

UNIVERSITÉ CATHOLIQUE DE LOUVAIN
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SYMMETRY AND SINGULARITIES FOR SOME SEMILINEAR ELLIPTIC PROBLEMS

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ELLIPTIC PROBLEMS

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Has not every regular variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied and provided also if need be that the notion of a solution shall be suitably extended ?

Hilbert's 20th problem, 1900.

Foreword

The thesis presents the results of our research on symmetry for some semilinear elliptic problems and on existence of solution for quasilinear problems involving singularities. The text is composed of two parts, each of which begins with a specific introduction.

The first part is devoted to symmetry and symmetry-breaking results. We study a class of partial differential equations involving radial weights on balls, annuli or $\mathbf{R}^N$ —where these weights are unbounded—. We show in particular that on unbounded domains, focusing on symmetric functions permits to recover compactness, which implies existence of solutions. Then, we stress the fact that symmetry-breaking occurs on bounded domains, depending both on the weights and on the nonlinearity of the equation. We also show that for the considered class of problems, the multibumps-solution phenomenon appears on the annulus as well as on the ball. These results have been published in [47, 48, 50].

The second part of the thesis is devoted to partial and ordinary differential equations with singularities. Using concentration-compactness tools, we show that a rather large class of functionals is lower semicontinuous, leading to the existence of a ground state solution. We also focus on the unicity of solutions for such a class of problems. These results will appear in [49].

Conventions

The results and equations are numbered within sections. When referring to another chapter, we prefix the number by the Roman numeral of the latter chapter.

Throughout the text, we use the same symbol C for different positive constants used in different lines of a reasoning.

Sequences $(u_n)_{n \in \mathbf{N}}$ and their subsequences are denoted in the same way, viz. (u_n) , to avoid too many subscripts.

If $\Omega \subset \mathbf{R}^N$ is a measurable set, then $|\Omega|$ denotes his Lebesgue measure in $\mathbf{R}^N$.

The following notations are extensively used in the text:

- $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent;
- $\text{supp } u$ is the support of the function u , i.e.

$$\text{supp } u := \{x \in \mathbf{R}^N : u(x) \neq 0\};$$

- $\mathcal{D}(\Omega)$ is the set of test functions of Ω , i.e.

$$\mathcal{D}(\Omega) := \{u \in \mathcal{C}^\infty(\Omega) : \text{supp } u \text{ is compact in } \Omega\};$$

- $\mathbf{R}_0^+$ denotes the positive real axis $(0, +\infty)$.

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Claudia is also a colleague, but first of all a grande amica. We passed together years of noisy life in Louvain-la-Neuve and Brussels: thanks for everything! I enjoyed the presence of many other friends during this time. An incomplete list include Benjamin, Fabienne, Marc, Rémi, Thomas, Nicolas, Jean-François, Bernadette, Léa, Jean-François, Malik and numerous Italians, such as Vale, Andrea, Erica and Mario. Christophe, who belongs to this list, was also my best skeptical spell checker.

A special thank to my father: his comments decisively improved the text. I hope he is happy to see a *doctor laboris causa* in the family. It is difficult to dissociate him from my mother; they gave me a precise arithmetical education.

The other members of my family also played a role they maybe ignore: Bonne Maman, André another *functional analyst*, Marie, Ivan who encouraged me eleven years ago to study physics, Catherine and all their families. A special thanks to Thaïs and the laundry machine.

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Part 1

Symmetry, breaking of symmetry and multibumps

Introduction

The calculus of variations

In 814 BC, Queen Dido of Tyre, fearing the ambitions of her younger brother Pygmalion, escaped to North Africa with her people. A local king gave her the permission to settle within the perimeter of an ox hide. She then cut the hide in thin strips and tied them together in order to form the longest rope possible. Also, she had to find how to place the rope so as to enclose the largest area possible. Using the seaside as a border, the solution found by Dido is the semi-circle [58]. This is usually considered as the first problem of the Calculus of Variations. It can be interpreted as a problem of optimization under constraint: the length of the boundary is the constraint, and the area enclosed is the quantity to maximize.

