

Economic Dynamics with Heterogeneous Capital Goods

Economic Dynamics with Heterogeneous Capital Goods

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To my parents

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Introduction.

Growth theory studies how the per capita of GDP grows in an economy. It is also used intensively to understand the differences in income across countries, and additionally it serves as an insightful basis to design strategies of economic development.

Historically, the first growth models are developed by R. Solow(1956), T. Swan(1956), and J. Meade(1961). In these theories the capital stock is taken as homogenous, and technological progress is neutral in the sense that it affects identically all the capital goods independently of their ages. In addition, the considered economies are closed and it is implicitly assumed that technological characteristics are the same in all places, so that there is no geography dimension in these models. In this framework, some important properties can be derived. At the steady-state equilibrium, all ratios – the capital-labor ratio, output per capita and consumption per capita – remain constants under constant technology progress. These models also entail a property of convergence: Countries starting with lower stock of capital should experience higher growth rates in the short run.

In this thesis, we will relax two major assumptions mentioned above. First of all, we will study growth models with heterogeneous capital goods, the so called *vintage capital models*: technological advances are not incorporated in all generations of capital goods and there is an optimal age distribution of the capital stocks. We will devote the three first chapters of this thesis to this class of models. Secondly, we will introduce explicitly the geographical dimension to the neoclassical growth models, which allow us to build a new class of models. We call them *geographic growth models*. In this framework, we will identify the consequences of capital mobility across space. In particular, we will examine the optimal stationary distribution of capital across space.

A. Vintage Capital Model

Vintage capital models, which were launched in the early 60s', have become increasingly popular in the economic literature. These models provide an approach for the analysis of investment volatility. The main difference between vintage capital model and the standard neoclassical growth model lies in the fact that in the former, new technological progress is embodied in new equipment, which gives rise to an endogenous process of creative destruction. Furthermore, as mentioned by Boucekkine, Germain, and Licandro(1997) and Benhabib and Rustichini(1991), optimal investment paths are no longer monotonic in contrast to the standard neoclassical growth model. In vintage capital models, the replacement of the old equipment is an economic decision, allowing for a formalization in the Schumpetrian vein. This replacement decision induces non-monotonic optimal paths for investment according to an echo principle (Boucekkine, Germain, and Licandro(1997) and Benhabib and Rustichini(1991)).

As in Boucekkine, Germain, and Licandro(1997), a vintage capital model with Harrod-neutral technical progress is employed in this thesis in order to consider a kind of balanced growth path, where all endogenous variables are growing at constant rates and endogenous scrapping is constant. In fact, as Hirofumi Uzawa(1961) demonstrated, Harrod-neutral technical progress is the only type of technical progress consistent with a stable steady-state ratio. This is due to the fact that only Harrod-neutral technical progress keeps the capital-output ratio constant over time as in the Solow-Swan.

In the framework of Boucekkine, Germain, and Licandro(1997), the approach is purely theoretical. More important and interesting extensions and applications to their work can be found, which are done in Chapter 1 and Chapter 2.

Our models incorporate two new elements with respect to the standard framework. First, we assume embodied technical progress in contrast to the typical neoclassical specification of neutral and disembodied technical progress. Second, we consider a vintage capital model, with endogenous scrapping decision. The standard models consider homogenous and infinitely lived capital stock.

In Chapter 1, we extended Boucekkine, Germain, and Licandro(1997)'s *AK*-type model

to a more general Cobb-Douglas case and apply it to answer the following question: Is it possible by using a new-energy saving technique to save economic growth, though there is a reduction of energy input in the long-run?

As argued by Smulders and de Nooij(2003), the reduction in energy use could have a negative impact on economic growth and development. Therefore, there is a clear trade-off between energy reduction and growth.

Carraro, Gerlagh and van der Zwaan(2003), among others, suggested that this trade-off could be less severe if energy conservation results from energy saving technologies.

The aim of this first chapter is to compare (or combine) the above two ideas. In particular, we focus on the previous trade-off in a general equilibrium setting with energy saving technical progress. We study an economy with exogenous energy saving technical progress embodied in the new equipment. As Baily (1981) observes, technical advances are typically incorporated to the economy through investment. Therefore, the old capital goods become less and less efficient over time, which might well induce the firms to scrap them (obsolescence). In our economy, we assume that different vintages of capital coexist in each period. Since new vintages are less energy consuming, firms may decide to replace the oldest and less efficient vintage. Indeed, if we model the idea of minimum energy requirement to use a machine by assuming complementarity between capital and energy inputs, finite scrapping time is optimal (Boucekkinine and Pommeret(2004)). This idea is implemented in this chapter and it is consisted with the empirical evidence put forward by Berndt and Wood(1975).

We perform a comparative study to contrast constant and decreasing returns to scale, for two possible scenarios: constant (optimistic) and decreasing (pessimistic) exogenous energy supply. We find that, under the assumption of existence of a balanced growth path, constant returns to scale achieve positive long run growth if the growth rate of the energy saving technical progress exceeds the decreasing rate of the energy supply. Otherwise, there is no long run growth.

Vintage capital models can successfully and explicitly explain the energy saving problem. In chapter 2, we employ it to study the environmental quality and pollution problem.

As early as in the 1970s, economists were facing the challenge of OPEC(Organization of the Petroleum Exporting Countries) and the dramatic predictions of the Club of Rome by introducing energy, natural resources and environmental pollution into neoclassical theory of growth (Aghion and Howitt(1998)). Since then the consideration between economic growth with environmental pollution and non-renewable resource attracted a huge attention in economic literature. This is especially true in the last decade. Grossman and Krueger(1995), for instance, study panel data, and find that there is a kind of Environmental Kuznet Curve (EKC): an inverted U-shaped relationship between per capita income and various types of pollution. This suggests that during the developing period, fast economic growth is companied by fast pollution, while pollution reduces when the economy reaches a “high enough” level of development. The World Bank Development Report(1992) shows similar observation. However there is hardly any theoretical support to this empirical observation. In their recent book, Aghion and Howitt(1998) put forward the question “ whether or not growth can be sustainable is the central question to which endogenous growth theory is addressed”. Almost at the same time Stocky(1998) raised the question as “ Will growth in developed countries eventually cease because the environmental costs of increased production more than offset the benefits of higher consumption? ”

However, there are several types of endogenous growth models and several ways to take into account the environmental problem. In chapter 2, we use a Schumpeterian framework through a vintage framework, where the law of motion of environmental quality follows Aghion and Howitt(1998).

In our vintage capital models, as mentioned above, new technology is embodied in new machines, which are more environmental friendly. The replacement of the old equipment is an economic decision, which, if taking into account environmental pollution, will guide policy maker to choose clean investment profile.

Along Ramsey’s Benthamian utilitarian way, a linear utility is used. In this case, the “bliss point” (in the sense of Ramsey(1928)) consists not only of the consumption level, but also of the environmental quality, which must be considered at the same time by policy makers. Sustainable development is possible under certain conditions. However a

disaster could happen in finite time if there is over-investment. Furthermore, and contrary to most vintage capital models, the transition economy is explicitly expressed in terms of endogenous scrapping age, which is a worthwhile contribution to the related literature, given this result is obtained via a linear utility function and an environmental evolution equation.

The above vintage capital models are successful in interpretation some of the endogenous growth issues, by using an endogenous scraping process and an exogenous technology change. However, the framework and approach are still more theoretical. A common assumption to these models is that investment is only allowed for the new vintage capital, a traditional assumption in the literature (see Malcomson(1975)). As a result, there is immediate new technology diffusion.

In reality, firms are used to invest in older capital goods. The main reason behind this is technology adoption. Technology adoption is costly in that it requires some specific vintage capital goods and involves learning or installation costs. Learning effects entail the concept of costly technology adoption, which motivates the investment in dominated technologies. Parente (1994) argued that the firms may be prevented from adopting new technologies as this supposes a loss in human capital (vintage specificity of human capital). However Parente did not work out his ideas in a vintage capital model. He only conducted a steady state analysis. Greenwood and Yorukoglu(1997) also combined vintage and learning effects in a computable dynamic general equilibrium model but they did so in a mostly ad-hoc way. Jovanovic and Nyarko(1996) explored the same idea in a stochastic framework, but in a partial equilibrium setting.

Is it possible to build a decentralized economic vintage technology model, in which investment is allowed in both new and existing vintage technological goods? This idea was already considered by Barucci and Gozzi (2001) and Feichtinger, Hartl, Kort and Veliov (2001). In a flexible mathematical framework (McKendrick systems), they studied the replacement-adoption problem within a partial equilibrium setting, where capital accumulation is captured by a first order partial differential equation.

In Chapter 3, we first generalizing previous partial equilibrium settings and, second, incorporating the adoption problem into the vintage framework.

So, instead of a technological leader innovates determining the exogenous path of technological progress for potential followers, we find an exogenously determined path of technical progress embodied in new vintage capital. Then the traditional problem (for followers) on whether to adopt new technologies or not, arises here as the problem on whether to invest in new vintage capital goods or not. Moreover, the other important decision concerning capital accumulation appears here transformed into the problem of how much to invest in existing technological capital goods.

Given such situation, in Chapter 3, in a decentralized economy, we establish the optimal investment rules, and consider the puzzles: (i) how much to invest in existing vintage capital goods (*improvement in the quality of investment*); (ii) and how much to invest in new vintage capital goods (*expanding variety of investment*). Our model considers such a trade-off.

Among our results, we obtain that the stock of capital is a *single hump-shaped* with respect to the vintage age for any fixed time t . In other words, there is *slow* technology diffusion.

B. Geographic Growth Model

The inclusion of the space dimension in economic analysis has regained relevance in the recent years. The emergence of the new economic geography is indeed one of the major events in the economic literature of the last decade (see Krugman(1991, 1993, 1996), Fujita, Krugman and Venables(1999), and Fujita and Thisse(2002)). The new economic geography models use general equilibrium frameworks with a refined specification of local and global market structures, and some precise assumptions on the mobility of production factors. Their usefulness in explaining the mechanics of agglomeration, the formation of cities, the determinants and implications of migrations, and more generally, the dynamics of the distribution of people and goods over space and time is undeniable, so undeniable that this discipline has become increasingly popular in the recent years.

Two main characteristics of the new economic geography contributions quoted just above are: (i) the discrete space structure, and (ii) the absence of capital accumulation.

Typically, the economic geographers use two-regions frameworks, mostly analogous to the two-country models usually invoked in trade theory. However, some continuous space ex-

tensions of these models have already been studied. In a continuous space extension of his 1993 two-region model, Krugman(1996) shows that the economy always displays regional convergence, in contrast to the two-region version in which convergence and divergence are both possible. Mossay(2003) proves that continuous space is not incompatible with regional divergence using a different migration scheme. In Krugman's model, migration follows utility level differentials, which in turn implies that local real wages provide the only incentive for moving (predominant regional convergence force). In Mossay, migrations **additionally** depend on idiosyncrasies in location taste, inducing a divergence force, which can balance the utility gradient force mentioned before. As a consequence, regional divergence is a possible outcome in this model.

Both models, however, ignore the role of capital accumulation in migrations: They both assume zero (individual) saving at any moment. Indeed, the zero saving assumption is a common characteristic of the new economic geography literature, especially in continuous space settings, with the notable exception of Brito(2004). This strong assumption is done to ease the resolution of the models, which are yet very complex with the addition of the space dimension.

Nonetheless, as capital accumulation is not allowed, the new economic geography models are losing a relevant determinant of migrations, and more importantly, an engine of growth. While a large part of growth theory is essentially based on capital accumulation, the new economic geography has mainly omitted this fundamental dimension so far. It seems however clear that many economic geography problems (eg. uneven regional development) have a preeminent growth component, and *vice versa*. Thus, there is an urgent need to unify in some way the two disciplines, or at least to develop some junction models.

Our modelling of space is done so as "to avoid simple but unrealistic boundary conditions" (Ten Raa(1986), page 528–530). Capital is perfectly mobile across space (and of course, across time through intertemporal substitution, as usual in a Ramsey-like model). Capital flows from regions with low returns to capital to regions with high returns. In such a case, it has been already shown by Brito(2004) that capital, the state variable of the optimal control problem, is governed by a parabolic partial differential equation. This is indeed the main difficulty of the problem compared to the traditional regional science approach,

as in Ten Raa(1986) and Puu(1982), where the considered fluid dynamics modelling gives rise to wave equations of income.

Before working on the optimal saving and growth path as in Ramsey's model, we first take for granted the Solow-Swan constant rate of savings in Chapter 4.

In line with Mossay(2003), we shall allow for both divergence and convergence forces. The convergence force is the well known neoclassical mechanism according to which poor regions attract capital because of decreasing returns to this factor. Divergence mechanisms are linked to space heterogeneity, given by region specific technology and/or saving rate. If a region produces, by using a more advanced technology, it attracts capital from less advanced regions, despite decreasing returns to scale. The same result holds for a region that saves, and therefore invests, at a larger rate.

Neoclassical economic theory predicts that regions will converge in the long run under perfect competition. However, this is not so. An argument that has been put forward is that technological transfers between regions is far from perfect. The lack of transferee expertise and poor training in the technology importing region, together with Government barriers may impede an effective technological transfer (see Niosi, Hanel and Fiset(1995)). Boucekkine, Martinez and Saglam(2003) point out the role of capital goods technological embodiment in technology adoption decisions. A developing country may not adopt the most sophisticated technique since it implies replacing existing capital and lose its technology-specific skills. The spatial Solow-Swan model allows to study the link between technology transfers and development. Indeed, it is flexible enough to study the existence of technological poles with partial transfer to neighboring regions, as well as more complicated patterns of knowledge diffusion across space and time.

Chapter 5 devotes to study more complex cases, in particular, a spatial Ramsey model. As in Chapter 4, space is continuous and infinite, and optimal consumption and capital accumulation are space dependent. We depart from the non-Benthamian Ramsey model of Brito(2004) by introducing spatial discounting. In our view, this is the most immediate and natural formulation of a spatial Ramsey model, and we will spend some paragraphs justifying spatial discounting, which follows the same idea as Ramsey introducing time discounting.

Taking a step further, we use an explicit integral representations of the solutions to the parabolic partial differential equations (see Pao(1992), for a nice textbook in the field, and Wen and Zou(2000, 2002)), we will clearly identify a serious problem with the optimal control of these equations: In contrast to the Ramsey model without space where there exists a one-to-one relationship between the initial value of the co-state variable, say $q(0)$, and the whole co-state trajectory, for a given capital stock path, this property does not hold at all in the spatial counterpart, that is $q(x, t)$, the co-state variable for location x at time t , is not uniquely defined by the data $q(0, x)$ because of the integral relationship linking $q(x, t)$ to $q(0, x)$. As a consequence, while the transversality conditions in the Ramsey model without space allows to identify a single optimal trajectory for the co-state variable, thus for the remaining variables of the model, there is no hope to get the same outcome with space. We are facing a typical *ill-posed problem* in the sense of Hadamard (1923): We cannot assure neither the existence nor the uniqueness of the solutions.

How to deal with this huge difficulty? In order to keep the possibility of comparison with the traditional Ramsey model's solution paths, we study the case of the Ramsey model with linear utility. In such a case, we are -as usual- able to disentangle the forward looking dynamics from the backward-looking, which ultimately allows us to use the available asymptotic literature on scalar initial-value parabolic equation. Depending on the initial capital distribution, optimal consumption per location can be initially corner or interior, and the dynamics of capital accumulation across space and time will be governed by a scalar parabolic equation. We shall study whether an initially "corner" location (i.e., with an initially corner consumption solution, which is in deed the idea of holding consumption to ensure the fastest reach of the optimal path) can converge to its interior regime or to any other regime to be characterized. The obtained results are notably different from the linear Ramsey model without space, due to the spatial dynamics induced by capital mobility, which are shown to enrich considerably the dynamics of the Ramsey model. In this sense, the linear spatial Ramsey model is rich enough to serve as a perfect illustration of how the spatial dynamics can interact with the typical mechanisms inherent to growth models.

Basic Notation and Terminology

All functions, arguments and parameters considered in this thesis are real.

$\mathbb{R}$ is real Euclidean space, $\mathbb{R}_+ = \{x \geq 0\}$ is a half space of $\mathbb{R}$.

t is an arbitrary point in $\mathbb{R}_+$, unless other stated.

$\dot{a}$ is define as derivative of function a with respect to t .

$L_+^\infty(\mathbb{R}) = \{y \in L^\infty(\mathbb{R}) | y(x) \geq 0 \text{ for almost every } x \in \mathbb{R}\}$, with norm defined as

$$\|u\|_{L^\infty} = \sup_R |u(x)|.$$

$C(R)$ is continuous function space, with norm defined the same as $\|u\|_C = \|u\|_{L^\infty}$.

$C^{2,1}(\mathbb{R} \times (0, T))$ consists all elements $u(x, t)$, which is second order continuous differentiable with respect to x and first order continuous differentiable with respect to t . The norm in $C^{2,1}(\mathbb{R} \times (0, T))$ is defined as

$$\|u\|_{C^{2,1}} = \|u\|_C + \left\| \frac{\partial u}{\partial x} \right\|_C + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_C + \left\| \frac{\partial u}{\partial t} \right\|_C.$$

“Theorem” denotes the results from pure mathematics.

“Proposition” states the proved results with economics assumptions.

“ Corollary ” is direct results from Theorem or Proposition.

Chapter 1

A comparative study of Energy Saving Technical Progress in a Vintage Capital Model

1.1 Introduction

Fossil fuel—more precisely petroleum and its refinery products—is an essential input in all modern economies. It has been argued that the limited availability of this basic input and the stabilization of greenhouse gases concentration call for a reduction of fossil fuel consumption. However, the reduction in petroleum consumption could have a negative impact on economic growth and development through cutbacks in energy use. Therefore, there is a clear trade-off between energy reduction and growth.

In this chapter, we re-examine the exhaustion problem of fossil fuel. In particular, we study the previous trade-off in a general equilibrium framework with energy saving technical progress, which is based on Boucekkine, Germain and Licandro (1997), considers an economy with exogenous energy saving technical progress embodied in the new equipment. In our economy, we assume that different vintages of capital coexist in each period. Since new vintages are less energy consuming, firms may decide to replace the oldest and less

efficient vintage. Indeed, if we model the idea of minimum energy requirement to use a machine by assuming complementarity between capital and energy inputs, finite scrapping time is optimal (Boucekkine and Pommeret(2004)). This idea is implemented in this chapter, and it is consisted with the empirical evidence put forward by Hudson and Jorgenson(1974).

The chapter is organized as follows. In section 2, we describe the general case model, with the representative consumer's problem and the rules that depicts both the optimal investment and the scrapping behavior of firms. The balanced growth path is presented in section 3, where we show the necessary conditions for its existence in both constant and decreasing returns to scale. Finally, some concluding remarks are considered in section 4.

1.2 The Model

Following Boucekkine, Germain and Licandro(1997), we consider an economy where the population is constant and there is only one good (the numeraire good), which can be assigned to consumption or investment. The good is produced in a competitive market by mean of a technology defined over vintage capital. Both constant and decreasing returns to scale are considered here. Also, we assume a competitive labor market and exogenously available energy supply.

1.2.1 Household

Let us assume that the representative household considers the following standard intertemporal maximization problem with a constant relative risk aversion (CRRA) instantaneous utility function

$$\max_{c(t)} \int_0^{\infty} \frac{c(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt, \quad (1.1)$$

subject to the budget constraint

$$\begin{aligned}\dot{a}(t) &= r(t)a(t) - c(t), \\ a(0) &\text{ given,} \\ \lim_{t \rightarrow \infty} a(t)e^{-\int_0^t r(z)dz} &= 0,\end{aligned}\tag{1.2}$$

with initial wealth a_0 , where $c(t)$ is per-capita consumption, $a(t)$ is per-capita asset held by the consumer at the interest rate $r(t)$ which is taken as given for the household. θ measures the constant relative risk aversion, and ρ is the time preference parameter (it is assumed to be a positive discount factor). Since in this chapter, we do not explicitly treat labor, we assume that it has no value leisure for the consumer. Then, in order to simplify the model, it is considered an inelastic labor supply normalized to one. The corresponding necessary conditions are $r(t) = \rho + \theta \frac{\dot{c}(t)}{c(t)}$, with $\lim_{t \rightarrow \infty} \lambda(t)a(t) = 0$, where $\lambda(t)$ is the co-state variable associated with the wealth accumulation equation.

1.2.2 Firms

The good is produced competitively by a representative firm solving the following optimal profit problem

$$\max_{y(t), i(t), T(t)} \int_0^\infty [y(t) - i(t) - e(t)P_e(t)] R(t)dt,\tag{1.3}$$

subject to output

$$y(t) = A \left(\int_{t-T(t)}^t i(z)dz \right)^\alpha, \quad 0 < \alpha \leq 1, \quad i(t) \geq 0,\tag{1.4}$$

and exogenous energy demand

$$e(t) = \int_{t-T(t)}^t i(z)e^{-\gamma z}dz, \quad 0 < \gamma < \rho,\tag{1.5}$$

where $i(t)$ is the investment of the representative firm the initial and conditions $i(z)$ is given for all $z \leq 0$. It is straight forward that there are different vintage capitals (that

is different time investment) co-exist to produce final goods¹. $e(t)$ is the demand of energy at a given price $P_e(t)$. The firm considers the energy price has given; however, it is endogenously determined in the energy market equalizing the demand and supply of energy.

Equation (1.4) is our technology defined over vintage capital. The energy demand is obtained by equation (1.5). Here $\gamma > 0$ represents the rate of energy saving technical progress and $T(t)$ is the age of the oldest operating machines or scrapping age. The discount factor $R(t)$ takes the form $R(t) = e^{-\int_0^t r(z)dz}$. Finally, we assume that $0 < \gamma < \rho$ to well define our integral².

Notice that the new technology is more energy saving. Certainly, each vintage $i(t)$ has an energy requirement $i(t)e^{-\gamma t}$ ³. Moreover, it is important to observe that we assume complementarity between capital and energy (Leontieff technology), which model the idea of minimum energy requirement to use a machine. Furthermore, this assumption is undeniable from numerous studies; for instance Hudson and Jorgenson(1974), or Berndt and Wood(1975). As we observed in the introduction, this complementarity ensures a finite optimal scrapping age⁴.

We define the capital stock

$$K(t) = \int_{t-T(t)}^t i(z)dz, \quad (1.6)$$

and the optimal life of machines of vintage t ⁵

$$J(t) = T(t + J(t)). \quad (1.7)$$

¹We can consider an alternative technology where decreasing returns to scale only affects new vintages (not all the active vintages) $y(t) = A \int_{t-T(t)}^t i(z)^\alpha dz$. Similarly to our case, it can be checked that the results remain the same.

²It is a standard assumption in the exogenous growth literature to have a bounded objective function.

³This assumption is central in the early vintage capital models. See Solow *et al.*(1966).

⁴ $T(t) < \infty$ is an essential and standard assumption considering the possibility of replacement ($T(t) \rightarrow \infty$ implies no scrapping). See, for example, d'Autume and Michel(1993) and Bardhan(1969).

⁵Notice that $T(t) = J(t - T(t))$.

From the first order condition (FOC)⁶ for $i(t)$, we get the *optimal investment rule*

$$\int_t^{t+J(t)} \alpha A \left(\int_{\tau-T(\tau)}^{\tau} i(z) dz \right)^{\alpha-1} e^{-\int_t^{\tau} r(z) dz} d\tau = 1 + \int_t^{t+J(t)} P_e(\tau) e^{-\gamma \tau} e^{-\int_t^{\tau} r(z) dz} d\tau, \quad (1.8)$$

where the left hand side (LHS) is the discounted marginal productivity during the whole lifetime of the capital acquired in t , 1 is the marginal purchase cost at t normalized to one, and the second term on the right hand side (RHS) is the discounted operation cost at t .

The optimal investment rule establishes that firms should invest at time t until the discounted marginal productivity during the whole lifetime of the capital acquired in t exactly compensates for both its discounted operation cost and its marginal purchase cost at t .

From the FOC for $T(t)$, we have the *optimal scrapping rule*

$$A\alpha \left(\int_{t-T(t)}^t i(z) dz \right)^{\alpha-1} = P_e(t) e^{-\gamma(t-T(t))}. \quad (1.9)$$

The optimal scrapping rule states that a machine should be scrapped as soon as its marginal productivity (which is the same for any machine whatever its age) no longer covers its operation cost (which rises with age).

Here the marginal productivity is given by $\alpha A (\int_{t-T(t)}^t i(z) dz)^{\alpha-1}$, and $P_e(t) e^{-\gamma(t-T(t))}$ represents the operation cost.

1.2.3 Decentralized equilibrium

The (decentralized) equilibrium of our economy is characterized by equation (1.2), the necessary condition of the household problem, equations (1.4)–(1.7), the optimal investment

⁶Following Boucekkine, Germain and Licandro(1997), we can consider an intermediary sector to create intermediate inputs for the final production. It is easy to observe that the case of symmetric equilibrium is exactly equivalent to our model without intermediate good sector. However, as Krusell(1998) observes, if the product-specific returns to R&D are strong enough, there may be asymmetric steady-state equilibria, in which large and small capital firms coexist. Nevertheless, this is not the case in our model with exogenous technical progress.

rule, the optimal scrapping rule, and the following two additional equations to close the model: $c(t) + i(t) = y(t)$ and the equilibrium condition in the energy market $e(t) = e_s(t)$. $e_s(t)$ is the available energy supply⁷, which in our model is assumed to be exogenous.

1.3 Balanced growth path

1.3.1 Definition of balanced growth path

Let us define our balanced growth path equilibrium as the situation where all the endogenous variables grow at constant rates, with constant and finite scrapping age $T(t) = J(t) = \bar{T}$ (Terborgh-Smith result). Boucekkine, Germain and Licandro(1997) considered a model equivalent to our case with constant returns to scale ($\alpha = 1$). They presented a sufficient condition for the existence of a particular balanced growth path with both constant scrapping age and constant available energy supply.⁸ For the case of decreasing returns to scale ($0 < \alpha < 1$), we find that an analytical proof of the existence of such a balanced growth path is not possible. Moreover, it is not difficult to check that an alternative balanced growth path, with non-constant scrapping age, is not compatible with constant growth of the other endogenous variables.

As a consequence, in order to compare constant and decreasing returns to scale, we present the necessary conditions of our balanced growth path for both constant and decreasing returns to scale.

⁷The available energy supply is a flow (exogenous) variable; for example, petrol or any petroleum refinery product to generate energy. Here we do not explicitly treat extraction sector either producer countries.

⁸They assume a technology that saves labor instead of energy, with constant (exogenous) labor supply.

1.3.2 Necessary Conditions

For the general case $0 < \alpha \leq 1$, we get from the necessary condition of the household problem and along the balanced growth path

$$r(t) = \rho + \theta\gamma_c = \text{constant} = r^*, \quad (1.10)$$

and

$$e^{-\int_t^\tau r(z)dz} = e^{-r^*(\tau-t)}, \quad (1.11)$$

where γ_c is the growth rate of consumption.

Taking (1.9) into (1.8), differentiating with respect t and rearranging terms, we obtain:

$$(e^{\gamma\bar{T}} - 1) - \frac{\gamma}{\gamma_{P_e} - r^*}(e^{(\gamma_{P_e} - r^*)\bar{T}} - 1) = \frac{r^*}{\bar{P}_e}e^{(\gamma - \gamma_{P_e})t}, \quad (1.12)$$

where γ_{P_e} and $\bar{P}_e$ are, respectively, the growth rate and the level of the energy prices. The LHS is constant for any t in the balanced growth path, and the RHS is a function of t . So the equality holds if and only if $\gamma = \gamma_{P_e}$. As in the standard growth model, this result states that, in terms of energy saving, energy prices grow at the same rate as productivity. Moreover, Boucekkine and Pommeret(2004)⁹ observe that this result can be justified in the context of intertemporal equilibrium model of optimal extraction of a non-renewable resources.

By definition of $K(t)$, we have along the balanced growth path that

$$K(t) = \begin{cases} \frac{\bar{i}}{\gamma_i}(1 - e^{-\gamma_i\bar{T}})e^{\gamma_i t}, & \text{if } \gamma_i > 0, \\ \bar{i}\bar{T}, & \text{if } \gamma_i = 0, \end{cases} \quad (1.13)$$

where $i(t) = \bar{i}e^{\gamma_i t}$. Then, the growth rate of investment, γ_i , and the growth rate of capital stock, γ_K , are equal.

⁹Our result $\gamma_{P_e}(= \gamma) < r^*$ is also consistent with the assumption made in Boucekkine and Pommeret(2004) about the growth rate of energy prices lower than the interest rate. Indeed, they point out that if $\gamma_{P_e} > r^*$ the firm would have an incentive to infinitely get in debt to buy an infinite amount of energy.

Moreover, by (1.9) and $\gamma = \gamma_{P_e}$,

$$A\alpha K(t)^{\alpha-1} = \overline{P_e} e^{\gamma \overline{T}}, \quad (1.14)$$

where $P_e(t) = \overline{P_e} e^{\gamma \overline{T}}$. Substituting (1.13) into (1.14) yields

$$e^{\gamma_i(\alpha-1)t} = \frac{\overline{P_e} e^{\gamma \overline{T}}}{A\alpha \overline{K}^{\alpha-1}}. \quad (1.15)$$

It is easy to see that (1.15) holds if and only if $\alpha = 1$ and/or $\gamma_i = 0$. Then, at this point, we have to distinguish between constant and decreasing returns to scale. If we have decreasing returns to scale ($0 < \alpha < 1$) then $\gamma_i = 0$. However, for the case of constant returns to scale ($\alpha = 1$) γ_i is undetermined *a priori*.

Case 1. Constant returns to scale

$\alpha = 1$ can not ensure long run growth in our model because of the endogenous scrapping decision and the minimum energy requirement to use a machine. Now, we are going to study the necessary conditions for our balanced growth path according to the different values of γ_i .

