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PROPAGATION OF INTERNAL MODES IN THREE-DIMENSIONAL OCEAN LEVEL MODELS

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The purpose of the present note is to briefly examine how the vertical discretization of ocean level models modifies the propagation of hydrostatic internal inertia-gravity waves. The results obtained herein may be regarded, in a certain sense, as a generalization of the part of Smith's work that was devoted to vertical modes in level models (Ocean Modelling 56, April 1984).

We consider small perturbations of a motionless reference state in a constant depth (H) ocean, where the stratification is characterized by a uniform value of the Brunt-Väisälä frequency N. The governing equations of those perturbations read:

$$\nabla \bullet \mathbf{u} + \frac{\partial w}{\partial z} = 0 , \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + f \mathbf{e}_z \times \mathbf{u} = - \nabla q , \tag{2}$$

$$\frac{\partial q}{\partial z} = b , \tag{3}$$

$$\frac{\partial b}{\partial t} + N^2 w = 0 , \tag{4}$$

where t, z and $\mathbf{e}_z$ denote the time, the vertical coordinate and the vertical unit vector, respectively; ∇ is the horizontal gradient operator; f represents the Coriolis parameter; $\mathbf{u}$, w, q and b are the horizontal velocity vector, the vertical velocity, the reduced pressure and the buoyancy, respectively.

As in Moore and Philander (1977), we assume that the solution of (1)-(4) may be obtained through separation of the vertical dependency. Accordingly, we write

$$(\mathbf{u}, w, q, b) = (\mathbf{u}'U(z), w'W(z), q'Q(z), b'B(z)) , \tag{5}$$

where the primed variables are function of time and horizontal coordinates only. Setting $U = Q = H\, \mathrm{d}W/\mathrm{d}z$ and $B = H\, \mathrm{d}Q/\mathrm{d}z$, substituting (5) into (1)-(4), we arrive at the separation equation

$$\frac{\mathrm{d}W^2}{\mathrm{d}z^2} + \frac{\alpha^2}{H^2} W = 0 . \tag{6}$$

Introducing the impermeability condition of the sea bottom and the sea surface — $W(z{=}0) = 0 = W(z{=}H)$ — into the general solution of (6), we see that the admissible values of the separation constant α are given by

$$\alpha_n = n \pi , \qquad n = 1, 2, 3, \dots , \tag{7}$$

where index n identifies the order of the vertical modes.

For each mode, we may define an equivalent depth

$$H_n \equiv \left(\frac{1}{\alpha_n^2}\frac{N^2 H}{g}\right) H = \left(\frac{1}{n^2 \pi^2}\frac{N^2 H}{g}\right) H , \qquad n = 1, 2, 3, \dots , \tag{8}$$

where g is the gravitational acceleration. Upon introducing an equivalent elevation $\eta = q'/g$, we show that the amplitudes $\mathbf{u}_n$ and η_n of mode n obey equations that are formally equivalent to the linearized shallow water equations, *i.e.*,

$$\frac{\partial \eta_n}{\partial t} + H_n \nabla \bullet \mathbf{u}'_n = 0 , \tag{9}$$

$$\frac{\partial \mathbf{u}'_n}{\partial t} + f \, \mathbf{e}_z \times \mathbf{u}'_n = - g \nabla \eta_n . \tag{10}$$

The phase speed of pure internal gravity waves is thus

$$c_{p,n} = \sqrt{g H_n} . \tag{11}$$

All this is well known.