Many physical problems studied in the XVIIIth century are of this nature: an example is the construction of the shortest path between two given points on a surface (geodesic problem); another example is the search of the curve between two points along which a mobile accelerated by gravity will move in the least time (brachistochrone problem). The gist of these problems is to find a profile, represented by a function u , minimizing a *functional* –a function of the function u – under some constraint. Using mathematical notations, these problems are of the form

$$(1) \quad \inf_{u \in X} J(u) \quad \text{with } J(u) := \int_{\Omega} f(x, u, \nabla u) dx$$

where $\Omega \subset \mathbf{R}^N$, $u : \Omega \rightarrow \mathbf{R}$, X is an appropriate functional space and J is called the *energy functional*.

This variational structure has a differential counterpart. If u is a minimizer of (1), then –modulo regularity of J – its first variation vanishes:

$$J'(u) = 0,$$

which is the *Euler equation* associated with the problem. As an example, a minimizer of the Dirichlet integral

$$\int_{\Omega} |\nabla u|^2 dx \quad \text{where} \quad u = u_0 \text{ on } \partial\Omega$$

is a solution of the *Laplace equation*

$$-\Delta u = 0 \text{ in } \Omega \quad \text{and} \quad u = u_0 \text{ on } \partial\Omega$$

when u_0 belongs to the suitable space $H^{\frac{1}{2}}(\partial\Omega)$.

We can also consider problems where the constraint is itself expressed by a functional I :

$$\inf_{\substack{u \in X \\ I(u)=0}} J(u).$$

In this case, we may use the Lagrange Multiplier Rule: if u is a minimizer, there exists $\lambda \in \mathbf{R}$ such that, for every element φ in the dual space X' ,

$$\langle J'(u), \varphi \rangle = \lambda \langle I'(u), \varphi \rangle,$$

which is the *Euler-Lagrange equation* associated with the problem.

Solving the associated equation in order to obtain the solution of the minimization problem is the *classical approach*. In modern analysis, we extensively use its counterpart: we solve the minimization problem in order to solve the associated partial differential equation.

The direct method

The most natural method to handle variational problems is the *direct method*. To simplify, suppose the dimension of X is finite. One way to obtain a minimizer is to find a minimizing sequence (u_n) for J belonging to a closed bounded set, which is compact as X is finite dimensional. Compactness of the set implies convergence, up to a subsequence, of (u_n) to some u , which is a minimizer when J is continuous. The hypothesis of continuity can be weakened: as we look for minimizers, lower semi-continuity suffices, i.e. $\liminf_{n \rightarrow \infty} J(u_n) \geq J(u)$ when $u_n \rightarrow u$. If the problem has a constraint, it remains to show that u satisfies it.

This method is extensively used for infinite-dimensional problems. For example, consider the following constrained variational problem:

$$(2) \quad \inf_{\substack{u \in H_0^1(\Omega) \\ I(u)=1}} J(u) = \|\nabla u\|_2^2 = \int_{\Omega} |\nabla u|^2 dx$$

where the constraint functional $I(u) := \|u\|_p = (\int_{\Omega} |u|^p dx)^{\frac{1}{p}}$ is the L^p -norm, the domain Ω is a regular open bounded subset of $\mathbf{R}^N$ and $2 < p < 2^* := \frac{2N}{N-2}$. Define

$$H^1(\Omega) := \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\},$$

where the norm $\|u\|$ is $(\|u\|_2^2 + \|\nabla u\|_2^2)^{\frac{1}{2}}$. The space $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega) := \{u \in C^\infty(\Omega) : \text{supp } u \text{ is compact in } \Omega\}$ with respect to the $H^1(\Omega)$ -norm. Roughly speaking, $H_0^1(\Omega)$ is the set of functions of $H^1(\Omega)$ vanishing on the boundary. As the functional J is positive, its infimum is real. So, consider a minimizing sequence (u_n) . We have to prove its convergence to a minimizer of J . The boundedness of the sequence is ensured as $\|\nabla u\|_2$ is an equivalent norm for $H_0^1(\Omega)$. Since we work in a reflexive Banach space, by virtue of Eberlein-Šmulian Theorem (see e.g. [9, Théorème III.28]), this implies that (u_n) has a *weak* limit up to a subsequence: $u_n \rightharpoonup u \in H_0^1(\Omega)$. Moreover, in Banach spaces, the norm is **weakly** lower semi-continuous, i.e. $u_n \rightharpoonup u$ merely implies