Let us assume that the energy market is in equilibrium along the balanced growth path, energy demand equals energy supply ($e_s(t)$). We make a distinction between two possible scenarios. There is an optimistic scenario with constant available energy supply. However, there is a gloomy situation where the available energy supply is decreasing¹⁰.

Case A: $\gamma_i = 0$

From equation (1.5) we get the energy demand along the balanced growth path

$$e(t) = \frac{\bar{i}}{\gamma} (e^{\gamma \overline{T}} - 1) e^{-\gamma t}. \quad (1.16)$$

A.1 Constant available energy supply $e_s(t) = \bar{e}_s$

Equalizing $e(t) = \bar{e}_s$ in equation (1.16), we get

$$\frac{\bar{i}}{\gamma} (e^{\gamma \overline{T}} - 1) e^{-\gamma t} = \bar{e}_s. \quad (1.17)$$

¹⁰There is no point in assuming increasing available energy supply, since we are considering resources subject to exhaustion. The most optimistic scenario is constant available energy supply.

Since RHS is constant, LHS has to be constant. This is impossible because $\gamma > 0$ and $\bar{T}$ is finite. Then, we get a contradiction. In case B we are going to study $\gamma_i \neq 0$.

A.2 Decreasing available energy supply $e_s(t) = \bar{e}_s e^{-\gamma_{e_s} t}$, with $\gamma_{e_s} > 0$.

Equalizing $e(t) = \bar{e}_s e^{-\gamma_{e_s} t}$ in equation (1.16) yields

$$\frac{\bar{i}}{\gamma}(e^{\gamma\bar{T}} - 1)e^{-\gamma t} = \bar{e}_s e^{-\gamma_{e_s} t}. \quad (1.18)$$

Then, to have balanced growth path γ has to equal γ_{e_s} . This means that the economy chooses a growth rate of energy saving technical progress equal to the decrease rate of energy supply. As a consequence:

$$i(t) = i^* = \frac{\bar{e}_s \gamma}{e^{\gamma\bar{T}} - 1}. \quad (1.19)$$

Since $\gamma_i (= \gamma_K) = 0$, from the production function $y(t) = AK(t)$, $\gamma_y = \gamma_K = 0$.

From the budget constraint $y(t) = c(t) + i(t)$, $\gamma_c = 0$.

Case B: $\gamma_i \neq 0$

Taking $i(t) = \bar{i}e^{\gamma_i t}$ in equation (1.5), the energy demand along the balanced growth path is given by

$$e(t) = \begin{cases} \frac{\bar{i}}{\gamma_i - \gamma}(1 - e^{-(\gamma_i - \gamma)\bar{T}})e^{(\gamma_i - \gamma)t}, & \text{if } \gamma_i \neq \gamma, \\ \bar{i}\bar{T}, & \text{if } \gamma_i = \gamma. \end{cases} \quad (1.20)$$

B.1 Constant available energy supply $e_s(t) = \bar{e}_s$

If $\gamma_i \neq \gamma$, equalizing $e(t) = \bar{e}_s$ in equation (1.20), we obtain

$$\bar{e}_s = \frac{\bar{i}}{\gamma_i - \gamma}(1 - e^{-(\gamma_i - \gamma)\bar{T}})e^{(\gamma_i - \gamma)t}. \quad (1.21)$$

Similar to the case A.1, we get a contradiction because $\gamma > 0$ and $\bar{T}$ is finite.

If $\gamma_i = \gamma$, equation (1.20) yields

$$\bar{e}_s = \bar{i}\bar{T}. \quad (1.22)$$

Hence, in this case we can have balanced growth path with $\gamma_i (= \gamma_K) = \gamma$. Moreover, as $y(t) = AK(t)$ then $\gamma_y = \gamma_K (= \gamma)$. From the budget constraint $y(t) = c(t) + i(t)$, we also achieve that $\gamma_c = \gamma$.

B.2 Decreasing available energy supply $e_s(t) = \bar{e}_s e^{-\gamma_{e_s} t}$, with $\gamma_{e_s} > 0$.

If $\gamma_i \neq \gamma$, equalizing $e_s(t) = \bar{e}_s e^{-\gamma_{e_s} t}$ in equation (1.20), we find

$$\bar{e}_s e^{-\gamma_{e_s} t} = \frac{\bar{i}}{\gamma_i - \gamma} (1 - e^{-(\gamma_i - \gamma)\bar{T}}) e^{(\gamma_i - \gamma)t}. \quad (1.23)$$

It is straightforward to see that a balanced growth path is possible if $\gamma_i = \gamma - \gamma_{e_s}$. Furthermore, if $\gamma > \gamma_{e_s}$ our economy has long run growth because $\gamma_i = \gamma - \gamma_{e_s} > 0$ ¹¹. However, if $\gamma = \gamma_{e_s}$ we get that $\gamma_i = 0$ which contradicts our initial statement. Finally, for the case $\gamma < \gamma_{e_s}$ our economy has no positive long run growth; indeed, $\gamma_i = \gamma - \gamma_{e_s} < 0$.

If $\gamma_i = \gamma$, from equation (1.20) we get

$$\bar{e}_s e^{\gamma_{e_s} t} = \bar{i}\bar{T}. \quad (1.24)$$

However, this is impossible because $\gamma_{e_s} > 0$ and $\bar{T}$ is finite.

To sum up our results, we establish the two following propositions for constant returns to scale:

Proposition 1.1 *Along the balanced growth path, assuming $\alpha = 1$, $e_s(t) = \bar{e}_s$ and $\gamma < \rho$,*

- 1) *the interest rate $r(t) = r^* = \rho + \theta\gamma$;*
- 2) *the growth rate of energy prices equals the growth rate of energy saving technical progress ($\gamma_{P_e} = \gamma$);*
- 3) *the growth rate of investment and capital stock are equal to the growth rate of energy saving technical progress ($\gamma_i = \gamma_K = \gamma$);*
- 4) *the growth rate of final good output equals the growth rate of energy saving technical progress ($\gamma_y = \gamma$);*
- 5) *the growth rate of consumption equals the growth rate of energy saving technical progress ($\gamma_c = \gamma$).*

¹¹ $\gamma_K = \gamma_i = \gamma - \gamma_{e_s} > 0$ Since $y(t) = AK(t)$, then $\gamma_y = \gamma_K = \gamma - \gamma_{e_s}$. From the budget constraint $y(t) = c(t) + i(t)$, we get that $\gamma_c = \gamma - \gamma_{e_s}$

Proposition 1.2 *Along the balanced growth path, assuming $\alpha = 1$, $e_s(t) = \bar{e}_s e^{-\gamma_{es} t}$ and $\gamma < \rho$,*

- 1) *the interest rate $r(t) = r^* = \rho + \theta\gamma_c$, where γ_c is given in the following 3) and 4);*
- 2) *the growth rate of energy prices equals the growth rate of energy saving technical progress ($\gamma_{P_e} = \gamma$);*
- 3) *if $\gamma = \gamma_{es}$, then*
 - (3.1) *there is no growth in the investment and the capital stock ($\gamma_i = \gamma_K = 0$),*
 - (3.2) *the growth rate of final good output is zero ($\gamma_y = 0$),*
 - (3.3) *there is no growth in the consumption ($\gamma_c = 0$);*
- 4) *if $\gamma > \gamma_{es}$, then*
 - (4.1) *there is positive growth in the investment and the capital stock ($\gamma_i = \gamma_K = \gamma - \gamma_{es} > 0$),*
 - (4.2) *the growth rate of final good output is $\gamma_y = \gamma - \gamma_{es}$,*
 - (4.3) *there is positive growth of consumption ($\gamma_c = \gamma - \gamma_{es}$);*
- 5) *if $\gamma < \gamma_{es}$, then the economy decreases in the rate $\gamma_{es} - \gamma$.*

We have to point out that constant returns to scale, in an optimistic scenario (*i.e.*, $e_s(t) = \bar{e}_s$), generates exogenous growth with the same rate as the growth of (exogenous) energy saving technical progress (γ). However, if a gloomy scenario is assumed (*i.e.*, $e_s(t) = \bar{e}_s e^{-\gamma_{es} t}$) our model can achieve long run growth with rate $\gamma - \gamma_{es}$. If the growth rate of the energy saving technical progress is greater than the decreasing rate of energy supply, the growth is positive. However, if the growth rate of the energy saving technical progress is equal or lower than the decreasing rate of energy supply, we get no or negative growth respectively. The reason is the following. First of all, we consider a balanced growth path with constant and finite scrapping age following the Terborgh-Smith result. Second, energy supply has two effects over the economy. On the one hand,

through the energy prices; and on the other, through the Leontieff production function, which model the idea of minimum energy requirement to use a machine. When we consider constant available energy supply (optimistic scenario), following our definition of balanced growth path, we are going to have long run growth because of the exogenous energy saving technical progress (in this case energy prices increases since the economy grows at the growth rate of the exogenous energy saving technical progress). However, in the case of decreasing available energy supply (gloomy scenario) the situation is quite different. Energy prices increase because of the decreasing available energy supply. Now, if the energy saving technical progress exactly compensates the decreasing available energy supply ($\gamma = \gamma_{es}$), then there is no long run growth, despite of having constant returns to scale (see equation (1.18)), because there is constant replacement and the energy supply is going to affect growth negatively through the minimum energy requirement (Leontieff technology). However, if the energy saving technical progress exceeds the decreasing energy supply, the second effect is also avoided and our economy can depict growth whose rate is the difference between the growth rate of energy saving technical progress and the decreasing rate of energy supply, $\gamma - \gamma_{es}$. Finally, if the energy saving technical progress is not stronger enough to offset the decreasing energy supply, our economy decreases.

Case 2. Decreasing returns to scale

As before, equation (1.15) holds if and only if $\alpha = 1$ and/or $\gamma_i = 0$. Since now we consider decreasing returns to scale ($0 < \alpha < 1$), the growth rate of investment has to be zero. As a consequence, $\gamma_K = 0$ because $\gamma_i = \gamma_K$. Furthermore, considering a canonical vintage capital model with Arrowian learning by doing technical progress, d'Autume and Michel(1993) get that decreasing returns to scale “kills” growth in the long run. Our model is mathematically close to their economy taking energy instead of labor.

1.4 Concluding remarks

We analyzed the hypothesis about the effectiveness of energy saving technologies to reduce the trade-off between economic growth and energy preservation. In order to incorporate

the role of technology replacement, we developed a general equilibrium model, where the output is produced by a vintage capital technology with endogenous scrapping rule. New vintages obsolete old machines because of their lower energy requirements. Constant and decreasing returns to scale are distinguished to develop a comparative study.

Under constant returns to scale and optimistic context (constant available energy supply), long run growth is possible (Proposition 1.1). In this case, the (exogenous) growth rate of the economy equals the (exogenous) growth rate of energy saving technical progress. However, considering a more realistic situation of gloomy scenario (decreasing available energy supply), our economy only can achieve long run growth if the growth rate of the energy saving technical progress excess the decreasing rate of the energy supply (Proposition 1.2). Here, the growth rate of our economy is given by the difference between the growth rate of energy saving technical progress and the decreasing rate of the energy supply (*i.e.*, $\gamma - \gamma_{e_s}$). Furthermore, when we assume decreasing returns to scale, the economy achieves balanced growth path only for the gloomy case; nevertheless, our economy does not exhibit growth in the long run. However, we could escape from the decreasing returns to scale incorporating additional elements such as learning-by-doing, human capital, subsidies, etc. The constant scrapping age and the reduced availability of energy (which affects the economy through the energy prices and the minimum energy requirements) explain our results in both constant and decreasing returns to scale.

We found some empirical values for the growth rate of energy saving technical progress (γ) and the decreasing rate of the energy supply (γ_e). About γ , the work of Azomahou, Boucekkine & Nguyen Van (2004) finds an annual growth rate of energy saving technical progress for the USA around 1.9%. As to γ_e , the Uppsala Hydrocarbon Depletion Study Group (Uppsala University) forecasts (up to 2050) an annual decreasing rate of oil supply around 1.8% for the total world ¹².

In conclusion, we could have compatibility between economic growth and energy preservation adopting energy saving technologies. However, the growth rate of the energy saving technical progress has to be greater than the decreasing rate of the energy supply to en-

¹²<http://www.isv.uu.se/nuclear.en.html>

sure a positive long run growth. Since in our model energy saving technical progress is exogenous, an interesting extension is considering that the economy endogenously decides the growth rate of energy saving technical progress. An R&D sector of energy saving technologies could be a good way to study this problem.

Chapter 2

Economic Growth and Environmental Quality in a Clean Vintage Capital Model

2.1 Introduction

Since the seminal paper by Grossman and Kruger(1991) there has been considerable academic interest in the relationship between economic development and environmental pollution. Importantly the authors showed that the link between these follows an inverted U-shaped pattern, now commonly referred to as the Environmental Kuznets Curve (EKC). This suggested that lower income regions are ‘too poor to be green’, but as countries become richer they will naturally reduce their generation of pollution. Several recent studies, in contrast, have put the existence and exact shape of an EKC into question.

In view of the recent policy developments, however, resolving this issue seems of particular importance. More precisely, the recent Kyoto Protocol has set reduction targets for pollutant emissions to which developed countries are expected to commit themselves to, but from which developing countries are at the first instance exempt. This would suggest that policymakers are of the view that wealth on its own does not result in a - or possibly

sufficient - reduction in pollution, a stance which as of date has not yet been substantiated in the academic literature.

Arguably one of the main reasons for the lack of consensus on the existence of an EKC can be attributed to the fact that the number of theoretical underpinnings is relatively sparse and hence that the mechanisms underlying the link between pollution and development are probably not yet well understood; see Dasgupta et al. (2002). The notable exceptions can broadly be categorized into neoclassical, overlapping generation, and endogenous growth models. For example, Selden and Song(1995) demonstrate the existence of an inverted U-shaped curve between pollution and output in a strictly neo-classical framework. John and Pecchenino (1994) and John and al.(1995), in contrast, use overlapping generation models to demonstrate the same. The use of endogenous growth models is particularly adapted to the kind of issues of the present study, as it allows to lay down the conditions of sustainable economic development. More recent contributions have introduced pollution in endogenous growth models in order to derive the EKC. An Ak model is used in Aghion and Howitt (1998) as well as in Stokey(1998).

Importantly, however, all of these models do not consider the decision of when to replace obsolete with newer technologies. Clearly, though, if one assumes that, as would be more realistic in most cases, older technologies are more environmentally unfriendly, then the decision when to scrap these is likely to be an important determinant of the extent of pollution generation.¹ In this paper we thus explicitly model how the decision to scrap obsolete technologies will affect the relationship between economic development and pollution. In order to do so we build on the Schumpeterian framework of Aghion and Howitt(1998) by introducing a vintage capital structure, where the law of motion of environmental quality will depend on the pollution flow and some upper limit on environmental quality that takes into account the exhaustibility of resources. Moreover, we explicitly assume that new technologies are more environmental friendly. We are thus able to shed light on the mechanisms through which the environmental quality affects growth performance following technological adoption.

¹In her paper, Stokey(1998) presents an Ak model of endogenous growth and a neoclassical framework with exogenous technological change..

Using our model we show that a reduction in environmental pollution during the industrialization process is only possible when the optimal rate of technological adoption has been reached. However, reaching this point will not necessarily guarantee that environmental quality improves. Rather, we identify three possible outcomes concerning the relationship between pollution and economic development, where these depend on the rate of investment relative to the rate of environmental friendliness technological improvement. Firstly, there is the case of sustainable development where investment, consumption and output, and environmental quality improve at a constant rate but different rate. However, feasibly there may also be the catastrophic outcome where the environmental quality's lower bound is reached in finite time. This case corresponds to a situation where the rate of investment outpaces the rate of technological improvement in environmental friendliness.²

Our theoretical predictions have important empirical implications in terms of seeking evidence for the EKC. Importantly, we find that there could be considerable heterogeneity across countries concerning their pollution-output relationship experience, depending on consumer tastes and the type of technological improvements. More precisely, countries may not only differ in the rate and when they reach the point along their development path at which they could potentially reduce their pollutant emissions, but this reduction is not guaranteed. In other words, the shape of the pollution-output relationship can differ widely across countries, and thus, the use of cross-country panel data sets to seek the existence of an EKC - which is now common practice in the literature - may be flawed. Instead it may be more insightful to study the pollution-outcome link by examining countries individually. Moreover if one wants to capture the full pattern of how industrialization affects environmental quality in individual countries, one is likely require long time series data, since the possibility of achieving a reduction in pollution may feasibly happen at very early age of economic development. As a first attempt in this direction, we use long time series on carbon dioxide emissions and an indicator of economic development for a number of developed and developing countries in order to in-

²Furthermore, and contrary to most vintage capital models, the transition dynamics are explicitly expressed in terms of endogenous scrapping age, which is a worthwhile contribution to the related literature, given this result is obtained via a linear utility function and an environmental evolution equation.

investigate the potential existence of an EKC for countries individually. In this regard, we rely on a nonparametric kernel regression estimator, and thus, places little restrictions on the functional form of the relationship between pollution and output. Our results do indeed provide evidence on considerable heterogeneity across the countries examined.

The rest of the paper is organized as follows. A vintage capital model is presented in section 2. In particular, we prove the existence of an optimal scrapping path, and show that it can be reached in finite time. Section 3, the empirical framework and results are displayed. A general discussion and a conclusion are provided in Section 4.

2.2 The model

In this section, we first present a standard vintage capital model, where we add an equation of motion representing environmental quality. We then derive the transition dynamics and the conditions under which a balanced growth path exists. In particular, we are able to fully characterize the optimal scrapping path, which is usually not the case in these types of models. This consequently allows us to provide parameter conditions under which an inverted U-shaped EKC will and will not exist.

2.2.1 A vintage capital structure

Consider an economy with a constant population level, where the labor market is perfectly competitive, and the production sector produces only one final good, which can be assigned to consumption or investment and plays the role of the numeraire.

Production Sector At time $t > 0$, per capita output $y(t)$ is assumed to follow a vintage capital rule

$$y(t) = \int_{t-T(t)}^t i(z) dz. \quad (2.1)$$

where $0 < T(t) < \infty$ represents the vintage of the oldest machine in use, and $i(z)$ is investment in a machine of age z . Define the life expectancy of a machine as $J(t) =$

$T(t + J(t))$, i.e., the expected life of a machine at time t is equal to the scrapping time $T(\cdot)$, evaluated at $t + J(t)$, which corresponds to the time when this new machine will be scrapped in the future.

As can be seen from (2.1), we, in contrast to Stokey(1998), do not consider the level of pollution as an input in the production sector. Instead, we allow pollution to enter consumers' utility function. Thus we are assuming that although the firm has the right to pollute, consumers also have the right to refuse buying goods from 'dirty' industries. Consequently, if a good is produced in such an industry, returns to capital will decrease with the employed technology. Hence, the firm is forced to scrap the old dirty machines and replace them by new cleaner ones, which in turn can be considered as an endogenous progress.

Environment Sector In this economy, household agents care not only about their per capita consumption level $c(t) > 0$, but also pay attention to environmental quality. Following Aghion and Howitt(1998, Chap.5), we assume that there is an upper limit of to environmental quality, denoted by $\bar{E}$. We measure $E(t)$ as the difference between the actual quality and this upper limit. Thus, environmental quality will always be negative. The equation of motion of environmental quality is given by

$$\dot{E}(t) = -qE(t) - \int_{t-T(t)}^t i(z)e^{-\gamma z} dz, \quad (2.2)$$

where $q > 0$ is the maximum potential rate of recovery of environment, $\gamma > 0$ is the constant rate, at which the pollution of investment of vintage declines, and $\int_{t-T(t)}^t i(z)e^{-\gamma z} dz$ measures pollution.³ One should note that from (2.2) pollution is a side-product of investment $i(z)$ in the production sector.⁴ Implicit in (2.2) is the assumption that new

³Here we use the concept of a "*cleaner*" *technological progress* instead of *abatement*. Copeland and Taylor (2003, Chap.2) however show that the two approaches are identical.

⁴The notion of rate of recovery of environment is used in Aghion and Howitt (1998) and refers to the self-regeneration capacity of every ecosystem, defined by Daly (1990, 1991) and adopted by the World Bank thereafter. The threshold for absorbing deterioration are defined as follows: waste emissions from a project should be within the assimilative capacity of the local environment to absorb without

machines are less polluting than older ones.⁵ Using a newer vintage leads henceforth to reduced pollution per input.

Further more, since our main point in this paper is sustainable development, we assume that environmental quality also has a lower limit, which we will refer to as the catastrophic threshold. This, in turn, implies that the optimal growth path, if it exists, will be constrained as follows

$$\underline{E} \leq E \leq 0.$$

Per capital output $y(t)$ can be consumed, $c(t)$, or invested in a vintage capital good, $i(t) \geq 0$,

$$y(t) = c(t) + i(t). \quad (2.3)$$

Central Planner The central planner's objective function will entail per capita consumption and environmental quality. More particularly, the planner will choose the paths of consumption and environmental quality in order to maximize the instantaneous utility of the infinitely lived representative household,

$$\max_c \int_0^\infty U(c, E) e^{-\rho t} dt = \max_c \int_0^\infty [\beta c(t) - (1 - \beta)E(t)] e^{-\rho t} dt, \quad (2.4)$$

subject to (2.2), (2.3), and

$$J(t) = T(t + J(t)), \quad (2.5)$$

where $\rho > 0$ is the constant time preference, $0 < \beta \leq 1$ is a weight parameter between consumption goods and environmental quality, and $i(z)$, $z \leq 0$ and $E(0)$ are given func-

unacceptable degradation of its future waste absorptive capacity or other important services. Harvest rates of renewable resource inputs should be within re-generative capacity of the natural system that generates them; depletion rates of nonrenewable resource inputs should be equal to the rate at which renewable substitutes are developed by human invention and investment.

⁵Grossman (1995) already noticed the effect of economic growth on the quality of the environment, since wealthier countries can afford to spend more on research and development, and thus, substitute dirty technologies with cleaner ones.

tions. Furthermore we assume that $0 < \gamma < \rho < 1$, which are necessary and sufficient conditions for the existence of balanced growth path in an exogenous growth models.⁶

2.2.2 Optimal Investment Strategy and Scrapping Rule

This section investigates the transitional dynamics from any initial investment profile and environmental condition towards the balanced growth path (BGP), if it exists. The BGP is defined as the path along which, consumption, investment, and output grow at the same constant rate, while the scrapping age T and optimal life expectancy J are finite constants. This subsection is organized as follows. First, we assume there exists a finite time $0 \leq t^* < \infty$, such that the interior solutions are beginning at t^* , and we derive optimal conditions that should be verified whenever we reach t^* . Second, we prove the existence of t^* . Third, we compute the transition dynamics during period $0 \leq t < t^*$, and characterize the optimal conditions on the investment and output, given the scrapping age and the BGP. Finally, we study sustainable growth and the pollution-output ratio and provide conditions under which an EKC exists.

After changing the order of integrals and rearranging the terms, first order conditions with respect to $i(t)$ and $J(t)$ are given by

$$\beta e^{-\rho t} \left(\frac{1 - e^{-\rho J(t)}}{\rho} - 1 \right) = e^{-\gamma t} \int_t^{t+J(t)} e^{-\rho z} \mu(z) dz; \quad (2.6)$$

$$\mu(t) = \beta e^{\gamma(t-T(t))}. \quad (2.7)$$

The co-state variable of $E(t)$ satisfies

$$\dot{\mu}(t) = (\rho + q)\mu(t) + (1 - \beta), \quad (2.8)$$

and the transversality condition is

$$\lim_{t \rightarrow \infty} e^{-\rho t} \mu(t) E(t) = 0.$$

⁶In order to obtain explicit solutions, we avoid more general utility functions. While general utility functions, would allow us to write down optimal conditions as in Ramsey type models, the equilibrium conditions for such an economy would give rise to a mixed-delay differential equation system with endogenous leads and lags (see Boucekine et al., (1997)).

(2.8) together with its transversality condition is Tobin's q in the sense of environmental quality, which describe the shadow value of environmental quality. As in the optimal investment profile, this shadow value determines the optimal investment strategy (2.6), and the optimal scrapping rule (2.7).

(2.6) states that the optimal investment strategy should make such that at time t the discounted marginal productivity during the whole lifetime of the capital acquired in t exactly compensates for both its discounted operation cost and its discounted environmental shadow value, where the first term on the left hand side is the discounted marginal productivity during the whole lifetime of the capital acquired in t , and the second term is the marginal purchase cost at t normalized to one, and the the right hand side is the discounted environmental shadow value at t .

The optimal scrapping rule (2.7) presents that a machine should be scrapped as soon as its operation cost with respect to consumption no longer covers its market value of environmental quality.

2.2.3 Balanced growth path

The BGP is defined by a constant optimal scrapping age and constant rates of growth for the other endogenous variables.

Substituting (2.7) into (2.6), we obtain straightforwardly

$$e^{-\rho t} \left(\frac{1 - e^{-\rho J(t)}}{\rho} - 1 \right) = e^{-\gamma t} \int_t^{t+J(t)} e^{-\rho z} e^{\gamma(z-T(z))} dz,$$

Deriving with respect to t , using (2.5) and rearranging terms, it follows that

$$e^{-\gamma T(t)} = 1 - \rho + \gamma - \frac{\gamma}{\rho} + \frac{\gamma}{\rho} e^{-\rho J(t)}. \quad (2.9)$$

Proposition 2.1 (Proof: see Boucekkine et al. (1997)) *With $0 < \gamma < \rho < 1$, for $t > t^*$, the unique differential interior solutions of $T(t)$ and $J(t)$ are given by*

$$J(t) = T(t) = T^*,$$

where T^* is the positive fixed-pointed of function $F(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with $\mathbb{R}_+ = \{x \geq 0\}$, and for any $x \geq 0$,

$$F(x) = -\frac{1}{\gamma} \ln \left(1 - \rho + \gamma - \frac{\gamma}{\rho} + \frac{\gamma}{\rho} e^{-\rho x} \right).$$

The optimal scrapping age does not depend on the weight between consumption goods and environmental quality. However, it depends on the consumers' time preference and on the technology program. Thus, different economies may highlight different optimal paths!

The above parameters and technological progress, do not grantee that a BGP can be reached. In the following, we are going to deduce the conditions, under which an optimal BGP could be reached.

Suppose investment grows at a constant rate g , with investment level $\bar{i}$, i.e. $i(t) = \bar{i}e^{gt}$. From this we can easily see that investment, consumption, and output grow at the same constant rate. Reconsidering equation (2.2), and using basic ordinary differential equation techniques, we obtain that, for $t > t^*$,

$$E(t) = E(t^*)e^{-qt} - e^{-qt} \int_{t^*}^t e^{qs} \int_{s-T^*}^s i(z)e^{-\gamma z} dz ds. \quad (2.10)$$

For a more explicit form of (2.10), see Appendix. From (2.1) and (2.10), we obtain the following results.

Proposition 2.2 *Suppose $0 < \gamma < \rho < 1$, and let $t > t^*$. (a) If furthermore $\gamma < \frac{\rho}{2}$, there is a BGP, where investment, consumption, and output grow at constant rate γ , and the growth rate of environmental quality is $q > 0$. Furthermore, sustainable growth is guaranteed and environmental quality will permanently improve, though never reach its upper bound. (b) When $g > \gamma$, i.e. in case of over-investment, there is no BGP and the economy converges towards the catastrophe outcome in finite time. (c) If $g < \gamma$, sustainable growth of output, investment, and consumption with growth rate g , with $0 < g < \min\{\gamma, 1 - e^{-gT^*}\}$, is a possible outcome. The steady state case, where output, investment, and consumption grow at a constant rate (i.e. $g = 0$) is also possible. In this*

case, environmental quality will constantly improve and tend to the upper bound in long run, but not at a constant rate of growth. We call this case the under-investment case.

In point (a), condition $\gamma < \frac{\rho}{2}$ ascertains an interior solution of consumption is achieved.⁷ The statement about the catastrophic outcome in point (b) directly results from the assumption concerning a lower bound in environmental quality. In other words, the environment can not infinitely worsen.

The intuition is obvious. Consider simultaneously consumption and environmental quality. If the constant scrapping time T^* is reached, the investment growth rate should be the same as the technological progress rate in order to keep sustainable growth of output, while environmental cleaning is done by nature's self regeneration ability. However, this self regeneration will never allow environmental quality to reach its upper bound (point (a)). In the case of over-investment in newer technologies, i.e. $g > \gamma$, the environment will be totally destroyed in finite time (catastrophic case, point (b)). Finally, if keeping investment and consumption at a steady state level (i.e. $g = 0$), since cleaner machine are contiguously employed, it is obvious that the environment will improve (point (c)).

Note that although T^* is reached does not imply that the most cleanest machine is found. New techniques always appear, which are more environmental friendly. However, scrapping the old machines earlier or later than T^* always worsens either consumption or environmental quality, or both!

2.2.4 Transition dynamics and optimal scrapping time

Two cases can be distinguished, corresponding to different levels of development. The first case corresponds to a situation where countries scrap too fast, i.e. $T(0) < T^*$. Intuitively, one could think of the *developed country* case, where economies start with a relatively high stock of machines. Conversely, the developing country case would correspond to the case where economies have initially a relatively low stock of machines.

⁷It can easily be shown that along the BGP, $c(t) = y(t) - i(t) = \bar{i}r^{\gamma t} [(1 - e^{-\gamma T^*}) / \gamma - 1] > 0$, implying $1 - e^{-\gamma T^*} > \gamma$, where γ is the growth rate of investment. Given (2.9), it follows that $1 - e^{-\gamma T^*} = \rho - \gamma + \frac{\gamma}{\rho}[1 - e^{-\rho T^*}] > \gamma$. A sufficient condition for this inequality to hold is $\rho > 2\gamma$.

