_\lambda}^{j_N} - 2\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} + \tilde{\pi}_{i+1,j_S,j_r-1,j_\lambda}^{j_N}}{\Delta r^2} & \text{else} \end{cases} \\
\partial_\lambda^2 \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N} &= \begin{cases} \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} - 2\tilde{\pi}_{i+1,j_S,j_r,j_\lambda-1}^{j_N} + \tilde{\pi}_{i+1,j_S,j_r,j_\lambda-2}^{j_N}}{\Delta\lambda^2} & \text{if } j_\lambda = M_\lambda \\ \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda+2}^{j_N} - 2\tilde{\pi}_{i+1,j_S,j_r,j_\lambda+1}^{j_N} + \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}}{\Delta\lambda^2} & \text{if } j_\lambda = 0 \\ \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda+1}^{j_N} - 2\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N} + \tilde{\pi}_{i+1,j_S,j_r,j_\lambda-1}^{j_N}}{\Delta\lambda^2} & \text{else} \end{cases} \\
(\tilde{\pi}^* - \tilde{\pi})_{i,j_S,j_r,j_\lambda}^{j_N} &= \begin{cases} 0 & \text{if } j_N = 0 \\ (\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N-1} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}) & \text{else} \end{cases}
\end{aligned}$$

$$\partial_{rS}^2 \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{jN} = \begin{cases} \frac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S,j_r-2,j_\lambda}^{jN} + \tilde{\pi}_{i+1,j_S-2,j_r-2,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S-2,j_r,j_\lambda}^{jN}}{4\Delta S\Delta r} & \text{if } j_S = M_S \text{ \& } j_r = M_r \\ \frac{\tilde{\pi}_{i+1,j_S,j_r+1,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S,j_r-1,j_\lambda}^{jN} + \tilde{\pi}_{i+1,j_S-2,j_r-1,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S-2,j_r+1,j_\lambda}^{jN}}{4\Delta S\Delta r} & \text{if } j_S = M_S \text{ \& } j_r \neq M_r, 0 \\ \frac{\tilde{\pi}_{i+1,j_S+1,j_r,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S+1,j_r-2,j_\lambda}^{jN} + \tilde{\pi}_{i+1,j_S-1,j_r-2,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S-1,j_r,j_\lambda}^{jN}}{4\Delta S\Delta r} & \text{if } j_S \neq M_S, 0 \text{ \& } j_r = M_r \\ \frac{\tilde{\pi}_{i+1,j_S+2,j_r,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S+2,j_r-2,j_\lambda}^{jN} + \tilde{\pi}_{i+1,j_S,j_r-2,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{jN}}{4\Delta S\Delta r} & \text{if } j_S = 0 \text{ \& } j_r = M_r \\ \frac{\tilde{\pi}_{i+1,j_S,j_r+2,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{jN} + \tilde{\pi}_{i+1,j_S-2,j_r,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S-2,j_r+2,j_\lambda}^{jN}}{4\Delta S\Delta r} & \text{if } j_S = M_S \text{ \& } j_r = 0 \\ \frac{\tilde{\pi}_{i+1,j_S+2,j_r+2,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S+2,j_r,j_\lambda}^{jN} + \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S,j_r+2,j_\lambda}^{jN}}{4\Delta S\Delta r} & \text{if } j_S = 0 \text{ \& } j_r = 0 \\ \frac{\tilde{\pi}_{i+1,j_S+2,j_r+1,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S+2,j_r-1,j_\lambda}^{jN} + \tilde{\pi}_{i+1,j_S,j_r-1,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S,j_r+1,j_\lambda}^{jN}}{4\Delta S\Delta r} & \text{if } j_S = 0 \text{ \& } j_r \neq 0, M_r \\ \frac{\tilde{\pi}_{i+1,j_S+1,j_r+2,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S+1,j_r,j_\lambda}^{jN} + \tilde{\pi}_{i+1,j_S-1,j_r,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S-1,j_r+2,j_\lambda}^{jN}}{4\Delta S\Delta r} & \text{if } j_S \neq 0, M_S \text{ \& } j_r = 0 \\ \frac{\tilde{\pi}_{i+1,j_S+1,j_r+1,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S+1,j_r-1,j_\lambda}^{jN} + \tilde{\pi}_{i+1,j_S-1,j_r-1,j_\lambda}^{jN} - \tilde{\pi}_{i+1,j_S-1,j_r+1,j_\lambda}^{jN}}{4\Delta S\Delta r} & \text{else} \end{cases}$$

Excluding the boundary conditions, the backward recursive scheme assumes the ensuing structure for $i \in \{0, 1, \dots, M_t - 1\}$

$$\begin{aligned} \tilde{\pi}_{i,j_S,j_r,j_\lambda}^{jN} &= b_{j_S,j_r} \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{jN} + c_{j_S,j_r}^+ \tilde{\pi}_{i+1,j_S+1,j_r,j_\lambda}^{jN} - c_{j_S,j_r}^- \tilde{\pi}_{i+1,j_S-1,j_r,j_\lambda}^{jN} + d_{j_r}^+ \tilde{\pi}_{i+1,j_S,j_r+1,j_\lambda}^{jN} - \\ & d_{j_r}^- \tilde{\pi}_{i+1,j_S,j_r-1,j_\lambda}^{jN} + (\theta_{i\Delta t, T-t^{(1)}}^{(1)} \frac{g_{j_r}}{2\Delta\lambda} + e_{j_r,j_\lambda}^+) \tilde{\pi}_{i+1,j_S,j_r,j_\lambda+1}^{jN} - (\theta_{i\Delta t, T-t^{(1)}}^{(1)} \frac{g_{j_r}}{2\Delta\lambda} + e_{j_r,j_\lambda}^-) \times \\ & \tilde{\pi}_{i+1,j_S,j_r,j_\lambda-1}^{jN} + g_{j_r} P_{i+1,j_S,j_r,j_\lambda}^{jN} + g_{j_r} (\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{jN-1} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{jN}) j_\lambda \Delta\lambda + \\ & \frac{\alpha_{i\Delta t}}{2} g_{j_r} \sqrt{S_{i+1,j_S,j_r,j_\lambda}^{jN} + (\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{jN-1} - \tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{jN})^2 j_\lambda \Delta\lambda}. \end{aligned}$$

$$\text{where } b_{j_S,j_r} = \frac{1 - \sigma_S^2 j_S^2 \Delta t - \frac{\sigma_{m_S}^2 \Delta t}{\Delta\lambda^2} - \frac{\sigma_r^2 \Delta t}{\Delta r^2}}{1 + (r_{min} + j_r \Delta r) \Delta t}, \quad c_{j_S,j_r}^{+-} = \frac{\frac{1}{2}(r_{min} + j_r \Delta r) j_S \Delta t + \frac{1}{2} \sigma_S^2 j_S^2 \Delta t}{1 + (r_{min} + j_r \Delta r) \Delta t},$$

$$d_{j_r}^{+-} = \frac{(\theta - a(r_{min} + j_r \Delta r) \Delta t) + \frac{\sigma_r^2 \Delta t}{2 \Delta r}}{1 + (r_{min} + j_r \Delta r) \Delta t}, e_{j_r, j_\lambda}^{+-} = \frac{\frac{1}{2} \mu_m j_\lambda \Delta t + \frac{1}{2} \frac{\sigma_m^2 \Delta t}{\Delta \lambda^2}}{1 + (r_{min} + j_r \Delta r) \Delta t}, g_{j_r} = \frac{\Delta t}{1 + (r_{min} + j_r \Delta r) \Delta t},$$

$$P_{i+1, j_S, j_r, j_\lambda}^{j_N} = \rho \sigma_r \sigma_S j_S \frac{\tilde{\pi}_{i+1, j_S+1, j_r+1, j_\lambda}^{j_N} - \tilde{\pi}_{i+1, j_S+1, j_r-1, j_\lambda}^{j_N} + \tilde{\pi}_{i+1, j_S-1, j_r-1, j_\lambda}^{j_N} - \tilde{\pi}_{i+1, j_S-1, j_r+1, j_\lambda}^{j_N}}{4 \Delta r} \text{ and } S_{i+1, j_S, j_r, j_\lambda}^{j_N} =$$