$$\inf_X J = \liminf_{n \rightarrow \infty} J(u_n) \geq J(u).$$

It remains to show that u satisfies the constraint $\|u\|_p = 1$. As $2 < p < 2^*$ and Ω is bounded, the embedding $H_0^1(\Omega) \subset L^p(\Omega)$ is compact (Rellich Theorem, see e.g. [59, Theorem 1.9]), i.e. weak convergence in $H_0^1(\Omega)$ implies strong convergence in $L^p(\Omega)$, so $\|u\|_p = \|u_n\|_p = 1$. The function u is then the solution of the constrained minimization problem.

By the Lagrange Multiplier Rule, there exists $\lambda \in \mathbf{R}$ such that, for all $\varphi \in X'$,

$$\langle J'(u), \varphi \rangle = \lambda \langle I'(u), \varphi \rangle.$$

Rescaling u by multiplication with $(J(u))^{\frac{1}{p-2}}$, we obtain a solution of the corresponding differential equation

$$-\Delta u = |u|^{p-2}u \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega.$$

In this scheme, using extensively the facts that Ω is bounded and $2 < p < 2^*$, we ensure the following features:

- the functional J has a lower bound, hence there exists a minimizing sequence (u_n) which weakly converges to an element u ;
- u minimizes J (by weak lower-semicontinuity);
- u satisfies the constraint (by compact embedding of spaces).

To tackle these issues can be hard in general, depending on the structure of the problem: if the domain is unbounded, the above framework breaks down as shown in the following example.

Example 1. Consider problem (2) with $\Omega = \mathbf{R}^N$, i.e. find a minimizer of $J(u) := \|\nabla u\|_2$ under the constraint $\|u\|_p = 1$ in $H^1(\mathbf{R}^N)$:

$$(2\text{bis}) \quad \inf_{\substack{u \in H^1(\mathbf{R}^N) \\ \|u\|_p = 1}} \|\nabla u\|_2^2.$$

Let $\varphi \in \mathcal{D}(B(0,1))$ such that $\|\varphi\|_p = 1$, and let $\varphi_n(x) := \varphi(x + (n, 0, \dots, 0))$. The elements of the sequence (φ_n) satisfy the constraint, but weakly converge to 0. Roughly speaking, the Rellich Theorem is not satisfied as the domain is unbounded: there is a *lack of compactness*.

In the sequel, we will develop some methods to overcome this lack of compactness on unbounded domains. Moreover, we will see that direct methods on subspaces of H^1 yield multiple solutions of the corresponding Euler-Lagrange equation.

Compactness thanks to symmetry

Compactness can be ensured by a restriction of the functional space to the subspace of functions invariant under specific symmetries. This method was first used by Strauss [55]. He proves that the subspace $H_r^1(\mathbf{R}^N)$ of radially symmetric functions of $H^1(\mathbf{R}^N)$ is compactly embedded in $L^p(\mathbf{R}^N)$ for every $2 < p < 2^* := \frac{2N}{N-2}$. This is done in two steps. First prove that for every $u \in H_r^1(\mathbf{R}^N)$,

$$|u(x)| \leq C \|u\|_2^{\frac{1}{2}} \|\nabla u\|_2^{\frac{1}{2}} |x|^{\frac{1-N}{2}} \quad \text{a.e. on } \mathbf{R}^N.$$

Then it suffices to integrate $|u|^p$ over $B^c(0,1)$ and to estimate this integral using the above inequality in order to obtain the compactness on $B^c(0,1)$; when the compactness on $B(0,1)$ is clear by Rellich Theorem.