In the Appendix, we show the following result

Proposition 2.3 *Given an investment profile of a country, if initially $T(0) < T^*$, then this economy should instantaneously jump to optimal scrapping path. In this case, $t^* = 0$. Conversely, if $T(0) > T^*$, there exists time t^0 , such that, $0 < t^0 < \infty$. Moreover,*

$$T(t) = \begin{cases} t - \frac{\rho + q}{\gamma}t - \frac{1}{\gamma} \ln \left(e^{-\gamma T(0)} + \frac{1 - \beta}{\beta(\rho + q)} (1 - e^{-(\rho + q)t}) \right), & 0 \leq t < t^0, \\ T^*, & t \geq t^0, \end{cases}$$

which is decreasing with time t , if $0 \leq t < t^0$, and $t^* = t^0$.

Remark 2.1 Under this kind of vintage capital setting, this is the first time to our knowledge that a clear and explicit transition dynamic is defined. For linear utility function, the immediate jump corresponds to the case of consuming all output for some while, whereas the other case would correspond to investing all output, until scrapping age T^* . That is, in finite time the corner solutions converge to interior solutions.

Intuitively, since the endogenous scrapping time $T(t)$ is increasing with the initial scrapping time $T(0)$, if an economy starts with a relatively high stock of machines, and scraps too fast, it is impossible, for any positive t , that $T(t)$ reaches T^* , the optimal path, given the endogenous scrapping program is decreasing with time. Henceforth, the only way to reach the optimal path is to immediately jump to this optimal path. Aghion and Howitt(1998, Chap.5) developed this idea under a Schumpeterian framework. However, they only show that there is some initial value of capital for this to happen, while we provide the explicit conditions under which this jump could indeed occur.

For the developing economy case, the instantaneous jump to the optimal path could be impossible (even though starting from corner solution, that is, zero consumption and all output invested in physical capital). Instead of an immediate technology adoption, which would compensate the initial low level of vintage physical capital, the new technology is supposed to be costly, implying the existence of time delays of adoption. For relatively poor economies, older and relatively environmental unfriendly machines still need to be employed for a certain period, until the optimal path is reached. Further, note that the

scrapping time $T(t)$ is increasing with the initial scrapping age $T(0)$. As a by-product, for two developing economies, the relatively richer one will reach the optimal path faster than the poorer one, provided all the other arguments are the same.

To show this result, let us suppose $0 < t^1 < t^*$, such that, $t^1 = T(t^1)$, and

$$t^1 = -\frac{1}{\rho + q} \ln \left(\frac{\beta(\rho + q)}{\beta(\rho + q) + 1 - \beta} \left(e^{-\gamma T(0)} + \frac{1 - \beta}{\beta(\rho + q)} \right) \right).$$

Then, starting from t^1 , $T(0)$ will be the age of the oldest machine in use. The effect of the oldest machine on output and pollution can be stated as

Proposition 2.4 *Assume initial investment profile of a country, and an initial scrapping time $T(0)$ are given. At time $t \geq t^1$, to keep on investing in the oldest machine in use, i.e. $i'(t - T(t)) \geq 0$, implies that both output and pollution are increasing functions of $T(0)$. Furthermore, output is concave with respect to $T(0)$, while pollution could be concave or convex, and the speed of pollution emission is faster than output,*

$$\frac{\partial^2 P(t; T(0))}{\partial T^2(0)} > \frac{\partial^2 y(t; T(0))}{\partial T^2(0)}, \quad (2.11)$$

where $P(t; T(0)) = P(t)$.

The proof is given in Appendix.⁸

The intuition of the above theorem goes as follows. The new techniques, which are more environmental friendly, are embodied in the new machines. So keeping old machines in use will not stop the growth of output, neither its by-product, pollution. However, too old machines will slow down the increasing speed of output, which will be overcome by the side-effect pollution.

2.2.5 Is there an Environmental Kuznets Curve?

In this section, we are going to study the relationship of pollution-output per capital.

⁸We only need to consider the case where $t \geq T(t)$, since in the case $t - T(t) < 0$, the oldest investment $i(t - T(t))$ is given by assumption at the beginning of the economy.

As mentioned in the previous section, when $T(0) > T^*$, the economy starts with a relatively low stock of capital. Then in order to reach the interior solution (and thus, the optimal path, if there is one) as fast as possible, one possibility is to invest all output as mentioned in Remark 2.1. In this case, starting from a corner solution, the economy subsequently pollutes, when $0 < t < t^*$,

$$P(t) = \int_{t-T(t)}^t y(z) e^{-\gamma z} dz.$$

Indeed, pollution increases with the accumulation of output $y(\cdot)$, during all the periods the machine is in use. Moreover, there is also a *delay effect* on pollution coming from output.

If $t > t^*$, that is, the optimal scraping age is reached, then difference cases appear.

If we are along the balanced growth path, that is, the growth rate of investment is the same as the rate of cleaning technique improvement, it follows

$$y(t) = \frac{\bar{i}e^{\gamma t}}{\gamma}(1 - e^{-\gamma T^*}), \quad P(t) = \bar{i}T^*.$$

In this case, pollution is independent of output and time, but output is increasing with time t .

However, if we are not on the balanced growth path, it follows

$$P(t) = \int_{t-T^*}^t \bar{i}e^{(g-\gamma)z} dz = \frac{\bar{i}}{g-\gamma}e^{(g-\gamma)t}(1 - e^{-(g-\gamma)T^*}),$$

and

$$\frac{P(t)}{y(t)} = \frac{g(1 - e^{-(g-\gamma)T^*})}{(g-\gamma)(1 - e^{-\gamma T^*})}e^{-\gamma t}.$$

It is easy to see that if this is the case of under-investment case, where the investment rate is lower than the rate of cleaning technique improvement, then pollution is decreasing over time due to the fact that $\gamma^* < \gamma$. However, if that is the case of over-investment, then the investment rate is higher than the rate of cleaning technology improvement, which leads to that the pollution is increasing with time, while the pollution-output ratio is always positive and decreasing over time.

Proposition 2.5 *Suppose that $0 < \gamma < \frac{\rho}{2}$.*

- (i) *The pollution-output ratio is decreasing over time.*
- (ii) *For $0 < t \leq t^*$, during the transition, pollution is increasing with respect to output.*
- (iii) *If $t > t^*$ and the BGP is reached, pollution is independent of output and time, and only depends on the optimal scrapping age and the turning point of investment in the economy.*
- (iv) *If $t > t^*$ and we are in the under-investment case mentioned in Proposition 2, then pollution is decreasing with time; while if we are in the over-investment case, the pollution is increasing with time.*

The assertion of decreasing pollution-output ratio of statement (i) is proved in the Appendix 2.5.4.

This result clearly improves our understanding of the standard EKC. When the BGP is reached, the pollution level is at its maximum level. However, for one country, when it can be considered as developed, it does however not decrease its pollution level which remains at this maximum level. Environmental quality improves mainly due to nature's self-regeneration ability. During the underdevelopment period, though pollution is increasing with respect to output, levels of pollution are lower than when the country is developed. These results will be further discussed and illustrated in the empirical section.

2.3 An empirical analysis

In the previous section, we have shown that (i) first, countries may have very different optimal scrapping age of old machines according to their consumer preferences, and (ii) second, after having reached T^* , the pollution-output ration could be positive, negative or zero. This has strong implications for the empirical analysis. In particular, these country-specific peculiarities will be accounted for, as well as the possibility of non-linearities in the pollution output relation. In the sequel, we will first describe the data and second, display the statistic results.

2.3.1 Data

The theoretical model has highlighted a change in the relation between pollution and output *before* and *after* the economy has reached the optimal age of changing technology. Moreover, under the conditions of an EKC, i.e. an inverted U shaped pollution-output curve, the turning point is reached at T^* , when the economy has attained the optimal scrapping age. This in turn means that our data has to cover the periods before and after T^* in order to capture a possible EKC. Most cross-country empirical studies on the EKC are based on sulfur and carbon dioxide emissions, stemming from the Historical Global Sulfur Emissions database (Lefohn et al., 1999), and the World Resource Institute (People and Ecosystems CD-rom) respectively, and cover the period following the WWII, essentially due to limitations in the availability of GDP/capita data. This is a strong limitation, in particular for developed countries, as we may suppose that these countries have attained the path of optimal scrapping age a long while ago. In the present study, we rely only on very long time series (up to 250 years). Although restricting the number of countries that can be screened (26 rather than about 80 in the existing studies), we can more confidently hypothesize that we capture periods before the potential turning point. Data on carbon dioxide stems from the Carbon Dioxide Information Analysis Center and has been compiled by Marland et al. (2003). The coverage period goes from 1751 for the earliest and 1901 for the latest to 2000 and the data represents national carbon dioxide emissions from fossil-fuel burning, cement manufacture, and gas flaring. In order to get carbon dioxide per capita figures, we have used population data from the Maddison (2001, 2003) database. The same source has been used for the GDP/capita data. Finally, for the USA, we have used GDP data from Johnston and Williamson (2003).

2.3.2 Statistic results

In Figures 1, we have represented the cases of the USA and the Canada, where we had data on 201 respectively 250 years. After a steep increase, the pollution-output relationship flattens in both cases, and even slightly decreases in the Canada case. The turning point, predicted as the period of optimal scrapping age in the theoretical model, is found to be

in between the second half of the 19th century and WWI. This result clearly supports the use of long time series if one is willing to capture a hypothetical EKC. A second important result is that in both countries, no inverted U-shaped curve can be detected. Rather so, in both cases the pollution-output relationship seems to flatten, which, in terms of our theoretical model, would mean that both countries have reached their BGP where consumption, investment, and output increase at a constant rate and the age of scrapping is constant. Moreover, environmental quality is supposed to improve, although never converging to its superior threshold.

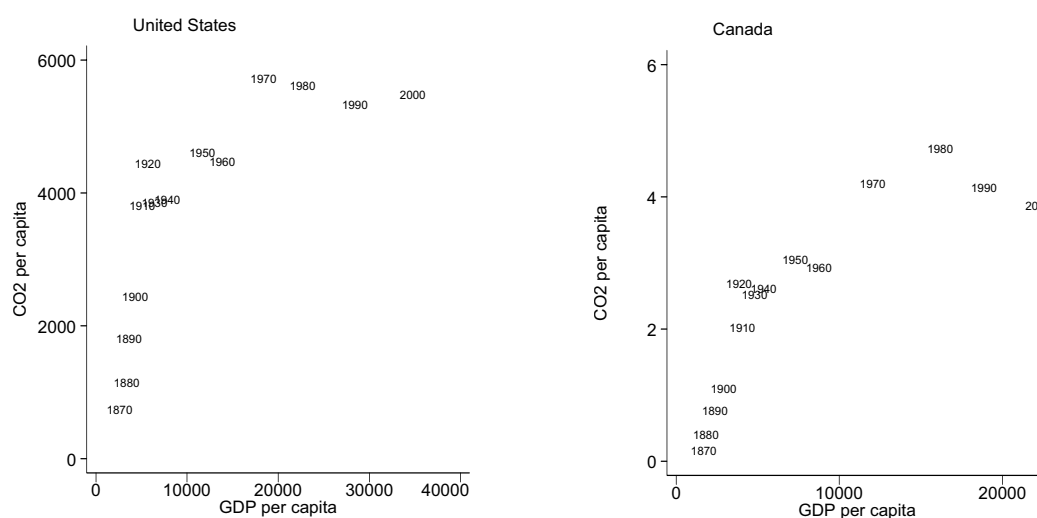


Figure 2.1: Flatten pollution-output relationship

In order to see whether this result can be retrieved for other cases, we show results in Figure 2 and Figure 3 for some developed country cases: Italy, Japan, Germany and

France, and Figure 4, for some developing country(low incomes) cases: India and Mexico.

In the case of Italy and Japan, the curve seems to steadily increase, which, according to our model, would mean that conditionally upon having reached their optimal scrapping age(comparing the income per capita with other developed counties, as Germany, France, Canada and so on), these two countries invest at a rate that is higher than the rate at which technology improves in terms of environmental friendliness⁹. A flattening of the curve can thus not be excluded.

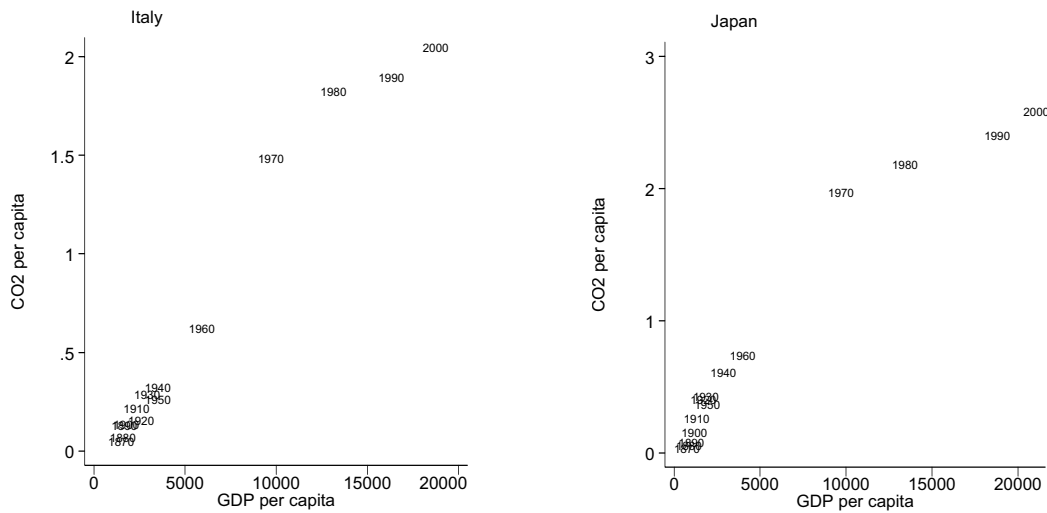


Figure 2.2: Increasing pollution-output relationship

Finally, Germany, and to some extent France, seem to highlight a downward sloping part

⁹A similar statement is more cumbersome for Norway and the Iberian countries (Spain and Portugal) given that although the curve is increasing, confidence bands are relatively broad

of their pollution-output curve, although again, for the latter cases, caution should be adopted given the confidence bands. In these cases, countries have a rate of investment such that environmental friendliness of the new investment evolve faster than the rate of investment.

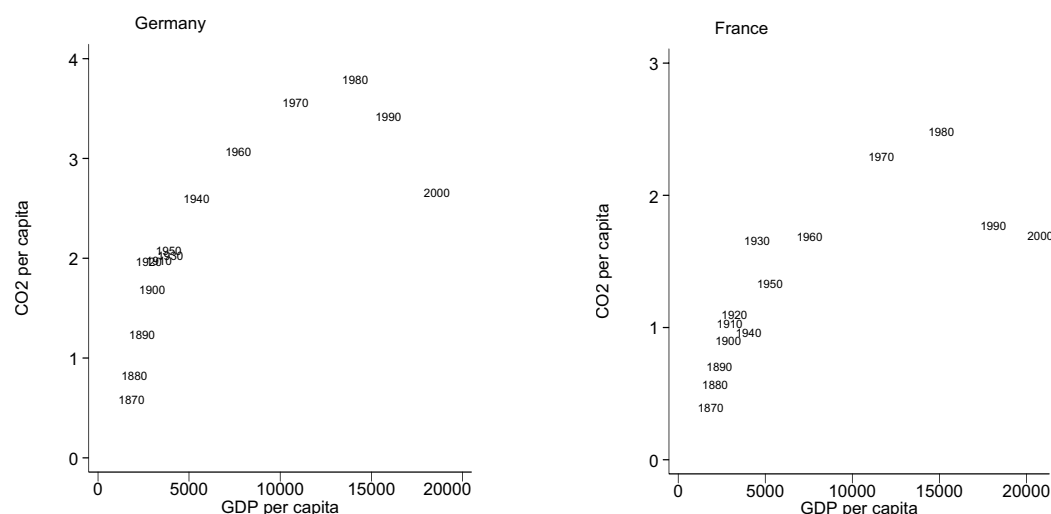


Figure 2.3: EKC pollution-output relationship

Coming to Figure 4, both countries highlight an upward sloping pollution-output curve. Again, referring to our theoretical model, one explanation could be that the rate of investment is higher than the rate at which technology improves in terms of environment. This explanation seems however dubious given the low rates of investment that can be observed in developing countries. For instance, in 2000, low income countries had rates of investment less than half compared to high income countries (Heston et al., 2002). A

more likely explanation would be that these countries have not yet reached their optimal scrapping age, i.e. the actual scrapping age is still longer than the optimal one. Actually, this explanation still holds if we consider that developing countries, although investing, they do so in older technologies respectively in second hand machinery.

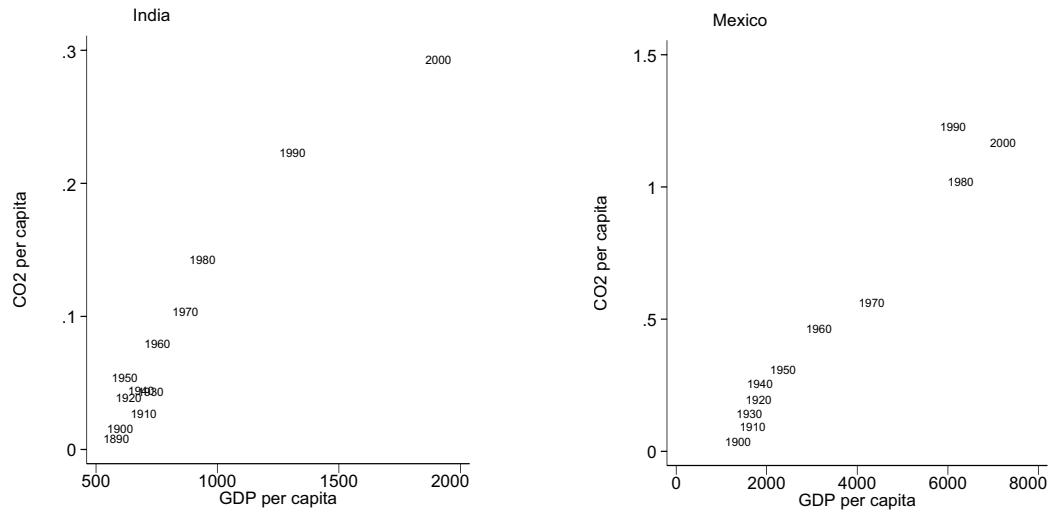


Figure 2.4: Developing economies increasing pollution-output relationship

2.4 Conclusion

The recent Kyoto Protocol has set reduction targets for pollutant emissions. Most developed countries have signed the protocol and thus committed to reduce their pollution generation. Furthermore, the protocol foresees that developing countries do not have

to comply to these reductions in a first instance. These two facts - developed countries having to reduce their pollution emissions, and developing countries being allowed to postpone their reduction - seem paradoxical in view of the literature studying the relationship between pollution and growth. solving thus the above-mentioned apparent paradox, it leaves open the more fundamental question of the underlying mechanism explaining the shape of the EKC.

Building on Ramsey's Benthamian utilitarian approach, we use a linear utility function. In this case, the "bliss point" will concern consumption levels and environmental quality, which will be considered simultaneously by the planner. We are able to derive conditions under which sustainable development is reached. However a country may reach a disastrous path where the environmental quality's lower bound is reached in finite time. This case corresponds to a situation of over-investment. Furthermore, and contrary to most vintage capital models, the transition dynamics are explicitly expressed in terms of endogenous scrapping age, which is a worthwhile contribution to the related literature.

The level of investment plays a crucial role in shaping the pollution-output ratio: this relationship can be upward sloping, downward sloping, or flat, according to whether countries are over or under-investing, i.e. whether investment rate is larger or smaller than the (cleaner) technology improvement rate.

2.5 Appendix

2.5.1 Proof of Proposition 2.3

In this appendix, we deduce the explicit form of $T(t)$, given the initial $T(0)$, and the optimal conditions (2.7) and (2.8).

Noting that equation (2.8) is a first order linear differential equation with initial condition $\mu(0) = \beta e^{-\gamma T(0)}$, we get

$$\mu(t) = e^{(\rho+q)t} \left[\beta e^{-\gamma T(0)} + \frac{1-\beta}{\rho+q} (1 - e^{-(\rho+q)t}) \right].$$

Combing this result of $\mu(t)$ with equation (2.7), we have

$$e^{-\gamma T(t)} = e^{(\rho+q-\gamma)t} \left[e^{-\gamma T(0)} + \frac{1-\beta}{\beta(\rho+q)} (1 - e^{-(\rho+q)t}) \right].$$

Hence,

$$T(t) = T(t; T(0)) = t - \frac{\rho+q}{\gamma} t - \frac{1}{\gamma} \ln \left(e^{-\gamma T(0)} + \frac{1-\beta}{\beta(\rho+q)} (1 - e^{-(\rho+q)t}) \right).$$

We can easily check that

$$\frac{\partial T(t; T(0))}{\partial T(0)} = \frac{e^{-\gamma T(0)}}{e^{-\gamma T(0)} + \frac{1-\beta}{\beta(\rho+q)} (1 - e^{-(\rho+q)t})} > 0,$$

and

$$T'(t) = 1 - \frac{\rho+q}{\gamma} - \frac{1}{\gamma} \frac{\frac{1-\beta}{\beta} e^{-(\rho+q)t}}{e^{-\gamma T(0)} + \frac{1-\beta}{\beta(\rho+q)} (1 - e^{-(\rho+q)t})} < 0. \diamond$$

2.5.2 Proof of Proposition 2.4: Output, pollution and $T(0)$

Using results from above, we show that

$$\frac{\partial^2 T(t; T(0))}{\partial T^2(0)} = \frac{-\gamma e^{-\gamma T(0)} \frac{1-\beta}{\beta(\rho+q)} (1 - e^{-(\rho+q)t})}{\left(e^{-\gamma T(0)} + \frac{1-\beta}{\beta(\rho+q)} (1 - e^{-(\rho+q)t}) \right)^2} < 0.$$

By the definition of $y(t)$ and $P(t)$, we straightforwardly get

$$\frac{\partial y(t; T(0))}{\partial T(0)} = i(t - T(t)) \frac{\partial T(t; T(0))}{\partial T(0)} > 0,$$

$$\frac{\partial P(t; T(0))}{\partial T(0)} = i(t - T(t)) e^{-\gamma(t-T(t))} \frac{\partial T(t; T(0))}{\partial T(0)} > 0,$$

$$\frac{\partial^2 y(t; T(0))}{\partial T^2(0)} = i(t - T(t)) \frac{\partial^2 T(t; T(0))}{\partial T^2(0)} - i'(t - T(t)) \left(\frac{\partial T(t; T(0))}{\partial T(0)} \right)^2 < 0,$$

due to assumptions $i(t) \geq 0$ and $i'(t - T(t)) \geq 0$. Furthermore,

$$\begin{aligned} \frac{\partial^2 P(t; T(0))}{\partial T^2(0)} &= e^{-\gamma(t-T(t))} \left[i(t - T(t)) \frac{\partial^2 T(t; T(0))}{\partial T^2(0)} - i'(t - T(t)) \left(\frac{\partial T(t; T(0))}{\partial T(0)} \right)^2 \right. \\ &\quad \left. + \gamma i(t - T(t)) \left(\frac{\partial T(t; T(0))}{\partial T(0)} \right)^2 \right] \\ &= e^{-\gamma(t-T(t))} \left[\frac{\partial^2 y(t; T(0))}{\partial T^2(0)} + \gamma i(t - T(t)) \left(\frac{\partial T(t; T(0))}{\partial T(0)} \right)^2 \right] > \frac{\partial^2 y(t; T(0))}{\partial T^2(0)}. \end{aligned}$$

In order to prove the last inequality, suppose the above direction of this inequality is incorrect. Then we have, for some $t > T(t)$,

$$e^{-\gamma(t-T(t))} \left[\frac{\partial^2 y(t; T(0))}{\partial T^2(0)} + \gamma i(t - T(t)) \left(\frac{\partial T(t; T(0))}{\partial T(0)} \right)^2 \right] \leq \frac{\partial^2 y(t; T(0))}{\partial T^2(0)},$$

and rearranging the terms, we obtain that

$$e^{-\gamma(t-T(t))} \gamma i(t - T(t)) \left(\frac{\partial T(t; T(0))}{\partial T(0)} \right)^2 \leq (1 - e^{-\gamma(t-T(t))}) \frac{\partial^2 y(t; T(0))}{\partial T^2(0)},$$

which is impossible for $t > T(t)$, given the left hand side is non-negative and the right hand side is strictly negative. $\diamond$

2.5.3 Explicit solution of the environmental equation (2.10)

Equation (2.10) can be rewritten

$$E(t) = \begin{cases} \xi(t^*) + \frac{\chi}{\theta(\gamma - \theta)} [e^{-\theta t} - e^{-(q\tau + \theta t^*)}] , & \text{if } 0 < g \neq \gamma, g \neq \gamma - q; \\ \xi(t^*) + \frac{\chi}{\theta} e^{-qt} \tau, & \text{if } 0 < g = \gamma - q; \\ \xi(t^*) + \frac{\psi}{-\gamma(\gamma - q)} [e^{-\gamma t} - e^{-(q\tau + \gamma t^*)}] , & \text{if } g = 0, q \neq \gamma; \\ \xi(t^*) + \frac{\psi}{\gamma} e^{-qt} \tau, & \text{if } g = 0, q = \gamma; \\ \xi(t^*) + \frac{\bar{i}T^*}{q} [1 - e^{q\tau}] , & \text{if } 0 < g = \gamma. \end{cases} \quad (2.A.1)$$

where $\xi(t^*) = E(t^*)e^{-qt}$, $\chi = \bar{i}(1 - e^{-(g-\gamma)T^*})$, $\psi = \bar{i}(1 - e^{\gamma T^*})$, $\theta = \gamma - g$, and $\tau = t - t^*$.

2.5.4 Proof of Proposition 2.5: Pollution-output decreasing ratio

Denote

$$R(t) = \frac{P(t)}{y(t)} = \frac{\int_{t-T(t)}^t y(z) e^{-\gamma z} dz}{\int_{t-T(t)}^t y(z) dz}.$$

The derivative of $R(t)$ with respect to t is,

$$\begin{aligned} R'(t) &= \frac{1}{y^2(t)} \left[y(t) \left(\int_{t-T(t)}^t y(z) dz e^{-\gamma t} - \int_{t-T(t)}^t y(z) e^{-\gamma z} dz \right) \right. \\ &\quad \left. - y(t - T(t))(1 - T'(t)) \left(\int_{t-T(t)}^t y(z) dz e^{-\gamma(t-T(t))} - \int_{t-T(t)}^t y(z) e^{-\gamma z} dz \right) \right] \\ &= \frac{1}{y^2(t)} \left[y(t) \int_{t-T(t)}^t y(z) (e^{-\gamma t} - e^{-\gamma z}) dz \right. \\ &\quad \left. - y(t - T(t))(1 - T'(t)) \int_{t-T(t)}^t y(z) (e^{-\gamma(t-T(t))} - e^{-\gamma z}) dz \right] \\ &< 0, \end{aligned}$$

where the last inequality comes from the fact that $T'(t) < 0$, and the fact that for $t - T(t) < z < t$, we have $e^{-\gamma t} < e^{-\gamma z} < e^{-\gamma(t-T(t))}$. $\diamond$

Chapter 3

Vintage Capital, Optimal investment and Technology Adoption

3.1 Introduction

As we notice from previous two chapters that the purely theoretical framework of Boucekkine, Germain, and Licandro(1997) are successful in interpretation some of the endogenous growth issues, by using an endogenous scraping process and an exogenous technology change. However, the framework and approach are still more theoretical. A common assumption to these models is that investment is only allowed for the new vintage capital, a traditional assumption in the literature (see Malcomson(1975)). As a result, there is immediate new technology diffusion.

In this chapter, we would like to go one step further and we tackle the issue *whether it is possible to build a market equilibrium vintage technology model, in which investment is allowed in both new and existing vintage technological goods?* We first generalize the previous partial equilibrium setting and, second, incorporate the adoption problem into the vintage framework.

So, instead of a technological leader that innovates and determines the exogenous path of technological progress for potential followers, we find an exogenously determined path

of technical progress embodied in new vintage technology. Then the traditional problem (for followers) on whether to adopt new technologies or not, arises here as the problem on whether to invest in new vintage technological goods or not. Moreover, the other important decision concerning capital accumulation appears here transformed into the problem of how much to invest in existing technological capital goods.

In this investment process, convex adjustment costs were considered. In general equilibrium literature, convex adjustment costs were introduced to help smooth the behavior of aggregate investment¹. Able(2002) shows that with linear homogenous convex adjustment costs it is possible to retain endogenous growth under AK framework, though the growth rate in the long-run is reduced due to the adjustment cost². However, without the property of linear homogeneity in the convex adjustment cost functions, we find that, it is impossible to obtain growth in the long-run.

This chapter is arranged as follows. In Section 2, a vintage capital model is introduced under a general equilibrium setting, while Section 3 shows Pontryagin's condition and its sufficiency. Section 4 provides the analysis of equilibria, technology diffusion and investment path.

3.2 The Decentralized Model

Suppose population is constant and there is only one final good, which can be assigned to consumption or (net) investment and plays the role of numeraire. The final good is produced in a competitive market by a constant returns to scale technology.

We presume that markets exist for technology of age $v \in (0, \bar{v})$, where $\bar{v}$ is the age of the oldest technology in use. After the adjustment cost (which is defined later) is paid, firms will not return to use the old technology again, and firms, that do not pay, face the problem of either continuing using old technology or switching to new technology by paying the adjustment cost.

¹See for example, Abel and Blanchard(1983) and Baxter and Crucini(1993)

²Special thanks to the anonymous referee noticing this point.

Per capita output $y(t)$ is produced by an accumulated capital stock, where different age of technologies co-exist at the same time with different productivity,

$$y(t) = \int_0^{\bar{v}} a(t, v)k(t, v)dv. \quad (3.1)$$

where $k(t, v)$ is per capita capital stock at time t with vintage technology $v \in [0, \bar{v})$ and productivity $a(t, v)$.