$$(\sigma_S j_S)^2 \frac{(\tilde{\pi}_{i+1, j_S+1, j_r, j_\lambda}^{j_N} - \tilde{\pi}_{i+1, j_S-1, j_r, j_\lambda}^{j_N})^2}{4} + \sigma_m^2 \frac{(\tilde{\pi}_{i+1, j_S, j_r, j_\lambda+1}^{j_N} - \tilde{\pi}_{i+1, j_S, j_r, j_\lambda-1}^{j_N})^2}{4 \Delta \lambda^2} + \sigma_r^2 \frac{(\tilde{\pi}_{i+1, j_S, j_r+1, j_\lambda}^{j_N} - \tilde{\pi}_{i+1, j_S, j_r-1, j_\lambda}^{j_N})^2}{4 \Delta r^2} +$$

$$\rho \sigma_r \sigma_S j_S \frac{\tilde{\pi}_{i+1, j_S+1, j_r+1, j_\lambda}^{j_N} - \tilde{\pi}_{i+1, j_S+1, j_r-1, j_\lambda}^{j_N}}{2 \Delta r} (\tilde{\pi}_{i+1, j_S+1, j_r, j_\lambda}^{j_N} - \tilde{\pi}_{i+1, j_S-1, j_r, j_\lambda}^{j_N}).$$

4.3 Numerical results

We consider a cohort of $n = 100$ females aged 35 contracting a pure endowment with a terminal bonus. Our numerical application follows the assumptions used in [Belhouari et al., 2024], where the parameters characterizing our financial-actuarial models can be summarized as follows

Parameters	Numerical Values
α_t	30%
i	0.9%
C	100000
T	10
P	0.9070291
r_0	1.6%
a	40%
θ	0.8%
σ_r	0.5%
σ_S	15%
ρ	20%
μ_S	5%
β	40.19%
λ_0	0.001080584
μ_m	0.0745323698
σ_m	0.0002248628

Prior to elucidating the numerical results, we must preliminarily verify the convergence of the finite difference method. Since the discretization step ΔN remains invariably equal to 1^{17} , it obviates the need for any calibration of the number of steps concerning the number of survivors. As depicted in Figure 1, the convergence of time is reached after 989 iterations, implying that approximately one iteration occurs every three days and one tier. Figure 2, Figure 3, and Figure 4 respectively investigate the convergence of interest rates, mortality rates, and the financial investment fund. The stability of the numerical values is deemed sufficient for $M_S = 35$, $M_r = 40$ and $M_\lambda = 40$. Figure 5 contrasts the time-consistent values of the Three-step method with its static counterparts and the best estimate calculated under $\mathbb{Z}_\theta^a \times \mathbb{Q}^f$. The dynamic Three-step method follows a monotonic trend analogous to that of the best estimate, reflecting a price upward over time. Furthermore, it demonstrates that the static values of the dynamic Three-step method exhibit a modest increase, ultimately resulting in a 3.7% cumulative growth by time 0. By including the diversifiable part, the dynamic Three-step method loses its time-consistent property, which it previously maintained with a valuation formula equivalent to $\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} [e^{-\int_t^T r_s ds} {}_t H_T]$, as referenced in lemma 1. Therefore, to better understand the effect of time-consistency, it is

¹⁷i.e. $M_N = n$.

important to compare the time-consistent Three-step prices, net of this best estimate, with the one-period Three-step prices, net again of this best estimate.¹⁸ As we can see in Figure 6, the static values undergo a substantial increase, ending in an 185% increase by time 0. To address this issue, we follow [Devolder and Lebègue, 2016] and adjust the $\alpha_{i\Delta t}$ safety loads applied in the backward recursive scheme. In our study, we propose to linearly decrease the safety loads over the last seven years as an initial attempt, and over the entire time-frame as a final attempt, within the backward algorithm¹⁹, as detailed below

$$\alpha_{i\Delta t} = \begin{cases} \frac{0.3}{770} \times i & 0 \leq i \leq 770 \mid 7 - \text{linear load decrease} \\ \frac{0.3}{989} \times i & 0 \leq i \leq 989 \mid 9 - \text{linear load decrease} \end{cases}$$

As illustrated in Figure 6, the first attempt successfully reduces the percentage increase due to time consistency in the instant 0 to a value below 100%, specifically 54%. More favorably, the last attempt, attains a 23% uplift in the one-step price at time 0.

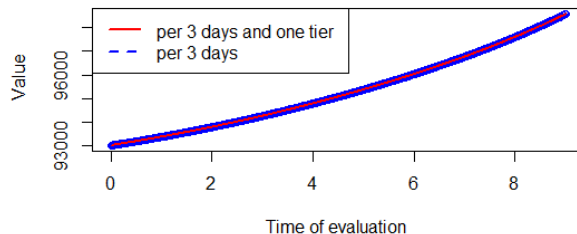


Figure 1: The Three-step method time-consistent dynamic with $M_S = 35$, $M_r = 40$, $M_\lambda = 40$, $M_N = n = 100$, and, $S_t = S_0$, $r_t = r_0$, $\lambda_t = \lambda_0$, $N_t = n$. ($t^{(1)} = 1$)

5 Conclusion

This paper explores the extension of the Three-step method to the dynamic context. To achieve this, we first refine the axiomatization of valuation operators, as well as the properties of market-consistency and actuarial-consistency, ultimately deriving a dynamic three-step decomposition. We defined a crucial concept from the literature known as the time-consistency. Furthermore, we presented an iterative backward procedure in a discrete model, enabling the enforcement of time-consistent dynamics where such consistency is previously absent. With a combination of any hybrid pure-endowment product, we identified that the diversifiable component of the Three-step method detrimentally induces the non-verification of the time-consistent property. Therefore, we applied the backward procedure through a dual approach to the Three-step

¹⁸In the one-period framework, the difference between the Three-step price and this best estimate is equal to the static value of the diversifiable component.

¹⁹[Devolder and Lebègue, 2016] examined the impact of various time-dependent evolutions, including linear, quadratic, and exponential, of security loads on the solvency capital requirement.

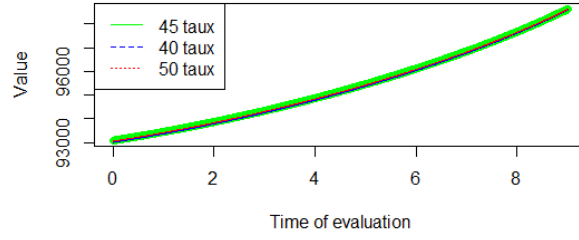


Figure 2: The Three-step method time-consistent dynamic with $M_t = 989$, $M_S = 35$, $M_\lambda = 40$, $M_N = n = 100$, and, $S_t = S_0$, $r_t = r_0$, $\lambda_t = \lambda_0$, $N_t = n$. ($t^{(1)} = 1$)

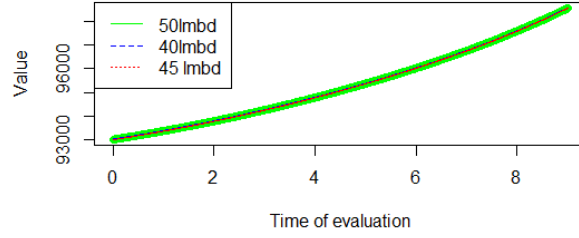


Figure 3: The Three-step method time-consistent dynamic with $M_t = 989$, $M_S = 35$, $M_r = 40$, $M_N = n = 100$, and, $S_t = S_0$, $r_t = r_0$, $\lambda_t = \lambda_0$, $N_t = n$. ($t^{(1)} = 1$)

