Economic growth and the use of non-renewable energy resources

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To the memory of my mother Pilar

Contents

Chapter 1

Introduction

This thesis is a contribution to the analysis of the relationship between the economic growth and the usage of non-renewable energy resources. More precisely, it is studied the conditions under which energy-saving technologies can sustain long-run growth, even if energy is mainly produced by means of non-renewable energy resources, such as fossil fuels.

1.1 Definition of non-renewable energy resources

First of all, let us provide the definition of non-renewable energy resources which will be considered along this thesis. To do that, it is needed to define the concept of natural resources. Following Hartwick (1986), the economic analysis defines natural resources as inputs derived from nature, which combined with labor, capital and raw materials produce goods and services. Moreover, as Rotillon (2005) observes, the economic viability of a natural resource depends on the available technology and the price of this resource. Then, a natural resource is deeply linked to both a specific technology and an economic environment. The theory of natural resources points out that these resources yield productive services over time (see for instance, Dasgupta and Heal (1979), Hartwick (1986), and Faucheux and Noël (1995)). Therefore, time is a central feature to study the usage of natural resources. Indeed, natural resource economics emphasizes the dynamical aspects of the economic problems involved by the usage of natural resources, as well as the decision-making in an inter-temporal setting.

Time allows us to identify different kinds of natural resources. The standard economic analysis distinguishes between renewable and non-renewable resources. Following the definition of Hartwick (1986) and Faucheux and Noël (1995), a renewable resource is a natural resource that can supply productive inputs to the economic system indefinitely. However, a non-renewable resource is a natural resource with a finite stock or supply. Nevertheless, the definition of renewable/non-renewable resource is not unique. Some authors apply the criterion of existence of a natural renewal capacity of the resource, *i.e.*, the resource reproduces itself without human intervention. Indeed, for authors such as Rotillon (2005), a renewable resource is defined as resource with natural renewal capacity. Otherwise, we call it non-renewable resource. However, Dasgupta and Heal (1979, page 3) prefer to use the concept of exhaustible resource: "a (natural) resource is exhaustible if it is possible to find a pattern of use which makes its supply dwindle to zero".

Hence, following the previous definitions, this thesis calls non-renewable energy resources to the natural resources for energy production, with finite stock and no natural renewal capacity, such as fossil (coal, oil and natural gas)[1] or nuclear fuels.

1.2 Energy and non-renewable resources

After establishing the definition of non-renewable energy resources, let us provide some empirical evidence showing the high dependence of the world economy on non-renewable energy resources as primary energy, which refers to energy in its initial form. To this aim, the next section studies the energy demand by analyzing the world primary energy demand data provided by the International Energy Agency (IEA (2004, 2006a)).

[1]Fossil fuels have natural renewal capacity but at geological time scale, *i.e.*, the speed of regeneration of the natural resource is too low to be significant for the economic system.

1.2.1 Energy demand

Following IEA (2004), in the absence of new government policies, world primary energy demand is projected to expand by almost 60% between 2002 and 2030, with an average annual growth rate of 1.7%[2], which is lower than in the past three decades (2% per year; see Table 1[3]). Moreover, the transport and power-generation sectors will absorb a growing share of global energy in line with past trends (see Table 2). Let us consider each fuel separately.

Fossil fuels (oil, natural gas and coal) dominate the global primary energy mix with a share about 80% (see Table 3, total fossil fuels). Moreover, according to IEA (2006a), fossil fuels will account for around 83% of the increase in world primary demand over the period 2004-2030.

Oil[4] is the largest fuel in global primary energy mix (see Table 3), with annual growth rate of its demand about 1.6% per year (see Table 1). However, IEA (2006a) predicts a decline, from 35.5% in 2002 to 33% in 2030. Oil use is concentrated in the transport sector, which is the main fuel in road, sea and air transportation. Indeed, IEA (2004) predicts that transport will account for two-thirds of the increase in total oil use.

Natural gas[5] is the third energy source in the world primary energy demand, with 21% of the total demand in 2002 (see Table 3). However, IEA (2004) forecasts that natural gas will be the second largest energy source after 2020 (24% in 2020, and 25% in 2030), overtaking coal[6]. Moreover, it is the fossil fuel with the highest growth rate (2.3% per year; see Table 1). The power sector will account for 60% of the increase in gas demand.

[2]IEA (2006a) estimates 1.6% over the period 2004-2030.

[3]All tables are provided at the end of the Introduction (Appendix).

[4]Oil includes crude oil, natural gas liquids, refinery feedstock and additives, other hydrocarbons and petroleum products.

[5]Natural gas includes natural gas associated and non-associated with petroleum deposits (but excluding natural gas liquids), and gas-works gas.

[6]Nevertheless, IEA (2006a) forecasts that this will not happen before 2030 due to higher estimates for coal.

Coal[7] is the second energy source, with 25.42% in 1971 and 24.75% in 2002 (see Table 3). However, IEA (2004) projects slower annual growth rate (1.5%)[8] than both oil (1.6%) and natural gas (2.3%) (see Table 1). Indeed, as it was just observed, coal will be overtaken by natural gas as the second largest energy source. Most of the increase in coal demand comes from power stations, but this will decrease in all OECD countries. However, industry, households and services will use more coal in non-OECD countries.

Nuclear fuels are exclusively used in power stations. However, the weight of nuclear power remains lower than fossil fuels (6.37% in 2004; see Table 3). Indeed, according to the IEA (2004), it will progressively decline because nuclear power has to compete with other technologies, and many countries decide to phase out nuclear power. However, IEA (2004, 2006a) also observes that these projections are subject to great uncertainty. Indeed, at this moment, the European Union (EU) shows great interest in nuclear power as a way to reduce both its dependence on imported natural gas and CO_2 emissions. The EU observes that nuclear fuels are more widely distributed than oil or natural gas, and they are CO_2-free[9].

Finally, let us briefly describe the size of renewable resources in the primary energy mix. Table 3 (total renewables) shows that the weight of renewable resources in the world primary energy demand is much lower than non-renewable resources (14.36% in 1971 and 13.16% in 2004). According to IEA (2004, 2006a), this trend will remain the same, with a

[7]Coal includes all kinds of coal. They are usually classified in hard coal and brown coal. Hard coal (coking and steam coal) has the highest gross calorific value (5700kcal/kg). However, brown coal contains a gross calorific value between 4156 kcal/kg and 5700 kcal/kg.

[8]1.6% according to IEA (2006a).

[9]Moreover, the EU wants to assume a leading position world-wide in the development of nuclear fusion. However, they recognize that important questions should be addressed, namely nuclear waste, nuclear safety and new future reactors. Indeed, the EU observes that nuclear fusion (if successful) only affects the european energy production in the long term (EU (2000)).

share around 13% in 2030[10]. The group called "other renewables" represents the energy source with the highest growth rate (see Table 1). Indeed, the IEA (2004) projects an annual growth rate of 5.7% until 2030. However, the share of these renewable resources remains very low (0.07% in 1971 and 0.51% in 2004) because they start from a very low level.

Summarizing, this section concludes that, regarding to energy, there is clear dependence of the world economy on non-renewable resources. Indeed, as it was shown before, non-renewable energy resources represents 86.84% of the world primary energy demand in 2004, with a share of fossil fuels about 80% (see Table 1, total non-renewables and total fossil fuels).

1.2.2 Energy reserves

As the previous section just observed, the world economy has high dependence on non-renewable energy resources. However, a key feature of non-renewable energy resource is that their stocks (reserves) are finite. Therefore, following Rotillon (2005), the usage of non-renewable energy resources calls attention to the "survival" (*i.e.*, economic growth and development) of the economic system due to the limited availability of these resources. Then, it is important to study the size of these reserves. Since 80% of the world primary energy demand is provided by means of fossil fuels, this section focuses on the analysis of fossil fuels reserves[11].

It is widely recognized that the estimation of fossil fuels reserves is extremely difficult. Indeed, IEA (2004, page 89) claims that "estimating (fossil fuels) reserves remains as much an art as a science". Then, before considering the data of these reserves, let us point out the following three estimation problems. Moreover, this will allow us to introduce the definitions of "proven", "probable" and "possible" reserves, which are

[10]However, the EU has set the target of doubling the share of renewable energy resources in the EU primary energy demand from 6% in 2002 to 12% in 2010 (EU (2000)).

[11]This thesis does not study neither nuclear fuels nor renewable energy resources.

concepts commonly used in the study of fossil fuel reserves.

First, the reserves of a fossil fuel are usually classified into "proven", "probable" and "possible" reserves. "Proven" reserves are accumulations of fuel that they have been discovered and there is a reasonable confidence that they can be extracted in economic profitable way[12]. When a discovered reserve is expected to be commercial, but with less certainty than "proven" reserves, it is called "probable" reserve. Moreover, if a discovered field is less likely to be recoverable than "probable" reserves, it is called "possible" reserve. However, problems arise because there are no legal international standards establishing how much proof is needed to demonstrate both the existence and economic profitability of a discovery. Nevertheless, in 1997, the Society of Petroleum Engineers (SPE) and the World Petroleum Congress established a benchmark definition of "proven", "probable" and "possible" reserves. Following the SPE (2000), "proven" reserves (or 1P) are reserves that the estimated quantity can be profitably extracted with probability of at least 90%. If this probability is of at least 50% then we talk about "proven plus probable" reserves (2P). Moreover, it is called "proven plus probable plus possible" reserves (3P) to those with a probability of at least 10%. However, there are no common information-disclosure requirements for reserves under the international accounting standards (Faucheux and Noël (1995), IEA (2004, 2006a) and Rotillon (2005)). Moreover, in practice, the methodologies to measure each category of reserves depend on their purpose. For example, the financial reporting standards (the Securities and Exchange Commission (SEC) is a well known case) are the most restrictive and only consider the lowest estimates with a confidence level above 90%, *i.e.*, they only consider "proven" reserves (1P) (see SEC (2003)).

A second problem with fossil fuels reserves is the reliability and lack of transparency of the reserves data reported by oil companies[13].

[12]Notice that this depends on the assumptions of extraction cost, technology and future prices.