Basing on different ages of technologies, we impose the following assumptions on the productivity functions.

Assumption A.1 *For any $t \geq 0$, $0 \leq v \leq \bar{v}$, there exists a positive constant M , such that,*

- 1° $0 \leq a(t, v) < M$, i.e., positive and bounded productivity can ensure the convergence of integral (3.1);
- 2° $\frac{\partial a(t, v)}{\partial v} < 0$, that is, “young” vintage technologies are more efficient than old ones.

In addition, the law of motion of per-capita capital stock $k(t, v)$ of a given technology of age $v(0 \leq v \leq \bar{v})$ at time t , satisfies the following first order partial differential equation³

$$\begin{cases} \frac{\partial k(t, v)}{\partial t} + \frac{\partial k(t, v)}{\partial v} = i(t, v) - \delta k(t, v), & t \in (0, \infty), v \in (0, \bar{v}], \\ k(t, 0) = i_0(t), & t \in (0, \infty), \\ k(0, v) = k_0(v), & v \in (0, \bar{v}], \end{cases} \quad (3.2)$$

where $\delta(\geq 0)$ is the natural depreciation rate of capital, $k_0(v)$ is the stock of capital at time 0, with technology v , $i(t, v)$ is gross investment in the existing vintage technology

³The partial differential equation expresses the transition equation generalized to the classical dynamical system $\dot{k}(t) = i(t) - \delta k(t)$. From the vintage capital point of view, during time period Δt at given time t , for given vintage technology v , the stock of capital will change from $k(t, v)$ to $k(t + \Delta t, v + \lambda \Delta t)$, where constant $\lambda \geq 0$ describes the relation between time t and vintage capital v . Simple manipulations allows us to impose $\lambda = 1$.

v at time t , that is, *the process of improving the quality of investment*, and $i_0(t)$ is gross investment in new technological goods at time t , that is, *expanding variety of investment*. Here, following Boucekkine et al (1997), we impose the ‘no hole’ assumption of the initial capital stock distribution,

$$\begin{cases} k_0(v) \geq 0, \text{ given } , \forall 0 < v \leq \bar{v}, \\ \int_0^{\bar{v}} k_0(v) dv > 0. \end{cases} \quad (4.a)$$

In other words, initially in this economy there must be some capital stock that can be used for some periods (second inequality), though that does not require that all existing technologies appear in this economy (first inequality).

The decentralized equilibrium model can be written as follows:

$$\max_{c(t)} \int_0^{\infty} c(t) e^{-\rho t} dt, \quad (3.3)$$

subject to the budget constraint, for any $t \geq 0$,

$$\begin{aligned} c(t) &= y(t) - \int_0^{\bar{v}} (l(t, v)i(t, v) + \alpha(t, v)i^2(t, v)) dv \\ &\quad - (l_0(t)i_0(t) + \alpha_0(t)i_0^2(t)), \end{aligned} \quad (3.4)$$

where $\rho(> 0)$ is the time preference parameter. The market clearing condition is given in (3.4). Thus the maximization programme consists in finding a per capita consumption profile $c(t)$, $\forall t > 0$, to solve a standard maximization problem with a linear instantaneous utility function⁴, As in Gould(1968), $l(t, v)i(t, v) + \alpha(t, v)i^2(t, v)$ represents the adjustment cost of gross investment of existing vintage technologies and $l_0(t)i_0(t) + \alpha_0(t)i_0^2(t)$ stands for the adjustment cost of gross investment of new vintage technologies. Lucas(1967) suggests that this cost behavior can be thought of as a sum of purchase costs (which is

⁴In order to obtain explicit solutions, we avoid more general utility functions. While general utility functions, would allow us to write down optimal conditions, the equilibrium conditions for such an economy would give rise to a mixed-delay differential equation system with endogenous leads and lags (see Boucekkine et al., (1997), (2005)).

extrinsic) and installation costs (which is *intrinsic*) to the firm. Following the assumption of Gould(1968) we take this (in general convex) adjustment costs as a quadratic function of gross investment, which leads to an explicit solution (see also Barucci and Gozzi (2001) and Feichtinger et al. (2001)).

For these known functions, the following assumptions are presumed.

Assumption A.2 *For every $0 \leq v \leq \bar{v}$ and for a fixed $\epsilon > 0$, we have*

(I)

$$l(t, v) \geq 0, \quad l(t, \bar{v}) = 0, \quad \text{and} \quad \alpha(t, v) \geq \epsilon,$$

(II)

$$\frac{\partial \alpha(t, v)}{\partial v} \leq 0, \quad \frac{\partial l(t, v)}{\partial v} \leq 0.$$

(III)

$$l_0(t) > \lim_{v \rightarrow 0} l(t, v), \quad \alpha_0 > \lim_{v \rightarrow 0} \alpha(t, v) \geq \epsilon.$$

Assumption A.2 indicates that there are positive adjustment costs for new and existing vintage technological capital goods, and the oldest technology is worthless (inequalities (I)). Moreover, ‘young’ vintage technologies are more expensive than those using ‘old’ technologies (inequalities (II)), and finally, new technologies are the most expensive (III) and are strictly more expensive than the existing ones.

Remark 1. Mathematically, from (5.8), $i(t, v)$ does not continue up to the boundary condition $i_0(t)$ at $v = 0$. As a result, the solution of (4) for $v > 0$ does not increase up to the boundary. Hence, this process has no accumulation of capital stocks equipped with new technology.

Before further solving the optimal control problem, we introduce some mathematical notation. Let $k(\cdot, v) : [0, \infty) \rightarrow L^2((0, \bar{v}]; \mathbb{R})$, where $L^2((0, \bar{v}]; \mathbb{R})$ denotes space of square integral functions, that is, for any fixed $v \in (0, \bar{v}]$,

$$(k(v))(t) = k(t, v) \in L^2(\mathbb{R}), \quad \int_0^\infty |k(t, v)|^2 dt < \infty.$$

Formally, (5.8) can be written as

$$\begin{cases} k'(t) = \mathcal{A}k(t) + i(t) + \delta_0 i_0(t), & t \in (0, \infty) \\ k(0) = k_0, \end{cases} \quad (3.5)$$

where $\mathcal{A} : L^2 \rightarrow L^2$ is a suitable operator defined as:

$$(-\mathcal{A}k(t))(v) = \frac{\partial k(t, v)}{\partial v} + \delta k(t, v), \text{ for any fixed } t,$$

and δ_0 is Dirac's delta at point 0,

$$\delta_0 = \begin{cases} 1, & \text{at } 0, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting (3.1) into (3.4), we have

$$\begin{aligned} c(t) &= \int_0^{\bar{v}} [a(t, v)k(t, v) - (l(t, v)i(t, v) + \alpha(t, v)i^2(t, v))] dv \\ &\quad - (l_0(t)i_0(t) + \alpha_0(t)i_0^2(t)). \end{aligned} \quad (3.6)$$

Substituting (3.6) into (5.7), the optimal control problem is equivalent to looking for optimal investment profiles $i(t, v), i_0(t)$

$$\begin{aligned} \max \quad & \int_0^\infty \left[\int_0^{\bar{v}} (a(t, v)k(t, v) - l(t, v)i(t, v) - \alpha(t, v)i^2(t, v)) dv \right. \\ & \left. - l_0(t)i_0(t) - \alpha_0(t)i_0^2(t) \right] e^{-\rho t} dt, \end{aligned} \quad (3.7)$$

subject to (3.5).

3.3 Investment Strategy and Pontryagin Condition

In this section, we are going to show explicitly the Tobin's q , and express explicitly the investment strategy through this q . We close this section by clearly expressing the capital stock due to the linear utility and quadratic adjustment cost functions.

Before introducing the value function, let us denote that $I_{\{v \geq t\}}$ is an indicator function of interval $v \geq t$, and $\mathcal{A}^*(: L^2 \rightarrow L^2)$ is an adjoint operator of $\mathcal{A}$, that is,

$$(\mathcal{A}^*k(\tau))(v) = \frac{\partial k(\tau, v)}{\partial v} - \delta k(\tau, v), \text{ for any fixed } t > 0.$$

Define the current value function of the maximization problem (3.7) and (3.5) as

$$\begin{aligned} V(t; k) &= \int_t^\infty \left[\int_0^{\bar{v}} (ak(\tau, v) - li(\tau, v) - \alpha i^2) dv - l_0 i_0(\tau) - \alpha_0 i_0^2(\tau) \right] e^{-\rho\tau} d\tau \\ &\quad - \int_t^\infty \langle q(\tau, v), k_t - (\mathcal{A}k(\tau) + i(\tau) + \delta_0 i_0(\tau)) \rangle_v e^{-\rho\tau} d\tau, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product, and q is costate variable. The first order conditions of the above value function with respect to the state variable states that

$$\begin{cases} \frac{\partial q(t, v)}{\partial v} = (\rho - \mathcal{A}^*)q(t, v) - a(t, v), \\ \lim_{t \rightarrow \infty} q(t, v)e^{-\rho t} = 0, \end{cases} \quad (3.8)$$

where the second equation serves as the transversality condition.

The above q is indeed Tobin's q , which presents the shadow value of capital. Through Tobin's q , we can obtain (by first order condition with respect to investment profiles) gross investments $i(\tau, v)$, and $i_0(\tau)$ as follows:

$$i(t, v) = \frac{q(t, v) - l(t, v)}{2\alpha(t, v)}, \quad (3.9)$$

$$i_0(t) = \frac{q(t, 0) - l_0(t)}{2\alpha_0(t)}. \quad (3.10)$$

The above analysis leads to the following results.

Proposition 1 (Pontryagin Condition and Optimal Investment Strategy). *The costate equation (shadow value of capital) given by (3.8) is Pontryagin's necessary condition, and the solution of this equation is*

$$\begin{aligned} q(t, v) &= \int_t^\infty e^{-\rho(\tau-t)} e^{-\delta(\tau-t)} a(\tau, v + (\tau - t)) I_{[0, \bar{v} - (\tau-t)]} d\tau \\ &= \int_v^{\bar{v}} e^{-(\rho+\delta)(\tau-v)} a(\tau - v + t, \tau) d\tau. \end{aligned} \quad (3.11)$$

Under assumption A.1 and A.2, at time t , there exists a unique optimal investment strategy $(i^*(t, v), i_0^*(t))$ for the optimal control problem. This strategy is given by (3.9) and (3.10).

The optimal investment strategy does not depend on the initial distribution of capital stock. The explicit solution of shadow value of capital, $q(t, v)$, depends on the time preference of consumers (*household profiles*), depreciation rate of capital and on the productivity of capital (*firm profiles*). With linear utility, this shadow value of capital is the real value of capital. At any time $t > 0$, $q(t, v)$ is the value of investment in a vintage technology $0 \leq v \leq \bar{v}$, equipped with productivity $a(t, v)$.

Because $\lim_{v \rightarrow \bar{v}} q(t, v) = 0$, once one technology is reaching its scrapping age, it has no value any more. Combining this and the fact that $l(t, \bar{v}) = 0$, we can assert that the investment in the oldest technology is zero.

Nonetheless, it is still possible that for an individual firm investment is negative, that is to say for $0 < v < \bar{v}$, $q(t, v) < l(t, v)$. This is the case of disinvestment and can be obtained when the capital is sold off (Gould (1968)). However, for an *aggregate economy* as in our case, this does not make sense, since there is no firm, who is willing to buy this costly capital stock. In order to avoid this disinvestment case, we follow Boucekkine, Germain, and Licandro (1997) who make the ‘piecewise continuous’ assumption on the productivity functions and adjustment cost functions.

Assumption A.3 (‘piecewise continuous’ assumption)

In the following, 1° always holds and at least 2° or 3° holds as well, where

1°

$$0 < \alpha(t, v), \alpha_0(v) \leq \bar{\alpha} < \infty,$$

$$q(t, v) = \int_v^{\bar{v}} e^{-(\rho+\delta)(\tau-v)} a(\tau - v + t, \tau) d\tau \geq l(t, v), \quad \forall v \in (0, \bar{v}],$$

$$q(t, 0) = \int_0^{\bar{v}} e^{-(\rho+\delta)\tau} a(\tau + t, \tau) d\tau \geq l_0(t);$$

2° There exists an interval $X \subset (0, \bar{v}]$, such that,

$$q(t, v) = \int_v^{\bar{v}} e^{-(\rho+\delta)(\tau-v)} a(\tau - v + t, \tau) d\tau > l(t, v), \quad \forall v \in X;$$

3°

$$q(t, 0) = \int_0^{\bar{v}} e^{-(\rho+\delta)\tau} a(\tau + t, \tau) d\tau > l_0(t).$$

The economic intuition of Assumption A.3 is the following. Any kind of technology adoption has finite adjustment costs and any existing vintage technology's value is no less than its purchasing cost (1°). While, if the value of gross investment in any existing technology is not strictly higher than the purchasing cost $l(t, v)$, $v \in (0, \bar{v})$, then there must be a new technology, with much higher productivity and the value of this new technology is strictly higher than the investment cost (2° and 3°).

It could be the case that, for some technology $v \in (0, \bar{v})$,

$$q(t, v) = \int_v^{\bar{v}} e^{-(\rho+\delta)(\tau-v)} a(\tau - v + t, \tau) d\tau = l(t, v),$$

because of the exogenous constant lifetime assumption (positive constant $\bar{v}$). When the technologies are no longer young, and if the purchasing costs are the same, after discounting the productivity, the value can only compensate this cost and nothing is left anymore.⁵ If this is the case, either we have to invest in new technological capital goods with higher productivity to replace the old ones, or adjust the assumption of lifetime $\bar{v}$, or both. In the present work, we consider the first case.

With Assumption A.1 and A.3, it is easy to check that the optimal gross investment given by (3.9) and (3.10) are non-negative and finite.

Solving partial differential state equation (5.8) (see Barucci and Gozzi (2001)), we obtain,

$$k(t, v) = e^{-\delta t} k_0(v - t) I_{\{v \geq t\}} + e^{-\delta v} i_0(t - v) I_{\{v < t\}} + \int_0^{t \wedge v} e^{-\delta \tau} i(t - \tau, v - \tau) d\tau, \quad (3.12)$$

with $t \wedge v = \min\{t, v\}$ and $i(t, v)$, $i_0(v)$ are given by (3.9) and (3.10).

⁵This is the case in the previous footnote, $\lambda > 1$, which means that this technology quickly outdates.

The above assumption A.3 implies that at least there exists $v_0 \in [0, \bar{v}]$, such that,

$$k(t, v_0) > 0, \quad \int_0^{\bar{v}} k(t, v) dv > 0.$$

Actually this productivity assumption is a key assumption that drives the economy, and provides the incentive to invest in new and more productive capital goods. Hence, in developed economies, investing in new and scrapping old vintage capital goods are necessary to keep the economy developing. Investing in the new and young technology, though they are more costly than old ones, is a kind of saving for the whole economy in long run perspective.

This leads us to the following propositions.

Proposition 2 (Necessary and sufficient condition). *Suppose that Assumptions A.1, A.2 and A.3 hold, then the above Pontryagin's conditions are not only necessary, but also sufficient for the original optimal control problem.*

The proof of sufficiency is given in Appendix 1.

Proposition 3. *Let Assumption A.3 hold. For any given investment strategy $i(t, v), i_0(t)$, there exists a unique solution of the capital accumulation equation (5.8), given by (3.12).*

Remark 2. If in one economy, there is only one market for new capital goods, we can assume $\alpha(t, v) = 0$, $\alpha_0(t) = 0$, and $l(t, v) \geq \epsilon > 0$, $l_0(t) \geq \epsilon > 0$, for any $t > 0$, and keep all the other conditions in Assumption A.2, then we can rebuild the results of Malcomson (1975) with exogenous constant scrapping rule, in a general equilibrium setting.

Additionally, choosing the optimal investment strategies above, there is an optimal capital stock.

Corollary 1. *Suppose that Assumption A.1, A.2 and A.3 hold, then there is a unique optimal investment strategy $(i^*(t, v), i_0^*(t))$, given by (3.9), (3.10), which is non-negative for $t > 0$ and $0 \leq v < \bar{v}$. Moreover, there is an optimal stock of capital given by,*

$$k^*(t, v) = e^{-\delta t} k_0(v - t) I_{\{v \geq t\}} + e^{-\delta v} i_0^*(t - v) I_{\{v < t\}} + \int_0^{t \wedge v} e^{-\delta \tau} i^*(t - \tau, v - \tau) d\tau.$$

3.4 Steady State

Boucekkine, Licandro, Puch and del Rio(2005) prove that without adjustment costs, vintage structures of AK -type technologies still lead with certainty to endogenous growth as in the standard neo-classical growth model. While using adjustment costs, provided they are not too large, Abel(2002) showed that growth may be endogenous in an AK model. However, one of the essential conditions in Abel (2002) is that the convex adjustment cost function must be linear homogenous.⁶ We can not gauge whether the quadratic adjustment cost is too large in the sense of Abel, but the quadratic function in our setting is obviously not linear homogenous. Nonetheless, the introduction of quadratic adjustment cost kills the engine of growth, due to the fact that the quadratic term breaks the balance of the market clearing equation (3.4).

Gould (1968, page 48) notices “ If the cost of adjustment function is such that average costs are greater for higher levels of I (gross investment) this will introduce dis-economies of scale thereby giving the firm a determinate size even if the production function is homogenous of degree one”.

Due to the above analysis and the constant population assumption in this aggregate economy, output and consumption will have a determinate size as well. It is easy to check that if the coefficients of quadratic terms go to zero, that is, $\alpha(t, v) = 0, \alpha_0(t) = 0$, then there is endogenous growth as demonstrated by Boucekkine et al.(2005).

Therefore in this section, we study the properties *around the equilibria*. For the exogenous functions, there could be two cases: time independent and time dependent. Indeed, Yorukoglu (1998, page 552) notes in his paper that IBM introduced its pentium PCs in the early 1990s at the same price as it introduced its 286 PC in the 1980s. Therefore it took less than a decade for the computing technology to improve on the order of 20 years in terms of both speed and memory capacities, without increasing the cost.

In this case, we may presume that the exogenous cost functions are time independent, except the productivity, which is an increasing function over time t . Additionally, we

⁶Actually, in an AK -model with adjustment cost, to guarantee endogenous growth, linear homogeneity of convex adjustment costs is one of the necessary and sufficient conditions.

will follow the belief that the technology is exponentially increasing with time t , that is, $a(t, v) = e^{a_1 t} a_2(v)$, where function $a_2(v)$ satisfies Assumption A.1 and $0 < a_1 < \rho$ (to assure the convergence of the integral). But all the functions related to adjustment costs only depend on the type of vintage technology v , rather than time t . Hence, we assume that $l(t, v) = l(v)$, $\alpha(t, v) = \alpha(v)$ for any $t \geq 0$ and $v \geq 0$, and l_0, α_0 are constants. Moreover the other conditions in Assumption A.2 and A.3 hold. Simple calculation from (3.9), (3.10), (3.8) and (3.12), lead to the shadow price of capital, and optimal investment as follows,⁷

$$\begin{cases} q^*(t, v) = \int_v^{\bar{v}} e^{-(\rho+\delta)(\tau-v)} a(\tau - v + t, \tau) d\tau \\ \quad = e^{(\rho+\delta-a_1)v} e^{a_1 t} \int_v^{\bar{v}} e^{-(\rho+\delta-a_1)\tau} a_2(\tau) d\tau, \\ q^*(t, 0) = e^{a_1 t} \int_0^{\bar{v}} e^{-(\rho+\delta-a_1)\tau} a_2(\tau) d\tau. \end{cases} \quad (3.13)$$

$$i^*(t, v) = \frac{q^*(t, v) - l(v)}{2\alpha(v)}, \quad i_0^*(t) = \frac{q^*(t, 0) - l_0}{2\alpha_0}, \quad (3.14)$$

and the capital accumulation is

$$\begin{cases} k^*(t, v) = e^{-\delta t} k_0(v - t) + \int_0^t e^{-\delta \tau} i^*(t - \tau, v - \tau) d\tau, & t \leq v, \\ k^*(t, v) = e^{-\delta v} i_0^*(t - v) + \int_0^v e^{-\delta \tau} i^*(t - \tau, v - \tau) d\tau, & t > v, \end{cases} \quad (3.15)$$

By the assumption of time independence of cost functions, it is easy to see that in (3.13) and (3.14), both the shadow value of capital and investments in the new and existing vintage capital goods are increasing functions with respect to time t for any fixed vintage capital. But even though the increasing investment in vintage technological capital goods does not match the classical results in the literature (see Boucekkine et al (1997), and Benhabib et al (1991)), they are rather consistent with the neoclassical growth model. The key reason is the time independent adjustment costs, which in fact have negative effects compared to the positive effects of the productivity of capital. It is not difficult to

⁷If the productivity function is also time independent, then in deed the steady state (determinate size of firm) is reached, so do the shadow value, output and consumption.

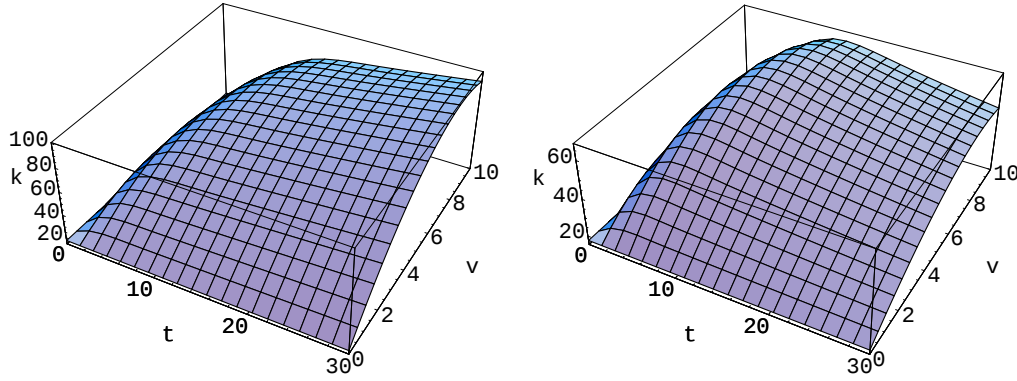


Figure 3.1: Left: Monotonic with time. Right: Non-monotonic with time

see (from (3.14)) that, if we also assume that the costs are increasing functions of time t (as Solow (1959)), then we lose the monotonicity of investment, see Example 1.

Example 1 Taking $\delta = 0.06$, $\rho = 0.04$, lifetime $\bar{v} = 10$, initial capital $k_0(v) = 1 - 0.1v$, new equipment costs are constant with respect to time, that is, $\alpha_0 = 0.5$, $l_0 = 3.98$. Moreover in the left figure, we take existing equipment costs are time independent, $\alpha(t, v) = 0.5e^{-0.1v}$ and $l(t, v) = 0.398(10 - v)$, but productivity function depends on time $a(t, v) = 4e^{0.01t}e^{-0.1v}$, we get a monotonic investment path and a monotonic capital accumulation. In the right Figure, we take $\alpha(t, v) = 0.125e^{0.03t-0.1v}$, $l(t, v) = 0.398(10 - v)$ and $a(t, v) = 4e^{0.01t}e^{-0.1v}$, then it is easy to see that there is a non-monotonic investment path and also the capital accumulation is not monotonic.

Actually, under the assumption of time dependent productivity and time independent cost functions, for the capital accumulation, our results are richer than in the literature so far. For existing vintage capital goods, if $t \leq v$, we have that

$$\begin{aligned} \frac{\partial k^*(t, v)}{\partial t} &= a_1 \int_0^t e^{-\delta\tau} \left[\frac{e^{-\delta\tau} q^*(t - \tau, v - \tau)}{2\alpha(v - \tau)} \right] d\tau \\ &\quad - e^{-\delta t} [\delta k_0(v - t) + k'_0(v - t)] + e^{-\delta t} \frac{q^*(0, v - t) - l(v - t)}{2\alpha(v - t)}. \end{aligned} \quad (3.16)$$

One can prove that for any fixed $v > 0$, and $t \leq v$, the second term in (3.16) is non-

negative (see Appendix 2) , and the first and the last terms are positive. As a result, the investment in the existing vintage capital has a positive effect on the capital accumulation, but with time passing, the old equipments storing slows down the capital accumulation process.

As for $v = 0$, we have

$$\frac{\partial k^*(t, 0)}{\partial t} = e^{-\delta v} (i^*)_0'(t - v) > 0,$$

and in the case for fixed v , satisfying, $0 < v < t$, we have that

$$\frac{\partial k^*(t, v)}{\partial t} = e^{-\delta v} (i_0^*)'(t - v) + \int_0^v e^{-\delta \tau} \frac{\partial i^*(t - \tau, v - \tau)}{\partial t} d\tau \geq 0. \quad (3.17)$$

That is, investment, in the new or young vintage capital good, will increase capital capacity.

We conclude the above analysis with Proposition 4.

Proposition 4 *For a given exponential productivity of capital and time independent cost function, there is a unique long-run optimal investment strategy, stationary steady state of capital accumulation and shadow value of capital, given by (3.14), (3.15) and (3.13).*

Furthermore, there is a monotonic investment path for new and existing vintage capital goods. Investing on existing vintage capital will always increase the capital accumulation, but the storing of old vintage capital goods will slow down this accumulation procedure.

Let us now study closer to the long-run steady state, that is $t > v$. We make the assumption that the following condition on the productivity and the adjustment cost functions is verified.

Assumption A.4

$$\frac{\frac{\partial q^*(t, v)}{\partial v} - l'(v)}{q^*(t, v) - l(v)} < \frac{\alpha'(v)}{\alpha(v)}, \text{ for any } v \in (0, \bar{v}]. \quad (3.18)$$

In order to give a clear economic idea of the above assumption, following Lucas(1967), we could interpret $l(v)$ as purchasing cost, and $\alpha(v)$ as installation cost. Remember the

Tobin's q , $q^*(t, v)$, is the real value in the linear utility case, thus the difference $q^*(t, v) - l(v)$ is in deed the possible profit, except installation cost, from investment in technology v . Hence the above Assumption A.4 indicates that investment in older technology, the growth rate of installation cost (*intrinsic factor*) with respect to technology age is higher than the growth rate of possible profit (*extrinsic factor*) over technology age. That is to say firms are reluctant to invest a lot in the old technologies, which directly leads to

$$\frac{di^*(t, v)}{dv} \leq 0, \text{ for any } v \in (0, \bar{v}], t > v.$$

With Assumption A.4, we have the following results (see proof in Appendix 3).

Proposition 5 *Let $t > v$, $0 < \delta < 1$, and Assumption A.4 hold. Suppose time dependent productivity and time independent cost functions, and there are vintage capital goods markets. We have that there exists a benchmark age $v^* \in (0, \bar{v})$, such that, at v^* , the investment in this vintage capital good compensates the depreciated capital ($i^*(t, v^*) = \delta k^*(t, v^*)$). Moreover, below this age, the optimal capital accumulation is an increasing function with respect to v , but above this age, the capital accumulation is a decreasing function of age. In other words, there is a single hump-shape of capital accumulation with respect to age. Hence there is slow new technology diffusion (see Figure 2).*

Furthermore, the optimal accumulation of vintage capital goods is increasing in terms of technology level a_1 , and decreasing with respect to $\delta, \rho, \alpha(v), l(v), l_0$ and α_0 .

Remark 3. As mentioned by Malcomson (1975), a firm buys only most recent vintage of equipment, (that is, only $v \rightarrow 0^+$ can happen), then there is immediate technology diffusion. In our model, there is a market, where vintage technological equipments are sold, therefore, there is not necessarily immediate technology diffusion, because of cost of information delay.

Example 2 We take the same functions as in Example 1, but changing is the direction of the axis.

From Figure 2, it is easy to see that there is single hump-shape of capital accumulation with respect to technology age in the two cases. Moreover at the steady state, there are

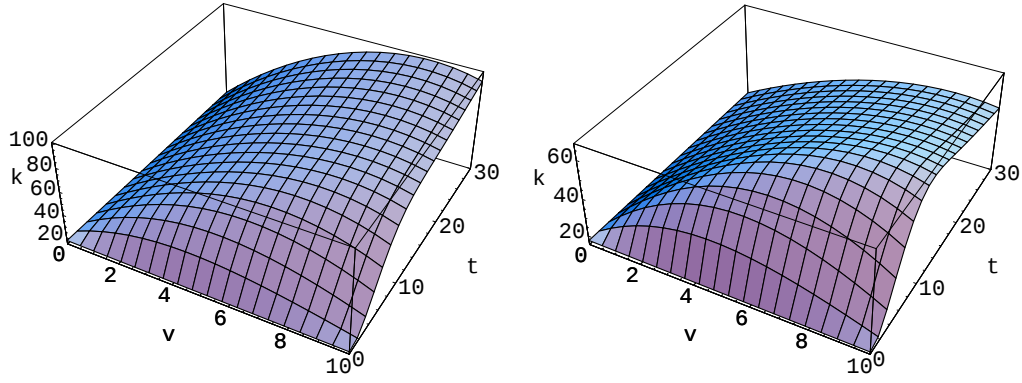


Figure 3.2: Single hum-shape of capital accumulation with age

less equipments with old or new technology than younger ones. Equipments with old technologies are no longer efficient enough, so they are scrapped out of the market. But the new ones are too expensive or need some time to be accepted by the market. Initially, there are some old equipments on the market, but with time passing, they are outdated and being scrapped off.