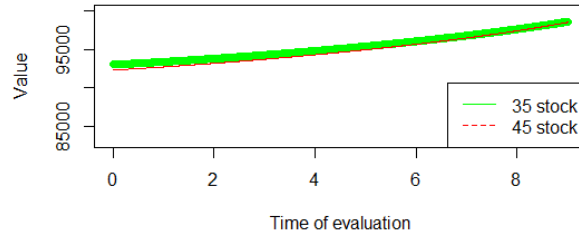


Figure 4: The Three-step method time-consistent dynamic with $M_t = 989$, $M_\lambda = 40$, $M_r = 40$, $M_N = n = 100$, and, $S_t = S_0$, $r_t = r_0$, $\lambda_t = \lambda_0$, $N_t = n$. ($t^{(1)} = 1$)

method to derive closed, time-consistent formulas using the purely actuarial Pure Endowment

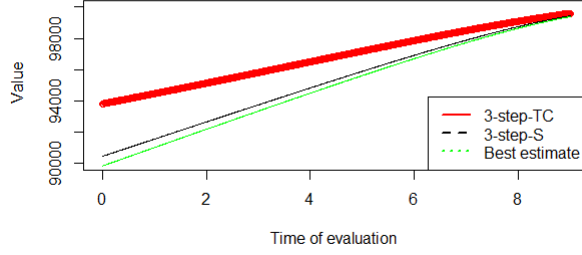


Figure 5: The dynamic Three-step method in the static versus time-consistent case and the best estimate under $\mathbb{Z}_\theta^a \times \mathbb{Q}^f$. ($t^{(1)} = 1$)

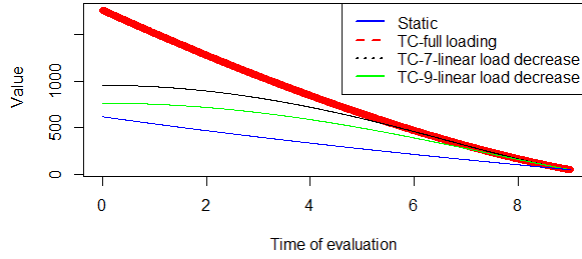


Figure 6: The dynamic Three-step method net of the best estimate $\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} [e^{-\int_t^T r_s ds} {}_t H_T]$ in the static versus time-consistent cases. ($t^{(1)} = 1$)

product. Thanks to the two-iteration closed-form solution, we observed that the diversifiable part influences the valuation of the non-diversifiable components of the subsequent iteration. Thus, achieving time-consistency by relying on traditional non-linear actuarial operators, such as the variance/standard-deviation premium principle, to assess the diversifiable part is not entirely viable. Immersed in the continuous-time limit, a partial differential equation governing the Three-step method time-consistent dynamic was derived. By specifying a financial terminal bonus as the financial payoff of the hybrid life Pure Endowment product, the PDE was numerically solved using the finite difference method. The obtained numerical results indicated that the effect of time led to a slight rise in the one-step prices of the full Three-step method. Nonetheless, the static prices of the diversifiable actuarial component have surged notably when compared to the time-consistent Three-step prices, net of the best estimate under $\mathbb{Z}_\theta^a \times \mathbb{Q}^f$. That being said, the time-consistent property, though vital for avoiding arbitrage opportunities, is not immune to criticism and may have specific downsides. To mitigate the resulting impact, the safety charges applied in the backward scheme were progressively reduced through a linear approach. In most cases, the actuary managing premium assessments has considerable flexibility in setting the level of security measures to apply. When necessary, a more risk-averse strategy can be employed by raising the security proportion loading, causing the premiums to increase. Conversely, a less cautious approach permits a reduction in the loading, resulting in lower pre-

miums. In our case, moderating the security loadings becomes a rational decision to ensure a suitable trade-off between time-consistency and the smooth progression of price changes. This indicates that careful consideration is required concerning the magnitude of security charges used in the backward scheme when applying non-linear actuarial premium principles to the valuation of the diversifiable part of the Three-step method.

APPENDIX.

Details about

$$\begin{aligned}\pi_t^{u,SD}({}_tH_u^2|\mathcal{F}_t) &\xrightarrow{N_t \rightarrow +\infty} 0, \\ \pi_t^{u,V}({}_tH_u^2|\mathcal{F}_t) &\xrightarrow{N_t \rightarrow +\infty} 0,\end{aligned}$$

for any $t \in [0, T]$.

Proof.

Let $0 \leq t \leq u \leq T$, such as $N_t > 0$.

Using the independence between mortality and finance, coupled with the application of the tower rule, we ascertain a zero value for the best estimate

$$\begin{aligned}\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^T r_s ds} {}_tH_u^2] &= \frac{1}{N_t} \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_s ds} h(S_u)] \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} \left[\sum_{j=1}^{N_t} (g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)]) \right] \\ &= \frac{1}{N_t} \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_s ds} h(S_u)] \sum_{j=1}^{N_t} (\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [g(\tau_j)] - \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)]) \\ &= 0.\end{aligned}$$

Regarding the loading term

$$\begin{aligned}\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tH_u^2] &= \frac{1}{N_t^2} \mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\sum_{j=1}^{N_t} (g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)]) h(S_u) \right] \\ &= \frac{1}{N_t^2} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2] \times \mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\sum_{j=1}^{N_t} (g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)]) \right].\end{aligned}$$

We further notice that

$$\begin{aligned}\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\sum_{j=1}^{N_t} (g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)]) \right] &= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\sum_{j=1}^{N_t} g(\tau_j) - \sum_{j=1}^{N_t} \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)] \right] \right] + \\ &\quad \mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\sum_{j=1}^{N_t} g(\tau_j) - \sum_{j=1}^{N_t} \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)] \right] \right].\end{aligned}$$

Hence

$$\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\sum_{j=1}^{N_t} (g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)]) \right] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\sum_{j=1}^{N_t} g(\tau_j) \right] \right] + 0.$$

Since the remaining lifetimes $(\tau_j)_{1 \leq j \leq N_t}$ are independent and identically distributed conditional on the filtration of systematic mortality $\mathfrak{S}_u$, this variance can be rewritten as

$$\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\sum_{j=1}^{N_t} (g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_j)]) \right] = N_t \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_1)] \right].$$

Thus

$$\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f}[_t H_u^2] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2] \times \frac{1}{N_t} \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_1)]].$$

Finally

$$\begin{cases} \pi_t^{u,V}(_t H_u^2 | \mathcal{F}_t) = \frac{\alpha_t}{2k} \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2] \times \frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_1)]]}{N_t} \times P(t, u). \\ \pi_t^{u,SD}(_t H_u^2 | \mathcal{F}_t) = \frac{\alpha_t}{2} \times \sqrt{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2]} \times \sqrt{\frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_u}^{\mathbb{P}^a \times \mathbb{P}^f} [g(\tau_1)]]}{N_t}} \times \sqrt{u-t} \times P(t, u). \end{cases}$$

Therefore $\pi_t^{u,SD}(_t H_u^2 | \mathcal{F}_t) \xrightarrow{N_t \rightarrow +\infty} 0$, and $\pi_t^{u,V}(_t H_u^2 | \mathcal{F}_t) \xrightarrow{N_t \rightarrow +\infty} 0$. $\square$

Details about remark 1.

Proof.

First, we seek to validate the assertion that the iterated property is inherent in the concept of time-consistency. Let $0 \leq t \leq u \leq T$, $0 < s \leq u - t$ and $\overline{h_u} \in \overline{\mathcal{H}_u^f}$.

The translation invariance axiom implies that,

$$\pi_{t+s}^{t+s}[\pi_{t+s}^u[\overline{h_u}]] = \pi_{t+s}^u[\overline{h_u}].$$

Since $\pi_{t+s}^u[\overline{h_u}] \in \overline{\mathcal{H}_{t+s}^f}$ and using (16) we reach the iterative definition.

Under the assumption of monotonicity, we embark on the verification of the converse direction.