[13]In 2004, Shell downgraded a fifth of its oil and natural gas reserves from "proven" to "probable" (or even to "possible"). Similarly, since the beginning of 2004, El Paso, (they revised down its "proven" North American gas reserves by 41%), Canada's Nexen and Husky Energy, Forest Oil, Vintage Petroleum and Western Gas Resources

Finally, as third problem, the reservoir engineers have to deal with great geotechnical and economic difficulties. For example, an increasing number of companies use seismic data to get estimations of their reserves. However, as it was just observed, the SEC does not fully trust on that technique and they forbid booking of reserves based on seismic data alone. Moreover, the price of fossil fuels plays important role. Indeed, when the price of fossil fuels rises, marginal fields become commercially viable and "possible" reserves can be shifted into "probable" or "proven" reserves. Moreover, investment in exploration is also affected by prices. As Appert (2004) observes, the elasticity of exploration and production expenditures to the crude oil price has averaged 0.5 in the last 15 years, *i.e.*, if the oil price rises 10%, the exploration and production expenditures increase by 5%, inducing new discoveries.

In the following, it is studied the data of fossil fuels reserves corresponding to oil, natural gas and coal. First, let us consider the oil reserves. Despite of the different approaches, the main data bases[14] provide an estimate about 1200 billions barrels of remaining "proven" reserves of oil for 2003[15]. However, for long-term studies, it is more convenient to consider the remaining ultimately recoverable resources, which includes "proven", "probable" and "possible" reserves from discovered fields, as well as oil not yet discovered. Nevertheless, very few estimates of this kind of reserves are available. Moreover, the existing estimates are subject to substantial uncertainty. The most reliable source for the estimation of global ultimately recoverable resources of oil is the US Geological Survey (USGS). In its most recent study (USGS (2000)), they announced the estimation of 3345 billions barrels of ultimate recoverable resources of conventional oil for 1996. However, IEA (2004), taking the IHS Energy database, observes that reserves additions from discoveries of new oilfields have fallen sharply since 1960s[16]. Knowing

revised their reserves concluding overestimation errors.

[14]Oil and Gas Journal, World Oil, Cedigaz, US Geological Survey (USGS), OPEC, IHS Energy, and BP.

[15]According to OGJ (2005), 1293 billion barrels at the end of 2005.

[16]Indeed, Rotillon (2005) notices that the current annual oil consumption is greater that the annual oil discoveries.

the estimated reserves, some studies calculate the ratio reserves divided by annual production of oil (R/P). This ratio provides the number of years that we can use a non-renewable resource if one keeps the same annual production of this resource. According to BP (2004), the R/P ratio at the end of 2003 was 41 years. However, since the reserves are constantly revised, R/P ratio usually changes on time (Rotillon (2005)). Indeed, one can observe that this ratio increased from 35 years in 1972 to 45 years in 1990[17].

Regarding to natural gas, Cedigaz (2006) observes that the "proven" reserves of this fossil fuel have increased steadily since the 1970s. Indeed, at the end of 2005, they estimate that these reserves are about 180 trillion cubic meters (tcm) (almost twice as high as twenty years ago), which are equivalent to 64 years of production at current rates (R/P ratio). The increase of these reserves emerges both from exploration and technical progress that have allowed existing reserves to be upgraded. Moreover, most of the new natural gas discoveries comes from oil exploration. However, as for oil, the recently discovered natural gas fields are generally smaller than in the past. Nevertheless, IEA (2004) observes that the potential gas resources are much greater than the "proven" reserves. As for oil, we use USGS (2000). The undiscovered reserves are estimated about 147 tcm, which accounts for 436 tcm of global ultimately recoverable resources of natural gas (in 2004, it was estimated that only 10% of these reserves were already used). It is estimated that 25% of the undiscovered natural gas reserves are associated with oil.

Finally, let us study the coal reserves. BP (2006) estimates that the world "proven" reserves of coal at the end of 2005 are about 909 billion tonnes, which accounts for a R/P ratio of 155 years at current production rates. Steam coal reserves are widespread; coking coal reserves are more limited; and brown coal is used by power plants situated close to

[17] As for the case of "proven" reserves, IEA (2004) notices that we can estimate a R/P ratio for ultimately recoverable resource and expected average production of oil. However, the rate at which the remaining ultimate resources can be converted to reserves is very uncertain.

mine, because its calorific capacity is low and coal, in general, is costly and difficult to transport.

Summing up, this section concludes that the reserves of oil and natural gas will last, respectively, 41 and 64 years at current rates. Moreover, the coal reserves will last around 155 years[18]. Hence, since the world economy shows a high dependence on fossil fuels, one concludes that the usage of non-renewable energy resources (in particular, fossil fuels) rises the question of the maintenance of economic growth and development in medium and long-term.

1.3 Energy-saving technical progress

The previous section observed that the usage of non-renewable energy resources (in particular, fossil fuels) calls attention to the "survival" of the economic system (Rotillon (2005)) in medium and long-term. Hartwick (1986), Faucheux and Noël (1995), and Rotillon (2005) notice that this concern was initially introduced in the 19^{th} century. Indeed, Jevons (1865) announced the end of the Industrial Revolution in the UK due to both the limited availability of coal reserves and the role of this energy resource for economic growth. In 20^{th} century, Forrester (1971) and Meadows *et al.* (1972) observed that the limited availability of non-renewable resources will stop economic growth. Later, Dasgupta and Heal (1974, 1979) also noticed that the usage of non-renewable energy resources implies a limit to economic growth and development of modern economies. Indeed, for Dasgupta and Heal (1979) a non-renewable resource is inessential if there is a feasible programme along which consumption is bounded away from zero. However, if feasible consumption must necessarily decline to zero in the long-run, the non-renewable resource is considered as essential[19]. Dasgupta and Heal (1974, 1979)

[18]However, as IEA (2004) observes, the usage of coal is limited due to pollution and high transport cost.

[19]As Pezzey and Withagen (1998) observe, Dasgupta and Heal (1974, page 4) provide a different definition of essentiality: "one regards a resource as being essential if output (...) is nil in the absence of the resource".

show the conditions under which, in absence of technical progress, non-renewable resources are essential or inessential. However, for the cases where resources are essential, they observe that essentiality can be overcome by means of technical progress. Indeed, a non-renewable energy resource regarded as essential in absence of technical progress, can be considered as inessential if technical progress is introduced[20]. Therefore, the same authors and others, namely Hartwick (1989), Newell *et al.* (1999), Löschel (2002), Carraro *et al.* (2003) and Smulders and Nooij (2003), believe in technical progress as a way to solve the trade-off between economic growth and the usage of non-renewable energy resources. Following these lines, technologies that use renewable energy resources have been proposed as a solution to this problem. However, the development of these technologies is slow. Indeed, as the last section observed, the share of renewable energy resources in the world primary energy demand is quite low (13.16% in 2002). Moreover, according to the IEA (2006a), this trend will remain the same, with a share around 13% in 2030.

However, Boucekkine and Pommeret (2004) point out that the usage of energy-saving technologies is an alternative and/or complement to the solution of renewable energy resources. Energy-saving technologies are technologies that reduce unnecessary energy-loss and improve the efficiency in both power-generation and end-uses. The idea behind energy-saving technical progress is energy efficiency, which is define as the inverse of the energy intensity (quantity of energy per unit of output). As the IEA (2004) observes, the world energy intensity decreases since 1970; and this trend will continue at annual decreasing rate about 1.5% for the period 2002-2030. Moreover, the IEA (2004, 2006a) recognizes the fundamental role of improving the energy efficiency of power-generation and end-uses, *i.e.*, energy-saving technologies. Furthermore, this interest is shared by all OECD countries (see for instance, Azomahou *et al.* (2004, 2006)), with particular attention of the EU through the Green Paper on energy efficiency "Doing more with less" (EU (2005)).

[20]This thesis studies two cases (respectively, Part I and II) where non-renewable energy resources are essential in absence of technical progress.

Indeed, the straight consequence of this Green Paper[21] was the Directive 2006/32/EC of the European Parliament and of the Council of 5 April 2006 on energy end-use efficiency and energy services.

EU (2005), and its subsequent Directive, observes that the EU could save at least 20% of its present energy consumption[22] by improving energy efficiency. There are many examples of energy-saving technologies (EU (2005)). An energy-saving electric bulb uses five time less current than a standard one. Replacing bulbs can easily save EUR 100 annually for an average household. The energy wasted by the stand-by control for lighting, heating, cooling and electric motors has increased during these days, because more and more appliances offer this feature. Electricity used in stand-by mode can reach between 5 and 10% of total electricity consumption in the residential sector (IEA). However, there are more efficient sleep modes that could be implemented. Regarding the car usage, the fleet of new passengers cars (more energy-saving) put on to the market in 2008/09 will reduce fuel consumption of around 25% compared to 1998. Furthermore, friction between tyres and the road accounts for up to 20% of a vehicle's consumption. Nevertheless, a properly performing tyres could reduce this consumption by 5%. In addition to that, an average driver using right tyres at the right pressure could save EUR 100 of the annual fuel bill (IEA).

1.4 Overview of the thesis

As it was established at the beginning of this introduction, the main target of this thesis is to study and emphasize the role of energy-saving technical progress as a solution to the trade-off between economic growth and the usage of non-renewable energy resources, such as fossil fuel. More

[21]Following the definition provided by European Commission, "Green papers are discussion papers published by the Commission on a specific policy area. Primarily they are documents addressed to interested parties who are invited to participate in a process of consultation and debate. In some cases they provide an impetus for subsequent legislation".

[22]This accounts for EUR 60 billion per year, which is equivalent to the present combined energy consumption of Germany and Finland.

precisely, this thesis studies the conditions under which energy-saving technologies can sustain long-run growth, even if energy is mainly produced by means of non-renewable energy resources. A general equilibrium framework is considered, giving special attention to the dynamical properties of the economy. In accordance with the well-known debate of complementarity *vs.* substitutability between physical capital and energy as production inputs, this thesis is divided into two parts.

The first part of this thesis assumes complementarity between physical capital and energy (Leontieff technology), which captures the idea of the existence of a minimum energy requirement to use a machine. Even if in contrast with the standard literature on non-renewable energy resources (see for instance, Dasgupta and Heal (1974, 1979), Solow (1974a,b), and Stiglitz (1974)), which assumes substitutability, the assumption of complementarity is indeed supported by various empirical studies (see for instance, Berndt and Wood (1974) and Hudson and Jorgenson (1975)). This relationship of complementarity allows us to introduce the assumption of different generations of machines coexisting in each period, *i.e.*, vintage capital model, by adding a new variable to the firm's problem: physical capital replacement. Indeed, in this first part of the thesis, it is provided a theoretical study of physical capital replacement, *i.e.*, vintage effect, which is an important environmental policy in modern economies when new machines are assumed to be more energy-saving. There are some few empirical studies dealing with the vintage effect linked to energy-saving technical progress. For instance, Azomahou *et al.* (2006), taking the Structural Analysis Database of the OECD and the energy databases of the IEA, empirically test (at sectoral level of OECD countries) the vintage effect considered in this thesis. Taking the 14 main sectors of the industry, they find empirical evidence of this vintage effect, with high heterogeneity among sectors.