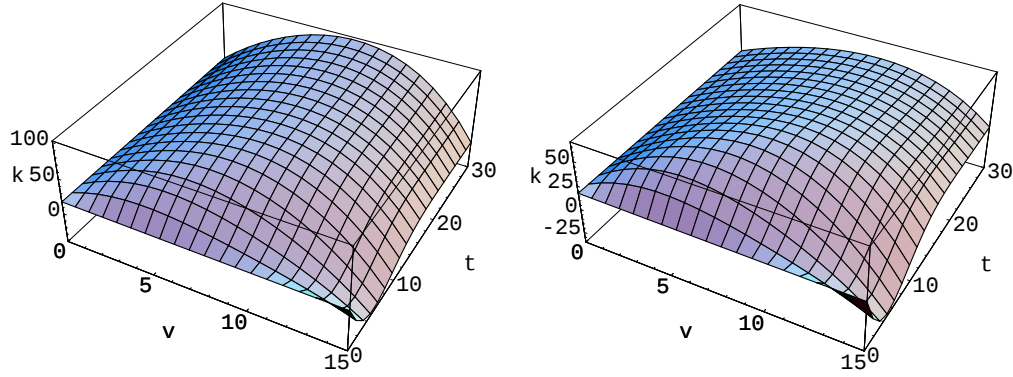


Figure 3.3: Prolong lifetime of vintage capital goods

Since both $q(t, v)$ and $k(t, v)$ are functions of lifetime $\bar{v}$, we can write $k(t, v), q(t, v)$ as $k(t, v; \bar{v}), q(t, v; \bar{v})$. Derivative $q(t, v; \bar{v})$ and $q(t, 0; \bar{v})$ with respect to lifetime $\bar{v}$, and sub-

stituting them into the partial derivative, $\frac{\partial k(t, v; \bar{v})}{\partial \bar{v}}$, we have that

$$\frac{\partial k(t, v; \bar{v})}{\partial \bar{v}} = \frac{e^{-\delta v} e^{-(\delta+\rho)\bar{v}} e^{a_1(t+\bar{v}-v)} a_2(\bar{v})}{2} \left[\int_0^v \frac{e^{(\rho+2\delta)\tau}}{\alpha(\tau)} d\tau + \frac{1}{\alpha_0} \right] > 0.$$

Hence increasing the lifetime of equipment will lead to increase the capital accumulation, i.e., less scrapping will save some capital. However, this procedure cannot always work out, since at the same time, we also have that

$$\begin{aligned} \frac{\partial^2 k(t, v; \bar{v})}{\partial \bar{v}^2} &= \frac{e^{-\delta v} e^{-(\delta+\rho-a_1)\bar{v}} e^{a_1(t-v)}}{2} \left[\int_0^v \frac{e^{(\rho+2\delta)\tau}}{\alpha(\tau)} d\tau + \frac{1}{\alpha_0} \right] \\ &\quad \times [-(\delta + \rho - a_1) a_2(\bar{v}) + a_2'(\bar{v})] < 0, \end{aligned}$$

due to the fact that, $a_1 < \rho$ and Assumption A.1 of a_2 .

As a conclusion, we have the following result.

Proposition 6 *Slowing down the exogenous scrapping rule will not reduce capital capacity, but keeping using old vintage technology will hinder the development of the economy. Hence, replacing the old technology by the new and more efficient ones is a process of creative destruction.*

Actually, this fact was already noticed by Salter (1960, page 72–73),

“In fact there is some evidence to suggest that one of the chief reasons for Anglo-American productivity differences lies in standards of obsolescence. It is a common theme in the Productivity Mission Reports that the productivity of the best plants in the United Kingdom is comparable with that of the best plants in the United States, and that the difference lies in a much higher proportion of plants employing outmoded methods in the United Kingdom— a much greater ‘tail’ of low-productivity plants. Such a situation is consistent with a higher standard of obsolescence in the United State which follows from a higher level of real wages .”

Camacho and Zou (2004) also showed that the region with higher technology will always be leading and this difference will not disappear until the same technology is employed everywhere.

Proposition 7 *For given time dependent productivity and time independent cost functions, there is unique long-run stationary steady state of optimal consumption, which is given by*

$$\begin{aligned} c^*(t) &= \int_0^{\bar{v}} \int_0^v a(t, v) e^{-\delta(v-\tau)} \frac{q^*(t-v+\tau, \tau) - l(\tau)}{2\alpha(\tau)} d\tau dv \\ &+ \int_0^{\bar{v}} a(t, v) e^{-\delta v} \frac{q^*(t-v, 0) - l_0}{2\alpha_0} dv \\ &- \int_0^{\bar{v}} \frac{(q^*)^2(t, v) - l^2(v)}{4\alpha(v)} dv - \frac{(q^*)^2(t, 0) - l_0^2}{4\alpha_0}. \end{aligned} \quad (3.19)$$

Moreover Assumption A.2 and $a_1 < \rho$ are sufficient conditions to achieve the interior solution of the above optimal consumption.

Furthermore, increasing the purchasing (or learning) costs, will harm the optimal consumption, while the installation costs of new and existing vintage capital goods have ambiguous effects on consumption.

Proof By substituting optimal investment and optimal capital accumulation (3.14) and (3.15) into the consumption function (3.4), we get (3.19).

The proof of admissible consumption $c(t) > 0$ is given in Appendix A4.

In that proof, we can see that Assumption A.2 and $a_1 < \rho$ are sufficient conditions for the interior solution of consumption (and not only for optimal consumption).

In the following we will study how the adjustment costs affect the optimal consumption.

Differentiating (3.19) with respect to the adoption cost of new technology, we have

$$\begin{aligned} \frac{\partial c^*(t)}{\partial \alpha_0} &= \frac{1}{4\alpha_0^2} \left[(q^*)^2(t, 0) - 2 \int_0^{\bar{v}} a(t, v) e^{-\delta v} q^*(t-v, 0) dv \right] \\ &+ \frac{1}{4\alpha_0^2} \left[2l_0 \int_0^{\bar{v}} a(t, v) e^{-\delta v} dv - l_0^2 \right], \end{aligned}$$

from which we can see that the sign of the above expression is not clear. As a result there are ambiguous effects of the installation cost on consumption. But it is easy to check that if $a_1 = 0$, that is, the technology does not improve with time t , then increasing the installation cost will hurt consumption. The reason of this is obvious: since the new

technology does not improve the productivity, there is no incentive to introduce the new ones, which are costly. But if the technological process increases with time $0 < a_1 < \rho$, it could be the case that even increasing the installation cost, the investment in this vintage technological capital good still can benefit the consumers, because the productivity is high enough to compensate the extra cost. But on the other hand, because of the embodied technology, as we see in Proposition 5, technology adoption will lead to have more and more capital accumulation and also increase the shadow price of capital. As a result, firms would like to invest more in the new and more efficient equipments, which leads to the conclusion that consumers could afford less consumption goods. Hence, strictly increasing the technology level maybe not beneficial to the consumers.

Derivative (3.19) with respect to the purchasing cost of new technology, we obtain

$$\frac{\partial c^*(t)}{\partial l_0} = \frac{1}{2\alpha_0} \left(l_0 - \int_0^{\bar{v}} a(v) e^{-\delta v} dv \right).$$

By 1° in Assumption A.2, it is easy to see that the difference on the right hand side of the above equation leads to $\frac{\partial c}{\partial l_0} \leq 0$. Hence, the purchasing cost of new vintage technological capital always has negative effects on the consumption.

In order to study the cost effect of the existing technology on consumption, we rewrite (3.19) as,

$$\begin{aligned} c^*(t) &= \int_0^{\bar{v}} \int_v^{\bar{v}} a(t, \tau) e^{-\delta(\tau-v)} \frac{q^*(t - \tau + v, v) - l(v)}{2\alpha(v)} d\tau dv \\ &+ \int_0^{\bar{v}} a(t, v) e^{-\delta v} \frac{q^*(t - v, 0) - l_0}{2\alpha_0} dv \\ &- \int_0^{\bar{v}} \frac{(q^*)^2(t, v) - l^2(v)}{4\alpha(v)} dv - \frac{(q^*)^2(t, 0) - l_0^2}{4\alpha_0}. \end{aligned} \quad (3.20)$$

Differentiating (3.20) with respect to the purchasing cost of existing vintage technology, we have

$$\frac{\partial c^*(t)}{\partial l(v)} = \int_0^{\bar{v}} \frac{1}{2\alpha(v)} \left[l(v) - \int_v^{\bar{v}} a(t, \tau) e^{-\delta(\tau-v)} d\tau \right] \leq 0.$$

Thus, increasing the purchasing cost of existing vintage will diminish consumption.

For the effect of the installation cost of the existing technological capital goods, differentiating (3.20) with respect to $\alpha(v)$, and rearrange the terms, we get,

$$\begin{aligned} \frac{\partial c^*(t)}{\partial \alpha(v)} &= \int_0^{\bar{v}} \frac{1}{4\alpha^2(v)} [(q^*(t, v))^2 - l^2(v) \\ &\quad - 2 \int_v^{\bar{v}} a(t, \tau) e^{-\delta(\tau-v)} (q^*(t - \tau + v, v) - l(v)) d\tau] dv, \end{aligned}$$

which has a similar effect on the installation cost of new technological equipments. $\diamond$

3.5 Conclusion

We translate the conceptual framework where the standard growth model studies the technology adoption problem into the vintage framework. Therefore in order to establish the optimal investment rules in such situation, firms have to consider two things: (i) how much to invest in existing vintage capital goods (*improvement in the quality of investment*), and how much to invest in new vintage capital goods (*expanding variety of investment*). The model analyzed here to study such a trade-off and the effect to the consumption.

Our framework is different from most of vintage models, which focus on the analysis of balanced growth path, not mention endogenous growth, Boucekkine et al(2005) is an exceptional contribution, where endogenous growth is reserved. By introducing adjustment cost, it could ‘kill’ growth if the adjustment cost is too ‘large’. As a consequence, an immediate(but not obvious) extension of this line of research is to include a linear homogenous convex adjustment cost function, which could save the endogenous growth, and hence the convergence to the balanced growth path.

3.6 Appendix

3.6.1 Proof of sufficient condition in Proposition 3.2

Denote

$$F_1(\tau, v; k) = e^{-\rho\tau} [a(\tau, v) + (q_t - \rho q + \mathcal{A}^*q)] k(\tau, v),$$

$$F_2(t, v; k) = e^{-\rho t} q(t, v)k(t, v),$$

$$F_3(\tau, v; i) = e^{-\rho\tau} [q(\tau, v)i(\tau, v) - l(\tau, v)i(\tau, v) - \alpha(\tau, v)i^2(\tau, v)],$$

$$F_4(\tau, v; i_0) = e^{-\rho\tau} [q(\tau, v)\delta_0 i_0(\tau)dv - l_0 i_0(\tau) - \alpha_0 i_0^2(\tau)].$$

Obviously, we have

$$\begin{aligned} V &\leq \int_t^\infty \int_0^{\bar{v}} F_1(\tau, v; k) dv d\tau + \int_0^{\bar{v}} F_2(t, v; k) dv \\ &\quad + \int_t^\infty \int_0^{\bar{v}} \max_{i, i_0} (F_3(\tau, v; i) + F_4(\tau, v; i_0)) dv d\tau. \end{aligned}$$

If for any $(\tau, v) \in [0, \infty) \times [0, \bar{v}]$, there is $(i^*(\tau, v), i_0^*(\tau))$, such that,

$$F_3(\tau, v; i^*(\tau, v)) = \max_i F_3(\tau, v; i), \quad F_4(\tau, v; i_0^*(\tau)) = \max_{i_0} F_4(\tau, v; i_0),$$

then the above $(i^*(\tau, v), i_0^*(\tau))$ is optimal for the original optimal control problem. In fact, we have

$$\begin{aligned} \frac{\partial F_3}{\partial i} &= q(\tau, v) - l(\tau, v) - 2\alpha(\tau, v)i(\tau, v), \\ \frac{\partial F_4}{\partial i_0} &= q(\tau, v) - l_0(\tau) - 2\alpha(\tau, v)i_0(\tau), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 F_3}{\partial i^2} &= -2\alpha(\tau, v) < 0, \\ \frac{\partial^2 F_4}{\partial i_0^2} &= -2\alpha(\tau, v) < 0, \end{aligned}$$

by Assumption C. Due to Assumption I,

$$\begin{aligned} \frac{\partial F_3(\tau, v; 0)}{\partial i} &\geq 0, \quad \frac{\partial F_4(t, v; 0)}{\partial i_0} \geq 0 \\ \frac{\partial F_3(\tau, v; \infty)}{\partial i} &< 0, \quad \frac{\partial F_4(t, v; \infty)}{\partial i_0} < 0, \end{aligned}$$

and in the first formula at least one of the inequality strictly holds, hence $F_3(\tau, v; i) + F_4(\tau, v; i_0)$ is strictly concave with respect to (i, i_0) . As a result there is unique maximum point, $(i^*(\tau, v), i_0^*(\tau))$, such that,

$$\frac{\partial F_3}{\partial i} = 0, \quad \frac{\partial F_4}{\partial i_0} = 0,$$

which is equivalent to (3.11) and (3.12). $\diamond$

3.6.2 Proof of Proposition 3.5

In the case $t > v$, for any fixed t , derivative the second equation in (3.18) with respect to v , we have

$$\begin{aligned} \frac{\partial k^*(t, v)}{\partial v} &= i^*(t, v) - \delta \left[e^{-\delta v} i_0^*(t - v) + \int_0^v e^{-\delta(v-\tau)} i^*(t - v + \tau, \tau) d\tau \right] \\ &\quad - a_1 \left[\frac{e^{-\delta v}}{2\alpha_0} q(t - v, 0) + \int_0^v \frac{e^{-\delta(v-\tau)}}{2\alpha(\tau)} q(t - v + \tau, \tau) d\tau \right] \\ &= i^*(t, v) - \delta k^*(t, v) \\ &\quad - a_1 \left[\frac{e^{-\delta v}}{2\alpha_0} q(t - v, 0) + \int_0^v \frac{e^{-\delta(v-\tau)}}{2\alpha(\tau)} q(t - v + \tau, \tau) d\tau \right]. \end{aligned} \tag{3.A.1}$$

Taking the limit for v going to zero, we obtain

$$\lim_{v \rightarrow 0^+} \frac{\partial k^*(t, v)}{\partial v} = i^*(t, 0^+) - \delta i_0^*(t) - \frac{a_1}{2\alpha_0} q(t, 0).$$

We claim that, for any $0 < \delta < 1$,

$$i^*(t, 0^+) \geq \delta i_0^*(t). \tag{3.A.2}$$

If so, then we have that

$$\lim_{v \rightarrow 0^+} \frac{\partial k^*(t, v)}{\partial v} + \frac{a_1}{2\alpha_0} q(t, 0) \geq 0. \tag{3.A.3}$$

On the other hand, defining

$$F(v) = a_1 \left[\frac{e^{-\delta v}}{2\alpha_0} q(t - v, 0) + \int_0^v \frac{e^{-\delta(v-\tau)}}{2\alpha(\tau)} q(t - v + \tau, \tau) d\tau \right],$$

we can easily get the following ordinary differential equation,

$$\begin{cases} F'(v) = -(\delta + a_1)F(v) + \frac{q(t, v)}{2\alpha(v)}, \\ F(0) = \frac{a_1}{2\alpha_0}q(t, 0). \end{cases}$$

Solving the above equation, we get

$$\begin{aligned} F(v) &= F(0) + e^{-(\delta+a_1)v} \left(\int_0^v \frac{q(t, \tau)}{2\alpha(\tau)} e^{(\delta+a_1)\tau} d\tau \right) \\ &\geq F(0) = \frac{a_1}{2\alpha_0}q(t, 0). \end{aligned}$$

Combing with (3.A.1), it follows

$$\frac{\partial k^*(t, v)}{\partial v} = i^*(t, v) - \delta k^*(t, v) - F(v) \leq i^*(t, v) - \delta k^*(t, v) - F(0),$$

and then

$$\frac{\partial k^*(t, v)}{\partial v} + F(0) \leq i^*(t, v) - \delta k^*(t, v). \quad (3.A.4)$$

Due to the definition of $q(t, v)$, there is $\lim_{v \rightarrow \bar{v}} q(t, v) = 0$. As a result,

$$\begin{aligned} \lim_{v \rightarrow \bar{v}} (i(t, v) - \delta k^*(t, v)) \\ = -\frac{l(\bar{v})}{2\alpha(\bar{v})} - \delta \left[e^{-\delta \bar{v}} i_0(t - \bar{v}) + \int_0^{\bar{v}} e^{-\delta \tau} i(t - \tau, \bar{v} - \tau) d\tau \right] < 0, \end{aligned} \quad (3.A.5)$$

because of Assumption I, and at least one of $i_0(t - \bar{v})$ and $i(t - \tau, \bar{v} - \tau)$ are strictly positive.

Combining (3.A.3), (3.A.4) and (3.A.5), we have that there exists a benchmark $v^* \in (0, \bar{v})$, such that

$$i^*(t, v^*) - \delta k^*(t, v^*) = 0,$$

and

$$\begin{cases} \frac{\partial k^*(t, v)}{\partial v} + \frac{a_1}{2\alpha_0}q(t, 0) \geq 0, & 0 < v < v^*, \\ \frac{\partial k^*(t, v)}{\partial v} + \frac{a_1}{2\alpha_0}q(t, 0) < 0, & v^* < v < \bar{v}. \end{cases}$$

Since $\frac{a_1}{2\alpha_0}q(t, 0)$ is independent of v for any time $t > 0$, therefore, $k^*(t, v)$ is single hum-shape function of v .

Now we proof the claim (3.A.2). Define

$$g(\delta) = i^*(t, 0^+) - \delta i_0^*(t) = \frac{q(t, 0) - l(t, 0^+)}{\alpha(0^+)} - \delta \frac{q(t, 0) - l_0}{\alpha_0}.$$

It is easy to check that

$$g(0) \geq 0, \quad g(1) \geq 0,$$

due to $\alpha_0 \geq \alpha(0^+)$ and $l_0 \geq l(0^+)$. Moreover combining with

$$g'(\delta) = -\frac{q(t, 0) - l_0}{2\alpha_0} - \frac{(1 - \delta)}{2\alpha_0} \int_0^{\bar{v}} a(t + s, s) e^{-(\rho + \delta)s} ds < 0,$$

we prove that

$$g(\delta) \geq 0. \quad \diamond$$

3.6.3 Proof of Proposition 3.7

We prove that under conditions of the Proposition, we have that $c(t) > 0, \forall t > 0$.

From direct calculation, we obtain that,

$$\begin{aligned} I_1 &= \int_0^{\bar{v}} a(t, v) e^{-\delta v} \frac{q^*(t - v, 0) - l_0}{2\alpha_0} dv \\ &= \frac{1}{2\alpha_0} \int_0^{\bar{v}} \int_0^{\bar{v}} e^{a_1 t} a_2(v) e^{-\delta v} e^{-(\rho + \delta)s} e^{a_1(s + t - v)} a_2(s) ds dv \\ &\quad - \frac{l_0}{2\alpha_0} \int_0^{\bar{v}} e^{a_1 t} a_2(v) e^{-\delta v} dv \\ &= \frac{q(t, 0)}{2\alpha_0} \int_0^{\bar{v}} e^{a_1(t - v)} a_2(v) e^{-\delta v} dv - \frac{l_0}{2\alpha_0} \int_0^{\bar{v}} e^{a_1 t} a_2(v) e^{-\delta v} dv, \end{aligned}$$

then

$$\begin{aligned}
I &= I_1 - \frac{q^2(t, 0) - l_0^2}{4\alpha_0} \\
&= \frac{q(t, 0)}{2\alpha_0} \left(\int_0^{\bar{v}} e^{a_1(t-v)} a_2(v) e^{-\delta v} dv - \frac{q(t, 0)}{2} \right) \\
&\quad + \frac{l_0}{2} \left(\frac{l_0}{2} - \int_0^{\bar{v}} e^{a_1 t} a_2(v) e^{-\delta v} dv \right) \\
&\geq \frac{q^2(t, 0)}{2\alpha_0} \left(e^{-a_1 \bar{v}} - \frac{1}{2} \right) + \frac{l_0}{2\alpha_0} \left(\frac{l_0}{2\alpha_0} - e^{(\rho-a_1)\bar{v}} q(t, 0) \right) \\
&\geq \frac{l_0 q(t, 0)}{2\alpha_0} [e^{-a_1 \bar{v}} + e^{(\rho-a_1)\bar{v}} - 1] \\
&\geq 0,
\end{aligned}$$

by the fact that $\rho > a_1$ and $q(t, 0) \geq l_0$, due to Assumption I.

Using the same argument but changing the order of the integration, we can get that

$$\begin{aligned}
II &= \int_0^{\bar{v}} \left[\int_0^v a(t, v) e^{-\delta \tau} i(t - \tau, v - \tau) d\tau - l(v) i(t, v) - \alpha(v) i^2(t, v) \right] dv \\
&\geq \int_0^{\bar{v}} \frac{l(v)}{2\alpha(v)} q(t, v) (e^{-a_1(\bar{v}-v)} + e^{(\rho-a_1)(\bar{v}-v)} - 1) dv \\
&\geq 0.
\end{aligned}$$

Consider Assumption I, at least one of the above two inequalities is a strict inequality.

Hence, we have that with condition $\rho > a_1$ and Assumption I,

$$c(t) > 0, \forall t > 0. \quad \diamond$$

Chapter 4

The Spatial Solow Model

4.1 Introduction

We mention in the introduction of this thesis that we would like to relax two major assumptions in the neo-classical growth model. One is about technology neutral and another one is that capital accumulations are location dependent. In the previous three chapters, we deal with the first assumption. In the present and the following chapter, we are going to focus on relaxing the second assumption, which is linked to new economic geography literature.

The new economic geography models use a general equilibrium framework with a refined specification of local and global market structures, and some precise assumptions on the mobility of production factors.

Two main characteristics of the new economic geography contributions are: (i) the discrete space structure, and (ii) the absence of capital accumulation. Since capital accumulation is not allowed, the new economic geography models are losing a relevant determinant of migrations, and more importantly, an engine of growth. It seems however clear that many economic geography problems (eg. uneven regional development) have a preeminent growth component, and *vice versa*. Thus, there is an urgent need to unify in some way

the new economic geography and the growth theory, or at least to develop some junction models.

This chapter constitutes a first step following exactly this line of research. We study the Solow-Swan model with one dimensional space, in which we *order the location* (note as $x \in \mathbb{R}$), such that, the central of the economy is $x = 0$. In other words, a region, which is far away from $x = 0$, is a very poor region, where there is no capital flow¹. Furthermore, as mentioned by Ten Raa (1986), page 528–530, “to avoid simple but unrealistic boundary conditions”, we consider that space is continuous and infinite, and that capital accumulation is space dependent. From macro-economic general equilibrium point of view, one dimension for space is reasonable, and, because of decreasing return to capital, the regions with the largest capital endowments are those which display the lowest return, hence allowing capital to flow to less capital endowments regions.

The spatial Solow-Swan model allows to study the link between technology transfers and development. Indeed, it is flexible enough to study the existence of technological poles with partial transfer to neighboring regions, as well as more complicated patterns of knowledge diffusion across space and time.

The chapter is organized as follows. Section 2 presents the spatial Solow-Swan problem. Section 3 is devoted to prove the existence of solutions, providing explicit solutions for the Ak case, and their convergence to a steady state. Section 4 presents different scenarios that bring out the relevance of initial conditions and of space dependent technology and savings. Section 5 concludes.

¹Without capital flow means $\lim_{x \rightarrow \pm\infty} \frac{\partial k(x,t)}{\partial x} = a$, where a is a constant. With a simple transformation, without loss of generality, we can assume $a = 0$. In this case, they do not have a surplus after consumption, so there is no trade. Mathematically, this is called Neumann’s problem. From the economic point of view, we also can impose a Dirichlet condition, that is, $\lim_{x \rightarrow \pm\infty} k(x,t) = b(t)$, with $b(t)$ is a constant cross space. Intuitively, it states that when a household is far away from the economic center, his stock of capital will be the same as his neighbors (since they do not have a surplus after consumption), though it could change with time. Except for the Pontryagin conditions in next chapter, the main results in this present and next chapter will no essential change.

4.2 The model

Assume that in an economy market, there is only one final good, which can be assigned to consumption or investment. In contrast to the standard Solow-Swan model, the law of motion of capital does not rely entirely on the saving capacity of the economy under consideration: the net flows of capital to a given location or space interval should also be accounted for. Suppose that households locate along the real line. At time $t > 0$, the technology at work in location $x \in \mathbb{R}$ produces output per capita $y(x, t) = A(x, t)f(k(x, t))$, where $k(x, t)$ is capital storing at location x at time $t > 0$, $A(x, t)(\geq 0)$ stands for total factor productivity at (x, t) , and $f(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the production function, which satisfies the following assumptions:

(A1) $f(\cdot)$ is non-negative, increasing and concave;

(A2) $f(\cdot)$ verifies the Inada conditions, that is,

$$f(0) = 0, \quad \lim_{k \rightarrow 0} f'(k) = +\infty, \quad \lim_{k \rightarrow +\infty} f'(k) = 0.$$

Moreover we assume that the production function is the same whatever is the location. Hence the budget constraint of household $x \in \mathbb{R}$ is

$$\frac{\partial k(x, t)}{\partial t} = s(x, t)A(x, t)f(k(x, t)) - \delta k(x, t) - \tau(x, t), \quad (4.1)$$

where δ is the depreciation rate of capital², $s(x, t)(\geq 0)$ is saving rate of household x at time $t(> 0)$, which could be homogenous or with respect to time and space, and $\tau(x, t)$ is the household's net trade balance of household x at time t , and also the capital account balance, by the assumption of homogenous depreciation rate of capital, and no arbitrage opportunities. Since the economy is closed, we have

$$\int_{\mathbb{R}} \left(\frac{\partial k(x, t)}{\partial t} - s(x, t)A(x, t)f(k(x, t)) + \delta k(x, t) + \tau(x, t) \right) dx = 0.$$

²Depreciation rate of capital is homogenous in time t , space x , and capital level k .

From (4.1), it is easy to see for any $[a, b] \subset \mathbb{R}$, it follows

$$\int_{[a,b]} \left(\frac{\partial k(x,t)}{\partial t} - s(x,t)A(x,t)f(k(x,t)) + \delta k(x,t) + \tau(x,t) \right) dx = 0. \quad (4.1')$$

On one side, the net trade balanced in region $[a, b]$ can be written as $\int_a^b \tau(x,t)dx$. On the other side, recall that capital movements tend to eliminate geographical differences. Suppose furthermore that there is no institution barriers to capital flow (or do not consider the adjustment speed)^{3 4}. Since without inter-regional arbitrage opportunities, the capital flows from regions with lower marginal productivity of capital to the higher ones, that is equivalent to saying that capital flows from regions with abundant capital toward the ones with relatively less capital. The above assumption leads to that for any region $X = [a, b]$, the capital flows through the boundary points a and b is $\left(\frac{\partial k(b,t)}{\partial x} - \frac{\partial k(a,t)}{\partial x} \right)$, which can be written as

$$- \left(\frac{\partial k(b,t)}{\partial x} - \frac{\partial k(a,t)}{\partial x} \right) = - \int_X \frac{\partial^2 k}{\partial x^2} dx.$$

Due to fact that the trade balance is equal to the capital flow in $[a, b]$, we obtain

$$\int_X \tau(x,t)dx = - \left(\frac{\partial k(b,t)}{\partial x} - \frac{\partial k(a,t)}{\partial x} \right).$$

Substitute the above equation into equation (4.1'), we have $\forall X \subset \mathbb{R}, \forall t$

$$\int_X \left(\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k}{\partial x^2} - (s(x,t)A(x,t)f(k(x,t)) - \delta k(x,t)) \right) dx = 0.$$

³We could assume that there is institution barrier (or adjustment speed) to capital flow depending on location and time(see Ten Raa(1986) and Puu(1982)). But if we presume they are independent of capital k and consumption c , then we can obtain a linear equation with coefficients in front of the Laplacean operator. After some affine transformation, we will get the similar result as below. But if the barriers (or adjustment speed) are functions of k and/or c , we are facing *nonlinear problems*, which we do not consider in this work.

⁴Nonetheless, we consider there is transportation cost. If transportation cost is a delay, then we are again facing very difficult *differential-difference* problem. Therefore as a first step, we will only consider the ice-cream type of transportation cost. That is, the transportation cost is proportional to output, then this term could be absorbed by the term of (gross) output function. As a result, the output term appears in the following is net output. We may study a more general case with space velocity, however, it will bring us so called (mathematically) *non-local problem*, which we will not deal in this setting.

By Hahn-Banach Theorem, therefore the budget constraint can be written as:

$$\frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k(x, t)}{\partial x^2} = s(x, t)A(x, t)f(k(x, t)) - \delta k(x, t), \quad \forall (x, t). \quad (4.2)$$

The initial distribution of capital, $k_0(x)$, is assumed to be known, bounded and continuous. Moreover, we assume that, if the location is far away from the origin, there is no capital flow, that is

$$\lim_{x \rightarrow \pm\infty} \frac{\partial k}{\partial x} = 0.$$

We can write the problem as, for any $(x, t) \in \mathbb{R} \times (0, \infty)$,

$$(P) \begin{cases} \frac{\partial k}{\partial t}(x, t) - \frac{\partial^2 k}{\partial x^2}(x, t) = s(x, t)A(x, t)f(k(x, t)) - \delta k(x, t), \\ k_0(x) > 0 \text{ given}, \\ \lim_{x \rightarrow \infty} \frac{\partial k}{\partial x} = \lim_{x \rightarrow -\infty} \frac{\partial k}{\partial x} = 0. \end{cases}$$

k_0 is defined in $L_+^\infty(\mathbb{R})$, where $L_+^\infty(\mathbb{R}) = \{y \in L^\infty(\mathbb{R}) | y(x) \geq 0 \text{ for almost every } x \in \mathbb{R}\}$.

4.3 Mathematical results

We shall start with a preliminary result, then clarify the point outlined just above.

4.3.1 Existence and uniqueness

Consider the general parabolic PDE in variable $u(x, t)$:

$$\frac{\partial u(x, t)}{\partial t} - \frac{\partial^2 u(x, t)}{\partial x^2} = G(u(x, t), z(x, t)), \quad (4.3)$$

where $G(\cdot)$ is any given continuous function, and $z(x, t)$ a forcing variable, with initial continuous function $u(x, 0) = u_0(x)$ given.