Lions [33] applied Strauss' technique to Sobolev spaces $W_r^{s,p}(\Omega)$ defined, for $s \in \mathbf{N}_0$ and $1 \leq p < \infty$, as the subset of radial functions of

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega), \quad \forall |\alpha| \leq s\}.$$

Lions' result is:

Let $N \geq 2$, $s > 0$, $p \in [1, +\infty[$ and

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } sp < N, \\ +\infty & \text{if } sp \geq N. \end{cases}$$

Then $W_r^{s,p}(\mathbf{R}^N)$ is compactly embedded in $L^q(\mathbf{R}^N)$ for every $p < q < p^*$.

So radially implies compactness. This phenomenon also appears in the case of more general symmetries. Let $N_i \geq 2$ for $i \in \{1, \dots, m\}$ with $\sum_{i=1}^m N_i = N$ and let $C := \mathbf{O}(N_1) \times \dots \times \mathbf{O}(N_m)$. The subspace $H_C^1(\mathbf{R}^N)$ of cylindrical symmetric functions of $H^1(\mathbf{R}^N)$, defined by

$$H_C^1(\mathbf{R}^N) := \{u \in H^1(\mathbf{R}^N) : gu = u \quad \forall g \in C\},$$

is compactly embedded in $L^p(\mathbf{R}^N)$ for every $2 < p < 2^*$, as proved by Lions [33, Section III]. For more general results, see [59, Theorem 1.24].

When compactness is ensured by symmetry, we can use classical methods to obtain critical points of the functional restricted to the symmetric subspace. To conclude the reasoning, it suffices to prove that the *restricted* critical point is also a critical point of the original functional. This is not generally true, as pointed out by Palais [40] where he gives exotic counterexamples. However, the result is valid on Hilbert spaces (or more generally Banach spaces) when the action of the symmetry group is isometric:

Let $u \in H_r^1(\mathbf{R}^N)$ and $J : H^1(\mathbf{R}^N) \rightarrow \mathbf{R}$.
If $J'|_{H_r^1}(u) = 0$, then $J'(u) = 0$.

The use of compactness for radially symmetric subspaces of $H^1(\mathbf{R}^N)$ is illustrated in Chapter I. We consider the problem

$$(3) \quad -\Delta u + |x|^a u = |x|^b u^{p-1}$$

on $\mathbf{R}^N$. This equation was first studied by Sirakov [51]. His main result is the existence of a nontrivial solution of (3) when $a > b \geq 0$ and $2 < p < p^\# := 2^* - \frac{4b}{a(N-2)}$. He obtained it using the classical Mountain-Pass Theorem of Ambrosetti-Rabinowitz [3] and a compact embedding of a suitable subspace of $H^1(\mathbf{R}^N)$ into $L^p(\mathbf{R}^N, \rho)$, spaces of functions u such that $\int_{\mathbf{R}^N} \rho |u|^p dx$ is finite. On the other hand, using Pohozaev's identity, Sirakov proved the non-existence of solutions when $p > \tilde{p} := 2^* + \frac{2b}{N-2}$. So a gap $[p^\#, \tilde{p}]$ remains between the existence and the non-existence regions of the space obtained using classical methods.

Independently, Schneider [45] and Sintzoff-Willem (Chapter I or [50]) investigated the problem using symmetry to obtain more compactness. The main idea underlying these works was to obtain the compact embedding of the functional space into a weighted L^p space.

Using Hardy-type and Hölder inequalities, Schneider shows the compact embedding of $\mathcal{D}_r^{1,2}(\mathbf{R}^N) \cap L^2(\mathbf{R}^N, |x|^a)$ in $L^p(\mathbf{R}^N, |x|^b)$ when $2 < p$

and $2 + 2\frac{b-a}{N-2} < p < \tilde{p}$. Note there is no relation of order between a and b . This leads to the existence of a radial solution of problem (3). Moreover, dealing accurately with Pohozaev-type tools, he obtains the non-existence of solutions when $p < 2 + 2\frac{b-a}{N+a}$, filling a part of the *Sirakov gap*.