Following the standard literature on non-renewable energy resources, this second part of the thesis assumes substitutability between capital and energy, where there is no minimum energy requirement to use a machine. A first implication of this assumption is that firms do not replace any machine[23]. Indeed, in this case, firms always keep the old equipment assigning to them an infinitely small amount of energy. In this second part of the thesis, it is observed that the standard literature on non-renewable energy resources gives central position to physical capital accumulation to offset the constraint on production possibilities due to non-renewable energy resources. This literature assumes the same technology for both physical capital accumulation and consumption, which implies (among other things) that the energy intensity of both sectors is the same. However, data do not support this implication (see for instance, Azomahou *et al.* (2006)) and suggest that physical capital accumulation is more energy-intensive than consumption. Following that, this second part of the thesis studies the implications of assuming that physical capital accumulation is relatively more energy-intensive than consumption, giving special attention to the role of energy-saving technologies.

1.4.1 Part I: Complementarity between capital and energy

This first part of the thesis contains two chapters[24]. The contribution of Chapter 2 is mainly methodological. Indeed, as far as we know, this is the first paper on energy resources that considers endogenous replacement in a general equilibrium framework. This paper is published in Resource and Energy Economics (Pérez-Barahona and Zou, 2006a). Chapter 3 applies the methodology developed in Chapter 2. The main contribution of Chapter 3 is to analyze the effect of a tax on the energy expenditure of firms as a way to encourage the replacement of the old equipment by new machines, which are assumed to be more energy-saving. This paper is published in the book Economic Modelling of Climate Change and Energy Policies (Pérez-Barahona and Zou, 2006b).

[23]Obviously, without replacement, some dynamical properties of the model are lost. However, one gains more tractability of the problem.

[24]Chapters 2 and 3 have been written in co-authorship with Benteng Zou.

Chapter 2: A comparative study of energy-saving technical progress in a vintage capital model

In this chapter, it is analyzed the hypothesis about the effectiveness of energy-saving technologies to reduce the trade-off between economic growth and energy preservation. In a general equilibrium vintage capital model with embodied energy-saving technical progress, it is proved that positive growth is only possible if the growth rate of the energy-saving technical progress is greater than the decreasing rate of the energy supply.

This model incorporates two new elements with respect to the standard framework. First, it is assumed embodied technical progress in contrast to the typical neoclassical specification of neutral and disembodied technical progress. Second, it is considered a vintage capital model, with endogenous scrapping decision. The standard models consider homogenous and infinitely lived capital stock.

Chapter 3: Energy-saving technical progress in a vintage capital model

Along the same lines as the previous chapter, Chapter 3 also studies the hypothesis of the trade-off between energy reduction and growth, which is less severe if energy conservation is raised by energy-saving technologies. However, it is studied now the effect of a tax on the energy expenditure of firms as a way to encourage investments in energy-saving technologies.

This model is an extension, to the general equilibrium case, of the work of Boucekkine and Pommeret (2004). Moreover, it is considered (exogenous) technical progress embodied in new capital goods, which are introduced in the economy through a vintage technology with endogenous obsolescence (scrapping) rule. The analysis of this paper is focused on the long-run consequences of modifications of a tax on the energy expenditure of firms. The methodology developed here is a combination of analytical and numerical methods. It is found that our model is very rich to capture the different elements that affect the long-run behavior of our economy. In particular, it is pointed out the frequently ignored issue of equipment replacement, which plays important role in energy-saving technical change. Indeed, an important consequence of considering such a replacement is that the increase of a tax on the energy expenditure of firms does not necessarily induce earlier replacement of machines if this tax is high enough. The reason is the following. To offset the effect of a higher energy expenditure tax, firms should replace earlier their equipment. Since replacement is costly, firms prefer to take higher energy expenditure than a faster (and too costly) equipment replacement.

1.4.2 Part II: Substitutability between capital and energy

Chapter 4 and 5 are collected in this second part of the thesis. At this point, following the standard literature on non-renewable energy resources, it is considered substitutability between physical capital and energy. However, in contrast with that literature (where it is also assumed the same technology for both physical capital accumulation and consumption), this second part studies the implications of assuming that physical capital accumulation is relatively more energy-intensive than consumption, giving special attention to the role of energy-saving technologies. Chapter 4 considers total depreciation of physical capital for each period. Then, in this chapter, physical capital is regarded as a flow variable. However, Chapter 5 eliminates this simplification, assuming physical capital as a stock variable, *i.e.*, a reproducible good.

Chapter 4: The problem of non-renewable energy resources in the production of physical capital

This chapter studies the possibilities of technical progress to deal with the growth limit problem imposed by the usage of non-renewable energy resources, when physical capital production is relatively more energy-intensive than consumption. In particular, this work presents the conditions under which energy-saving technologies can sustain long run growth, even if energy is produced by means of non-renewable energy resources. The mechanism behind is energy efficiency.

Standard papers on non-renewable energy resources (see for instance, Dasgupta and Heal (1974, 1979), Stiglitz (1974), Solow (1974a), Hartwick (1989), and Smulders and Nooij (2003)) assume the same technology for both physical capital production and consumption (which implies that the energy intensity of both sectors is the same), as well as the existence of reasonable substitution possibilities between energy and physical capital. Moreover, this literature also claims that if energy is produced by means of non-renewable resources, physical capital accumulation could offset the constraints on production possibilities due to non-renewable energy resources (Dasguptan and Heal (1979)). Nevertheless, Chapter 4 argues that this explanation about physical capital accumulation as a solution to the problem of non-renewable energy resources is not totally satisfactory since there is empirical evidence showing that physical capital production is relatively more energy-intensive than consumption (Azomahou *et al.* (2006)). Indeed, under this hypothesis, if energy were produced by means of non-renewable energy resources, this would limit growth through physical capital accumulation. Then, all solution based on rising physical capital accumulation can have dubious effectiveness to solve the trade-off between economic growth and the usage of non-renewable energy resources if one does not analyze the role of energy in physical capital accumulation, *i.e.*, equipment good production. Following that idea, this chapter shows that, when physical capital production is relatively more energy-intensive than consumption, technical progress (in particular, energy-saving technical progress) plays a central role to guarantee long-run growth. Indeed, it is established that if the growth rate of technical progress is high enough there is balanced growth path (BGP) with positive growth of all the endogenous variables. Otherwise, the economy decreases at constant rate, with all the endogenous variables converging to zero asymptotically. This result contrasts with the claim of Stiglitz (1974) and Dasgupta and Heal (1979). These authors argue that, even without technical progress, capital accumulation can offset the constraint on production possibilities due to non-renewable energy resources. However, Chapter 4 shows that the economy decreases (converging to zero asymptotically) if there is no growth of technical progress.

Chapter 5: Capital accumulation and non-renewable energy resources: a Special Functions case

On the same lines as Chapter 4, this chapter studies the implications of assuming different technologies for physical capital accumulation and consumption. The main difference with respect to Chapter 4 is that the assumption of total depreciation of physical capital for each period is eliminated, considering physical capital as a stock instead of a flow. The four main contributions of this chapter are the following.

First, as Chapter 4 shows, in contrast with the claim of Stiglitz (1974) and Dasgupta and Heal (1979), technical progress plays a crucial role to sustain long-run growth in this economy. Indeed, Chapter 4, assuming that physical capital accumulation is more energy-intensive than consumption, and total capital depreciation (*i.e.*, physical capital as a flow variable), shows that the economy decreases at constant rate (with all the endogenous variables converging to zero asymptotically) if there is no growth of technical progress. Moreover, considering physical capital as a stock variable, Chapter 5 proves that neither balanced growth path (BGP) nor steady state equilibrium exist if there is no growth of technical progress.

The second contribution of this chapter is mainly methodological. This model entails some technical difficulties. In particular, one has to deal with an optimal control problem with mixed constraints and two states variables: capital accumulation and non-renewable energy resource stock. Moreover, the capital accumulation law is a non-linear function with respect to investment. However, it is possible to provide a close-form solution of the optimal solution paths of our variables in levels for every time t by applying the technique of Special Functions representation. This is a very well known method in the field of mathematical physics to describe the behavior of dynamical systems. This tool provides a full analytical characterization of both short and long-run dynamics, without using qualitative techniques, such as phase diagrams. This chapter uses a family of Special Functions called Gauss Hypergeometric functions (see Boucekkine and Ruiz-Tamarit (2004) for an application to the Lucas-Uzawa model).

Applying the previously presented technique of Special Functions representation, it is provided the necessary and sufficient conditions for existence and uniqueness of BGP, which is the third contribution of this paper. Moreover, it is proved that the economy asymptotically converges to the BGP. This result contrast with Chapter 4, where the economy does not present dynamical transition. The reason is that Chapter 4 assumes physical capital as a flow variable. Indeed, Chapter 5 eliminates that simplification by considering physical capital as a durable good.

Finally, since the Special Functions representation provides a close-form solution of the optimal solution paths of our variables in levels, the fourth contribution of Chapter 5 is that one can explicitly study the monotonicity properties of the optimal trajectories in levels, which is typically unallowed in standard approaches. It is obtained that the assumption of physical capital as relatively more energy-intensive sector than consumption, and the technical progress (in particular, energy-saving technical progress), introduce new results on the non-monotonicity of the optimal trajectories in levels, which contrast with Dasgupta and Heal (1974)'s results about the monotonicity of consumption and extraction of non-renewable energy resources.

Appendix

Table 1

Average annual growth rate of world primary energy demand (%)

Energy	1971-2002	2002-2010	2002-2020	2002-2030
Coal	1.70	1.8	1.6	1.50
Oil	1.4	2	1.8	1.6
Natural gas	2.9	2.7	2.6	2.3
Nuclear	10.8	1.5	0.6	0.4
Hydro	2.5	2.6	2	1.8
Biomass and waste	1.6	1.5	1.4	1.3
Other renewables[a]	8.5	8	6.2	5.7
Total primary energy	2	2.1	1.9	1.7

[a]Geothermal, solar, wind, tidal and wave energy.

Source: World Energy Outlook. IEA (2004)

Table 2

Sectoral shares in world primary energy demand (%)

Energy	1971	1990	2002	2004	2010	2020	2030
Power-generation	21.88	32.07	36.38	36.89	37.83	39.08	40.21
Industry	27.19	24.44	21.61	22.41	21.14	20.82	20.46
Transport	15.46	16.43	17.66	17.57	18.29	19.13	19.85
Own use & losses	9.50	10.05	9.91	9.50	9.87	9.77	9.61
Other[a]	25.98	17.01	14.43	13.63	12.87	11.21	9.86
Total	100	100	100	100	100	100	100

[a]This includes residential, services, public and agriculture sectors.