Pao(1992) proves the existence of solutions to this kind of equations⁵. In order to ensure uniqueness, we need a further assumption on growth for $x \rightarrow \pm\infty$.

⁵We also give a proof in appendix 4.6

(A3) For any given finite T , if $(x, t) \in \mathbb{R} \times (0, T]$, there exist constants $z_0 > 0$, $u_0 > 0$ and $b < \frac{1}{4T}$, such that, as $x \rightarrow \pm\infty$

$$0 < z(x, t) \leq z_0 e^{b|x^2|}, \quad 0 < u_0(x) \leq u_0 e^{b|x^2|}.$$

Theorem 4.1 *Let assumption (A3) hold and $z(x, t) \in C^{2,1}(\mathbb{R} \times (0, T))$. Then problem (5.5) has a unique solution $u \in C^{2,1}(\mathbb{R} \times (0, T])$, given by*

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}} \Gamma_0(x - y, t) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma_0(x - y, t - \tau) [G(u(y, \tau), z(y, \tau))] dy d\tau. \end{aligned} \tag{4.4}$$

Moreover,

$$|u| \leq K e^{b'|x|^2}, \text{ as } x \rightarrow \infty,$$

where K is a positive constant, which depends only on z_0 , u_0 , T , and $b' \leq \min\{b, \frac{1}{4T}\}$,

$$\Gamma_0(x, t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}, & t > 0, \\ 0, & t < 0. \end{cases}$$

Furthermore if z, u_0 are bounded functions, then the above unique solution is also bounded.

Notice that (4.4) is a kind of explicit representation of the solution paths of the typical parabolic PDE (4.3); it involves some “canonical” functions $\Gamma(x, t)$ just like the general characterization of the solutions to ordinary differential equation involve exponential terms. Let us keep this solution representation in mind from now on. It considerably helps clarifying the peculiarity of our problem.

For a backward parabolic equation with terminal condition,

$$\begin{cases} \mathcal{L}^* w = w_t + w_{xx} = H(w(x, t), h(x, t)), & x \in \mathbb{R}, \quad t \in [0, T), \\ w(x, T) = w_1(x), \text{ given, } & x \in \mathbb{R}. \end{cases}$$

let $v(x, t) = w(x, T - t)$, then we have similar results.

Corollary 4.1 Suppose $H(\cdot)$ is a continuous function, $h(x, t) \in C^{2,1}(\mathbb{R} \times (0, T))$ and for any given finite T , if $(x, t) \in \mathbb{R} \times (0, T]$, there exist constants $h_1 > 0$, $w_1 > 0$ and $b_1 < \frac{1}{4T}$, such that, as $x \rightarrow \pm\infty$

$$0 < h(x, t) \leq h_1 e^{b_1 |x^2|}, \quad 0 < w_1(x) \leq w_1 e^{b_1 |x^2|}.$$

Then the solution at (x, t) is

$$\begin{aligned} w(x, t) &= \int_{\mathbb{R}} \Gamma_0(x - y, T - t) w_1(y) dy \\ &\quad - \int_t^T \int_{\mathbb{R}} \Gamma_0(x - y, T - \tau) H(w(y, T + t - \tau), h(y, T + t - \tau)) dy d\tau. \end{aligned}$$

More refinements on the explicit representations of the solutions to parabolic PDEs can be found in Wen and Zou(2000, 2002).

Proposition 4.1 If s, A are continuous bounded function and if f verifies (A1) and (A2), there exists a unique global continuous nonnegative bounded solution to problem (P).

Proof: If $(x, t) \in \mathbb{R} \times (0, T)$, where $0 < T < \infty$, it is obtained in Theorem 4.1 that there exists a unique bounded solution to problem (P), provided function s, A and k_0 are bounded. Following Hofbauer and Simon(2001), we obtain the global existence, uniqueness and bound of the solution to (P) in $\mathbb{R} \times (0, \infty)$.

By the definition of output function, negative function cannot be a solution. $\diamond$

The following theorem gives an explicit solution for the Ak model.

Theorem 4.2 Suppose that the production function, $f(k(x, t)) = k(x, t)$.

If A and s are constants, then the solution to problem P is given by

$$k(x, t) = e^{(sA - \delta)t} \int_{\mathbb{R}} \Gamma_0(x - y, t) k_0(y) dy. \quad (4.5)$$

If $A = A(x, t)$, $s = s(x, t)$, the solution to problem (P) is given by

$$k(x, t) = \int_{\mathbb{R}} \Gamma(x - \xi, t) k_0(\xi) d\xi,$$

where Γ is defined as

$$\Gamma(x - \xi, t - \tau) = \Gamma_0(x - \xi, t - \tau) + \int_{\tau}^t \int_{\mathbb{R}} \Gamma_0(x - \eta, t - \sigma) \Phi(\eta - \xi, \sigma - \tau) d\eta d\sigma,$$

and Φ satisfies

$$\Phi(\eta - \xi, \sigma - \tau) = \sum_{\nu=1}^{\infty} (\mathcal{L}\Gamma_0)_{\nu}(\eta - \xi, \sigma - \tau).$$

The operator $\mathcal{L}$ is recursively defined, and it is given by

$$\begin{aligned} (\mathcal{L}\Gamma_0)_1 &= \mathcal{L}\Gamma_0 = (s(x, t)A(x, t) - \delta)\Gamma_0(x, t), \\ (\mathcal{L}\Gamma_0)_{\nu+1}(\eta - \xi, \sigma - \tau) &= \int_{\tau}^{\sigma} \int_{\mathbb{R}} ((\mathcal{L}\Gamma_0)(\eta - y, \sigma - s))(\mathcal{L}\Gamma_0)_{\nu}(y - \xi, s - \tau) dy ds. \end{aligned}$$

Proof: See Page 261–265 of Ladyzenskaja, Solonnikov and Ural’ceva(1968) and/or page 14 of Friedman(1983). $\diamond$

Remark As proved by Ladyzenskaja, Solonnikov and Ural’ceva(1968), if $k_0(x)$ does not increase too rapidly for $|x| \rightarrow +\infty$ (for example, not faster than e^{x^2}), then the integral in (4.5) converges. Hence we can get the same order of growth rate as in the standard Solow model.

Theorem 4.1 allows to clearly study the long run behavior of $k(x, t)$ when $A(x, t) = A$ and $s(x, t) = s$, where s and A are constants. For if $sA \leq \delta$, then from (4.5) one can check that

$$\lim_{t \rightarrow \infty} k(x, t) = 0.$$

If an economy does not save at least to compensate for depreciation, then it will decay until no capital is left.

If, on the contrary, $sA > \delta$, we obtain that:

$$\lim_{t \rightarrow \infty} k(x, t) = \infty.$$

This implies that, as in the 1-dimensional case, the spatial Ak model does not have a steady state.

4.3.2 Steady State and Convergence

We define a steady state solution to (4.1) by the standard conditions $\frac{\partial k(x,t)}{\partial t} = 0$, $A(x, t) = A(x)$ and $s(x, t) = s(x)$:

$$(P_S) \begin{cases} \frac{\partial^2 k(x)}{\partial x^2} + s(x)A(x)f(k(x)) - \delta k(x) = 0, \\ \lim_{x \rightarrow \infty} \frac{\partial k}{\partial x} = \lim_{x \rightarrow -\infty} \frac{\partial k}{\partial x} = 0, \end{cases}$$

We can reduce the problem into an ordinary differential equation 2-dimensional system:

$$\begin{aligned} \frac{\partial k(x)}{\partial x} &= w(x), \\ \frac{\partial w(x)}{\partial x} &= \frac{\partial^2 k(x)}{\partial x^2} = -s(x)A(x)f(k(x)) + \delta k(x). \end{aligned}$$

then a solution to this system is given by,

$$\begin{aligned} k(x) &= \int_{-\infty}^x w(z)dz, \\ w(x) &= \int_{-\infty}^x [-s(x)A(x)f(k(z)) + \delta k(z)]dz. \end{aligned}$$

Any solution $(k(x), w(x))_{x \in \mathbb{R}}$ must also verify that,

$$\lim_{x \rightarrow \infty} \frac{\partial k}{\partial x}(x) = \lim_{x \rightarrow -\infty} \frac{\partial k}{\partial x}(x) = 0, \quad (4.6)$$

this implies that, $\lim_{x \rightarrow \pm\infty} w(x) = 0$. If k verifies that

$$-s(x)A(x)f(k(x)) + \delta k(x) = 0, \quad \forall x, \quad (4.7)$$

then, the boundary conditions are verified and k is a particular solution of (P_S) . Unfortunately, the stationary solutions are not unique.

Proposition 4.2 *If the production function f verifies (A1) and (A2), then the nonnegative solution $k(t, x; k_0)$ to problem P converges to a stationary solution as $t \rightarrow \infty$.*

Proof: The proof requires some minor changes to the proof provided in Bandle, Pozio and Tesi(1987) for a similar problem. $\diamond$

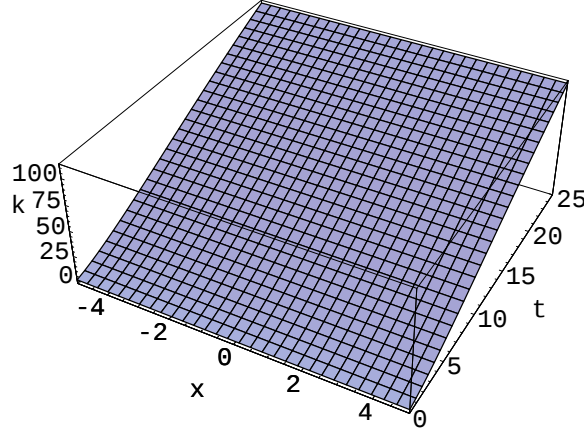


Figure 4.1: Simulation results when space is homogenous.

4.4 Dynamic simulations

We illustrate in this section the behavior of solutions to (P) under different scenarios. In particular, we simulate the spatial Solow model with a Cobb-Douglas production function $f(k) = k^\alpha$. We consider various cases depending on initial conditions and on whether A and s are constants or space-dependent.

Example 4.1 (Homogeneous in initial capital and technology) . We shall assume in this first example that, initially, all households are equally endowed with one unit of physical capital, they share the same saving rate $s(x, t) = 0.2$ and the technological coefficient $A(x, t) = 10$, and the capital share in the production production, $\alpha = 1/3$ and physical capital depreciation $\delta = 0.05$.

Simulated capital reproduces a neoclassical growth path (see Figure 4.1). Marginal productivity of capital is the same along the real line, so that investors are indifferent among all locations. Since there is no source of heterogeneity, all points produce and grow at the same rates.

Example 4.2 (Homogeneous in technology, but heterogeneous in initial capital) . We introduce heterogeneity at the initial endowment of capital to study whether differences

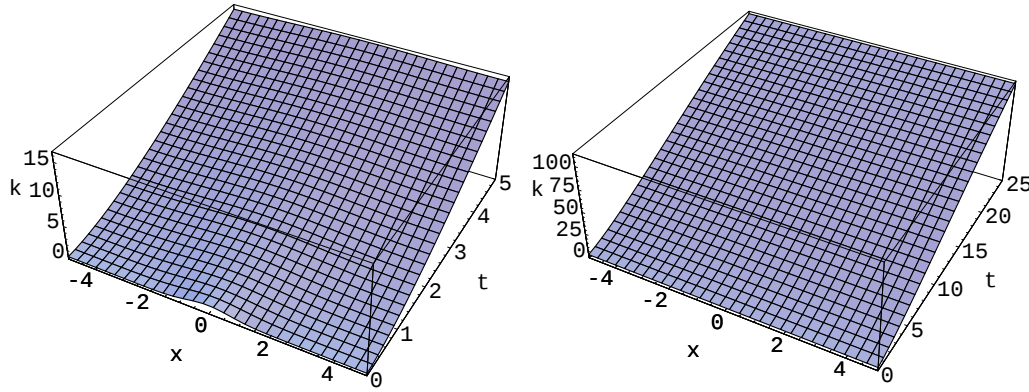


Figure 4.2: Simulation results for different time horizons. On the left $t \in [0, 5]$ and on the right $t \in [0, 25]$.

across regions may persist in the long run. We assume that $k(x, 0) = e^{-x^2}$. The rest of parameters take the same values as in Example 4.1. Figure 4.2 shows that after some iterations, initial differences are smoothed out and that, in the long run, all points in space will be equally rich.

Example 4.3 (Heterogeneous in technology). In this example, $A(x, t) = e^{-x^2}$, that is, the central region uses a more advanced technology. Since there is no technology transfer, as a result, they remain leaders forever. The first graph in Figure 4.3 shows the growth path when the initial condition is spatially homogenous, $k(x, 0) = 1$. In the second graph, $k(x, 0) = e^{-x^2}$, which adds a further source of heterogeneity. Results show that whichever the initial condition, any difference in technology which is not subject to modification through time (i.e. if there are no technological spill-overs from the center to the periphery), leads to a non homogenous steady state.

Example 4.4 (Heterogeneous in technology). In Example 4.3, we notice that technology will dominate the initial endowment different. However, the above example keeps the same shapes of initial capital distribution and technology distribution across space. In this example, we take different shapes between initial capital endowment and technology, where technology is the same as in Example 4.3 and $k_0(x) = e^{x/100}$. We could see that

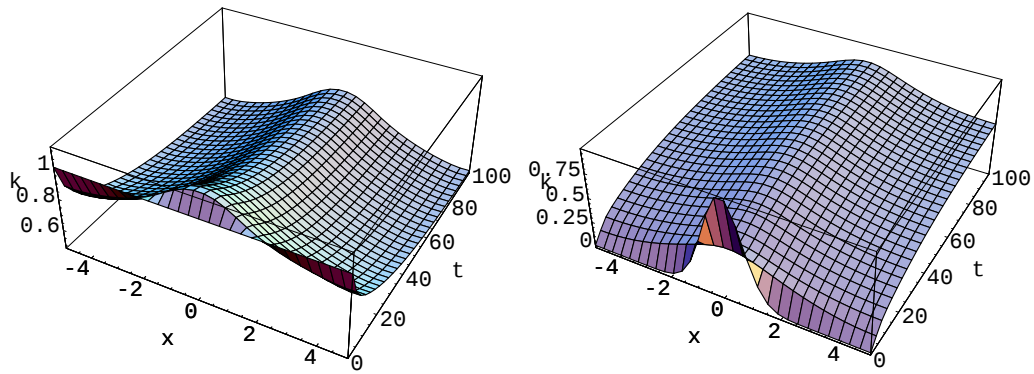


Figure 4.3: Left: $k(x, 0) = 1, \forall x$. Right: $k(x, 0) = e^{-x^2}, \forall x$

initially, there is higher initial capital stock at the right boundary point in Figure 4.4. While with time going, this advantage disappears and technology dominates the initial difference.

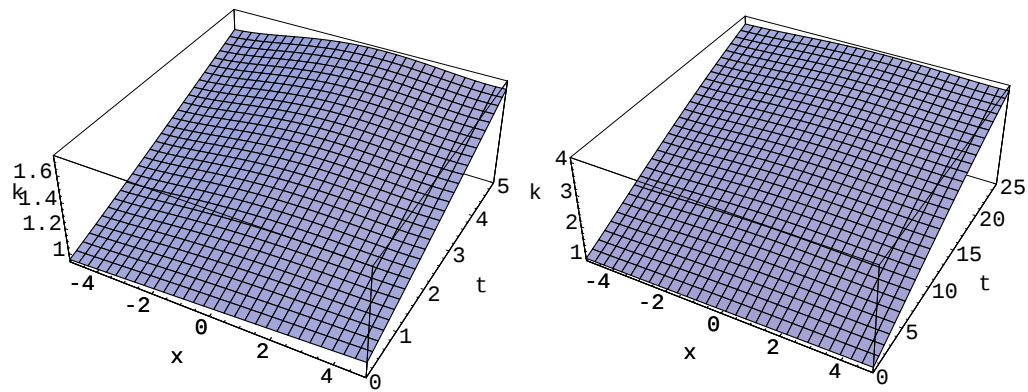


Figure 4.4: Left: $t = 5, \forall x$. Right: $t = 25, \forall x$

4.5 Conclusion

In this chapter, we solve a Solow-Swan model in continuous time and space. We prove at the same time, the existence of a solution to the problem and the convergence to a stationary solution. Results coincide with the non-spatial neoclassical intuition. We obtain that in the Ak case, the model does not have a steady state; furthermore, with a standard neoclassical production function, this steady state exists and we prove convergence. If space is homogenous, i.e. if all locations produce using the same technology and they save at the same rate, then at the steady state, all locations have the same level of physical capital. This is true whatever the initial condition. However, if spatial heterogeneity is introduced at the level of the technology or savings rate, regional differences persist.

Further research in this field should lead to the generalization of our results. A natural continuation is the extension to the Ramsey model, which will show in the next chapter.

4.6 Appendix

4.6.1 Proof of Theorem 4.1

Let $(x, t) \in \mathbb{R} \times (0, T]$. Define a sequence $\{u^{(n)}\}$, $(n \geq 1)$ successively from the iteration process

$$\begin{cases} \mathcal{L}u^{(n)} = u_t^{(n)} - u_{xx}^{(n)} = G(u^{(n-1)}(x, t), z(x, t)), & \text{in } \mathbb{R} \times (0, T], \\ u^{(n)}(x, 0) = u_0(x), & \text{in } \mathbb{R}, \end{cases}$$

with $u^{(0)}(x, t) = u_0(x)$. Then this sequence is well defined, if assumption (A1)- (A3) hold. Due to Theorem 7.1.1 of Pao(1992), a unique solution sequence $\{u^{(n)}\} \in C^{2,1}(\mathbb{R} \times (0, T])$ exists and is given by

$$\begin{aligned} u^{(n)}(x, t) &= \int_{\mathbb{R}} \Gamma_0(x - y, t) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma_0(x - y, t - \tau) [G(u^{(n-1)}(y, \tau), z(y, \tau))] dy d\tau, \end{aligned} \tag{4.A.1}$$

where $\Gamma(x, t)$ is the fundamental solution to parabolic operator $\mathcal{L}$. It is well known that this fundamental solution takes form as

$$\Gamma_0(x, t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}, & t > 0, \\ 0, & t < 0. \end{cases}$$

(see for example, page 261–265, Ladyzenskaja, Solonnikov and Ural'ceva(1968) or page 14, Frideman(1983)). Furthermore, from Theorem 7.1.1, Pao(1992), for each n , the solution satisfy the growth condition

$$|u^{(n)}| \leq K' e^{b|x|^2}, \text{ as } x \rightarrow \pm\infty,$$

for some positive constant K' . Notice that, the sequence staring from u_0 , and then K' does not depend on n . Hence, we obtain that for $t \in (0, T]$, for any x , there is estimate

$$|u^{(n)}| \leq K'' e^{b'|x|^2},$$

for some positive constant K'' .

Then there is a subsequence, denoted as $u^{(n_j)}$, which converges to a function $\tilde{u} \in C^{2,1}(\mathbb{R} \times (0, T])$, and satisfies

$$|\tilde{u}| \leq M e^{b|x|^2}, \quad x \in \mathbb{R},$$

with some positive constant M .

Due to the uniqueness solution of the linear equation(see Theorem 7.1.1, Pao(1992)), it is not difficult to prove that in fact the whole sequence converges to $\tilde{u}$. In (4.A.1), taking limit for $n \rightarrow \infty$ on both sides, we have that

$$\begin{aligned} \tilde{u}(x, t) &= \int_{\mathbb{R}} \Gamma_0(x - y, t) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma_0(x - y, t - \tau) [G(\tilde{u}(y, \tau), z(y, \tau))] dy d\tau. \end{aligned}$$

By the well known fundamental solution result, $\tilde{u}$ is the solution of problem (4.3) for $(x, t) \in \mathbb{R} \times (0, T]$. Denote this above solution as u , then u satisfies the growth condition

$$0 < u \leq K e^{b|x|^2}, \text{ as } x \rightarrow \pm\infty. \quad \diamond$$

Chapter 5

Bridging the gap between growth theory and the new economic geography: The spatial Ramsey model

5.1 Introduction

This chapter will continue the work in Chapter 4. We study the Ramsey model with space. Space is continuous and infinite, and optimal consumption and capital accumulation are space dependent. In particular, we introduce spatial discounting. In our view, this is the most immediate and natural formulation of a spatial Ramsey model, and we will spend some paragraphs justifying spatial discounting.

Establishing the Pontryagin conditions in our parabolic case with infinite time and infinite space is not a very difficult task, using the most recent advances in the related mathematical discipline, notably Raymond and Zidani(1998), and Lenhart and Yong(1992). Unfortunately, the asymptotic properties of the resulting dynamic systems are by now still unsolved in the mathematical literature. Actually, the asymptotic literature of partial dif-

ferential equations (see for example, Bandle, Pozio and Tesi(1987)) has only addressed the case of scalar (or system of) equation(s) with initial values. In a Ramsey-like model, the intertemporal optimization yields a forward variable, consumption, and a transversality condition. As a result, the obtained dynamic system is no longer assimilable to a Cauchy problem, and it turns out that there is no natural transformation allowing to recover the characteristics of a Cauchy problem, specially for the asymptotic assessment. In this chapter, we take a step further. Using explicit integral representations of the solutions to parabolic partial differential equations (see Pao(1992), for a nice textbook in the field, and Wen and Zou(2000, 2002)), we will clearly identify a serious problem with the optimal control of these equations: In contrast to the Ramsey model without space where there exists a one-to-one relationship between the initial value of the co-state variable, say $q(0)$, and the whole co-state trajectory, for a given capital stock path, this property does not hold at all in the spatial counterpart, that is $q(x, t)$, the co-state variable for location x at time t , is not uniquely defined by the data $q(0, x)$ because of the integral relationship linking $q(x, t)$ to $q(0, x)$. As a consequence, while the transversality conditions in the Ramsey model without space allows to identify a single optimal trajectory for the co-state variable, thus for the remaining variables of the model, there is no hope to get the same outcome with space. We are facing a typical *ill-posed problem* in the sense of Hadamard(1923): We cannot assure neither the existence nor the uniqueness of the solutions.

For the reader familiar with parabolic differential equations, this is not surprise: with or without the optimal control context, such equations are well-known to be very hard to analyze, and as we will see in this chapter, even the asymptotic and the stationary equilibria of a scalar general parabolic equations are not completely characterized.

How to deal with this huge difficulty? In order to keep the possibility to compare with the traditional Ramsey model's solution paths, we study the case of the Ramsey model with linear utility. In such a case, we are able to disentangle the forward looking dynamics from the backward-looking, which ultimately allows us to use the available asymptotic literature on scalar initial-value parabolic equation. Depending on the initial capital distribution, optimal consumption per location can be initially corner or interior, and

the dynamics of capital accumulation across space and time will be governed by a scalar parabolic equation. We shall study whether an initially “corner” location (ie. with an initially corner consumption solution) can converge to its interior regime or to any other regime to be characterized. The obtained results are notably different from the linear Ramsey model without space, due to the spatial dynamics induced by capital mobility, which are shown to enrich considerably the dynamics of the Ramsey model. In this sense, the linear spatial Ramsey model is rich enough to serve as a perfect illustration of how the spatial dynamics can interact with the typical mechanisms inherent to growth models.

The chapter is organized as follows. Section 2 states our general spatial Ramsey model with some economic motivations. It also derives the associated Pontryagin conditions using the recent related mathematical literature. Section 3 is one of the most crucial contributions of this chapter: we study the existence and uniqueness of solutions to the dynamic system induced by the Pontryagin conditions and show via explicit integral representations of the solutions, that the latter problem is ill-posed. In the left sections we study the linear utility case. Section 4 is the detailed analysis of the linear utility case. The interior and corner solutions are first characterized. Section 5 studies the convergence from below and from above the interior solution, assuming that *all* the locations start either below or above their interior regime. In Section 6, we consider the situation in which space is partitioned into two half-spaces, one above the corresponding interior regime and the other below. We study in depth the consequences of capital mobility on the asymptotic capital distribution across space. Section 7 concludes.

5.2 The general spatial Ramsey model

We describe here the ingredients of our Ramsey model, formulate the corresponding optimal control problem and give the associated Pontryagin conditions.

5.2.1 General specifications

We consider in this chapter the following central planner problem

$$\max_c \int_0^\infty \int_{\mathbb{R}} U((c(x, t), x)) e^{-\rho t} dt dx, \quad (5.1)$$

where $c(x, t)$ is the consumption level of a representative household located at x at time t , $x \in \mathbb{R}$ and $t \geq 0$, $U(c(x, t), x)$ is the instantaneous utility function and $\rho > 0$ stands for the time discounting rate. For a given location x , the utility function is standard, i.e., $\frac{\partial U}{\partial c} > 0$, $\frac{\partial^2 U}{\partial^2 c} < 0$, and checking the Inada conditions. Our specification of the objective function can be interpreted in two ways. First, the preferences depend on the location of the household, which is by no way inconsistent with the geography literature which typically report different attitudes towards consumption as we move from a region to another. Another plausible interpretation of the specification is the following. Suppose that $U(c, x)$ is separable, $U(c, x) = V(c) \psi(x)$, with $V(\cdot)$ a strictly increasing and concave function, and $\psi(x)$ ¹ an integrable and strictly positive function such that $\int_{\mathbb{R}} \psi(x) = 1$. In such case, the presence of x *via* $\psi(x)$ in the integrand of the objective function stands for the weight assigned to location x by the central planner in a world of homogenous individual preferences. Again, this assumption is most acceptable if one has in mind that in many cases the governments' concerns and actions are not uniform in space. For example, if the government is concerned with uneven regional development, she should assign more weight to the poor regions. Further assumptions on the shape of preferences with respect to x will be done along the way.

We now turn to describe how capital flows from a location to another². Hereafter we denote by $k(x, t)$ the capital stock held by the representative household located at x at date t . In contrast to the standard Ramsey model, the law of motion of capital does not rely entirely on the saving capacity of the economy under consideration: The net flows of capital to a given location or space interval should also be accounted for. Suppose that

¹Another interpretation is that $\psi(x)$ could also be interpreted as a population density function as Clark (Clark 1951).

²See also a latter papers by Brito(2004) for another intuition.

the technology at work in location x is simply $y(x, t) = A(x, t)f(k(x, t))$, where $A(x, t)$ stands for total factor productivity at location x and date t , and $f(\cdot)$ is the standard neoclassical production function, which satisfies the following assumptions:

(A1) $f(\cdot)$ is non-negative, increasing and concave;

(A2) $f(\cdot)$ verifies the Inada conditions, that is,

$$f(0) = 0, \quad \lim_{k \rightarrow 0} f'(k) = +\infty, \quad \lim_{k \rightarrow +\infty} f'(k) = 0.$$

Moreover we assume that the production function is the same whatever is the location. Hence the budget constraint of household $x \in \mathbb{R}$ is

$$\frac{\partial k(x, t)}{\partial t} = s(x, t)A(x, t)f(k(x, t)) - \delta k(x, t) - \tau(x, t), \quad (5.2)$$

where δ is the depreciation rate of capital³, $s(x, t) (\geq 0)$ is saving rate of household x at time $t (> 0)$, which could be homogenous or with respect to time and space, and $\tau(x, t)$ is the household's net trade balance of household x at time t , and also the capital account balance, by the assumption of homogenous depreciation rate of capital, no arbitrage opportunities. Since the economy is closed, we have

$$\int_{\mathbb{R}} \left(\frac{\partial k(x, t)}{\partial t} - s(x, t)A(x, t)f(k(x, t)) + \delta k(x, t) + \tau(x, t) \right) dx = 0.$$

From (5.2), it is easy to see for any $[a, b] \subset \mathbb{R}$, it follows

$$\int_{[a, b]} \left(\frac{\partial k(x, t)}{\partial t} - s(x, t)A(x, t)f(k(x, t)) + \delta k(x, t) + \tau(x, t) \right) dx = 0. \quad (5.2')$$

On one side, the net trade balanced in region $[a, b]$ can be written as $\int_a^b \tau(x, t) dx$. On the other side, recall that capital movements tend to eliminate geographical differences. Suppose furthermore that there is no institution barriers to capital flow (or do not consider

³Depreciation rate of capital is homogenous in time t , space x , and capital level k .

the adjustment speed)^{4 5}. Since without inter-regional arbitrage opportunities, the capital flows from regions with lower marginal productivity of capital to the higher ones, that is equivalent to saying that capital flows from regions with abundant capital toward the ones with relatively less capital. The above assumption leads to that for any region $X = [a, b]$, the capital flows through the boundary points a and b is $\left(\frac{\partial k(b, t)}{\partial x} - \frac{\partial k(a, t)}{\partial x} \right)$, which can be written as

$$- \left(\frac{\partial k(b, t)}{\partial x} - \frac{\partial k(a, t)}{\partial x} \right) = - \int_X \frac{\partial^2 k}{\partial x^2} dx.$$

Due to fact that the trade balance is equal to the capital flow in $[a, b]$, we obtain

$$\int_X \tau(x, t) dx = - \left(\frac{\partial k(b, t)}{\partial x} - \frac{\partial k(a, t)}{\partial x} \right).$$

Substitute the above equation into equation (5.2'), we have $\forall X \subset \mathbb{R}, \forall t$

$$\int_X \left(\frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k}{\partial x^2} - (s(x, t)A(x, t)f(k(x, t)) - \delta k(x, t)) \right) dx = 0.$$

By Hahn-Banach Theorem, therefore the budget constraint can be written as:

$$\frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k(x, t)}{\partial x^2} = s(x, t)A(x, t)f(k(x, t)) - \delta k(x, t), \quad \forall (x, t). \quad (5.3)$$

The initial distribution of capital, $k_0(x)$, is assumed to be known, bounded and continuous. Moreover, we assume that, if the location is far away from the origin, there is no capital

⁴We could assume that there is institution barrier (or adjustment speed) to capital flow depending on location and time(see Ten Raa(1986) and Puu(1982)). But if we presume they are independent of capital k and consumption c , then we can obtain a linear equation with coefficients in front of the Laplacean operator. After some affine transformation, we will get the similar result as below. But if the barriers (or adjustment speed) are functions of k and/or c , we are facing *nonlinear problems*, which we do not consider in this work.