Roughly speaking, lack of compactness can appear for unbounded domains both at infinity by dilation and at the origin by a concentration phenomenon balanced by the weight $|x|^b$. The control at infinity is ensured by condition $p < \tilde{p}$ whereas the inequalities $2 < p$ and $2 + 2\frac{b-a}{N-2} < p$ permit to avoid the lack of compactness at the origin.

Meanwhile, we studied (3) also looking for compact embedding results. Instead of using a Hardy-type inequality, we tried to adapt Strauss' method. Our main result –Theorem I.2.3– ensures the existence of a radial solution to the problem for $2 < p$ and $2 + 2\frac{b-a}{N-1+\frac{a}{2}} < p < \tilde{p}$. This better result leaves however a gap between $p = 2 + 2\frac{b-a}{N+a}$ and $p = 2 + 2\frac{b-a}{N-1+\frac{a}{2}}$.

Figure 1 summarizes the results for the case $N = 3$.

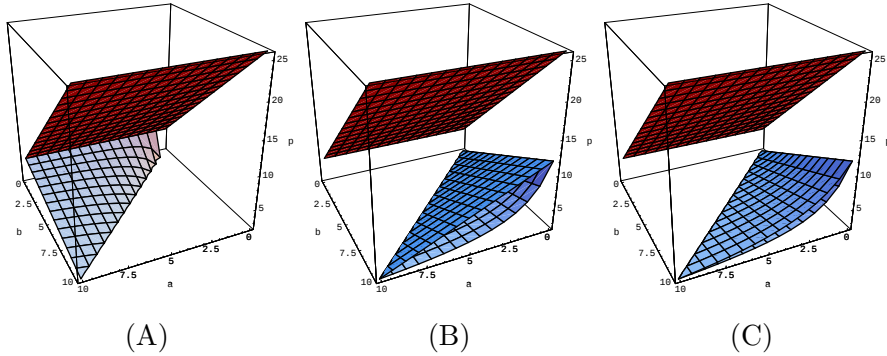


FIGURE 1. The existence domains respectively obtained by (A) Sirakov, (B) Schneider and (C) Sintzoff-Willem.

The zones of non-existence and existence of solutions are determined as follows, where $\tilde{p} := 2^* + \frac{2b}{N-2}$ and $p^\# := 2^* - \frac{4b}{a(N-2)}$:

- (A) Non-existence above the upper surface $p = \tilde{p}$; existence below the lower one $p = p^\#$; the Sirakov gap is the space between the upper and lower surfaces.
- (B) Non-existence above the upper surface $p = \tilde{p}$ and below the lower one $p = 2 + 2\frac{b-a}{N+a}$; existence between the upper surface

and the middle surface $p = 2 + 2\frac{b-a}{N-2}$; the Sirakov gap is reduced to the slice between the middle surface and the lower one.

- (C) Non-existence as in (B); existence between the upper surface $p = \tilde{p}$ and the middle one $p = 2 + 2\frac{b-a}{N-1+\frac{a}{2}}$; the Sirakov gap is further reduced to the slice between the middle surface and the lower one $p = 2 + 2\frac{b-a}{N+a}$. Note the middle and lower surfaces here are very close.

Figure 2 represents the same regions as in Figure 1 for $a, b \in [0, 100]$.

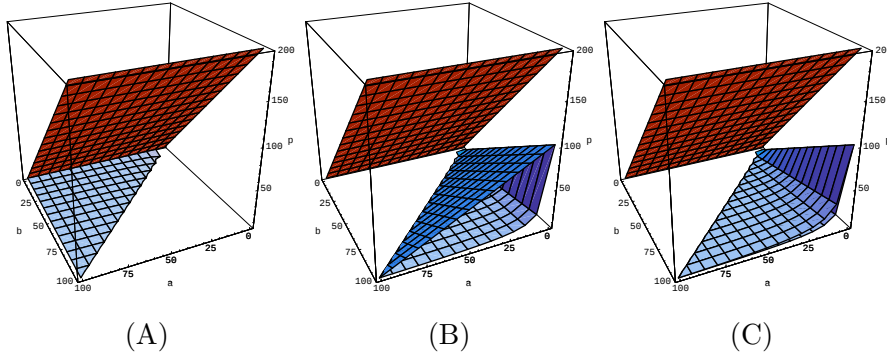


FIGURE 2. The existence domains for $a, b \in [0, 100]$.