Source: World Energy Outlook. IEA (2004, 2006a)

Table 3

World primary energy demand (%)

Energy	1971	1990	2002	2004	2010	2020	2030
Coal	25.42	25.00	23.09	24.75	22.66	22.17	21.84
Oil	43.59	36.43	35.53	35.17	35.33	35.22	34.97
Natural gas	16.11	19.24	21.17	20.55	22.17	23.96	25.05
Nuclear	0.52	6.01	6.69	6.37	6.38	5.39	4.63
Hydro	1.88	2.12	2.17	2.16	2.26	2.23	2.21
Biomass and waste	12.41	10.57	10.82	10.50	10.37	9.91	9.73
Other renewables[a]	0.07	0.64	0.53	0.51	0.83	1.12	1.55
Total primary energy	100	100	100	100	100	100	100
Total fossil fuels[b]	85.12	80.66	79.80	80.46	80.16	81.35	81.86
Total non-renewables[c]	85.64	86.67	86.49	86.84	86.54	86.73	86.50
Total renewables[d]	14.36	13.33	13.51	13.16	13.46	13.27	13.50

[a]Geothermal, solar, wind, tidal and wave energy.
[b]Coal + oil + natural gas.
[c]Fossil fuels + nuclear.
[d]Hydro + biomass and waste + other renewables.

Source: World Energy Outlook. IEA (2004, 2006a)

Chapter 2

A comparative study of energy-saving technical progress in a vintage capital model

(Pérez-Barahona A. and Zou B., 2006. A comparative study of energy saving technical progress in a vintage capital model. Resource and Energy Economics. Volume 28, Issue 2 , May 2006. Pages 181-191)

2.1 Introduction

Fossil fuel–more precisely petroleum and its refinery products–is an essential input in all modern economies. It has been argued that the limited availability of this basic input and the stabilization of greenhouse gases concentration call for a reduction of fossil fuel consumption. However, the reduction in petroleum consumption could have a negative impact on economic growth and development through cutbacks in energy use (Smulders and de Nooij (2003)). Therefore, there is a clear trade-off between energy reduction and growth.

Some authors (see for instance, Carraro *et al.* (2003)) suggest that this trade-off could be less severe if energy conservation is raised by energy-saving technologies. In this chapter, we re-examine the exhaustion problem of fossil fuel. In particular, we study the previous trade-off in a general equilibrium framework with energy-saving technical progress. This model based on Boucekkine *et al.* (1997), considers an economy with exogenous energy-saving technical progress embodied in the new equipment. As Baily (1981) observes, technical advances are typically incorporated into the economy through investment. Therefore, the old capital goods get less and less efficient over time, which might well induce the firms to scrap them (obsolescence). In our economy, we assume that different vintages of capital coexist in each period. Since new vintages are less energy consuming, firms may decide to replace the oldest and less efficient vintage. Indeed, if we model the idea of minimum energy requirement to use a machine by assuming complementarity between capital and energy inputs, finite scrapping time is optimal (Boucekkine and Pommeret (2004)). This idea is implemented in this chapter, and it is consisted with the empirical evidence put forward by Hudson and Jorgenson (1974), or Berndt and Wood (1975). Notice that the standard literature on non-renewable energy resources (see for instance, Dasgupta and Heal (1974, 1979), Solow (1974a,b), and Stiglitz (1974)) assumes substitutability between capital and energy inputs. Indeed, under this assumption, it is never optimal to scrap equipment, *i.e.*, infinite scrapping age[1].

Our model incorporates two new elements with respect to the standard framework. First, we assume embodied technical progress in contrast to the typical neoclassical specification of neutral and disembodied technical progress. Second, we consider a vintage capital model, with endogenous scrapping decision. The standard models consider homogenous and infinitely lived capital stock.

We perform a comparative study to contrast constant and decreasing returns to scale, for two possible scenarios: constant (optimistic) and

[1]Sustainability implies that it is always optimal to assign an infinitely low amount of energy to the less efficient machines.

decreasing (pessimistic) exogenous energy supply. We find that, under the assumption of existence of balanced growth path (BGP) defined by constant growth rate of all the endogenous variables and constant scrapping age, constant returns to scale achieves positive long run growth if the growth rate of the energy-saving technical progress is greater than the decreasing rate of the energy supply. As we will show, this is not a trivial result. Indeed, constant return to scale can not ensure long run growth in our model because of the endogenous scrapping decision and the minimum energy requirement to use a machine.

The chapter is organized as follows. In Section 2.2, we describe the general case model, with the representative consumer's problem and the rules that depicts both the optimal investment and the scrapping behavior of firms. The BGP is presented in Section 2.3, where we show the necessary conditions for its existence and uniqueness for both constant and decreasing returns to scale. Finally, some concluding remarks are considered in Section 2.4.

2.2 The model

Following Boucekkine *et al.* (1997), we consider an economy where the population is constant and there is only one good (the numeraire good), which can be assigned to consumption or investment. The good is produced in a competitive market by means of a technology defined on vintage capital. Both constant and decreasing returns to scale are considered here. We also assume competitive labor market and exogenously available energy supply.

2.2.1 Household

Let us assume that the representative household considers the following standard inter-temporal maximization problem with constant relative risk aversion (CRRA) instantaneous utility function

$$\max_{c(t)} \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} \exp(-\rho t)dt, \tag{1}$$

subject to the budget constraint

$$\begin{gathered}\dot{a}(t) = r(t)a(t) - c(t),\\ a(0) \text{ given},\\ \lim_{t\to\infty} a(t)\exp\left(-\int_0^t r(z)dz\right) = 0,\end{gathered} \tag{2}$$

with initial wealth a_0, where $c(t)$ is per-capita consumption, and $a(t)$ is per-capita asset held by the consumer at interest rate $r(t)$ which is taken as given for the household. θ measures the constant relative risk aversion, and ρ is the time preference parameter (it is assumed to be a positive discount factor). Since in this chapter we do not explicitly treat labor, we assume that it has not value leisure for the consumer. Then, in order to simplify the model, it is considered inelastic labor supply normalized to one. The corresponding necessary conditions are $r(t) = \rho + \theta\frac{\dot{c}(t)}{c(t)}$, with $\lim_{t\to\infty}\lambda(t)a(t) = 0$, where $\lambda(t)$ is the co-state variable associated with the wealth accumulation equation.

2.2.2 Firms

The good is produced competitively by a representative firm solving the following optimal profit problem

$$\max_{y(t),i(t),T(t)} \int_0^\infty [y(t) - i(t) - e(t)P_e(t)]\, R(t)dt, \tag{3}$$

subject to output

$$y(t) = A\left(\int_{t-T(t)}^t i(z)dz\right)^\alpha, \quad 0 < \alpha \le 1, \quad i(t) \ge 0, \tag{4}$$

and energy demand

$$e(t) = \int_{t-T(t)}^t i(z)\exp(-\gamma z)dz, \quad 0 < \gamma < \rho, \tag{5}$$

where $i(t)$ is the investment of the representative firm, and the initial conditions $i(z)$ are given for all $z \le 0$. It is straight forward that there are different capital vintages (that is, different time investment), which co-exist to produce final goods[2]. $e(t)$ is the demand of energy at a price $P_e(t)$. The firm considers the energy price as given; however, it is endogenously determined in the energy market equalizing the demand and supply of energy.

Equation (4) is our technology defined on vintage capital. The energy demand is obtained from equation (5). Here, $\gamma > 0$ represents the growth rate of energy-saving technical progress and $T(t)$ is the age of the oldest operating machine or scrapping age. The discount factor $R(t)$ takes the form $R(t) = \exp\left(-\int_0^t r(z)dz\right)$. Finally, we assume that $0 < \gamma < \rho$ to well define our integral[3].

Notice that the new technology is more energy-saving. Certainly, each vintage $i(t)$ has an energy requirement $i(t)\exp(-\gamma t)$ [4]. Moreover, it is important to observe that we assume complementarity between capital and energy (Leontieff technology), which models the idea of minimum energy requirement to use a machine. Furthermore, this assumption is undeniable from numerous studies; see for instance, Hudson and Jorgenson (1974), or Berndt and Wood (1975). As we observed in the introduction, this complementarity ensures a finite optimal scrapping age[5].

We define the capital stock as

$$K(t) = \int_{t-T(t)}^{t} i(z)dz, \tag{6}$$

[2]We can consider an alternative technology where decreasing returns to scale only affects new vintages (not all the active vintages): $y(t) = A\int_{t-T(t)}^{t} i(z)^{\alpha}dz$. Similarly to our case, it can be checked that the results remain the same.

[3]It is a standard assumption in the exogenous growth literature to have a bounded objective function.

[4]This assumption is central in the early vintage capital models. See Solow *et al.* (1966).

[5]$T(t) < \infty$ is an essential and standard assumption considering the possibility of replacement ($T(t) \rightarrow \infty$ implies no scrapping). See for example, d'Autume and Michel (1993) and Bardhan (1969).

and the optimal life of machines of vintage t [6]

$$J(t) = T(t + J(t)). \tag{7}$$

From the first order condition (FOC)[7] for $i(t)$, we get the *optimal investment rule*

$$\begin{aligned} &\int_t^{t+J(t)} \alpha A \left(\int_{\tau - T(\tau)}^{\tau} i(z)dz \right)^{\alpha - 1} \exp\left(- \int_t^{\tau} r(z)dz \right) d\tau = \\ &\quad 1 + \int_t^{t+J(t)} P_e(\tau) \exp(-\gamma t) \exp\left(- \int_t^{\tau} r(z)dz \right) d\tau, \end{aligned} \tag{8}$$

where the left hand side (LHS) is the discounted marginal productivity during the whole lifetime of the capital acquired in t, 1 is the marginal purchase cost at t normalized to one, and the second term on the right hand side (RHS) is the discounted operation cost at t.

The optimal investment rule establishes that firms should invest at time t until the discounted marginal productivity during the whole lifetime of the capital acquired in t exactly compensates for both its discounted operation cost and its marginal purchase cost at t.

From the FOC for $T(t)$, we have the *optimal scrapping rule*

$$A\alpha \left(\int_{t-T(t)}^{t} i(z)dz \right)^{\alpha - 1} = P_e(t) \exp[-\gamma(t - T(t))]. \tag{9}$$

The optimal scrapping rule states that a machine should be scrapped as soon as its marginal productivity (which is the same for any machine whatever its age) no longer covers its operation cost (which rises with age).