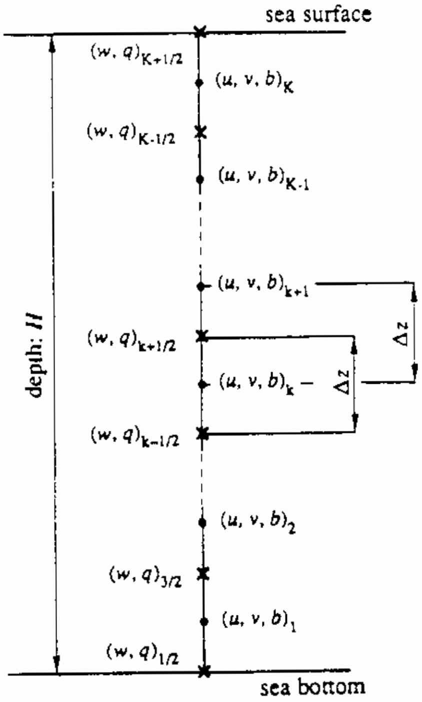

Figure 1. Arrangement of the discretized dependent variables in the vertical direction.

Many numerical techniques for solving (9)-(10) have been examined. However, it is not clear that they are of any relevance to internal waves since it must first be shown that the splitting into numerical modes results in discretized analogues of (9)-(10). We are going to demonstrate that.

We consider a vertical staggering of variables that is in agreement with that of the most classical ocean models (Bryan, 1969; Semtner, 1986) (see Figure 1). We take K vertical grid boxes of constant height Δz. The finite difference form of (1)-(4) is

$$\nabla^{\mathrm{N}} \bullet \mathbf{u}_{\mathrm{k}} + \frac{\delta w_{\mathrm{k}}}{\Delta z} = 0 , \tag{12}$$

$$\partial_t^{\mathrm{N}} \mathbf{u}_{\mathrm{k}} + f \mathbf{e}_z \times \mathbf{u}_{\mathrm{k}} = - \nabla^{\mathrm{N}} \overline{q}_{\mathrm{k}} , \tag{13}$$

$$\frac{\delta q_{\mathrm{k}}}{\Delta z} = b_{\mathrm{k}} , \tag{14}$$

$$\partial_t^{\mathrm{N}} b_{\mathrm{k}} + N^2 \overline{w}_{\mathrm{k}} = 0 , \tag{15}$$

where ∇^{N} and ∂_t^{N} are the numerical counterparts of ∇ and $\partial/\partial t$. Discrete operators "—" and "δ" are defined by

$$\overline{r} = [r(z+\Delta z/2) + r(z-\Delta z/2)] / 2 , \tag{16}$$

$$\delta = r(z+\Delta z/2) - r(z-\Delta z/2) . \tag{17}$$

Assuming that variable separation (5) still applies, that the numerical operators (16) and (17) commute with ∇^{N} and ∂_t^{N}, and that

$$U_{\mathrm{k}} = \delta W_{\mathrm{k}} = \overline{Q}_{\mathrm{k}} , \tag{18}$$

$$\delta Q_{\mathrm{k}} = B_{\mathrm{k}} , \tag{19}$$

we obtain

$$\partial_t^{\mathrm{N}} \eta_{\mathrm{n}} + H_{\mathrm{n}}^{\mathrm{N}} \nabla^{\mathrm{N}} \bullet \mathbf{u}'_{\mathrm{n}} = 0, \tag{20}$$

$$\partial_t^{\mathrm{N}} \mathbf{u}'_{\mathrm{n}} + f \mathbf{e}_z \times \mathbf{u}'_{\mathrm{n}} = -g \nabla^{\mathrm{N}} \eta_{\mathrm{n}} . \tag{21}$$

The latter equations are the discretized counterparts of (9)-(10), except that the separation constant $\alpha_{\mathrm{n}}^{\mathrm{N}}$, and thus the equivalent depth $H_{\mathrm{n}}^{\mathrm{N}}$, now derive from the following difference equation:

$$W_{k+3/2} - 2\frac{4-(\alpha^N/K)^2}{4+(\alpha^N/K)^2}W_{k+1/2} + W_{k-1/2} = 0\,, \tag{22}$$

which holds for $k = 1, 2, \ldots, K-1$. Taking into account the impermeability conditions $W_{1/2} = 0 = W_{K+1/2}$, the solution of (22) is (Bender and Orszag, 1978)