Minimax methods

The direct method permits to reach the global minimizer of the functional, possibly under constraints. But in general the energy functional has other critical points. It is interesting to find them when we are looking for multiple solutions or when the functional is not bounded from below. Various methods were developed in this context. The simplest one is the Mountain-Pass Theorem due to Ambrosetti-Rabinowitz [3]. The underlying idea is that when the functional has a volcano shape, there exists a pass at a level c between the bottom of the crater (said 0) and the outside (said e), and c is a critical value of the functional:

Let X a Banach space, J a $\mathcal{C}^1$ functional defined on X , $e \in X$ and $r \in \mathbf{R}$ such that $\|e\| > r$, $J(0) = 0 = J(e)$ and $\inf_{\|u\|=r} J(u) > 0$ and let $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$ where

$$\Gamma := \{\gamma \in \mathcal{C}([0, 1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

If J satisfies the Palais-Smale condition at level c , i.e. every sequence (u_n) such that $J(u_n) \rightarrow c$ and $J'(u_n) \rightarrow 0$ has a convergent subsequence, then there exists a critical point $u \in X$ with $J(u) = c$.

Other geometries can entail the existence of critical points. Let us mention the Saddle-Point and Linking Theorems due to Rabinowitz or the Fountain Theorem of Bartsch (see e.g. [59, Theorems 2.11, 2.12 and 3.6]).

There also exist variants of the Mountain-Pass Theorem: a result of Brezis-Nirenberg, quoted in [5, Theorem 10], leads to a critical point belonging to a certain subclass of the Banach space and permits to obtain positive solutions. Rabinowitz [42, Theorem 9.12] proves that when the hypotheses of Mountain-Pass Theorem are satisfied on every finite-dimensional subspaces and the functional is even, the functional possesses a sequence (u_n) of critical points with energy tending to $+\infty$.

After having studied problem (3) by minimization, we worked on its non-quadratic variant (see Chapter II)

$$(4) \quad -\Delta u + |x|^a |u|^{q-2} u = |x|^b |u|^{p-2} u$$

on $\mathbf{R}^N$. As the left-hand term is not homogeneous, we cannot handle this problem using constraint minimization as done for (3): the Lagrange multiplier cannot be eliminated by rescaling. So we use the result of Rabinowitz in combination with a compact embedding of radial subspaces obtained as in Chapter I; leading to the existence of infinitely many radial solutions of the problem. This is carried out in Theorem II.2.7.

Breaking of symmetry

We have shown that the use of symmetries can overcome the lack of compactness, but this method has a defect: the solution obtained is not necessarily a global minimizer of the energy functional; global minimizers are often called “ground states” in physical problems. However, one can think that, when a problem has a symmetry (e.g. problem (3) is radially symmetric), the solution inherits it.

Symmetry inheritance is not true in general, and symmetry breaking already appears in very simple problems. As an example, consider the problem of joining the four vertices of a square. The problem has four axes of symmetry—as shown in Figure 3—whereas the solution has only two axes of symmetry: symmetry is broken.

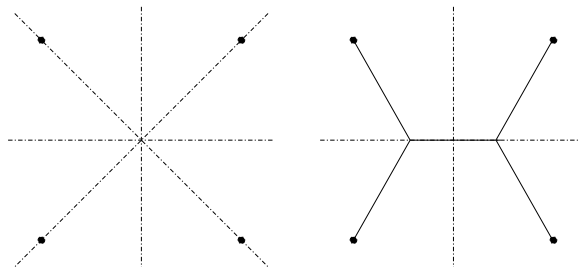


FIGURE 3. A simple example of symmetry breaking.