Here the marginal productivity is given by $\alpha A(\int_{t-T(t)}^{t} i(z)dz)^{\alpha-1}$, and $P_e(t)\exp[-\gamma(t - T(t))]$ represents the operation cost.

[6]Notice that $T(t) = J(t - T(t))$.

[7]Following Boucekkine *et al.* (1997), we can consider an intermediary sector to create intermediate inputs for the final production. It is easy to observe that the case of symmetric equilibrium is exactly equivalent to our model without intermediate good sector. However, as Krusell (1995) observes, if the product-specific returns to R&D are strong enough, there may be asymmetric steady-state equilibria, in which large and small capital firms coexist. Nevertheless, this is not the case in our model with exogenous technical progress.

2.2.3 Decentralized equilibrium

The (decentralized) equilibrium of our economy is characterized by equation (2), the necessary conditions of the household's problem, equations (4)–(7), the optimal investment rule, the optimal scrapping rule, and the following two additional equations to close the model: $c(t) + i(t) = y(t)$ and the equilibrium condition of the energy market $e(t) = e_s(t)$. $e_s(t)$ is the available energy supply[8], which in our model is assumed to be exogenous.

2.3 Balanced growth path

2.3.1 Definition of BGP

Let us define our BGP equilibrium as the situation where all the endogenous variables grow at constant rates, with constant and finite scrapping age $T(t) = J(t) = \overline{T}$ (Terborgh-Smith result[9]). Boucekkine *et al.* (1997) considere a model equivalent to our case with constant returns to scale ($\alpha = 1$). They provide a sufficient condition for existence of a particular BGP with both constant scrapping age and constant available energy supply.[10] For the case of decreasing returns to scale ($0 < \alpha < 1$), we find that the analytical proof of existence of such a BGP is not possible. Moreover, it is not difficult to check that an alternative BGP, with non-constant scrapping age, is not compatible with constant growth of the other endogenous variables.

As a consequence, in order to compare constant and decreasing returns to scale, we present the necessary conditions for existence and uniqueness of our BGP for both constant and decreasing returns to scale.

[8]The available energy supply is a flow (exogenous) variable; for example, petrol or any petroleum refinery product to generate energy. Here we do not explicitly treat neither extraction sector nor producer countries.

[9]See Terborg (1949) and Smith (1961).

[10]They assume a technology that saves labor instead of energy, with constant (exogenous) labor supply.

2.3.2 Necessary conditions

For the general case $0 < \alpha \leq 1$, taking the necessary conditions of the household's problem along the BGP, we get that

$$r(t) = \rho + \theta\gamma_c = \text{constant} = r^*, \tag{10}$$

and

$$\exp\left(-\int_t^\tau r(z)dz\right) = \exp[-r^*(\tau - t)], \tag{11}$$

where γ_c is the growth rate of consumption.

Taking equation (9) into equation (8), differentiating with respect to t and rearranging terms, we obtain:

$$[\exp(\gamma\overline{T}) - 1] - \frac{\gamma}{\gamma_{P_e} - r^*}\{\exp[(\gamma_{P_e} - r^*)\overline{T}] - 1\} =$$

$$\frac{r^*}{\overline{P}_e}\exp[(\gamma - \gamma_{P_e})t], \tag{12}$$

where γ_{P_e} and $\overline{P}_e$ are, respectively, the growth rate and the level of the energy prices. The LHS is constant for any t along the BGP, and the RHS is a function of t. So the equality holds if and only if $\gamma = \gamma_{P_e}$. As in the standard growth model, this result states that, in terms of energy-saving, energy prices grow at the same rate as productivity. Moreover, Boucekkine and Pommeret (2004)[11] observe that this result can be justified in the context of intertemporal equilibrium model of optimal extraction of non-renewable resources.

By definition of $K(t)$, we have that, along the BGP,

$$K(t) = \begin{cases} \frac{\bar{i}}{\gamma_i}[1 - \exp(-\gamma_i\overline{T})]\exp(\gamma_i t), & \text{if } \gamma_i > 0, \\ \bar{i}\overline{T}, & \text{if } \gamma_i = 0, \end{cases} \tag{13}$$

where $i(t) = \bar{i}\exp(\gamma_i t)$. Then, the growth rate of investment, γ_i, and the growth rate of capital stock, γ_K, are equal.

[11]Our result $\gamma_{P_e}(= \gamma) < r^*$ is also consistent with the assumption made in Boucekkine and Pommeret (2004) about the growth rate of energy prices lower than the interest rate. Indeed, they point out that if $\gamma_{P_e} > r^*$ the firm would have an incentive to infinitely get in debt to buy an infinite amount of energy.

Moreover, from equation (9) and $\gamma = \gamma_{P_e}$,

$$A\alpha K(t)^{\alpha-1} = \overline{P_e}\exp(\gamma\overline{T}), \tag{14}$$

where $P_e(t) = \overline{P}_e \exp(\gamma\overline{T})$. By substituting equation (13) into equation (14), it yields

$$\exp[\gamma_i(\alpha-1)t] = \frac{\overline{P_e}\exp(\gamma\overline{T})}{A\alpha\overline{K}^{\alpha-1}}. \tag{15}$$

It is easy to see that equation (15) holds if and only if $\alpha = 1$ and/or $\gamma_i = 0$. Then, at this point, we have to distinguish between constant and decreasing returns to scale. If we have decreasing returns to scale ($0 < \alpha < 1$) then $\gamma_i = 0$. However, for the case of constant returns to scale ($\alpha = 1$) γ_i is *a priori* undetermined.

Constant returns to scale

$\alpha = 1$ can not ensure long run growth in our model because of the endogenous scrapping decision and the minimum energy requirement to use a machine. Now, we are going to study the necessary conditions for our BGP according to different values of γ_i.

Let us assume that the energy market is in equilibrium along the BGP, *i.e.*, energy demand equals energy supply ($e_s(t)$). We make a distinction between two possible scenarios. There is an optimistic scenario with constant available energy supply. However, there is a gloomy situation where the available energy supply is decreasing[12].

Case A: $\gamma_i = 0$

From equation (5) we get the energy demand along the BGP

$$e(t) = \frac{\bar{i}}{\gamma}[\exp(\gamma\overline{T}) - 1]\exp(-\gamma t). \tag{16}$$

Case A.1: constant available energy supply $e_s(t) = \overline{e}_s$

Equalizing $e(t) = \overline{e}_s$ in equation (16), we get

$$\frac{\bar{i}}{\gamma}[\exp(\gamma\overline{T}) - 1]\exp(-\gamma t) = \overline{e}_s. \tag{17}$$

[12]There is no point in assuming increasing available energy supply, since we are considering resources subject to exhaustion. The most optimistic scenario is constant available energy supply.

Since RHS is constant, LHS has to be constant. This is impossible because $\gamma > 0$ and $\overline{T}$ is finite. Then, we get a contradiction. In case B we are going to study $\gamma_i \neq 0$.

Case A.2: decreasing available energy supply $e_s(t) = \overline{e}_s \cdot \exp(-\gamma_{e_s} t)$**, with** $\gamma_{e_s} > 0$.

Equalizing $e(t) = \overline{e}_s \exp(-\gamma_{e_s} t)$ in equation (16) yields

$$\frac{\overline{i}}{\gamma}[\exp(\gamma\overline{T}) - 1]\exp(-\gamma t) = \overline{e}_s \exp(-\gamma_{e_s} t). \tag{18}$$

Then, to have BGP γ has to equal γ_{e_s}. This means that the economy chooses a growth rate of energy-saving technical progress equal to the decreasing rate of energy supply. As a consequence:

$$i(t) = i^* = \frac{\overline{e}_s \gamma}{\exp(\gamma\overline{T}) - 1}. \tag{19}$$

Since $\gamma_i (= \gamma_K) = 0$, from the production function $y(t) = AK(t)$, $\gamma_y = \gamma_K = 0$.

From the budget constraint $y(t) = c(t) + i(t)$, $\gamma_c = 0$.

Case B: $\gamma_i \neq 0$

Taking $i(t) = \overline{i}\exp(\gamma_i t)$ in equation (5), the energy demand along the BGP is given by

$$e(t) = \begin{cases} \frac{\overline{i}}{\gamma_i - \gamma}\{1 - \exp[-(\gamma_i - \gamma)\overline{T}]\}\exp[(\gamma_i - \gamma)t], & \text{if } \gamma_i \neq \gamma, \\ \overline{i}\overline{T}, & \text{if } \gamma_i = \gamma. \end{cases} \tag{20}$$

Case B.1: constant available energy supply $e_s(t) = \overline{e}_s$

If $\gamma_i \neq \gamma$, equalizing $e(t) = \overline{e}_s$ in equation (20), we obtain

$$\overline{e}_s = \frac{\overline{i}}{\gamma_i - \gamma}\{1 - \exp[-(\gamma_i - \gamma)\overline{T}]\}\exp[(\gamma_i - \gamma)t]. \tag{21}$$

Similar to the case A.1, we get a contradiction because $\gamma > 0$ and $\overline{T}$ is finite.

If $\gamma_i = \gamma$, equation (20) yields

$$\overline{e}_s = \overline{i}\overline{T}. \tag{22}$$

Hence, this case yields BGP with $\gamma_i (= \gamma_K) = \gamma$. Moreover, as $y(t) = AK(t)$, then $\gamma_y = \gamma_K (= \gamma)$. From the budget constraint $y(t) = c(t) + i(t)$, we also achieve that $\gamma_c = \gamma$.

Case B.2: decreasing available energy supply $e_s(t) = \overline{e}_s \exp(-\gamma_{e_s} t)$, with $\gamma_{e_s} > 0$.

If $\gamma_i \neq \gamma$, equalizing $e_s(t) = \overline{e}_s \exp(-\gamma_{e_s} t)$ in equation (20), we find

$$\overline{e}_s \exp(-\gamma_{e_s} t) = \frac{\overline{i}}{\gamma_i - \gamma} \{1 - \exp[-(\gamma_i - \gamma)\overline{T}]\} \exp[(\gamma_i - \gamma)t]. \qquad (23)$$

It is straightforward to see that BGP is possible if $\gamma_i = \gamma - \gamma_{e_s}$. Furthermore, if $\gamma > \gamma_{e_s}$ our economy has long run growth because $\gamma_i = \gamma - \gamma_{e_s} > 0$[13]. However, if $\gamma = \gamma_{e_s}$ we get that $\gamma_i = 0$ which contradicts our initial statement. Finally, for the case $\gamma < \gamma_{e_s}$ our economy has not positive long run growth; indeed, $\gamma_i = \gamma - \gamma_{e_s} < 0$.