Let $0 \leq t \leq u, v \leq T$, $0 < s \leq u - t$, $0 < s \leq v - t$, $\overline{h_u} \in \overline{\mathcal{H}_u^f}$, $\overline{g_v} \in \overline{\mathcal{H}_v^f}$ such as $\pi_{t+s}^u[\overline{h_u}] \leq \pi_{t+s}^v[\overline{g_v}]$.

Since $\pi_{t+s}^u[\overline{h_u}] \in \overline{\mathcal{H}_{t+s}^f}$, $\pi_{t+s}^v[\overline{g_v}] \in \overline{\mathcal{H}_{t+s}^f}$, then by monotonicity

$$\pi_t^{t+s}[\pi_{t+s}^u[\overline{h_u}]] \leq \pi_t^{t+s}[\pi_{t+s}^v[\overline{g_v}]].$$

Hence using (17),

$$\pi_t^u[\overline{h_u}] \leq \pi_t^v[\overline{g_v}].$$

$\square$

Details about lemma 1.

Proof.

Let $0 \leq t \leq u \leq T$ such us $0 < s \leq u - t$ and $N_{t+s} > 0$.

$$\begin{aligned} \pi_t^{t+s,2}[\frac{N_{t+s}}{N_t} \times \pi_{t+s}^{u,2}[_{t+s} H_u]] &= \pi_t^{t+s,2}[\frac{N_{t+s}}{N_t} \times (\pi_{t+s}^{u,f}[_{t+s} H_u^h] + \pi_{t+s}^{u,n}[_{t+s} H_u^{sy}])] \\ &= \pi_t^{t+s,2}[\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \times \frac{N_{t+s}}{N_t} \\ &\quad (\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] + \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] - \\ &\quad \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}])] \\ &= \pi_t^{t+s,2}[\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \times \frac{N_{t+s}}{N_t} \\ &\quad \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}]]. \end{aligned}$$

Let us set ${}_tLY_{t+s} = \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \times \frac{N_{t+s}}{N_t} \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}]$, then its two-step decomposition is

$$\begin{cases} {}_tLY_{t+s}^h = \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+s}^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tLY_{t+s}] = \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}]] \times 1_{N_t > 0}, \\ {}_tLY_{t+s}^{sy} = \left({}_tLY_{t+s} - \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}]] \right) \times 1_{N_t > 0}. \end{cases}$$

Using the independence between mortality and finance and tower rule property under $\mathbb{Z}_\theta^a$ and $\mathbb{Q}^f$, we obtain

$$\begin{aligned} \pi_t^{t+s,2} \left[\frac{N_{t+s}}{N_t} \times \pi_{t+s}^{u,2} [{}_{t+s}H_u] \right] &= \pi_t^{t+s,2} \left[\frac{N_{t+s}}{N_t} \times \left(\pi_{t+s}^{u,f} [{}_{t+s}H_u^h] + \pi_{t+s}^{u,n} [{}_{t+s}H_u^{sy}] \right) \right] \\ &= \pi_t^{t+s,f} [{}_tLY_{t+s}^h] + \pi_t^{t+s,n} [{}_tLY_{t+s}^{sy}] \\ &= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_v dv} h(S_u)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}]] + \\ &\quad \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_v dv} h(S_u)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}] - \\ &\quad \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_v dv} h(S_u)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}]] \\ &= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^u r_v dv} h(S_u)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}] \\ &= \pi_t^{u,f} [{}_tH_u^h] + \pi_t^{u,n} [{}_tH_u^{sy}] \\ &= \pi_t^{u,2} [{}_tH_u]. \end{aligned}$$

Thus, the time-consistent valuation at time $t \in [0, T]$ can be calculated as $\pi_t^2 [{}_tH_T] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^T r_v dv} h(S_T)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^T \lambda_v dv}]$. $\square$

Details about lemma 2.

Proof.

For the proof, we rely on the variance premium principle as the operator for evaluating the diversifiable risk. Notably, the result holds true under the standard deviation premium principle, and the proof can be readily adapted.

Let $0 \leq t \leq u \leq T$ such us $0 < s \leq u - t$ and $N_{t+s} > 0$. Since $\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u r_v dv} (\frac{N_u}{N_{t+s}} -$

$e^{-\int_{t+s}^u \lambda_v dv}] = 0$, and $g(\tau_1)|_{\mathcal{F}_{t+s} \vee \mathfrak{S}_u} \sim \text{Bernoulli}(e^{-\int_{t+s}^u \lambda_v dv})$, we have

$$\begin{aligned}
\pi_t^{t+s,3} \left[\frac{N_{t+s}}{N_t} \times \pi_{t+s}^{u,3} [{}_{t+s}H_u] \right] &= \pi_t^{t+s,3} \left[\frac{N_{t+s}}{N_t} \times \left(\pi_{t+s}^{u,f} [{}_{t+s}H_u^1] + \pi_{t+s}^{u,V} [{}_{t+s}H_u^2] + \pi_{t+s}^{u,n} [{}_{t+s}H_u^3] \right) \right] \\
&= \pi_t^{t+s,3} \left[\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \frac{N_{t+s}}{N_t} \times \left(\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] + \right. \right. \\
&\quad \left. \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] - \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] \right) + \frac{\alpha_{t+s}}{2k} \times P(t+s, u) \times \\
&\quad \left. \mathbb{V}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_u}{N_{t+s}} - e^{-\int_{t+s}^u \lambda_v dv} \right] \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2] \times \frac{N_{t+s}}{N_t} \right] \\
&= \pi_t^{t+s,3} \left[\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \frac{N_{t+s}}{N_t} \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] + \right. \\
&\quad \left. \frac{\alpha_{t+s}}{2k} \times P(t+s, u) \times \frac{\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv} (1 - e^{-\int_{t+s}^u \lambda_v dv})]}{N_{t+s}} \times \right. \\
&\quad \left. \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2] \times \frac{N_{t+s}}{N_t} \right].
\end{aligned}$$

Let us set ${}_tLY_{t+s} = \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \frac{N_{t+s}}{N_t} \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] + \frac{\alpha_{t+s}}{2k} \times P(t+s, u) \times \frac{\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv} (1 - e^{-\int_{t+s}^u \lambda_v dv})]}{N_t} \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2]$, then its three-step decomposition is

$$\left\{ \begin{aligned}
{}_tLY_{t+s}^1 &= \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+s}^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tLY_{t+s}] = (\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}]] \\
&\quad + \frac{\alpha_{t+s}}{2k} \times \frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv} (1 - e^{-\int_{t+s}^u \lambda_v dv})]}{N_t} \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2] \times P(t+s, u)) \times 1_{N_t > 0}, \\
{}_tLY_{t+s}^2 &= {}_tLY_{t+s} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+s} \vee \mathcal{F}_{t+s}^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tLY_{t+s}] = \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \left(\frac{N_{t+s}}{N_t} \times \right. \\
&\quad \left. \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] - \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}] \right) \times 1_{N_t > 0}, \\
{}_tLY_{t+s}^3 &= \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+s} \vee \mathcal{F}_{t+s}^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tLY_{t+s}] - \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+s}^f}^{\mathbb{P}^a \times \mathbb{P}^f} [{}_tLY_{t+s}] = (\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)] \times \\
&\quad \left[\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}] - \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}]] \right] + \frac{\alpha_{t+s}}{2k} \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2] \times \\
&\quad P(t+s, u) \left[\frac{\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv} (1 - e^{-\int_{t+s}^u \lambda_v dv})]}{N_t} - \frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv} (1 - e^{-\int_{t+s}^u \lambda_v dv})]}{N_t} \right]) \times 1_{N_t > 0}.
\end{aligned} \right.$$