A more classical problem is the *Newton problem* consisting in the construction of a body of minimal resistance in a sparse liquid: the particles of the liquid interact only once with the body and do not interact with each other. We search a bounded shape represented by a function u , with support $B(0, 1)$, minimizing the resistance in the liquid. Newton modeled the resistance of the body by the functional

$$J(u) := \int_{B(0,1)} \frac{1}{1 + |\nabla u|^2} dx,$$

with $u : B(0, 1) \rightarrow \mathbf{R}$ vanishing on the boundary, concave and upper bounded by a constant M . Concavity is necessary otherwise the infimum of R is 0, taking profiles u with arbitrarily large oscillations in order to have $|\nabla u|$ arbitrarily large. The upper bound is also necessary as taking arbitrarily sharp cone-shaped profiles u implies the same kind of problem.

As $J(u)$ is invariant under rotation, Newton supposed in 1685 [37] that the solution was radially symmetric too. So he used a reduction to a one-dimensional problem and found the “solution”, see Figure 4.

But in 1996, Brock-Ferone-Kawohl [12] proved that the actual minimizer is not radially symmetric! Guasoni gives in his master thesis [27] a shape having less resistance than the Newton’s one, but still not optimal, see Figure 5. In fact, Newton’s problem remains open to this day.

Inheritance of radial symmetry is a fruitful field of research in the theory of partial differential equations. A major result in this area is the well-known work of Gidas-Ni-Nirenberg [26]. They consider

$$-\Delta u = f(|x|, u)$$

in $B(0, 1)$ with Dirichlet boundary conditions, i.e. with $u = 0$ on the boundary. By means of the Maximum Principle and the Moving Plane

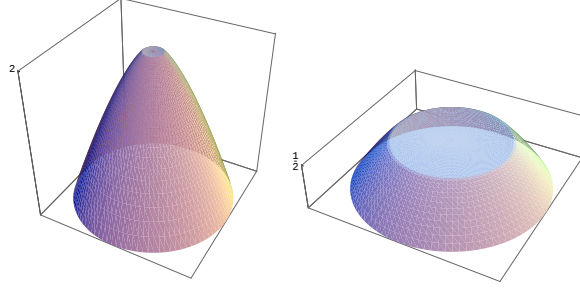


FIGURE 4. Optimal radial profiles obtained by Newton for an upper bound equal to 2 or $\frac{1}{2}$.

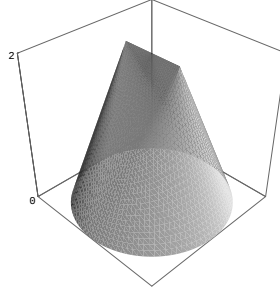


FIGURE 5. A better, non-radial but suboptimal profile discovered by Guasoni.

Method, they proved that the solution is radially symmetric and decreases with respect to $|x|$ provided f is sufficiently regular and decreases with respect to $|x|$.

Gidas-Ni-Nirenberg's result applies to equation

$$-\Delta u = \lambda u + u^{p-1}, \quad u > 0$$

on Ω , with Dirichlet boundary conditions when Ω is a ball. However, working on an annulus, Brezis-Nirenberg [11] proved that for λ sufficiently small there exist both radial and nonradial solutions. To obtain this result, they compare the infimum of the energy functional under constraint on the whole space $H_0^1(\Omega)$ and on its restriction to radially symmetric elements; showing that, for λ sufficiently small, the general minimizer is strictly lower than the radial one. Their method is also applicable to the problem $-\Delta u = u^{p-1}$; which admits both radial and nonradial solutions for p sufficiently close to (and lower than) 2^* .