If $\gamma_i = \gamma$, from equation (20) we get

$$\overline{e}_s \exp(\gamma_{e_s} t) = \overline{i}\overline{T}. \qquad (24)$$

However, this is impossible because $\gamma_{e_s} > 0$ and $\overline{T}$ is finite.

To sum up our results, we establish the two following propositions for constant returns to scale:

Proposition 1. *Along the BGP, assuming $\alpha = 1$, $e_s(t) = \overline{e}_s$ and $\gamma < \rho$,*

1) *the interest rate $r(t) = r^* = \rho + \theta\gamma$;*

2) *the growth rate of energy prices equals the growth rate of energy-saving technical progress ($\gamma_{P_e} = \gamma$);*

3) *the growth rate of investment and capital stock are equal to the growth rate of energy-saving technical progress ($\gamma_i = \gamma_K = \gamma$);*

[13] $\gamma_K = \gamma_i = \gamma - \gamma_{e_s} > 0$. Since $y(t) = AK(t)$, then $\gamma_y = \gamma_K = \gamma - \gamma_{e_s}$. From the budget constraint of the economy, $y(t) = c(t) + i(t)$, we get that $\gamma_c = \gamma - \gamma_{e_s}$.

4) the growth rate of final good output equals the growth rate of energy-saving technical progress ($\gamma_y = \gamma$);

5) the growth rate of consumption equals the growth rate of energy-saving technical progress ($\gamma_c = \gamma$).

Proposition 2. *Along the BGP , assuming $\alpha = 1$, $e_s(t) = \overline{e}_s \exp(-\gamma_{e_s} t)$ and $\gamma < \rho$,*

1) the interest rate $r(t) = r^* = \rho + \theta\gamma_c$, where γ_c is given in the following 3) and 4);

2) the growth rate of energy prices equals the growth rate of energy-saving technical progress ($\gamma_{P_e} = \gamma$);

3) if $\gamma = \gamma_{e_s}$, then

(3.1) there is not growth in the investment and the capital stock ($\gamma_i = \gamma_K = 0$),

(3.2) the growth rate of final good output is zero ($\gamma_y = 0$),

(3.3) there is not growth in the consumption ($\gamma_c = 0$);

4) if $\gamma > \gamma_{e_s}$, then

(4.1) there is positive growth in the investment and the capital stock ($\gamma_i = \gamma_K = \gamma - \gamma_{e_s} > 0$),

(4.2) the growth rate of final good output is $\gamma_y = \gamma - \gamma_{e_s}$,

(4.3) there is positive growth of consumption ($\gamma_c = \gamma - \gamma_{e_s}$);

5) if $\gamma < \gamma_{e_s}$, then the economy decreases at the rate $\gamma_{e_s} - \gamma$.

We have to point out that constant returns to scale, in an optimistic scenario (*i.e.*, $e_s(t) = \overline{e}_s$), generates exogenous growth at the same rate as the growth of (exogenous) energy-saving technical progress (γ). However, if a gloomy scenario is assumed (*i.e.*, $e_s(t) = \overline{e}_s \exp(-\gamma_{e_s} t)$) our model can achieve long run growth at the rate $\gamma - \gamma_{e_s}$. If the growth rate of the energy-saving technical progress is greater than the decreasing rate of energy supply, the growth is positive. However, if the growth rate of the energy-saving technical progress is equal or lower than the decreasing rate of energy supply, we get zero or negative growth respectively. The reason is the following. First of all, we consider a BGP with constant and finite scrapping age following the Terborgh-Smith result. Second, energy supply has two effects on the economy. On the one hand, through the energy prices; and on the other, through the Leontieff production function, which models the idea of minimum energy requirement to use a machine. When we consider constant available energy supply (optimistic scenario), following our definition of BGP, we have long run growth because of the exogenous energy-saving technical progress (in this case energy prices increase since the economy grows at the growth rate of the exogenous energy-saving technical progress). However, in the case of decreasing available energy supply (gloomy scenario) the situation is quite different. Energy prices increase because of the decreasing available energy supply. Now, if the energy-saving technical progress exactly compensates the decreasing available energy supply ($\gamma = \gamma_{e_s}$), then there is not long run growth, despite of having constant returns to scale (see equation (18)), because there is constant replacement and the energy supply has a negative effect on growth due to the minimum energy requirement (Leontieff technology). However, if the growth rate of energy-saving technical progress is greater than the decreasing rate of energy supply, the second effect is also avoided and our economy can depict growth, where the growth rate is the difference between the growth rate of energy-saving technical progress and the decreasing rate of energy supply, $\gamma - \gamma_{e_s}$. Finally, if the energy-saving technical progress is not strong enough to offset the decreasing energy supply, our economy decreases.

Decreasing returns to scale

As before, equation (15) holds if and only if $\alpha = 1$ and/or $\gamma_i = 0$. Since now we consider decreasing returns to scale ($0 < \alpha < 1$), the growth rate of investment has to be zero. As a consequence, $\gamma_K = 0$ because $\gamma_i = \gamma_K$.

Furthermore, considering a canonical vintage capital model with Arrowian learning by doing technical progress, d'Autume and Michel

(1993) get that decreasing returns to scale "kills" growth in the long run. Our model is mathematically close to their economy taking energy instead of labor.

2.4 Concluding remarks

We analyzed the hypothesis about the effectiveness of energy-saving technologies to reduce the trade-off between economic growth and energy preservation. In order to incorporate the role of technology replacement, we developed a general equilibrium model, where the output is produced by means of vintage capital technology with endogenous scrapping rule. New vintages obsolete old machines because of their lower energy requirements. Constant and decreasing returns to scale are distinguished to develop a comparative study.

Under constant returns to scale and optimistic context (constant available energy supply), there is long run growth (Proposition 1). In this case, the (exogenous) growth rate of the economy equals the (exogenous) growth rate of energy-saving technical progress. However, considering a more realistic situation of gloomy scenario (decreasing available energy supply), our economy only achieves long run growth if the growth rate of energy-saving technical progress is greater than the decreasing rate of the energy supply (Proposition 2). Here, the growth rate of our economy is given by the difference between the growth rate of energy-saving technical progress and the decreasing rate of the energy supply (*i.e.*, $\gamma - \gamma_{e_s}$). Furthermore, when we assume decreasing returns to scale, the economy achieves BGP only for the gloomy case; nevertheless, our economy does not exhibit growth in the long run. However, we could escape from the decreasing returns to scale incorporating additional elements such as learning-by-doing, human capital, subsidies, etc. The constant scrapping age and the reduced availability of energy (which affects the economy through the energy prices and the minimum energy requirements) explain our results in both constant and decreasing returns to scale.

As the introduction of this chapter notices, the question of endogenous replacement of the old equipment by the new less energy consuming machinery was not previously studied, neither theoretically nor empirically, by the standard literature on energy resources. In this chapter, we studied this issue from a theoretical perspective. However, we found some few empirical studies dealing with this question. Azomahou *et al.* (2006), taking the Structural Analysis Database of the OECD and the energy databases of the IEA, empirically test (at sectoral level of OECD countries) the vintage effect considered in this chapter[14]. Taking the 14 main sectors of the industry, they find empirical evidence of this vintage effect (as well as induced-innovation hypothesis), with high heterogeneity among sectors.

Regarding to the growth rate of energy-saving technical progress (γ), Azomahou *et al.* (2004), find a value of the annual growth rate of energy-saving technical progress for the USA around 1.9%. As we established in this chapter, one should compare this value with the decreasing rate of the energy supply (γ_e). Since in this chapter we study long run growth, one should consider forecast studies about the energy supply (in particular, fossil fuels). However, as it is very well known in the empirical literature of non-renewable energy resources, the reliability of fossil fuels reserves data reported by oil companies is low. Indeed, the IEA claims that these data "have been called into question by a series of large downward revisions" (IEA (2006a), page 81). Nevertheless, as an empirical illustration, we can consider the studies of the Uppsala Hydrocarbon Depletion Study Group (Uppsala University)[15]. They forecast (up to 2050) an annual decreasing rate of oil supply around 1.8% for the total world. Then, according to our model (see Proposition 2), the economy depicts positive growth along the BGP (if it exists). However, notice that this chapter only considers energy-saving technical progress. Indeed, if we include other kinds of technical progress (for example, disembodied technical progress, or learning-by-doing), the growth rate of

[14] They also study the (energy) induced-innovation hypothesis, *i.e.*, the rise of energy prices induces energy-saving technical progress. However, in our model, energy-saving technical progress is assumed as exogenous variable.

[15] http://www.isv.uu.se/nuclear.en.html

the economy along the BGP will be higher.

Finally, we conclude this chapter pointing out that there is compatibility between economic growth and energy preservation by adopting energy-saving technologies. However, the growth rate of the energy-saving technical progress has to be greater than the decreasing rate of the energy supply to ensure a positive long run growth. Since in our model energy-saving technical progress is exogenous, an interesting extension is considering that the economy endogenously decides the growth rate of energy-saving technical progress. An R&D sector of energy-saving technologies could be a good way to study this problem.

Chapter 3

Energy-saving technical progress in a vintage capital model

(Pérez-Barahona A. and Zou B., 2006b. Energy-saving technical progress in a vintage capital model. Economic Modelling of Climate Change and Energy Policies. Edward Elgar. Pages 166-179)

3.1 Introduction

As IEA (2004, 2006a) observes (see Chapter 1, Table 3), the global primary energy demand is dominated by non-renewable energy resources (fossil and nuclear fuels), which account for 86.84% of the world primary energy demand in 2004, where the share of fossil fuels (coal, oil and natural gas) is around 80.5%. Moreover, according to the predictions of the IEA (2006a, 2004), fossil fuels will account for around 83% of the increase in the world primary energy demand over the period 2004-2030, rising the share of fossil fuels from 80% to 82%. Hence, regarding to energy, there is a clear dependence of the world economy on non-renewable energy resources. However, a key feature of this kind of energy resources is that their stocks (reserves) are finite. Then, as Rotillon (2005) points out, the usage of these energy resources calls attention to the survival of the economic system (in particular, economic growth and development) due to the exhaustibility characteristic of the reserves of fossil fuels. Indeed, authors like Dasgupta and Heal (1974), Hartwick (1989) and Smulders and Nooij (2003), point out that the usage of non-renewable energy resources (in particular, fossil fuels) *a priori* implies a limit to economic growth and development of modern economies. Nevertheless, authors like Newel *et al.* (1999), Carraro *et al.* (2003), and Boucekkine and Pommeret (2004) suggest that this trade-off can be (at least) less serve if energy conservation were increased by means of energy-saving technologies.