⁵Nonetheless, we consider there is transportation cost. If transportation cost is a delay, then we are again facing very difficult *differential-difference* problem. Therefore as a first step, we will only consider the ice-cream type of transportation cost. That is, the transportation cost is proportional to output, then this term could be absorbed by the term of (gross) output function. As a result, the output term appears in the following is net output. We may study a more general case with space velocity, however, it will bring us so called (mathematically) *non-local problem*, which we will not deal in this setting.

flow, that is

$$\lim_{x \rightarrow \pm\infty} \frac{\partial k}{\partial x} = 0.$$

We can write the problem as, for any $(x, t) \in \mathbb{R} \times (0, \infty)$,

$$(P) \begin{cases} \frac{\partial k}{\partial t}(x, t) - \frac{\partial^2 k}{\partial x^2}(x, t) = s(x, t)A(x, t)f(k(x, t)) - \delta k(x, t), \\ k_0(x) > 0 \text{ given}, \\ \lim_{x \rightarrow \infty} \frac{\partial k}{\partial x} = \lim_{x \rightarrow -\infty} \frac{\partial k}{\partial x} = 0. \end{cases}$$

k_0 is defined in $L_+^\infty(\mathbb{R})$.

The new term $\frac{\partial^2 k(x, t)}{\partial x^2}$ is the spacial ingredient of the dynamics of capital accumulation, it simply captures capital mobility across space. It is a parabolic partial differential equation, and as argued in the introduction, it complicates tremendously the treatment of the associated optimal control problem. We shall precisely identify the source of this complication. Before let us present briefly our optimal control problem.

5.2.2 The optimal control problem

We can write our optimal control problem as follows

$$\max_c \int_0^\infty \int_{\mathbb{R}} U((c(x, t), x)) e^{-\rho t} dx dt.$$

subject to:

$$\begin{cases} \frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k}{\partial x^2}(x, t) = A(x, t)f(k(x, t)) - c(x, t), & (x, t) \in \mathbb{R} \times [0, \infty), \\ k(x, 0) = k_0(x) > 0, & x \in \mathbb{R}, \\ \lim_{x \rightarrow \pm\infty} \frac{\partial k}{\partial x} = 0. \end{cases}$$

The initial distribution of capital, $k(x, 0) \in C(\mathbb{R})$, is assumed to be known positive bounded function. Moreover, we assume that, if the location is far away from the origin, there is no capital flow, that is $\lim_{x \rightarrow \pm\infty} \frac{\partial k}{\partial x} = 0$.

Here comes the definition of an optimal solution.

Definition 5.1 *A trajectory $(c(x, t), k(x, t))$, with $k(x, t)$ in $C^{2,1}(\mathbb{R} \times [0, \infty))$ and $c(x, t)$ piecewise- $C^{2,1}(\mathbb{R} \times [0, \infty))$, is admissible if $k(x, t)$ is a solution of the state equation with control $c(x, t)$ on $t \geq 0$, $x \in \mathbb{R}$, and if the integral objective function (5.1) converges. A trajectory $(c^*(x, t), k^*(x, t))$, $t \geq 0$, $x \in \mathbb{R}$ is an optimal solution of optimal control problem (5.1), if it is admissible and it is optimal in the set of admissible trajectories, i.e., for any admissible trajectory $(c(x, t), k(x, t))$, the value of the integral (5.1) is not greater than its value corresponding to $(c^*(x, t), k^*(x, t))$.*

It is not very hard to see that the shape of preferences is crucial for the convergence of the integral when space is **unbounded**. In effect, suppose that the preferences are homogenous or that the central planner assigns the same weight to all locations, ie. the utility level at x only depends on the consumption level at x , then the integral is clearly divergent for a uniformly distributed consumption. Brito(2004) noticed this fact, and to get rid of it, he considered a different objective function, namely average utility function in space instead of our Benthamian type functional. We prefer to take another approach, and notably to maintain the Benthamian functional as the natural extension of the original Ramsey model. We could have simplified our treatment by having space bounded but we finally prefer to address the pure case of infinite space and infinite time.

But considering that space is infinite just like time imposes a kind of symmetric handling of both to get admissible solutions. In particular, just like time discounting is needed to ensure the convergence of the integral objective function in the standard Ramsey model, we need a kind of *space discounting*. Hence, a natural choice of $U(c, x)$ is to take it **rapidly decreasing** with respect to the second variable. That is, $U(c, x)$, for any fixed c , defined as,

$$\{U(c, \cdot) \in C(\mathbb{R}) \mid \forall m \in \mathbb{Z}_+, |x^m U(c, x)| \leq M_m, \forall x \in \mathbb{R}, M > 0\}.$$

A possible choice of $U(c, x)$ checking the above mentioned characteristic is⁶ $U(c, x) = V(c) \frac{\rho'}{2} e^{-\rho'|x|}$, where $V(\cdot)$ is strictly increasing and concave in c , and $\rho' > 0$. Whatever

⁶In this case, it is in deed the density function in population in the sense of Clark(1951).

is the interpretation, Heterogeneous individual preferences or non-uniform weighting of homogenous preferences by the central planner, the specification above depicts a kind of **preference for the center of the space**. The idea of spatial discounting is not new in the economic geography literature. Fujita and Thisse(2002), chapter 7, present some firms problems with infinite space location and spatial discounting. Applied to our set-up, this idea amounts to assume that there is a single shopping center situated at $x = 0$ from where the consumption good can be bought. Travelling to $x = 0$ implies a loss in utility, and this loss can take the exponential form given above. Indeed, Fujita and Thisse refer to this property by **spatially discounted accessibility**.

5.2.3 The Pontryagin conditions

The Pontryagin conditions corresponding to the control of a parabolic partial differential equation are rigorously studied in Raymond and Zidani(1998, 2000). Using the same kind of variational methods, in Appendix 5.8.1, we can establish the first-order conditions fitting our specific problem. These conditions are:

$$\left\{ \begin{array}{l} \frac{\partial q(x, t)}{\partial t} + \frac{\partial^2 q}{\partial x^2}(x, t) + q(x, t) A(x, t) f'(k(x, t)) = 0, \quad (x, t) \in \mathbb{R} \times [0, \infty), \\ q(x, t) = e^{-\rho t} \frac{\partial U}{\partial c(x, t)}, \quad (x, t) \in \mathbb{R} \times [0, \infty), \\ \lim_{t \rightarrow \infty} q(x, t) = 0, \\ \lim_{x \rightarrow \pm \infty} \frac{\partial q}{\partial x} = 0. \end{array} \right.$$

The first equation is the expected adjoint equation, with $q(x, t)$ playing the role of the co-state variable. As in the standard Ramsey model, the latter is equal to discounted marginal utility of consumption at the optimum, this should be true for every x and t in our spatial extension. The three last limit conditions are respectively the usual (time) transversality condition for infinite horizon discounted problems, and the two (space) transversality conditions implied by the asymptotic constraints on capital flow, $\lim_{x \rightarrow \pm \infty} \frac{\partial k}{\partial x} = 0$. Notice the adjoint equation is also (non-surprisingly) a parabolic PDE. However in contrast to the state equation, which is of the Cauchy type, the adjoint equation has no initial

value $q_0(x) = q(x, 0)$, but this is also a property of the adjoint equation in the standard non-spatial Ramsey model. Finally, one should mention that generally the above conditions are not only necessary, they are also sufficient under the typical concavity conditions like our conditions on the utility and production function across space. See for example Gozzi and Tessitore(1998). So that solving for optimal trajectories amounts in principle to solving the following system:

$$\left\{ \begin{array}{l} \frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k}{\partial x^2}(x, t) = A(x, t)f(k(x, t)) - c(x, t), \quad (x, t) \in \mathbb{R} \times [0, \infty), \\ \frac{\partial q(x, t)}{\partial t} + \frac{\partial^2 q}{\partial x^2}(x, t) + q(x, t) A(x, t) f'(k(x, t)) = 0, \quad (x, t) \in \mathbb{R} \times [0, \infty), \\ q(x, t) = e^{-\rho t} \frac{\partial U}{\partial c(x, t)}, \quad (x, t) \in \mathbb{R} \times [0, \infty), \\ \lim_{t \rightarrow \infty} q(x, t) = 0, \\ \lim_{x \rightarrow \pm \infty} \frac{\partial q}{\partial x} = 0, \\ \lim_{x \rightarrow \pm \infty} \frac{\partial k}{\partial x} = 0, \end{array} \right. \quad (5.4)$$

for given continuous $k_0(x)$. While establishing the existence of solutions to the corresponding problem in the standard Ramsey (also referred to as the Hamiltonian system or the Cass-Shell system) is far from obvious (see first proof in Gaines, 1976), the task is incomparably harder with the space dimension. As we will see in the next sub-section, there is a key difference with respect to the standard Ramsey model which makes our elementary spatial extension amazingly more complicated.

5.3 The existence and uniqueness problem

We shall start with a preliminary result, then clarify the point outlined just above.

5.3.1 A preliminary result

Consider the general parabolic PDE in variable $u(x, t)$:

$$\frac{\partial u(x, t)}{\partial t} - \frac{\partial^2 u}{\partial x^2}(x, t) = G(u(x, t), z(x, t)), \quad (5.5)$$

where $G(\cdot)$ is any given continuous function, and $z(x, t)$ a forcing variable, with initial continuous function $u(x, 0) = u_0(x)$ given.

Pao(1992) proves the existence of solutions to this kind of equations⁷. In order to ensure uniqueness, we need a further assumption on growth for $x \rightarrow \pm\infty$.

(A3) For any given finite T , if $(x, t) \in \mathbb{R} \times (0, T]$, there exist constants $z_0 > 0$, $u_0 > 0$ and $b < \frac{1}{4T}$, such that, as $x \rightarrow \pm\infty$

$$0 < z(x, t) \leq z_0 e^{b|x^2|}, \quad 0 < u_0(x) \leq u_0 e^{b|x^2|}.$$

Theorem 5.1 *Let assumption (A3) hold and $z(x, t) \in C^{2,1}(\mathbb{R} \times (0, T))$. Then problem (5.5) has a unique solution $u \in C^{2,1}(\mathbb{R} \times (0, T])$, given by*

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}} \Gamma_0(x - y, t) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma_0(x - y, t - \tau) [G(u(y, \tau), z(y, \tau))] dy d\tau. \end{aligned} \quad (5.6)$$

Moreover,

$$|u| \leq K e^{b'|x|^2}, \text{ as } x \rightarrow \infty,$$

where K is a positive constant, which depends only on z_0 , u_0 , T , and $b' \leq \min\{b, \frac{1}{4T}\}$,

$$\Gamma_0(x, t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}, & t > 0, \\ 0, & t < 0. \end{cases}$$

Furthermore if z, u_0 are bounded functions, then the above unique solution is also bounded.

⁷We also give a proof in Appendix 4.6.1

Notice that (5.6) is a kind of explicit representation of the solution paths of the typical parabolic PDE (5.5); it involves some “canonical” functions $\Gamma(x, t)$ just like the general characterization of the solutions to ordinary differential equation involve exponential terms. Let us keep this solution representation in mind from now on. It considerably helps clarifying the peculiarity of our problem.

For a backward parabolic equation with terminal condition,

$$\begin{cases} \mathcal{L}^*w = w_t + w_{xx} = H(w(x, t), h(x, t)), & x \in \mathbb{R}, \quad t \in [0, T), \\ w(x, T) = w_1(x), \text{ given,} & x \in \mathbb{R}. \end{cases}$$

let $v(x, t) = w(x, T - t)$, then we have similar results.

Corollary 5.1 Suppose $H(\cdot)$ is a continuous function, and for any given finite T , if $(x, t) \in \mathbb{R} \times (0, T]$, there exist constants $h_1 > 0$, $w_1 > 0$ and $b_1 < \frac{1}{4T}$, such that, as $x \rightarrow \pm\infty$

$$0 < h(x, t) \leq h_1 e^{b_1 |x^2|}, \quad 0 < w_1(x) \leq w_1 e^{b_1 |x^2|}.$$

Then the solution at (x, t) is

$$\begin{aligned} w(x, t) &= \int_{\mathbb{R}} \Gamma_0(x - y, T - t) \phi(y) dy \\ &\quad - \int_t^T \int_{\mathbb{R}} \Gamma_0(x - y, T - \tau) H(w(y, T + t - \tau), h(y, T + t - \tau)) dy d\tau. \end{aligned}$$

More refinements on the explicit representations of the solutions to parabolic PDEs can be found in Wen and Zou(2000, 2002).

5.3.2 Why the control of parabolic PDEs hurts?

To make better the point, let us come back to the standard Ramsey model. The adjoint equation is:

$$q'(t) + q(t) A(t) f'(k(t)) = 0,$$

with obvious notations. Integrating the induced ordinary differential equation from 0 to t , one gets:

$$q(t) = q(0) e^{-\int_0^t A(s) f'(k(s)) ds}.$$

Obviously, $q(0)$ is not known; however, there exists a one-to-one relationship between $q(0)$ and $q(t)$ for a fixed capital trajectory. To any $q(0)$ is associated a single $q(t)$, and to any $q(t)$, one can only identify a unique compatible $q(0)$ value. Typically, $q(0)$ is uniquely determined by the transversality condition $\lim_{t \rightarrow \infty} q(t) = 0$, which establishes uniqueness of optimal trajectories in the Ramsey model. Unfortunately, the same trick does not work in the spatial extension.

Consider our adjoint equation:

$$\frac{\partial q(x, t)}{\partial t} + \frac{\partial^2 q}{\partial x^2}(x, t) + q(x, t) A(x, t) f'(k(x, t)) = 0,$$

for a given capital and technology paths across time and space. Suppose by Theorem 1 that provided these paths are bounded in the sense of Assumption 3, the solution to this PDE can be represented as:

$$\begin{aligned} q(x, t) &= \int_{\mathbb{R}} \Gamma_0(x - y, t) q_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma_0(x - y, t - \tau) [-q(y, \tau) A(y, \tau) f'(k(y, \tau))] dy d\tau. \end{aligned}$$

Because $q_0(x)$ enters an integral, we lose the one-to-one relationship between the initial value- here $q_0(x)$ - and the whole trajectory $q(x, t)$. If $q_0(x)$ were known, then we can fix a unique path $q(x, t)$, but the reverse is evidently WRONG. Unfortunately, the transversality conditions will not be helpful to identify a unique $q_0(x)$ precisely because of the integral representation displayed in Theorem 1. In particular, the usual “economic” transversality condition $\lim_{t \rightarrow \infty} q(x, t) = 0$ will not help identifying the “good” $q_0(x)$, nor the remaining space transversality conditions can solve the problem, simply because the unknown $q_0(x)$ are inside the integrals and not outside. In the language of the PDE literature, our problem is called “ill-posed” (see definition in Hadamard J., (1923)): we cannot assure neither the existence nor the uniqueness of solution. Some “extra” information is needed

to get rid of this. The other way to surmount it is to take linear utility, which induce a degenerescent adjoint equation. We try this strategy in the remaining sections.

5.4 The linear spatial Ramsey model

From now on, we will concentrate on the linear Ramsey model, the special case with linear utility. The objective function becomes:

$$\max_c \int_0^\infty \int_{\mathbb{R}} (c(x, t) \psi(x)) e^{-\rho t} dt dx,$$

Further assumptions on the shape of preferences with respect to x are required:

(A0) $\psi(x) > 0$, $\rho\psi(x) - \psi''(x) > 0$ for all $x \in \mathbb{R}$ and $\int_{\mathbb{R}} \psi(x) = 1$.

As it will be clear in a few paragraphs, this condition is needed in our linear case to assure the positivity of the capital trajectory. The corresponding optimal control problem is:

$$\max_c \int_0^\infty \int_{\mathbb{R}} c(x, t) \psi(x) e^{-\rho t} dx dt. \quad (5.7)$$

subject to,

$$\begin{cases} \frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k}{\partial x^2}(x, t) = A(x, t) f(k(x, t)) - c(x, t) - \delta k(x, t), & (x, t) \in \mathbb{R} \times [0, \infty), \\ k(x, 0) = k_0(x) > 0, & x \in \mathbb{R}, \\ \lim_{x \rightarrow \pm\infty} \frac{\partial k}{\partial x} = 0. \end{cases} \quad (5.8)$$

In the sequel of this section, we consider the optimal control problem (5.7), subject to state equation (5.8), the solution of which is given by Theorem 1. In this linear case, $U(c, x) = c(x, t) \psi(x)$. Then from a direct calculation, we have⁸

$$f'(k) = \frac{(\rho + \delta) \psi(x) - \psi''(x)}{A(x, t) \psi(x)}, \quad (5.9)$$

⁸If there is no location different, $\psi''(x) = 0$, we obtain the same result as in the classical Ramsey models.

and the corresponding capital (which is an **interior solution**) is

$$k^i(x, t) = (f')^{-1} \left(\frac{(\rho + \delta)\psi(x) - \psi''(x)}{A(x, t)\psi(x)} \right).$$

From (5.8), we have that

$$c(x, t) = A(x, t)f(k) - \delta k + k_{xx} - k_t, \quad (5.10)$$

which gives the dynamics of the economy starting from the initial condition $k_0(x)$ to the solution as (5.9). In the rest of this work, we consider a time independent technology, i.e. $A(x, t) = A(x)$. Then from (5.9), the interior solution for capital is also time independent.

The optimal consumption decision can lay in one of the following cases:

$$c(x, t) = \begin{cases} 0, \\ A(x)f(k), \\ \in (0, A(x)f(k)). \end{cases}$$

In the first case, consumption is zero and all the output less capital depreciation is used for investment. In the second one there is no investment, all output is consumed. These are the two corner solutions for consumption. The third case covers the interior solution. The next sub-sections are devoted to study the optimal dynamics, starting with any corner regime. Particular attention will be paid to the conditions under which the economy moves from the corner to the interior regimes (per location), as it is traditional in the optimal control problems which are linear in the control variables.

Before tackling this issue, we introduce some preliminary important definitions. Precisely, we define the steady state (or stationary) solutions and the upper and lower solutions of the steady state problems. The latter concept is extremely useful in the literature of PDEs.

The steady state of problem (P) is defined as:

$$(P_S) \begin{cases} -\frac{\partial^2 k}{\partial x^2}(x) = A(x)f(k(x)) - c(x) - \delta k(x), & x \in \mathbb{R}, \\ \lim_{x \rightarrow \pm\infty} \frac{\partial k}{\partial x}(x) = 0. \end{cases} \quad (5.11)$$

Now we recall the mathematical definition of upper and lower solutions.

Definition 5.2 *A function $k^u(x)$ is an upper solution of (P_s) if it satisfies that*

$$-\frac{\partial^2 k}{\partial x^2}(x) \geq A(x)f(k(x)) - c(x) - \delta k(x), \quad x \in \mathbb{R}.$$

Similarly, we say that a function $k_l(x, t)$ is a lower solution of problem (P_s) if the inequality above is verified with sign $\leq$.

The next Section 5.5 studies the convergence from below and from above the interior solution, assuming that *all* the locations start either below or above their interior regime. We shall refer to this space configuration as the “one-world” case. This case is the simplest one and it already allows to capture the main idea of this chapter, that is the spatial dynamics, induced by perfect capital mobility, enrich considerably the asymptotic of the Ramsey model. Section 5.6 considers a more complicated case, that is the situation where space is partitioned into two half-spaces, one above the corresponding interior regime and the other below. We shall refer to this space configuration as the North-South case.

5.5 Convergence in the one-world case

We shall study the case where the initial capital stock in the whole space is typically lower (Resp. higher) than the interior value, which corresponds to the case of a “too” high (Resp. low) return to capital. As we will see along the way, the mathematical treatment of these two polar cases is everything but symmetric!

5.5.1 High marginal productivity case

Suppose that at $t = 0$,

$$f'(k_0(x)) > \frac{(\rho + \delta)\psi(x) - \psi''(x)}{A(x)\psi(x)}, \quad (5.12)$$

and that

$$-(k_0(x))_{xx} \leq A(x)f(k_0(x)) - \delta k_0(x), \quad \forall x \in \mathbb{R}. \quad (5.13)$$

That is, initially the marginal productivity is higher than the marginal cost. As a result, it is optimal to keep on investing until the capital stock satisfies the optimal rule (5.9) if possible, and $c = 0$. By assumption (A1) $f''(k) < 0$, so in this case

$$k_0(x) < k^i(x), \quad \forall x \in \mathbb{R}.$$

Hence the dynamics of the state equation are

$$\begin{cases} \frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k}{\partial x^2}(x, t) = A(x, t)f(k(x, t)) - \delta k(x, t), & (x, t) \in \mathbb{R} \times [0, \infty), \\ k(x, 0) = k_0(x) > 0, & x \in \mathbb{R}, \\ \lim_{x \rightarrow \pm\infty} \frac{\partial k}{\partial x} = 0. \end{cases} \quad (5.8')$$

Condition (5.13) actually ensures that $k_0(x)$ is a *lower solution* of the stationary equation,

$$-k(x)_{xx} = A(x)f(k(x)) - \delta k(x), \quad \forall x \in \mathbb{R}. \quad (5.14)$$

The convergence results heavily rely on condition (5.13), as the following Theorem clearly shows (with proof in Appendix).

Proposition 5.1 *Suppose (A1), (A2) hold and A is a bounded function. Moreover, we assume that marginal productivity is no less than the marginal depreciation of capital, $Af'(k) \geq \delta$.*

(a) *And if k_0 is a lower solution to (5.14), then the solution to (5.8), with $c = 0$, is nondecreasing with respect to t and*

$$k_0(x) \leq k(x, t), \quad \forall x \in \mathbb{R}, \quad \forall t \geq 0.$$

(b) *If k_0 is a lower solution of (5.13) and it is not a solution to (5.14), then $k(x, t)$ is strictly increasing in t .*

From (b), we can easily get the following result.

Proposition 5.2 *Suppose (A1), (A2) hold and A is a positive bounded function.*

- (a) If k_0 checks (5.13) and is not a solution to (5.14), moreover marginal productivity is no less than the marginal depreciation of capital, $Af'(k) \geq \delta$, then there exist finite time $T_1 < \infty$, such that, the capital accumulation process achieves the interior solution at T_1 , that is,

$$k^i(x) = e^{-\delta T_1} \int_{\mathbb{R}} \Gamma(x-y, T_1) k_0(y) dy + e^{-\delta T_1} \int_0^{T_1} \int_{\mathbb{R}} \Gamma(x-y, T_1-s) A(y) f(k^i(y, s)) dy ds.$$

- (b) Otherwise, we have

$$\lim_{t \rightarrow \infty} k(x, t) = \bar{k}(x),$$

for all $x \in \mathbb{R}$, where $\bar{k}(x)$ is a stationary solution of (5.8') which is positive in $\mathbb{R}$.

Proof (a) is a direct result from Theorem 2, and (b) is a reformulation of Corollary 3.1, Bandle et al.(1987). $\diamond$

Theorem 5.5.1 features the different asymptotic outcomes. Starting below the interior solution, the economy may or may not reach the interior solution. If $k_0(x)$ is a solution of (5.14), that is a positive steady state solution to (5.11), then by Theorem 5.1 and Proposition 5.5-(b), we should have $k(x, t) = k_0(x) = \bar{k}(x)$, $\forall t$. Hence, $k(x, t) < k^i(x)$ forever. On the other hand, if $k_0(x)$ does not check condition (5.14), then convergence to the interior solution is not reached either, see the simulation results in Section 3.3.

In contrast, if (a) holds, that is if $k_0(x)$ checks (5.13) and is not a solution to (5.14), then at time T_1 , the optimal rule (5.9) is restored, and the economy is again driven by the original state equation, with $c \in (0, Af(k))$. Since the linearity of the objective function and the constraints with respect to the decision variable are necessary and sufficient conditions for (5.9) to be optimal. Therefore, at T_1 the economy achieves the optimal capital level, so does the consumption, which is given as:

$$f'(k^i(x)) = \frac{(\rho + \delta)\psi(x) - \psi''(x)}{A(x)\psi(x)}, \quad c_i(x) = A(x)f(k^i) + (k^i)_{xx} - \delta k^i. \quad (5.15)$$

Notice that in this case,⁹ capital and consumption form a steady state solution of (5.7').

⁹Which is indeed due to our assumption of time-independent technical progress.

5.5.2 Low marginal productivity case

Suppose initially that $k_0(x)$ satisfies

$$f'(k_0(x)) < \frac{(\rho + \delta) - \psi(x) - \psi''(x)}{A(x)\psi(x)}.$$

Productivity in this economy is too low, the marginal cost is higher than marginal productivity. As a result, the economy stops investing at any location and consume all the output of the location until (5.9) holds (if possible): $c(x, t) = A(x)f(k(x, t))$. By the concavity of the production function, we have that actually $k_0(x) > k^i(x)$, $\forall x \in \mathbb{R}$.

Then, the capital dynamics are described by:

$$\begin{cases} \frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k}{\partial x^2}(x, t) = -\delta k(x, t), & (x, t) \in \mathbb{R} \times [0, \infty), \\ k(x, 0) = k_0(x) > 0, & x \in \mathbb{R}, \\ \lim_{x \rightarrow \pm\infty} \frac{\partial k}{\partial x} = 0. \end{cases} \quad (5.7'')$$

Before stating the main convergence theorem, we first study a special case, with no capital depreciation rate, $\delta = 0$. We have the following result.

Proposition 5.3 *Suppose (A1), (A2) and A is a bounded function. Furthermore, let $\delta = 0$.*

(a) *Then the solution of (5.8), with $c(x, t) = A(x)f(k(x, t))$, is non-increasing with respect to t and*

$$k_0(x) \geq k(x, t), \quad \forall x \in \mathbb{R}, \quad \forall t \geq 0.$$

(b) *If $k_0(x)$ is an upper solution and not a real solution of the steady state, then $k(x, t)$ is strictly decreasing in t . Furthermore, there exist finite time $T_2 < \infty$, such that, the capital accumulation process achieves the interior solution at T_2 , that is,*

$$k^i(x) = \int_{\mathbb{R}} \Gamma(x - y, T_2) k_0(y) dy.$$

If $k_0(x)$ is not an upper-solution, convergence to the interior solution is not at all granted. Trivially, if for example, the initial capital distribution is uniform across space, that is

$$k_0(x) = k_0 = \text{constant} > k^i(x), \quad \forall x,$$

then, it is easy to see that $k(x, t) = k_0(x), \forall x, \forall t > 0$.

Proposition 5.3 does not deliver any conclusion when the initial capital distribution is strictly convex or convex-concave. We will shed light on this issue with some numerical exercises later. A natural question around is whether these mixed results still hold under nonzero capital depreciation. Indeed with nonzero capital depreciation, things are quite different.

Let $u(x, t) = k(x, t)e^{\delta t}$, after simple calculations, we obtain

$$\begin{cases} u_t - u_{xx} = (k_t - k_{xx})e^{\delta t} + \delta k e^{\delta t} = 0, \\ u(x, 0) = k_0(x). \end{cases}$$

So the theoretical part of the solution will not change. However notice that $k(x, t) = u(x, t)e^{-\delta t}$ is the product of two terms: a bounded term $u(x, t)$ and a second term that converges to zero as t goes to infinity. Then, there exist a point in time, t_1 , such that, $k(x, t_1)$ equals the interior solution. At this point, consumption changes to its interior value. This means that from t_1 onwards, the solution equals the interior solution, i.e. **capital converges** to the interior solution.

To conclude the above analysis, we write as the main convergence result.

Proposition 5.4 *Suppose (A1), (A2) and A is a bounded function. In the case of low marginal productivity, for any initial capital, the existence of non-zero depreciation ensures convergence to the interior solution in finite time.*

5.5.3 Numerical experiments

We illustrate now the main results of this section. We have simulated some examples of convergence and non-convergence to the interior solution, depending on the marginal

productivity of the region and the shape of the initial condition. To do so, we have used *Mathematica*, version 4. We assume that $\rho = 0.3$ and we choose the following exponential space discount function: $\psi(x) = \frac{\phi}{2}e^{-\phi|x|}$ with $\phi = 0.1$. This function verifies condition (A0) almost everywhere. In all examples, we have chosen a Cobb-Douglas production function, $f(k) = k^{\frac{1}{3}}$.

For simplicity reasons, in the first two subsections, we assume $\delta = 0$.

5.5.4 High marginal productivity, non convergence

Let us give an example of a poor region, with high marginal productivity and zero consumption, which does not reach the interior solution. This region does not converge because its initial physical capital endowment is not a lower solution to the Steady State of equation (5.14).

For this example, the technological progress coefficient A equals 1. With this parameters and functions:

$$k^i(x) = (f')^{-1} \left(\frac{\rho\psi(x) - \psi''(x)}{A(x,t)\psi(x)} \right) = \left(\frac{A\alpha}{\rho - \phi} \right)^{\frac{2}{3}} \simeq 1.40572.$$

The initial condition for capital is

$$k_0(x) = \begin{cases} 0, & x \leq 0, \\ x^{\frac{1}{4}}, & 0 < x \leq 1, \\ 1, & x > 1. \end{cases}$$

To make the simulation as clear as possible and without loss of generality, we select now the range of space in which this function is an upper solution to the steady state of equation (5.14). This characteristic makes the solution non-increasing as Figure 5.1 shows. Physical capital decreases till it becomes zero everywhere.