Based on the assumed independence between mortality and finance, using tower rule property

under $\mathbb{Z}_\theta^a$ and $\mathbb{Q}^f$ and after completing some necessary calculations

$$\begin{aligned}
\pi_t^{t+s,3} \left[\frac{N_{t+s}}{N_t} \times \pi_{t+s}^{u,3} [t+s H_u] \right] &= \pi_t^{t+s,3} \left[\frac{N_{t+s}}{N_t} \times \left(\pi_{t+s}^{u,f} [t+s H_u^1] + \pi_{t+s}^{u,V} [t+s H_u^2] + \pi_{t+s}^{u,n} [t+s H_u^3] \right) \right] \\
&= \pi_t^{t+s,f} [t LY_{t+s}^1] + \pi_t^{t+s,V} [t LY_{t+s}^2] + \pi_t^{t+s,n} [t LY_{t+s}^3] \\
&= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^{t+s} r_v dv} h(S_u)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^{t+s} \lambda_v dv}] + \frac{\alpha_{t+s}}{2k} \times \\
&\quad \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^{t+s} r_v dv} P(t+s, u)] \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2] \times \\
&\quad \frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv} (1 - e^{-\int_{t+s}^u \lambda_v dv})]]}{N_t} + \\
&\quad \frac{\alpha_t}{2k} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)]^2] \times \mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+s}}{N_t} \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] \right. \\
&\quad \left. - \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^u \lambda_v dv}] \right] \times P(t, t+s).
\end{aligned}$$

Since $\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^{t+s} r_v dv} h(S_u)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_t^{t+s} \lambda_v dv}] = \pi_t^{u,f} [t H_u^1] + \pi_t^{u,n} [t H_u^3]$, $g(\tau_1)|_{\mathcal{F}_t \vee \mathfrak{G}_{t+s}} \sim \text{Bernoulli}(e^{-\int_t^{t+s} \lambda_v dv})$ and through additional refinement, we arrive at

$$\begin{aligned}
\pi_t^{t+s,3} \left[\frac{N_{t+s}}{N_t} \times \pi_{t+s}^{u,3} [t+s H_u] \right] &= \pi_t^{t+s,3} \left[\frac{N_{t+s}}{N_t} \times \left(\pi_{t+s}^{u,f} [t+s H_u^1] + \pi_{t+s}^{u,V} [t+s H_u^2] + \pi_{t+s}^{u,n} [t+s H_u^3] \right) \right] \\
&= \pi_t^{u,f} [t H_u^1] + \pi_t^{u,n} [t H_u^3] + \frac{\alpha_t}{2k} \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_{t+s}^u r_v dv} h(S_u)]^2] \times \\
&\quad \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\left(\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv}] \right)^2 e^{-\int_t^{t+s} \lambda_v ds} \times (1 - e^{-\int_t^{t+s} \lambda_v dv}) \right] \times \\
&\quad P(t, t+s) + \frac{\alpha_{t+s}}{2k} \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^{t+s} r_v dv} P(t+s, u)] \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [h(S_u)^2] \\
&\quad \times \frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t+s}^u \lambda_v dv} (1 - e^{-\int_{t+s}^u \lambda_v dv})]]}{N_t} \\
&\neq \pi_t^{u,f} [t H_u^1] + \pi_t^{u,V} [t H_u^2] + \pi_t^{u,n} [t H_u^3].
\end{aligned}$$

In the end, we conclude that the dynamic Three-step method $(\pi_t^{u,3})_{0 \leq t \leq u \leq T}$ is not time-consistent. $\square$

Details about lemma 3.

Proof.

We implement the backward scheme referenced in (22), specifically concentrating on two iterations. That is, we set $\Delta t^* = T - t^*$, $s = \Delta t^*$ and $\tilde{s} = 2\Delta t^* = T$. The value at the instant t^* is

$$\begin{aligned}
\tilde{\pi}_{t^*}^3 [t^* H_T | \mathcal{F}_{t^*}] &= \pi_{t^*}^f [t^* H_T^1] + \pi_{t^*}^V [t^* H_T^2] + \pi_{t^*}^n [t^* H_T^3] \\
&= P(t^*, T) \left(\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f} [e^{-\int_{t^*}^T \lambda_s ds}] + \frac{\alpha_{t^*}}{2k} \times \frac{\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-\int_{t^*}^T \lambda_s ds} (1 - e^{-\int_{t^*}^T \lambda_s ds})]}{N_{t^*}} \right) \\
&= P(t^*, T) \left(e^{-\lambda_{t^*} l_{t^*, T} + \frac{1}{2} s_{t^*, T}^2} e^{-\theta_{t^*, T}^{(1)} s_{t^*, T}^2} + \frac{\alpha_{t^*}}{2k} \times \frac{e^{-\lambda_{t^*} l_{t^*, T} + \frac{1}{2} s_{t^*, T}^2} - e^{-2\lambda_{t^*} l_{t^*, T} + 2s_{t^*, T}^2}}{N_{t^*}} \right).
\end{aligned}$$

Secondly, the time-consistent valuation at the instant 0 can be calculated as follows

$$\begin{aligned}\tilde{\pi}_0^3[{}_0H_T|\mathcal{F}_0] &= \pi_0^{t^*,3}\left[\frac{N_{t^*}}{n} \times \tilde{\pi}_{t^*}^3[{}_tH_T|\mathcal{F}_{t^*}]|\mathcal{F}_0\right] \\ &= \pi_0^{t^*,3}\left[P(t^*, T)\left(\frac{N_{t^*}}{n}\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\int_{t^*}^T \lambda_s ds}] + \frac{\alpha_{t^*}}{2k} \times \frac{\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-\int_{t^*}^T \lambda_s ds}(1 - e^{-\int_{t^*}^T \lambda_s ds})]}{n}\right)\right].\end{aligned}$$

Let us set ${}_0LY_{t^*} = P(t^*, T) \times \left(\frac{N_{t^*}}{n} \times \mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\int_{t^*}^T \lambda_s ds}] + \frac{\alpha_{t^*}}{2k} \times \frac{\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-\int_{t^*}^T \lambda_s ds}(1 - e^{-\int_{t^*}^T \lambda_s ds})]}{n}\right)$, then its three-step decomposition is

$$\left\{ \begin{aligned} {}_0LY_{t^*}^1 &= \mathbb{E}_{\mathcal{F}_0 \vee \mathcal{F}_{t^*}^f}^{\mathbb{P}^a \times \mathbb{P}^f}[{}_0LY_{t^*}] = P(t^*, T) \times \left(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\int_0^T \lambda_s ds}]] + \frac{\alpha_{t^*}}{2k} \times \frac{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-\int_0^T \lambda_s ds}(1 - e^{-\int_0^T \lambda_s ds})]}{n}\right), \\ {}_0LY_{t^*}^2 &= {}_0LY_{t^*} - \mathbb{E}_{\mathcal{F}_0 \vee \mathfrak{S}_{t^*} \vee \mathcal{F}_{t^*}^f}^{\mathbb{P}^a \times \mathbb{P}^f}[{}_0LY_{t^*}] = P(t^*, T) \times \left(\frac{N_{t^*}}{n} \times \mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\int_{t^*}^T \lambda_s ds}] - \mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\int_0^T \lambda_s ds}]\right), \\ {}_0LY_{t^*}^3 &= \mathbb{E}_{\mathcal{F}_0 \vee \mathfrak{S}_{t^*} \vee \mathcal{F}_{t^*}^f}^{\mathbb{P}^a \times \mathbb{P}^f}[{}_0LY_{t^*}] - \mathbb{E}_{\mathcal{F}_0 \vee \mathcal{F}_{t^*}^f}^{\mathbb{P}^a \times \mathbb{P}^f}[{}_0LY_{t^*}] = P(t^*, T) \times \left(\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\int_0^T \lambda_s ds}] - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\int_0^T \lambda_s ds}]] + \frac{\alpha_{t^*}}{2k} \left[\frac{\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-\int_{t^*}^T \lambda_s ds}(1 - e^{-\int_{t^*}^T \lambda_s ds})]}{n} - \frac{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-\int_0^T \lambda_s ds}(1 - e^{-\int_0^T \lambda_s ds})]}{n}\right]\right). \end{aligned} \right.$$