There is a growing evidence that energy-saving technical progress has been significant in the last two decades. Newell *et al.* (1999) study how the increase of the energy cost in recent years induced energy-savings innovation in the US. Moreover, Boucekkine and Pommeret (2004) analyze the optimal pace of capital accumulation, at the firm level, when technical progress is energy-saving. They develop a partial equilibrium model with embodied energy-saving technical progress, describing obsolescence by means of vintage capital technology with endogenous scrapping (replacement) decision of the old technology, and complementarity between capital and energy inputs. Indeed, Azomahou *et al.* (2006), taking the Structural Analysis Database of the OECD and the energy databases of the IEA, find (at sectoral level) empirical evidence of this vintage effect associated to energy-saving technical progress.

On the same lines as Chapter 2 (Pérez-Barahona and Zou (2006a)), Chapter 3 extends Boucekkine and Pommeret (2002)'s contribution to the general equilibrium case. The general equilibrium framework is extremely interesting because it allows us to study the global effect of environmental policies on the economy, and its relation with the limited energy supply of non-renewable resources and the diffusion of energy-saving technologies. In this chapter, we consider (exogenous) energy-saving technical progress embodied in new capital goods, which are introduced in the economy through vintage technology with endogenous obsolescence (scrapping) rule. We find that our model captures a wide range of effects with long-run implications. Moreover, we achieve to the conclusion that the frequently ignored issue of equipment replacement plays an important role in energy-saving technical change, at least, in the long-run. Indeed, one important consequence of assuming endogenous equipment replacement is that an increase of a tax on the energy expenditure of firms does not necessarily induce earlier equipment replacement if this tax is high enough. The intuition is the following. Replacement can overcome a rise of the energy expenditure tax. However, if this tax is already high, the replacement should be very often. Since frequent scraping can be too costly with respect to the change of energy prices, firms decide to keep their old equipment, which is assumed to be more energy demanding.

The chapter is organized as follows. In Section 3.2, we describe the model, the household's behavior and the rules of both optimal investment and scrapping behavior of the firms. The necessary conditions for balanced growth path (BGP) are presented in Section 3.3. Section 3.4 does the comparative static analysis. Finally, some concluding remarks are considered in Section 3.5.

3.2 The model

Let us consider an economy with constant population level, where the labor market is perfectly competitive. The production sector produces only one final good (numeraire), which can be assigned to consumption or investment. One important difference with respect to the model of Boucekkine *et al.* (1997) is that, in our case, intermediate inputs are produced by means of non-linear technology. Moreover, this technology is defined by vintage capital. The intermediate inputs market is supposed to be monopolistically competitive to allow for a concave profit function in the intermediate inputs sectors. Furthermore, we assume that the energy supply is exogenous.

3.2.1 Household

The household solves the following standard inter-temporal maximization problem with constant relative risk aversion (CRRA) instantaneous utility function:

$$\max_{c(t)} \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} \exp(-\rho t) dt, \tag{1}$$

subject to the budget constraint and the corresponding transversality condition:

$$\begin{gathered} \dot{a}(t) = r(t)a(t) - c(t), \\ a(0) \text{ given}, \\ \lim_{t\to\infty} a(t) \exp\left(-\int_0^t r(z)dz\right) = 0, \end{gathered} \tag{2}$$

where $c(t)$ is per-capita consumption, $a(t)$ is per-capita assets held by the household and the interest rate $r(t)$ is taken as given by the household. $\theta(> 0)$ measures the constant relative risk aversion, and $\rho(> 0)$ is the time preference parameter. It is easy to check that, along the optimal path, the consumption follows

$$r(t) = \rho + \theta \frac{\dot{c}(t)}{c(t)}. \tag{3}$$

3.2.2 Final good firm

The final good is produced competitively by a representative firm solving the following optimal profit problem:

$$\max_{y_j(t)} \left\{ y(t) - \int_0^1 p_j(t) y_j(t) dj \right\},$$

where the per-capita production $y(t)$ is given by following constant elasticity of substitution (CES) production technology:

$$y(t) = \left(\int_0^1 y_j(t)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}},$$

which is defined over a continuum of intermediate inputs $y_j(t)$ with $j \in [0, 1]$. Moreover, we assume constant elasticity of substitution $\epsilon > 1$. Furthermore, prices are given by

$$p_j(t) = \left(\frac{y_j(t)}{y(t)} \right)^{-\frac{1}{\epsilon}},$$

from the standard monopolistic competition economy (see Dixit and Stiglitz (1977)), and they are taken as given by the final good firm.

3.2.3 Intermediate input firm

Producing in a monopolistically competitive market, the representative intermediate input-j firm maximizes its profits

$$\max_{y_j(t),i_j(t),T_j(t),p_j(t)} \int_0^{\infty} \left[p_j(t)y_j(t) - i_j(t) - e_j(t)Pe(t)(1+Z)\right] R(t)dt, \qquad (4)$$

subject to

$$y_j(t) = A\left(\int_{t-T_j(t)}^{t} i_j(z)dz\right)^{\alpha}, \quad 0 < \alpha < 1, \qquad (5)$$

$$e_j(t) = \int_{t-T_j(t)}^{t} i_j(z)\exp(-\gamma z)dz, \quad 0 < \gamma < r, \qquad (6)$$

$$p_j(t) = \left(\frac{y_j(t)}{y(t)}\right)^{-\frac{1}{\epsilon}}, \qquad (7)$$

where

$$R(t) = \exp\left(-\int_0^t r(z)dz\right),$$

with the initial conditions $i_j(t)$ given for all $t \leq 0$ and $j \in [0,1]$.

$e_j(t)$ and $P_e(t)$ are, respectively, the demand and the price of energy, which are endogenous. Z is a tax on the the energy expenditure of firms, which is established by the government[1]. $i_j(t)$ is the investment of the representative intermediate input-j firm. The output and the price of intermediate input j are respectively represented by $y_j(t)$ and $p_j(t)$. The price of intermediate input j and the final good production per-capita, $y(t)$, are taken as given by the monopoly. Equation (5) is our non-linear technology defined over vintage capital. The energy demand is obtained by equation (6). Here, $\gamma > 0$ represents the growth rate of energy-saving technical progress and $T_j(t)$ is the age of the oldest operating machine

[1]Notice that, in our model, there is no externality. Indeed, the aim of this tax (Z) is to induce greater replacement of the old technology, which is assumed to be more energy demanding. However, in a pollution set-up, this tax can also be useful to correct pollution externalities by compensating the households.

or scrapping age. Considering monopolistic competition, the inverse demand function is given by equation (7).

Notice that new machines are assumed to be more energy-saving than the old equipment. Moreover, it is important to observe that, as in Chapter 2, we assume complementarity between capital and energy (Leontieff technology). This assumption describes the idea of minimum energy requirement to use a machine. Indeed, each vintage t has an energy requirement of $i_j(t)\exp(-\gamma t)$. This assumption is standard in numerous studies (see for instance, Hudson and Jorgenson (1974), and Berndt and Wood (1975)).

Let us define the capital stock as

$$K(t) = \int_{t-T(t)}^{t} i(z)dz,$$

and the optimal lifetime of machines of vintage t as a function

$$J_j(t) = T_j[t + J_j(t)]. \tag{8}$$

Notice that $T_j(t) = J_j[t - T_j(t)]$.

Substituting equations (5)-(8) into equation (4), and considering the symmetric case[2], we can simplify our problem as follows:

$$\max_{i(t),T(t)} \int_{t}^{t+J(t)} \exp\left(-\int_0^t r(z)dz\right)$$
$$\cdot\left[A\left(\int_{\tau-T(\tau)}^{\tau} i(z)dz\right)^{\alpha} - i(\tau) - P_e(\tau)(1+Z)\int_{\tau-T(\tau)}^{\tau} i(z)\exp(-\gamma z)dz\right]d\tau, \tag{9}$$

where $r(t)$ is given by equation (3).

From the first order condition (FOC) for $i(t)$, we obtain the *optimal investment rule*:

$$\int_{t}^{t+J(t)} \alpha A\left(\int_{\tau-T(\tau)}^{\tau} i(z)dz\right)^{\alpha-1} \exp\left(-\int_0^t r(z)dz\right)d\tau =$$

[2]See Chapter 2, footnote 7.

$$1+\int_t^{t+J(t)}(1+Z)P_e(\tau)e^{-\gamma t}\exp\left(-\int_0^t r(z)dz\right)d\tau. \quad (10)$$

where the left-hand side (LHS) is the discounted marginal productivity during the whole lifetime of the capital acquired in t; 1 is the marginal purchase cost at t, which is normalized to one; and the second term on the right-hand side (RHS) is the discounted operation cost at t.

The optimal investment rule establishes that firms should invest at time t until the discounted marginal productivity during the whole lifetime of the capital acquired in t exactly compensates for both their discounted operation cost and their marginal purchase cost at t.

The FOC for $T(t)$ leads to the *optimal scrapping rule*:

$$A\alpha\left(\int_{t-T(t)}^t i(z)dz\right)^{\alpha-1}=P_e(t)(1+Z)\exp\left\{-\gamma[t-T(t)]\right\}. \quad (11)$$

The optimal scrapping rule states that a machine should be scrapped as soon as its marginal productivity (which is the same for any machine whatever its age) no longer covers its operation cost (which rises with age)[3].

To summarize, the (decentralized) equilibrium of our economy is characterized by equations (2)-(3) (household side), equations (5)-(8), the optimal investment rule, the optimal scrapping rule, and the following three additional equations to close the model: $c(t)+i(t)=y(t)$, $i(t)=\dot{a}(t)$ and $e(t)=e_s(t)$, which is the equilibrium condition in the energy market. $e_s(t)$ is the energy supply. As in Chapter 2, we want to describe the effect of both endogenous energy prices and limited stock (reserves) of non-renewable energy resources. Due to the difficult analytical tractability of vintage capital models, we do not explicitly consider the extraction sector[4]. Therefore, in this chapter, we assume energy supply as exogenous variable. Indeed, one can interprete this assump-

[3]Notice that, the marginal productivity is given by $\alpha A(\int_{t-T(t)}^t i(z)dz)^{\alpha-1}$, and $(1+Z)P_e(t)\exp\{-\gamma[t-T(t)]\}$ represents the operation cost.

[4]Notice that a realistic model of an extraction sector involves issues such as market power, refinery activities and available reserves.

tion as a small economy which faces a given flow of energy each period t.