$$W_{k-1/2} = i\,2\,A\,\sin[(k-1)\,\theta]\,, \tag{23}$$

where A is a real constant and $i = \sqrt{-1}$. The admissible values of θ are

$$\theta_n = \frac{n\,\pi}{K}, \qquad n = 1, 2, \ldots, K-1\,. \tag{24}$$

Notice that there exist only $K-1$ discrete vertical modes. After some calculations, we get

$$\alpha_n^N = \frac{2\,K}{[1 + (\mathrm{cotg}\frac{n\pi}{K})^2]^{1/2} + \mathrm{cotg}\frac{n\pi}{K}}\,, \qquad n = 1, 2, \ldots, K-1\,. \tag{25}$$

so that

$$H_n^N = \left\{\frac{\left\{[1 + (\mathrm{cotg}\frac{n\pi}{K})^2]^{1/2} + \mathrm{cotg}\frac{n\pi}{K}\right\}^2}{4\,K^2}\,\frac{N^2\,H}{g}\right\} H\,, \qquad n = 1, 2, \ldots, K-1\,. \tag{26}$$

Of course, when n, the order of the mode, is small compared with the number of grid boxes in the vertical direction, it is possible to show that H_n^N is asymptotic to H_n. Indeed, when $(n\pi)/K \to 0$, $\mathrm{cotg}[(n\pi)/K] \sim K/(n\pi)$, so that $H_n^N \sim H_n$.

To compare H_n^N with H_n, we found it appropriate to examine the ratio

$$\frac{c_{p,n}^N}{c_{p,n}} = \frac{\sqrt{g\,H_n^N}}{\sqrt{g\,H_n}} = \frac{n\,\pi}{2\,K}\left\{[1 + (\mathrm{cotg}\frac{n\pi}{K})^2]^{1/2} + \mathrm{cotg}\frac{n\pi}{K}\right\}, \qquad n = 1, 2, \ldots, K-1\,, \tag{27}$$

where $c_{p,n}^N$ is the phase speed of pure gravity waves related to mode "n" in the case where the truncation errors associated with time and horizontal finite differencing are negligible, *e.g.*, for very long horizontal wavelengths.

The ratio (27) is calculated in Table 1 for all modes for which $2 \le K \le 10$. We see that $c_{p,n}^N/c_{p,n}$ is an decreasing function of n/K and that $c_{p,n}^N/c_{p,n}$ never exceeds 1.

When n decreases, the vertical length scale of the corresponding mode increases. Thus, the smaller n/K is, the better the corresponding mode is resolved by the vertical discretization.

To conclude, one may say that the vertical discretization results in a relative decrease of the equivalent depth that is all the more significant when the mode considered is less adequately resolved on the vertical grid. In the numerical model, this implies a slower propagation of the internal gravity and inertia-gravity waves. But, this slowdown depends *in fine* on the particularities of time and horizontal differencing and the ratio. Therefore, with the exception of very special cases where the time and horizontal discretizations are very accurate, $c^N_{p,n}/c_{p,n}$ only gives a rough indication about the actual propagation speed error of numerical gravity waves.

order of the mode: n

number of grid points: K

	1	2	3	4	5	6	6	8	9
2	.785	XX	XX	XX	XX	XX	XX	XX	XX
3	.907	.605	XX	XX	XX	XX	XX	XX	XX
4	.948	.785	.488	XX	XX	XX	XX	XX	XX
5	.967	.865	.685	.408	XX	XX	XX	XX	XX
6	.977	.907	.785	.605	.351	XX	XX	XX	XX
7	.983	.932	.844	.716	.540	.307	XX	XX	XX
8	.987	.948	.882	.785	.656	.488	.273	XX	XX
9	.990	.959	.907	.832	.732	.605	.445	.246	XX
10	.992	.967	.925	.865	.785	.687	.560	.408	.224

Table 1. The ratio $c^N_{p,n}/c_{p,n}$ as a function of K and n. Notice that K and n are integers such that $K \geq 2$ and $1 \leq n \leq K-1$.

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