With tower rule property under $\mathbb{Z}_\theta^a$ and $\mathbb{Q}^f$, as well as the constancy of interest rates, and $g(\tau_1)|_{\mathcal{F}_0 \vee \mathfrak{S}_{t^*}} \sim \text{Bernoulli}(e^{-\int_0^{t^*} \lambda_s ds})$, we obtain

$$\begin{aligned}\tilde{\pi}_0^3[{}_0H_T|\mathcal{F}_0] &= \pi_0^{t^*,f}[{}_0LY_{t^*}^1] + \pi_0^{t^*,V}[{}_0LY_{t^*}^2] + \pi_0^{t^*,n}[{}_0LY_{t^*}^3] \\ &= P(0, T) \left(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\int_0^T \lambda_s ds}] + \frac{\alpha_{t^*}}{2k} \frac{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-\int_{t^*}^T \lambda_s ds}(1 - e^{-\int_{t^*}^T \lambda_s ds})]]}{n}\right) \\ &\quad + \frac{\alpha_0}{2k} \times P(0, t^*)P(t^*, T)^2 \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\left(\mathbb{E}_{\mathcal{F}_{t^*}}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\int_{t^*}^T \lambda_s ds}]\right)^2 e^{-\int_0^{t^*} \lambda_s ds} \times (1 - e^{-\int_0^{t^*} \lambda_s ds})\right] \\ &= P(0, T) \left(e^{-\lambda_0 l_{0,T} - s_{0,T}^2 (\theta_{0,T}^{(1)} - \frac{1}{2})} + \frac{\alpha_{t^*}}{2k} \frac{e^{\frac{1}{2}s_{t^*,T}^2} \mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-\lambda_{t^*} l_{t^*,T}}] - e^{2s_{t^*,T}^2} \mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a \times \mathbb{P}^f}[e^{-2\lambda_{t^*} l_{t^*,T}}]}{n}\right) \\ &\quad + \frac{\alpha_0}{2k} \times P(0, t^*)P(t^*, T)^2 e^{-2s_{t^*,T}^2 (\theta_{t^*,T}^{(1)} - \frac{1}{2})} \left(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-2\lambda_{t^*} l_{t^*,T}} \times e^{-\int_0^{t^*} \lambda_s ds}] - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-2\lambda_{t^*} l_{t^*,T}}] \times e^{-2\int_0^{t^*} \lambda_s ds}\right).\end{aligned}$$

The terms $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f}[e^{-\lambda_{t^*} l_{t^*,T}}]$ and $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f}[e^{-2\lambda_{t^*} l_{t^*,T}}]$ are solved using the following density

$$\frac{d\mathbb{Z}_\theta^a}{d\mathbb{P}^a}\bigg|_{\mathcal{F}_0} = \frac{e^{\int_0^{t^*} \theta_{0,t^*}^{(1)} \lambda_s ds}}{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{\int_0^{t^*} \theta_{0,t^*}^{(1)} \lambda_s ds}]} = e^{\int_0^{t^*} \theta_{0,t^*}^{(1)} \sigma_m \frac{e^{\mu m(t^* - u)} - 1}{\mu m} dW_m(u) - \frac{1}{2} \int_0^{t^*} (\theta_{0,t^*}^{(1)} \sigma_m \frac{e^{\mu m(t^* - u)} - 1}{\mu m})^2 du},$$

whereas, the derivation of the terms $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-2\lambda_{t^*} l_{t^*,T}} \times e^{-2 \int_0^{t^*} \lambda_s ds}]$ and $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f} [e^{-2\lambda_{t^*} l_{t^*,T}} \times e^{-\int_0^{t^*} \lambda_s ds}]$ involves the corresponding density

$$\frac{d\bar{\mathbb{P}}^a}{d\mathbb{P}^a} \Big|_{\mathcal{F}_0} = \frac{e^{-2\lambda_{t^*} l_{t^*,T}}}{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-2\lambda_{t^*} l_{t^*,T}}]} = e^{-\frac{1}{2} \int_0^{t^*} F(u)^2 du - \int_0^{t^*} F(u) dW_m(u)},$$

$$F(u) = 2 \times l_{t^*,T} \sigma_m e^{\mu_m(t^* - u)}.$$

After performing the necessary computations, we arrive at the following final formula

$$\begin{aligned} \tilde{\pi}_0^3[{}_0H_T|\mathcal{F}_0] &= P(0, T) \left(e^{-\lambda_0 l_0, T - s_{0,T}^2 (\theta_{0,T}^{(1)} - \frac{1}{2})} + \frac{\alpha_t}{2k} \frac{e^{\frac{1}{2} s_{t^*}^2, T} \tilde{f}_1 - e^{2s_{t^*}^2, T} \tilde{f}_4}{n} \right) + \frac{\alpha_0}{2k} \times P(0, t^*) \times \\ &P(t^*, T)^2 e^{-2s_{t^*}^2, T (\theta_{t^*,T}^{(1)} - \frac{1}{2})} ({}_1f_1^2 - {}_2f_4^8). \end{aligned}$$

□

Proof of Theorem 1.

Proof.

To mitigate notational ambiguity, we opt to exclude the "3" notation from $\tilde{\pi}_t^3$, though the derivation of the PDE especially for the Three-step method time-consistent dynamic remains implicit.

The risk ${}_tLY_{t+dt} = \frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}$, considered at time $t + dt$, is decomposed into the three components in order to evaluate it at time t , resulting in the following

$$\begin{cases} {}_tLY_{t+dt}^1 = \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right], \\ {}_tLY_{t+dt}^2 = \frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right], \\ {}_tLY_{t+dt}^3 = \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] - \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]. \end{cases}$$

Therefore, the constraint $\tilde{\pi}_t = \pi_t^{t+dt} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]$ is equivalent to

$$\begin{aligned} \tilde{\pi}_t &= \pi_t^{t+dt, f} [{}_tLY_{t+dt}^1] + \pi_t^{t+dt, SD} [{}_tLY_{t+dt}^2] + \pi_t^{t+dt, n} [{}_tLY_{t+dt}^3] \\ &= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^{t+dt} r_s ds} \times \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]] + \\ &\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^{t+dt} r_s ds} \times \left(\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] \right)] \\ &+ \frac{\alpha_t}{2} \sqrt{\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] \right]} \times \sqrt{dt} \times P(t, t + dt) + \\ &\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} [e^{-\int_t^{t+dt} r_s ds} \times \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]] - \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} [e^{-\int_t^{t+dt} r_s ds} \times \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]]. \end{aligned}$$

Purely actuarial risks in $\mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]$ are $\mathcal{F}_t$ -measurable, which allows them to be factored out of expectations under $\mathbb{Z}_\theta^a$ or $\mathbb{P}^a$ measure. Hence

$$\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} [e^{-\int_t^{t+dt} r_s ds} \times \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} [e^{-\int_t^{t+dt} r_s ds} \times \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]].$$

Secondly, since the computation of $\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]$ does not depend on any measure, then $\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] = \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]$. Consequently, applying tower rule property leads us to

$$\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f} \left[e^{-\int_t^{t+dt} r_s ds} \times \left(\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] \right) \right] = 0.$$

Thirdly, using the same previous argument, we have that $\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}^3 \right] = \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right]$. By further applying tower rule property, we have

$$\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[e^{-\int_t^{t+dt} r_s ds} \times \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] \right] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[e^{-\int_t^{t+dt} r_s ds} \times \frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right].$$

By combining all these simplifications, stretching dt towards 0, we derive the following

$$lm^{(1)} + \frac{\alpha_t}{2} \times lm^{(2)} = 0,$$

where $lm^{(1)} = \lim_{dt \rightarrow 0} \frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[e^{-\int_t^{t+dt} r_s ds} \frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] - \tilde{\pi}_t}{dt}$ and

$$lm^{(2)} = \lim_{dt \rightarrow 0} P(t, t+dt) \times \sqrt{\frac{\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] \right]}{dt}}.$$