5.5.5 High marginal productivity, convergence

Let us assume a high marginal productivity region, whose initial capital endowment is homogenous, $k_0(x) = 1, \forall x$ and the technological coefficient $A = 3$. Since this initial

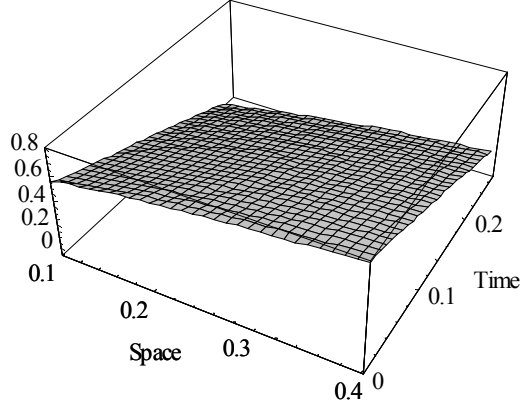


Figure 5.1: Initial capital is not a lower but an upper solution

condition verifies condition (5.12) and it is not a true solution of (5.13), Proposition 5.4 ensures convergence to the interior solution.

In this example, the interior solution is:

$$k^i(x) = 2.92402$$

Further, we can compute the exact time at which the interior solution is reached. This relies on the fact that space is homogenous. Then, dynamics can be reduced to just:

$$\frac{\partial k(x, t)}{\partial t} = A(x, t) f(k(x, t)), \quad \forall (x, t).$$

The solution to this differential equation is:

$$k(x, t) = (1 + 6t + 12t^2 + 8t^3)^{\frac{1}{2}}.$$

Actually, we can very easily compute the exact time at which this solution reaches the interior solution. At this moment, T_1 , $k(x, T_1) = k^i(x, T_1)$. Therefore, in order to compute T_1 we just need to solve:

$$k(x, T_1) = (1 + 6T_1 + 12T_1^2 + 8T_1^3)^{\frac{1}{2}} = \left(\frac{\alpha A}{\rho - \phi} \right)^{\frac{2}{1-\alpha}}.$$

The solution to this equation provides the desired value, $T_1 = 2$.

5.5.6 Low marginal productivity, non convergence

We provide now two examples in which there is **zero** capital depreciation and capital does not converge to the interior solution. This two examples show the essential role of the depreciation rate of physical capital for the convergence, as well the important of concavity of the initial condition with zero depreciate rate.

In the first example, initial physical capital is constant: $k_0(x) = 50, \forall x \in \mathbb{R}$. We can compute the solution as:

$$k(x, t) = \int_{\mathbb{R}} \Gamma(x - y, t) k_0(y) dy = 50 \int_{\mathbb{R}} \Gamma(x - y, t) dy = 50, \forall (x, t) \in \mathbb{R} \times \mathbb{R}^+.$$

For the second example we have picked the following initial condition:

$$k_0(x) = 60 + 10\sin(x).$$

This function is by no means concave. The simulation of the solution to the dynamical problem shows the non convergence to the interior solution. Physical capital grows in every location, thus becoming larger and getting further from k_i .

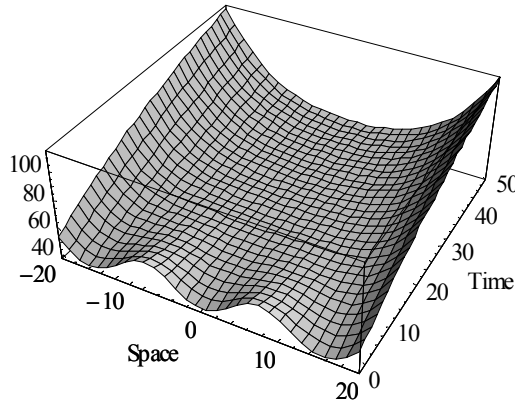


Figure 5.2: Non concave, non constant initial physical capital endowment

5.6 Convergence in the North-South case

We study now an economy where locations with high and low marginal productivity coexist. Since initial capital is continuous, there is at least one region $\bar{x}$, such that, (5.9) holds at $\bar{x}$. Without loss of generality, we can assume that $\bar{x} = 0$, and that locations $x \in \mathbb{R}_+$ have low productivity and high productivity locations are in $\mathbb{R}_-$. That is,

$$k_0(x) \leq k^i(x), \quad x \in \mathbb{R}_- \quad \text{and} \quad k_0(x) \geq k^i(x), \quad x \in \mathbb{R}_+.$$

Consider problem (5.8), with the two corner consumption cases, then the solution is given by

$$\begin{aligned} k(x, t) = & e^{-\delta t} \int_{\mathbb{R}} \Gamma(x - y; t) k_0(y) dy \\ & + e^{-\delta t} \int_0^t \int_{\mathbb{R}_+} \Gamma(x - y; t - \tau) A(y) f(k(y, \tau)) dy d\tau, \quad (x, t) \in \mathbb{R} \times [0, \infty). \end{aligned} \quad (5.16)$$

To analyze the dynamics and asymptotic behavior of this equation, we shall need some specific tools, which are developed in the next sub-section.

5.6.1 Theoretical result

De Mottoni et al.(1984) proved that under certain assumptions the solution (5.16) lies between an upper and a lower bounds that we show in the following theorem.

Notation. $k(x, t; k_0)$ stands for the solution to problem (P) with initial condition k_0 evaluated at (x, t) .

Theorem 5.2 *Assume $(A1)$ and $(A2)$.*

- i) *Suppose k_l (resp. k^u) to be a lower (resp. upper) solution of (P_S) , then the corresponding solution $k(x, t; k_l)$ of (P) is a non-decreasing (resp. increasing) function of t .*

- ii) Suppose there exist a lower solution k_l and an upper solution k^u of (P_S) . Then the corresponding solutions $k(x, t; k_l)$ and $k(x, t; k^u)$ of P are defined for any $t \in [0, T)$ and satisfy

$$k_l(x) \leq k(x, t; k_l) \leq k(x, t; k^u) \leq k^u(x),$$

where $k(x, t; k_0)$ stands for the solution to problem (P) with initial condition k_0 and $t \in [0, T)$.

- iii) Under the assumptions of ii) $k(x, t; k_l)$, $k(x, t; k^u)$ converge as $t \rightarrow \infty$ to limits k_* , k^* respectively, which are solutions of (P_S) and satisfy $k_* < k^*$. In addition, k_* , k^* are minimal, respectively maximal solutions of (P_S) in space $K = \{k \in L_+^\infty | k_l \leq k \leq k^u\}$, where

$$L_+^\infty(\mathbb{R}) = \{u \geq 0 : \sup_{x \in \mathbb{R}} u(x) < \infty\}.$$

- iv) Let k^u be an upper solution of (P_S) , and k_l be a lower solution of (P_S) , such that $Af'(k(\cdot; k_l)) > \delta$. If none of them are real solutions of (P_S) , then the corresponding solution $k(x, t; k_l)$ of (P) is a strictly increasing and $k(x, t; k^u)$ is a strictly decreasing function of t . Furthermore, there exists a finite time $0 < T_3 < \infty$, such that, the time dependent solutions reach the interior solution.

Proof: i)–iii) see Lemma 1.1, de Mottoni et al. (1984), except that at $x = 0$, the right hand side of the steady state equation is not C^1 , but that can be handled by taking a smooth function, which converges to the original problem or by using a mollifier function to the right hand side. iv) can be proved similar to Proposition 5.2 and Proposition 5.4. $\diamond$

Theorem 5.2 is very important in that it sheds light on the behavior of the solutions paths of (5.8) both in the short and long run. Property i) implies that if the equation is initialized with an upper solution (to (P_S)) then the corresponding solution path is non-increasing, which already provides monotonic solutions to the system. Property ii) establishes that if the Cauchy equation is initialized with an upper or lower solution to (P_S) , then the obtained solutions paths lay between the lower and upper solutions. Property i) gives monotonicity and Property ii) gives bounds. This allows to obtain relatively general convergence results as depicted in the result iii). Effectively, it is finally

shown that the path obtained with an initial condition equal to an upper (resp. lower) solution to (P_S) will converge as the time horizon increases to a maximal (resp. minimal) solution of the set of stationary solutions. However, we can only promise convergence to the interior solution under special conditions (see the counter-examples in the previous subsections).

Theorem 5.2 guarantees the convergence of the time-dependent solution to a steady state solution when the initial condition is either an upper solution or a lower solution of (P_S) . Theorem 5.3 gives the convergence property of the solution for a relatively larger class of initial functions. We shall need the following definition:

Definition 5.3 *An interval $[w_1, w_2] := \{w \in L_+^\infty(\mathbb{R}) : w_1 \leq w \leq w_2\}$ is called L^∞ -attractive if there is a set $Q \subset L_+^\infty(\mathbb{R})$, such that,*

- (i) $[w_1, w_2] \subset Q$;
- (ii) *for any $w_0 \in Q$, the solution $w(., t; w_0)$ of problem (P) exists and satisfies that $\text{dist}\{w(., t; w_0), [w_1, w_2]\} \rightarrow 0$ in L^∞ .*

Theorem 5.3 *Assume (A1) and (A2). Let $\underline{k}$, $\bar{k}$ be bounded lower and upper solutions of (P_S) , and let k_*, k^* be the respective minimal and maximal solutions in $[\underline{k}, \bar{k}]$. Then $[k_*, k^*]$ attracts all solutions $k(x, t; k_0)$ of problem (P) with any initial capital condition $k_0 \in [\underline{k}, \bar{k}]$. When $k_* = k^*$, $k(x, t)$ converges to the unique steady state solution as $t \rightarrow \infty$ for any $k_0 \in [\underline{k}, \bar{k}]$.*

Proof. See Theorem 11.3, Chapter 7, C.V. Pao(1992), or Theorem 3.1, Bandle et al.(1987). $\diamond$

Clearly, the previous theorem does not say anything about the attainability of the interior solutions, nor it is conclusive for all initial capital distributions. We shall complement this analysis with some numerical experiments just below.

5.6.2 Numerical solution

We simulate system (P) using a finite difference approximation. The idea of the finite difference method is to replace the second derivative with respect to space with a central

difference quotient in x , and replace the derivative with respect to time with a forward difference in time. For this purpose we set up a grid: we consider a finite but “large enough” space and time horizon $[0, J] \times [0, N]$, where boundaries do not affect results. Points in this space are $(j\Delta x, n\Delta t)$ for $j = 0, 1, \dots, J$ and $n = 1, 2, \dots, N$. Then, if v is a function defined on the grid, we denote by $v(j\Delta x, n\Delta t) = v_j^n$.

A general finite difference approximation for the problem $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ is:

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{1}{\Delta x^2} \left(\theta (v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}) + (1 - \theta) (v_{j+1}^n - 2v_j^n + v_{j-1}^n) \right), \quad (5.17)$$

where $0 \leq \theta \leq 1$. $\theta = 0$ gives the explicit scheme, $\theta = 1/2$ the Crank-Nicolson, and $\theta = 1$ a fully implicit backward time-difference method. As proved in Smith G.D. (1974) p. 24, the finite difference approximation method is stable and convergent for $1/2 \leq \theta \leq 1$, but for $0 \leq \theta \leq 1/2$ we must have

$$r = \frac{\delta t}{(\delta x)^2} \leq \frac{1}{2(1 - 2\theta)}.$$

In practical terms, in an explicit method there exist a formula to obtain v_j^{n+1} for every n in terms of known values whereas with an implicit method we have to solve a system of equations to advance to the next time level. Although the explicit method may be computationally simple it has one serious drawback. The time step Δt is necessarily very small because the process is valid only for $0 < \Delta t/(\Delta x)^2 \leq 1/2$. On the other hand, the implicit method is more costly per time step, but in return we obtain a substantial benefit in the stability and convergence properties, which allows us to use a much larger time step and thus will cut the overall computing costs. Furthermore, the implicit method is unconditionally stable, meaning that it is stable without restrictions on the relative size of Δt and Δx (see Golub G.H. and Ortega J.M., (1992), p. 264).

For these reasons, we choose the implicit method to solve numerically our problem. Then, if v is a function defined on the grid, we denote by $v(j\Delta x, n\Delta t) = v_j^n$. Then, problem (P) can be written in terms of finite difference elements as:

$$\begin{cases} \frac{v_j^{n+1} - v_j^n}{\Delta t} - \frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{\Delta x^2} = A_j^n (v_j^n)^\alpha - \delta v_j^n - c_j^n, \\ v_j^0 \text{ given,} \\ v_0^{n+1} = v_2^{n+1}, \quad v_J^{n+1} = v_{J-2}^{n+1}, \end{cases} \quad (5.18)$$

where A_j^n is the technology matrix that describes the technological state of all points $(j\Delta x, n\Delta t)$. The last condition gives the boundary condition for space. Since the central planner is maximizing agents' utility, consumption is equal to zero in places with high marginal productivity of capital; it is equal to production if marginal productivity of capital is high and equal to production minus depreciation if capital is at its interior solution.

The algorithm of solving (5.18) is showed in Appendix 5.8.3.

Unless otherwise indicated, the parameter values and functional forms used in this sections are those in table 1.

$f(k(x, t)) = k(x, t)^\alpha$	$\alpha = 1/3$
$\psi(x) = e^{-\phi x }$	$\phi = 0.1$
$x_0 = 50$	
$\delta = 0.03$	
$\rho = 0.3$	

Table 5.1: Functional specifications and parameter values for the numerical exercise

We present some examples where the solution to (P) does converge to the interior solution and others where it does not converge to the interior solution but to a different Steady State. In each case, we compare the results we obtain with the case where a trade barrier is introduced.

5.6.3 Convergence to the interior solution

Let us assume that technology is homogenous across space, $A = 10$. Then, the interior solution is:

$$k^i(x, t) = (f')^{-1} \left(\frac{(\rho + \delta)\psi(x) - \psi''(x)}{A(x, t)\psi(x)} \right) = \left(\frac{\rho + \delta - \phi^2}{A\alpha} \right)^{\frac{1}{(\alpha-1)}}.$$

The initial condition is:

$$k_0(x) = \begin{cases} k^i(x)e^{(x-x_0)/100}, & x \leq x_0, \\ k^i(x)(x-x_0)^{0.3}, & x > x_0. \end{cases}$$

Points to the left of x_0 have an initial endowment below k^i , whereas points to the right of x_0 have an initial endowment above k^i . $k_0(x_0)$ is equal to $k^i(x_0)$.

Simulation results are shown in Figure 5.3.

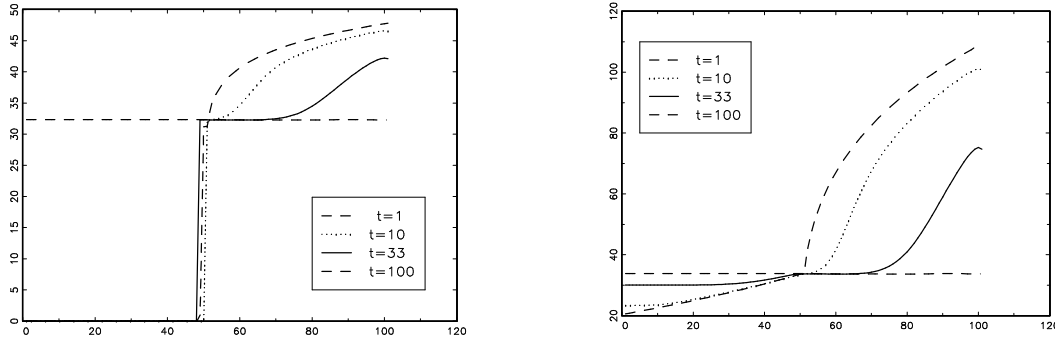


Figure 5.3: Left: Consumption. Right: Physical capital.

Space is represented on the horizontal axis and physical capital is on the vertical axis. We show the level of physical capital at different times, $t = 0$, $t = 10$, $t = 33$ and $t = 100$. For $t = 100$ the interior solution has been reached.

5.6.4 Trade barrier with homogenous technology

Let us assume now that in the same framework as in the previous example, x_0 is a trade barrier. That is, there is no capital flow across this point. Then, the results mimic those of the one-world configuration seen before. Indeed, due to this trade barrier space is divided into two autarkic regions. There is one region to the left of x_0 with high marginal productivity of capital and zero consumption that will converge to the interior solution.

To the right of x_0 there is a region with low marginal productivity of capital that consumes the entire production, and ends up far from the corresponding interior solution. These results are shown in Figure 5.4:

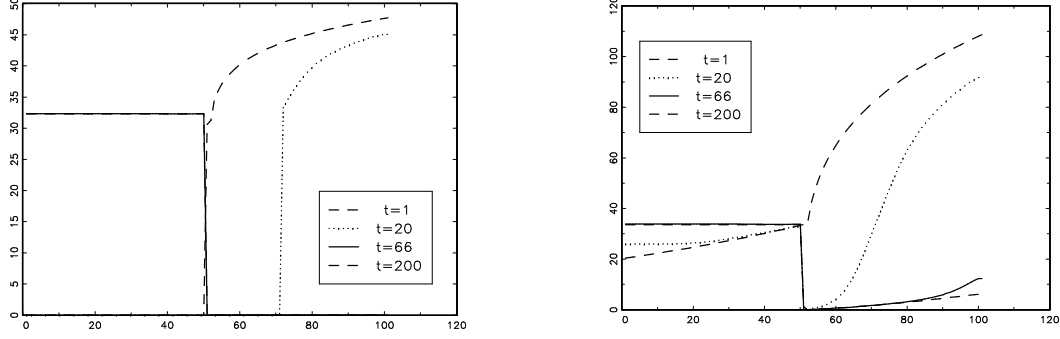


Figure 5.4: Left: Consumption. Right: Physical capital.

Notice that convergence to the Steady State compared with the example before, is slower. The Steady State is no longer constant for the region to the right of x_0 . This is due to trade. Indeed, places do benefit from their neighbors. That is why for the closest places to the trade barrier the level of physical capital is almost zero since they do not have neighbors to the left. This also explains why points far away from x_0 , even if they do not invest in their own capital, keep a positive level of capital benefiting from their neighbor's foreign investment.

5.6.5 Heterogeneous Technology and convergence to a Steady State

Let us assume that technology is described as:

$$A(x) = \begin{cases} 10, & x \leq x_0, \\ 10e^{\frac{x-x_0}{50}}, & x > x_0. \end{cases}$$

Independently of a place's level of capital, if this place is located below x_0 it will produce with a low technique forever; whereas points above x_0 will always produce with a high technique.

In this case, the interior solution is:

$$k^i(x, t) = (f')^{-1} \left(\frac{(\rho + \delta)\psi(x) - \psi''(x)}{A(x, t)\psi(x)} \right) = \left(\frac{\rho + \delta - \phi^2}{A(x)\alpha} \right)^{\frac{1}{(\alpha-1)}}.$$

The initial condition keeps the same relationship with the interior solution as in the previous examples:

$$k_0(x) = \begin{cases} k^i(x)e^{(x-x_0)/100}, & x \leq x_0, \\ k^i(x)(x-x_0)^{0.05}, & x > x_0. \end{cases}$$

Simulation results are shown in Figure 5.5.

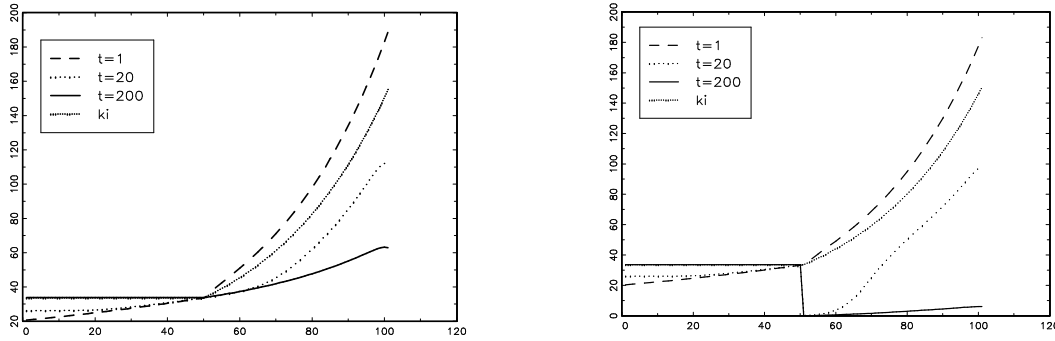


Figure 5.5: Left: Physical capital when there is no trade barrier. Right: Physical capital with a trade barrier at the central location.

We show the level of physical capital and consumption at different times, $t = 0$, $t = 20$ and $t = 200$. A Steady State solution to (P) different from the interior solution is reached. Let us compare the interior solution to this problem and the Steady State. At the interior solution, technologically advanced places retain their advantageous position due to their technology level. On the other hand, the Steady State solution provides a lower level of

physical capital for these places to the right of x_0 . Still, physical capital in these points is larger than in the region below x_0 .

It is remarkable that in the case of the trade barrier, the region initially better endowed ends up with a level of physical capital below that of the least-favored region, despite the never-end technology advantage.

5.7 Conclusion

In this chapter, we have tried to formulate a prototype of spatial Ramsey model with continuous space. In particular, we have departed from the non-Benthamian Ramsey model of Brito(2004) by introducing spatial discounting. We have studied the induced dynamic problem and shown why the optimal control of the resulting parabolic partial differential equations finally gives rise to an ill-posed problem. Our detailed analysis of the linear Ramsey model, which is clearly a way to escape from the ill-posed problem, has the advantage to highlight the tremendous complexity of spatial dynamics even in this linear case.

Two main conclusions can be drawn from our work: first of all, the spatial dimension in a Ramsey framework clearly “adds something” to the story of the neoclassical growth models, with much less mechanical asymptotic results and convergence properties, and more case studies. Second, there is still a tremendous effort to do in order to understand completely what is going on in these models. In particular, we should try to reach a much better understanding of the structure of the stationary solutions. In this respect, developing new analytical and/or computational tools sounds as a minimal prior condition. These technical tasks should be undertaken before tackling more interesting economic extensions of the model, notably migrations.

5.8 Appendix

5.8.1 Obtain first order condition

Suppose that (c^*, k^*) is one optimal solution of the optimal control problem and consider the perturbed path (c, k) , such that, for any arbitrary functions p , h , and $\epsilon > 0$,

$$p, h : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R},$$

and

$$c = c^* + \epsilon h, \quad k = k^* + \epsilon p.$$

The corresponding value function V is defined as

$$V = \int_0^\infty \int_{\mathbb{R}} U(c(x, t), x) e^{-\rho t} dt dx \\ - \int_0^\infty \int_{\mathbb{R}} q(x, t) \left(\frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k}{\partial x^2}(x, t) - A(x, t) f(k(x, t)) + c(x, t) \right) dx dt.$$

As usual, to achieve the maximum of V , the first order condition $\frac{\partial V}{\partial \epsilon} = 0$ should hold.

Let us study first:

$$\int_0^\infty \int_{\mathbb{R}} q(x, t) \frac{\partial k(x, t)}{\partial t} dx dt, \quad \text{and} \quad \int_0^\infty \int_{\mathbb{R}} q(x, t) \frac{\partial^2 k(x, t)}{\partial x^2} dx dt.$$

Integration by parts yields

$$\int_0^\infty \int_{\mathbb{R}} q(x, t) \frac{\partial k(x, t)}{\partial t} dx dt = \int_{\mathbb{R}} k q|_0^\infty dx - \int_0^\infty \int_{\mathbb{R}} k \frac{\partial q}{\partial t} dx dt,$$

and

$$\int_0^\infty \int_{\mathbb{R}} q(x, t) \frac{\partial^2 k(x, t)}{\partial x^2} dx dt = \int_0^\infty q \frac{\partial k}{\partial x} |_{-\infty}^\infty dt - \int_0^\infty \int_{\mathbb{R}} \frac{\partial q}{\partial x} \frac{\partial k}{\partial x} = \\ = \int_0^\infty q \frac{\partial k}{\partial x} |_{-\infty}^\infty dt - \int_0^\infty k \frac{\partial q}{\partial x} |_{-\infty}^\infty + \int_0^\infty \int_{\mathbb{R}} k \frac{\partial^2 q}{\partial x^2}.$$

As a consequence,

$$\frac{\partial V}{\partial \epsilon} = \int_0^\infty \int_{\mathbb{R}} (\psi(x) e^{-\rho t} - q) h dx dt -$$

$$\begin{aligned}
& - \int_0^\infty \int_{\mathbb{R}} p(x, t) \left(-\frac{\partial q}{\partial t} - \frac{\partial^2 q}{\partial x^2} \right) dx dt - \int_{\mathbb{R}} p(x, t) q(x, t) \Big|_0^\infty dx + \\
& + \int_0^\infty q \frac{\partial}{\partial \epsilon} \frac{\partial k}{\partial x} \Big|_{-\infty}^\infty dt - \int_0^\infty p \frac{\partial q}{\partial x} \Big|_{-\infty}^\infty + \int_0^\infty \int_{\mathbb{R}} qp Af'(k(x, t)) dx dt.
\end{aligned}$$

Then, $\frac{\partial V}{\partial \epsilon} = 0$, provided

$$q(x, t) = e^{-\rho t} U_1'(c, x),$$

and for admissible $c(x, t)$ and $k(x, t)$ in $C^{2,1}(\mathbb{R} \times [0, \infty))$,

$$(Q) \begin{cases} \frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + q(x, t) Af'(k(x, t)) = 0, \\ q(x, t) = e^{-\rho t} U_1'(c, x), \\ \lim_{t \rightarrow \infty} q(x, t) = 0, \\ \lim_{x \rightarrow \infty} \frac{\partial q}{\partial x} = \lim_{x \rightarrow -\infty} \frac{\partial q}{\partial x} = 0. \end{cases} \quad \diamond$$

5.8.2 Proof of Proposition 5.1.

(a) Let $w(x, t) = k(x, t) - k_0(x)$, then $w(x, t)$ satisfies the following equation

$$w_t - w_{xx} = A(x)f(k) + (k_0)_{xx} \geq A(x)f(k) - Af(k_0(x)) = A(x)f'(\eta)w$$

where η is between $k(x, t)$ and $k_0(x)$, and the inequality comes from assumption of lower solution. Also initially, $w_0(x) = k(x, 0) - k_0(x) = 0$. Then by a comparison theorem of linear parabolic equation, we obtain that

$$w \geq 0, \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}.$$

As a result, $k(x, t) \geq k_0(x)$.

Now, we prove that $k(x, t)$ is nondecreasing in t . For any fixed constant $\delta > 0$, denote $k_\delta(x, t) = k(x, t + \delta)$ consider function $v(x, t) = k(x, t + \delta) - k(x, t)$. It is easy to check that $v(x, t)$ satisfies

$$\begin{cases} v_t - v_{xx} = Af(k_\delta) - Af(k) = Af'(\zeta)v, & \forall t \geq 0, \quad \forall x \in \mathbb{R}, \\ v(x, 0) = k(x, \delta) - k_0(x) \geq 0, & \forall x \in \mathbb{R}. \end{cases}$$

Again by a comparison theorem of linear parabolic equation and the fact that v is bounded as $|x| \rightarrow \infty$, we have that

$$v(x, t) \geq 0, \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}.$$

In another word,

$$k(x, t + \zeta) \geq k(x, t), \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}.$$

(b) If $k_0(x)$ is not a true solution of (5.14), then

$$-(k_0)_{xx} \leq A(x)f(k_0(x)), \quad \text{in } [a, b], \quad (5.A.2)$$

for every sufficiently large bounded interval $[a, b] \subset \mathbb{R}$, and strictly inequality in (5.A.2) at least at one point. Define $w(x, t) = k(x, t + \delta) - k(x, t)$, for any constant $\delta > 0$. From the prove of (a), it is easy to see that

$$w(x, t) \geq 0, \quad \text{on } \{a\} \cup \{b\} \times (0, \infty),$$

and

$$w(x, t) \geq 0, \quad \text{in } [a, b],$$

then the positivity lemma,(see, for example, Lemma 2.1, Chapter 2, C.V. Pao(1992)) implies that either

$$w(x, t) \equiv 0, \quad \text{or } w(x, t) > 0, \quad \text{in } [a, b] \times (0, \infty).$$

However, if $w(x, t) \equiv 0$, then

$$k(x, t) = k_0(x), \quad \text{in } [a, b] \times (0, \infty),$$

which implies

$$-(k_0)_{xx} = A(x)f(k_0(x)), \quad \text{in } [a, b],$$

which contradicts the fact that strict inequality in (5.A.2) holds at least at one point in $[a, b]$.

Hence, $w(x, t) > 0$, and then $k(x, t + \delta) > k(x, t)$, for any $\delta > 0$. And since this is true for any large interval, it follows that for each $x \in \mathbb{R}$, $k(x, t)$ is strictly increasing. $\diamond$

5.8.3 Simulation Algorithm for Section 5.6

The resolution algorithm:

Step 1: *Initialization*

Choose an initial consumption matrix $C_0 = \{c0_j^n\}_{n=1,\dots,N;j=1,\dots,J}$ and a stopping parameter ϵ . Compute the induced capital $K_0 = \{k_j^n\}_{n=1,\dots,N;j=1,\dots,J}$ using (5.18).

Step 2: *Central planner policy*

For $m = 1, \dots, M$, where M is the maximum number of iterations, we define $C_m = \{cm_j^n\}_{n=1,\dots,N;j=1,\dots,J}$ and $K_m = \{km_j^n\}_{n=1,\dots,N;j=1,\dots,J}$ as the consumption and capital matrix of the m -th iteration.

Compare $K_{(m-1)}$ with the interior solution $Ki = \{ki_j^n\}_{n=1,\dots,N;j=1,\dots,J}$. We build the next iteration consumption matrix $C_m = \{cm_j^n\}_{n=1,\dots,N;j=1,\dots,J}$ as follows: if $k(m-1)_j^n < ki_j^n$ then $cm_j^n = 0$; if $k(m-1)_j^n > ki_j^n$ then $cm_j^n = A_j^n(k(m-1)_j^n)^\alpha$. Finally, if $k(m-1)_j^n = ki_j^n$ then $cm_j^n = A_j^n(k(m-1)_j^n)^\alpha - \delta k(m-1)_j^n$.

Compute the Euclidean distance between C_m and C_{m-1} . If this distance is smaller than ϵ , then STOP and return K_m and C_m .

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