Another symmetry-breaking result was obtained by Esteban [22]. She studied equation

$$(5) \quad -\Delta u + u = |u|^{p-2}u$$

on the exterior of a ball $B^c[0, 1]$ with Neumann boundary conditions, i.e. with $\frac{\partial u}{\partial n} = 0$ on the boundary. First, she had to overcome the lack of compactness induced by unboundedness of the domain to obtain the existence of solutions. This was done using the concentration-compactness theory developed by Lions [34]. Next, she proved that the ground state solution of the problem cannot be radially symmetric. The idea of the proof is similar to the one in [11], but more difficult to handle: compare the energy functional of both radial and nonradial solutions having same L^p norm, and show that the nonradial one has to be smaller.

This method of comparison between energies of radial and non-radial functions is the main idea underlying several works in this direction, as we will see here below.

Symmetry breaking appears for the problem (3) on balls $B(0, R)$ sufficiently large. This comes from the fact that for $2 + 2\frac{b-a}{N-1+\frac{a}{2}} < p$, $2 < p < 2^*$ and $ap < 2b$, there exists a radial minimizer of the energy, but not a global minimizer on the whole space $\mathbf{R}^N$. So, for R large enough, we show in Theorem I.3.1 that when the radius R tends to $+\infty$ the radial minimizer has more energy than the general minimizer.

We also obtain this kind of transition results by varying p when the exponent a and the radius of the ball are fixed, said $R = 1$. Theorems II.3.3 and II.4.4 show that for p close to 2, the solution tends to be radially symmetric for every b : there is no breaking of symmetry. But when p tends to 2^* , the minimizer tends to be nonradially symmetric for every b . There exists a limiting exponent b^* for b above which the solution is asymmetric, and the limit of b^* is $+\infty$ when p tends to 2 and is 0 when p tends to 2^* .

Multibump solutions

On $\mathbf{R}^N$, symmetry implies compactness and ensures the existence of infinitely many radial solutions. On the other hand, we have seen that some symmetry breaking can appear, depending on factors such as the radius of the ball or the exponent b .

Sometimes, this phenomenon of symmetry breaking can degenerate dramatically, leading to the existence of infinitely many nonequivalent nonradial solutions; where equivalent means that these solutions are the same, modulo a rotation of the domain. This phenomenon was

first observed by Coffman in 1984 [16]. Following Esteban, he studied equation

$$(5) \quad -\Delta u + u = |u|^{p-2}u,$$

but on $\Omega = D(R, d) := B(0, R + d) \setminus B[0, R]$, viz. the annulus of radius R and thickness d . He considered the Rayleigh quotient

$$Q(u) := \frac{\int_{\Omega} |\nabla u|^2 + u^2}{(\int_{\Omega} u^p)^{\frac{2}{p}}}$$

associated with the problem and restricted to subspaces of invariant functions under rotations of fixed amplitudes, exhibiting admissible functions having smaller Rayleigh quotients than the radial minimizer. And, when the radius R tends to infinity, more and more subclasses have less energy than the radial minimizer, which implies his main result.

Coffman's method works in dimension 2 for $p > 2$ or in higher even dimensions for $2 < p < 2^*(1 - \frac{1}{N})$. In 1990, Li [32] completed the analysis for $N \geq 4$ and $2 < p < 2^*$, and in 1997 Byeon [13] completed the study for the three-dimensional case.

The difference between equation (3) and (5) is the presence of the weights $|x|^a$ and $|x|^b$. So, working on an annulus, one can try to figure out how this new potential may modify the situation. When the ratio of d to R tends to 0, the weights tend to be constant on the annulus, so we can obtain the same result as Coffman. This is achieved in Section III.3.

Moreover, consider the similar equation on the unit ball. The presence of the weights $|x|^a$ and $|x|^b$ *pushes* the solution near the boundary. This creates an *annulus effect*, i.e. the bulk of the mass of the solution is located on an annulus. One can think that, at this point, multibump solutions can appear for the ball as well as in the case of the annulus. This is indeed true: working in a similar way, we exhibit the existence of infinitely many nonradial solutions when the exponents tends to ∞ , see Section III.2. As noted before, this kind of result has been established for the annulus in the eighties and nineties, but as far as we know, the case of the ball is new.