Let us denote $\tilde{dW}^\rho(s) = \rho d\tilde{W}_r(t) + \sqrt{1 - \rho^2} d\tilde{W}_S(t)$ and $dW^\rho(s) = \rho dW_r(t) + \sqrt{1 - \rho^2} dW_S(t)$. Before delving into the application of Ito's lemma, a prerequisite is to determine the dynamics of the stochastic mortality rate λ_t under $\mathbb{Z}_\theta^a$. The density described in (15) can be restated anew as follows,

$$\frac{d\mathbb{Z}_\theta^a}{d\mathbb{P}^a} \Big|_{\mathcal{F}_0} = e^{\int_0^T \theta_{0,T}^{(1)} \sigma_m \frac{e^{\mu_m(T-u)} - 1}{\mu_m} dW_m(u) - \frac{1}{2} \int_0^T (\theta_{0,T}^{(1)} \sigma_m \frac{e^{\mu_m(T-u)} - 1}{\mu_m})^2 du}.$$

Hence, it becomes apparent that the transition to $\mathbb{Z}_\theta^a$ through Girsanov's theorem can be accomplished by positing $(dW_m^*(t))_{t \geq 0}$ as $dW_m^*(t) = dW_m(t) - \theta_{0,T}^{(1)} \sigma_m \frac{e^{\mu_m(T-t)} - 1}{\mu_m} dt$. Therefore, we reach the dynamics of λ_t under $\mathbb{Z}_\theta^a$, as follows

$$d\lambda_t = (\overline{\theta_{t,T}^{(1)}} + \mu_m \lambda_t) dt + \sigma_m dW_m^*(t),$$

where $\overline{\theta_{t,T}^{(1)}} = \theta_{0,T}^{(1)} \sigma_m^2 \times \frac{e^{\mu_m(T-t)} - 1}{\mu_m}$.

By employing Ito's lemma with jump process under $\mathbb{Z}_\theta^a \times \mathbb{Q}^f$, we obtain

$$\begin{aligned} \tilde{\pi}_{t+dt} &= \tilde{\pi}_t + \int_t^{t+dt} \partial_t \tilde{\pi}_s ds + \int_t^{t+dt} \partial_S \tilde{\pi}_s r_s S_s ds + \sigma_S \int_t^{t+dt} \partial_S \tilde{\pi}_s S_s dW^\rho(s) \\ &\quad + \int_t^{t+dt} (\theta - ar_s) \partial_r \tilde{\pi}_s ds + \sigma_r \int_t^{t+dt} \partial_r \tilde{\pi}_s dW_r(s) + \int_t^{t+dt} (\overline{\theta_{s,T}^{(1)}} + \mu_m \lambda_s) \partial_\lambda \tilde{\pi}_s ds + \\ &\quad \sigma_m \int_t^{t+dt} \partial_\lambda \tilde{\pi}_s dW_m^*(s) + \frac{1}{2} \sigma_S^2 \int_t^{t+dt} \partial_S^2 \tilde{\pi}_s S_s^2 ds + \frac{1}{2} \sigma_m^2 \int_t^{t+dt} \partial_\lambda^2 \tilde{\pi}_s ds + \\ &\quad \frac{1}{2} \sigma_r^2 \int_t^{t+dt} \partial_r^2 \tilde{\pi}_s ds + \rho \sigma_r \sigma_S \int_t^{t+dt} \partial_{rS}^2 \tilde{\pi}_s S_s ds + \int_t^{t+dt} (\tilde{\pi}_s^* - \tilde{\pi}_s) dN_s, \end{aligned}$$

where $\tilde{\pi}_s^*$ denotes the time-consistent value just before the jump in the number of survivors, i.e. $\tilde{\pi}_s^*[\cdot] = \tilde{\pi}_s[\cdot | N_s - 1]$.

By applying the discount factor multiplication $e^{-\int_t^{t+dt} r_s ds}$, we subsequently engage in the calculation of the mathematical expectation to reach ²⁰

$$\begin{aligned}
lm^{(1)} &= \tilde{\pi}_t \lim_{dt \rightarrow 0} \frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} \times e^{-\int_t^{t+dt} r_s ds} \right] - 1}{dt} + \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} \times e^{-\int_t^{t+dt} r_s ds} \partial_t \tilde{\pi}_s \right] ds}{dt} + \\
&\lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s ds} \partial_S \tilde{\pi}_s r_s S_s \right] ds}{dt} + \\
&\sigma_S \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s ds} \partial_S \tilde{\pi}_s S_s dW^\rho(s) \right]}{dt} \\
&+ \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s ds} \partial_r \tilde{\pi}_s (\theta - ar_s) \right] ds}{dt} + \\
&\sigma_r \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s ds} \partial_r \tilde{\pi}_s dW_r(s) \right]}{dt} + \\
&\lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} \times e^{-\int_t^{t+dt} r_s ds} (\overline{\theta_{s,T}^{(1)}} + \mu_m \lambda_s) \partial_\lambda \tilde{\pi}_s \right] ds}{dt} + \\
&\frac{\sigma_S^2}{2} \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s ds} \partial_S^2 \tilde{\pi}_s S_s^2 \right] ds}{dt} + \\
&\frac{\sigma_m^2}{2} \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[e^{-\int_t^{t+dt} r_s ds} \frac{N_{t+dt}}{N_t} \partial_\lambda^2 \tilde{\pi}_s \right] ds}{dt} + \\
&\frac{\sigma_r^2}{2} \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s ds} \partial_r^2 \tilde{\pi}_s \right] ds}{dt} + \\
&\rho \sigma_S \sigma_r \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s ds} \partial_{Sr}^2 \tilde{\pi}_s S_s \right] ds}{dt} \\
&+ \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s ds} (\tilde{\pi}_s^* - \tilde{\pi}_s) dN_s \right]}{dt} + \\
&\sigma_m \lim_{dt \rightarrow 0} \frac{\int_t^{t+dt} \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} \left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s ds} \partial_\lambda \tilde{\pi}_s dW_m(s) \right] ds}{dt}. \\
&= -r_t \tilde{\pi}_t + \partial_t \tilde{\pi}_t + r_t S_t \partial_S \tilde{\pi}_t + (\theta - ar_t) \partial_r \tilde{\pi}_t + (\overline{\theta_{t,T}^{(1)}} + \mu_m \lambda_t) \partial_\lambda \tilde{\pi}_t + \frac{1}{2} \sigma_S^2 S_t^2 \partial_S^2 \tilde{\pi}_t + \\
&\frac{1}{2} \sigma_m^2 \partial_\lambda^2 \tilde{\pi}_t + \frac{1}{2} \sigma_r^2 \partial_r^2 \tilde{\pi}_t + \rho \sigma_r \sigma_S S_t \partial_{rS}^2 \tilde{\pi}_t + (\tilde{\pi}_t^* - \tilde{\pi}_t) \lambda_t.
\end{aligned}$$

²⁰We remind that $\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f} [dN_t] = \lambda_t dt$.

Solving for the limit $lm^{(2)}$ requires expanding the variance term as a preliminary step

$$\begin{aligned}
\text{Vaiance terme} &\triangleq \mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] \right] \\
&= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] \right] \\
&= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\left(\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right)^2 \right] - \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\left(\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\tilde{\pi}_{t+dt} \right] \right)^2 \right] \\
&= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\left(\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right)^2 \right] - (\tilde{\pi}_t)^2 - \\
&\quad \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\left(\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] - \tilde{\pi}_t \right)^2 \right] \\
&\quad - 2\tilde{\pi}_t \times \left[\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] - \tilde{\pi}_t \right].
\end{aligned}$$

Hence $\lim_{dt \rightarrow 0} \frac{\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] \right]}{dt} = lm_1^{(2)} - 2\tilde{\pi}_t lm_2^{(2)}$, where

$$lm_1^{(2)} = \lim_{dt \rightarrow 0} \frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\left(\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right)^2 \right] - (\tilde{\pi}_t)^2}{dt} \text{ and } lm_2^{(2)} = \lim_{dt \rightarrow 0} \frac{\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} \right] - \tilde{\pi}_t}{dt}.$$

By employing Ito's lemma with jump process once more, this time specifically under $\mathbb{P}^a \times \mathbb{P}^f$, we obtain

$$\begin{aligned}
\tilde{\pi}_{t+dt} &= \tilde{\pi}_t + \int_t^{t+dt} \partial_t \tilde{\pi}_s ds + \int_t^{t+dt} \partial_S \tilde{\pi}_s \mu_S S_s ds + \sigma_S \int_t^{t+dt} \partial_S \tilde{\pi}_s S_s d\tilde{W}^\rho(s) + \\
&\quad + \int_t^{t+dt} (b - ar_s) \partial_r \tilde{\pi}_s ds + \sigma_r \int_t^{t+dt} \partial_r \tilde{\pi}_s d\tilde{W}_r(s) + \mu_m \int_t^{t+dt} \partial_\lambda \tilde{\pi}_s^{SD} \lambda_s ds + \\
&\quad \sigma_m \int_t^{t+dt} \partial_\lambda \tilde{\pi}_s d\tilde{W}_m(s) + \frac{1}{2} \sigma_S^2 \int_t^{t+dt} \partial_S^2 \tilde{\pi}_s S_s^2 ds + \frac{1}{2} \sigma_m^2 \int_t^{t+dt} \partial_\lambda^2 \tilde{\pi}_s ds + \\
&\quad \frac{1}{2} \sigma_r^2 \int_t^{t+dt} \partial_r^2 \tilde{\pi}_s ds + \rho \sigma_r \sigma_S \int_t^{t+dt} \partial_{rS}^2 \tilde{\pi}_s S_s ds + \int_t^{t+dt} (\tilde{\pi}_s^* - \tilde{\pi}_s) dN_s.
\end{aligned}$$

Through a computation highly analogous to that applied to $lm^{(1)}$, brings us to

$$\begin{aligned}
lm_2^{(2)} &= \partial_t \tilde{\pi}_t + \mu_S S_t \partial_S \tilde{\pi}_t + (b - ar_t) \partial_r \tilde{\pi}_t + \mu_m \lambda_t \partial_\lambda \tilde{\pi}_t + \frac{1}{2} \sigma_S^2 S_t^2 \partial_S^2 \tilde{\pi}_t + \\
&\quad \frac{1}{2} \sigma_m^2 \partial_\lambda^2 \tilde{\pi}_t + \frac{1}{2} \sigma_r^2 \partial_r^2 \tilde{\pi}_t + \rho \sigma_r \sigma_S S_t \partial_{rS}^2 \tilde{\pi}_t + (\tilde{\pi}_t^* - \tilde{\pi}_t) \lambda_t.
\end{aligned}$$

In our final application of Ito's lemma with jump process, focusing on $(\tilde{\pi}_{t+dt})^2$ under $\mathbb{P}^a \times \mathbb{P}^f$, we obtain

$$\begin{aligned}
(\tilde{\pi}_{t+dt})^2 &= (\tilde{\pi}_t)^2 + \int_t^{t+dt} \partial_t (\tilde{\pi}_s)^2 ds + \int_t^{t+dt} \partial_S (\tilde{\pi}_s)^2 \mu_S S_s ds + \sigma_S \int_t^{t+dt} \partial_S (\tilde{\pi}_s)^2 S_s d\tilde{W}^\rho(s) + \\
&\quad + \int_t^{t+dt} (b - ar_s) \partial_r (\tilde{\pi}_s)^2 ds + \sigma_r \int_t^{t+dt} \partial_r (\tilde{\pi}_s)^2 d\tilde{W}_r(s) + \mu_m \int_t^{t+dt} \partial_\lambda (\tilde{\pi}_s)^2 \lambda_s ds + \\
&\quad \sigma_m \int_t^{t+dt} \partial_\lambda (\tilde{\pi}_s)^2 d\tilde{W}_m(s) + \frac{1}{2} \sigma_S^2 \int_t^{t+dt} \partial_S^2 (\tilde{\pi}_s)^2 S_s^2 ds + \frac{1}{2} \sigma_m^2 \int_t^{t+dt} \partial_\lambda^2 (\tilde{\pi}_s)^2 ds + \\
&\quad \frac{1}{2} \sigma_r^2 \int_t^{t+dt} \partial_r^2 (\tilde{\pi}_s)^2 ds + \rho \sigma_r \sigma_S \int_t^{t+dt} \partial_{rS}^2 (\tilde{\pi}_s)^2 S_s ds + \int_t^{t+dt} ((\tilde{\pi}_s^*)^2 - (\tilde{\pi}_s)^2) dN_s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
lm_1^{(2)} &= \partial_t(\tilde{\pi}_t)^2 + \mu_S S_t \partial_S(\tilde{\pi}_t)^2 + (b - ar_t) \partial_r(\tilde{\pi}_t)^2 + \mu_m \lambda_t \partial_\lambda(\tilde{\pi}_t)^2 + \\
&\quad \frac{1}{2} \sigma_S^2 S_t^2 \partial_S^2(\tilde{\pi}_t)^2 + \frac{1}{2} \sigma_m^2 \partial_\lambda^2(\tilde{\pi}_t)^2 + \frac{1}{2} \sigma_r^2 \partial_r^2(\tilde{\pi}_t)^2 + \rho \sigma_r \sigma_S S_t \partial_r \partial_S(\tilde{\pi}_t)^2 + \\
&\quad ((\tilde{\pi}_t^*)^2 - (\tilde{\pi}_t)^2) \lambda_t. \\
&= 2\partial_t \tilde{\pi}_t \tilde{\pi}_t + 2\mu_S S_t \partial_S \tilde{\pi}_t \tilde{\pi}_t + 2(b - ar_t) \partial_r \tilde{\pi}_t \tilde{\pi}_t + 2\mu_m \lambda_t \partial_\lambda \tilde{\pi}_t \tilde{\pi}_t \\
&\quad + \sigma_S^2 S_t^2 [\partial_S \tilde{\pi}_t]^2 + \partial_S^2 \tilde{\pi}_t \tilde{\pi}_t + \sigma_m^2 [(\partial_\lambda \tilde{\pi}_t)^2 + \partial_\lambda^2 \tilde{\pi}_t \tilde{\pi}_t] + \sigma_r^2 [(\partial_r \tilde{\pi}_t)^2 + \\
&\quad \partial_r^2 \tilde{\pi}_t \tilde{\pi}_t] + 2\rho \sigma_S \sigma_r S_t [\partial_r \partial_S \tilde{\pi}_t \tilde{\pi}_t + \partial_S \tilde{\pi}_t \partial_r \tilde{\pi}_t] + ((\tilde{\pi}_t^*)^2 - (\tilde{\pi}_t)^2) \lambda_t.
\end{aligned}$$

$$\text{Finally, } \lim_{dt \rightarrow 0} \frac{\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_t + dt}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{E}_{t+dt} \vee \mathcal{F}_{t+dt}^f}^{\mathbb{P}^a \times \mathbb{P}^f} \left[\frac{N_t + dt}{N_t} \times \tilde{\pi}_{t+dt} \right] \right]}{dt} = \sigma_S^2 S_t^2 (\partial_S \tilde{\pi}_t)^2 + \sigma_m^2 (\partial_\lambda \tilde{\pi}_t)^2 + \sigma_r^2 (\partial_r \tilde{\pi}_t)^2 + 2\rho \sigma_S \sigma_r S_t \times \partial_S \tilde{\pi}_t \partial_r \tilde{\pi}_t + (\tilde{\pi}_t - \tilde{\pi}_t^*)^2 \lambda_t.$$

The proof is concluded by considering the continuity property of the root function, and applying the rule that the the product of two existing finite limits equals the limit of the product.

□

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No potential conflict of interest was reported by the authors.

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8 Data availability statement

This research did not use any data, hence data availability is not applicable.

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