3.3 Balanced growth path

Let us define our balanced growth path (BGP) equilibrium as the situation where all the endogenous variables grow at constant rate[5], with constant and finite scrapping age $T(t) = J(t) = T$ (Terborgh-Smith result)[6]. In Chapter 2 we showed that there is no long-run growth for output, investment and consumption for the case $0 < \alpha < 1$. Moreover, since energy market equilibrium requires energy demand to be equal to energy supply ($e_s(t)$), they also observe that (for this case) BGP is only possible under a gloomy scenario, *i.e.*, $e_s(t) = \overline{e}_s \exp(-\gamma_{e_s} t)$, with $\gamma_{e_s} = \gamma$. In the following, we provide the necessary conditions for BGP equilibrium. These conditions will be used in the next section of the chapter.

It is easy to prove that, at equilibrium, the interest rate is constant, *i.e.*, $r(t) = \rho$. Differentiating equation (10) and rearranging terms, from equation (11) we obtain

$$[\exp(\gamma T) - 1] - \frac{\gamma}{\gamma_{P_e} - \rho} \left\{\exp[(\gamma_{P_e} - \rho)J] - 1\right\} =$$
$$\frac{\rho}{(1+Z)\overline{P}_e} \exp\left\{(\gamma - \gamma_{P_e})t\right\},$$

where γ_{P_e} and $\overline{P}_e$ are, respectively, the growth rate and the level of the energy price. The LHS is constant for any t in the equilibrium, and the RHS is a function of t. So the equality holds if and only if $\gamma = \gamma_{P_e}$. As in Chapter 2, this result states that, in terms of energy-saving, energy price grow at the same rate as productivity. Following Chapter 2, $\gamma_i = 0$ and

$$K(t) = \overline{i}\overline{T}. \tag{12}$$

The reason for the absence of long-run growth of investment is the following. The effect of both scrapping age and energy-saving technical

[5]That is $x(t) = \overline{x}\exp(\gamma_x t)$, where γ_x is the growth rate of variable x.

[6]See, for instance, Bardhan (1969) and Boucekkine *et al.* (1997).

progress is not strong enough to overcome the negative effect of decreasing returns to scale ($0 < \alpha < 1$). More precisely, our framework considers a CRRA instantaneous utility function. As a consequence, the interest rate is constant along BGP. Then, following Terborgh-Smith result, the scrapping age is also constant. Taking the optimal investment rule along the BGP, we get

$$\frac{A\alpha(\bar{i}\overline{T})^{\alpha-1}}{\rho}[1-\exp(-\rho\overline{T})] = 1+\frac{(1+Z)\overline{P}_e}{\rho-\gamma}[1-\exp\{-(\rho-\gamma)\overline{T}\}]. \quad (13)$$

From Chapter 2, it is straightforward that the discounted operation cost is constant because the decreasing energy supply offsets the effect of the energy-saving technical progress (γ). Hence, as the marginal purchase cost ($= 1$) remains constant, the investment has to be constant along the BGP. The economic interpretation of this result is the following. Two elements are crucial to provide the interpretation. First, the scrapping age is assumed to be constant and finite, following the Terborgh-Smith result. Second, the energy supply has two effects on the economy. On the one hand through energy price, and on the other through the Leontieff production function, which describes the idea of minimum energy requirements to use a machine. The energy-saving technical progress (γ) offsets the effect of energy price, which increases due to the energy supply (gloomy scenario). But the effect of minimum energy requirements to use a machine is not compensated since the growth rate of energy-saving technical progress can not be greater than γ_{e_s} (otherwise, there is no BGP). Hence, since the scrapping age is constant, investment does not grow along the BGP.

To finish this section, we point out that our findings are not standard results. Indeed, the neoclassical framework would achieve exogenous growth. For example, Solow-Swan and Ramsey models, with exogenous technical progress, describe economies which grow at the growth rate of both population and exogenous technical progress. However, in this chapter we find that the reduced availability of energy (non-renewable resource) offsets the (exogenous) energy-saving technical progress. As a consequence, since we consider BGP equilibrium with constant scrapping age, our economy does not achieve long-run growth. Finally, we have to observe that our result is consistent with the partial equilibrium model of Boucekkine and Pommeret (2004), which also depicts no growth along the BGP.

3.4 Comparative static analysis

In this section, we study the effects of modifications in the parameters of our economy on the endogenous variables along the BGP, *i.e.*, comparative static analysis. In our model these effects are mainly explained by the behavior of the scrapping age and investment. The behavior of the optimal investment is given by equation (13), and the optimal scrapping age is described by the following expression:

$$A\alpha(\overline{iT})^{\alpha-1} = (1+Z)\overline{P}_e \exp(\gamma\overline{T}). \tag{14}$$

Since along the BGP the optimal investment rule (equation (13)) and optimal scrapping rule (equation (14)) are functions of $\bar{i}$, $\overline{T}$ and $\overline{P}_e$, we need a third equation to completely describe the behavior of both the scrapping age and investment. We find this equation from the equilibrium condition of the energy market. From equation (6), the energy demand is given by the following equation:

$$e(t) = \frac{\bar{i}}{\gamma}\exp(-\gamma t)[\exp(\gamma\overline{T}) - 1]. \tag{15}$$

Since we assume exogenous long-run energy supply, which is equal to $\bar{e}_s \exp(-\gamma t)$ to have BGP, then we get the expression

$$\bar{e}_s = \frac{\bar{i}}{\gamma}[\exp(\gamma\overline{T}) - 1] \tag{16}$$

by equalizing energy demand (equation (15)) and supply. Therefore, along the BGP, the values of the optimal investment ($\bar{i}$), scrapping age ($\overline{T}$), and level of energy price ($\overline{P_e}$) are given by the equations (13), (14) and (16), which is a static (simultaneous) system of non-linear equations, taken the values of the parameters as given.

3.4.1 Energy expenditure tax (Z)

In this section we study the effect of an increase of the energy tax of firms. More precisely, this section describes the effects of an increase of Z on the optimal scrapping age ($\overline{T}$), investment ($\bar{i}$) and output ($\overline{y}$) of our economy along the BGP. The most natural way to proceed is the following. From equation (16), we provide the investment ($\bar{i}$) as a function of the scrapping age ($\overline{T}$). Applying this result into equation (14), and differentiating the resulting expression with respect to Z, we get a function $F_1(\partial\overline{T}/\partial Z, \partial\overline{P}_e/\partial Z)$. However, we need a second relationship between these two derivatives to be able to determine their value. This expression, $F_2(\partial\overline{T}/\partial Z, \partial\overline{P}_e/\partial Z)$, arises by differentiating the optimal investment rule along the BGP (equation (13)) with respect to Z, after considerable manipulations. Hence, taking these two functions, it is possible to provide the analytical expression of both derivatives. However, due to the complexity of these expressions, we can not determine the sigh of these derivatives, neither for general values of the parameter, nor by imposing restrictions on some of them. Therefore, we propose an alternative approach.

Let us consider the following strategy. Knowing $\partial\overline{P}_e/\partial Z$, we get $\partial\overline{T}/\partial Z$ from function $F_1(\bullet)$. Moreover, taking equation (16), we obtain $\partial\bar{i}/\partial Z$. Since $\overline{y} = A(\bar{i}\overline{T})^{\alpha}$, one easily gets $\partial\overline{y}/\partial Z$. This strategy allows us to identify the different effects of an increase of Z on the economy, which is one of the main objectives of this chapter. However, to determine the value of $\partial\overline{P}_e/\partial Z$, as well as the strength of these effects (*i.e.*, the net effect), we have to solve numerically the system of non-linear equations given by the equations (13), (14) and (16), for different values of Z. For the numerical solution, we need to provide values to the parameters involved in the previous non-linear system. From the economic literature, it is widely accepted a value for the discount rate (ρ) about 3%, as well as 1/3 for α. Following the empirical study of Azomahou *et al.* (2004), we consider $\gamma = 0.02$. Regarding to the energy expenditure tax (Z), there is no available measurement of Z as an aggregation of different fuels. However, we can use the data of tax on unleaded gasoline provided the IEA (2000, 2006a) as a proxy of Z. According to the IEA, this tax was about 76% in France, Germany and UK for the period 1990-2000, with rising trend. Indeed, in 2006, this tax was 80% in France, 83% in Germany and 81% in the UK. Following that, we consider a value of Z about 80%. Finally, we choose A and $\overline{e}_s$ to fit a ratio investment/output about 16%, which is commonly accepted by the economic literature.

Optimal scrapping age ($\overline{T}$)

Taking the function $F_1(\bullet)$, we get:

$$\frac{\partial \overline{T}}{\partial Z} =$$

$$-\left(\frac{1}{\overline{\overline{T}}} - \frac{\gamma \exp(\gamma \overline{T})}{\exp(\gamma \overline{T}) - 1} + \frac{\gamma}{1-\alpha}\right)^{-1} \frac{1}{1-\alpha}\left(\frac{1}{\overline{P}_e}\frac{\partial \overline{P}_e}{\partial Z} + \frac{1}{1+Z}\right). \quad (17)$$

Following equation (17), we distinguish two effects of the energy expenditure tax on the scrapping age: direct and indirect effect.

The **direct effect** is the effect of a modification of the energy tax on the scrapping age for a fixed level of energy price:

$$\frac{\partial \overline{T}}{\partial Z}_{|\overline{P}_e fixed} = -\left(\frac{1}{\overline{\overline{T}}} - \frac{\gamma \exp(\gamma \overline{T})}{\exp(\gamma \overline{T}) - 1} + \frac{\gamma}{1-\alpha}\right)^{-1} \frac{1}{1-\alpha}\frac{1}{1+Z}.$$

The term $\frac{1}{1-\alpha}\frac{1}{1+Z}$ is always positive because $0 < \alpha < 1$ and $0 < Z < 1$. Considering $\gamma > 0$, $\rho > 0$ and $\overline{T} > 0$, it is easy to prove (see Appendix A) that the term between brackets is positive. Then, the direct effect has a negative effect on the scrapping age, *i.e.*, $\partial\overline{T}/\partial Z_{|\overline{P}_e fixed} < 0$. So, if the energy expenditure tax increases, the scrapping age is reduced for a fixed level of energy price. The interpretation of this effect is the following. If the energy tax increases, the operation cost rises for a fixed level of energy price. Consequently, firms decide to replace their equipment earlier. We can verify this explanation taking the scrapping and investment rule along the BGP. When the tax increases, firms can modify their scrapping and investment decision. The net result is given by substituting the scrapping rule (equation (14)) into the investment rule (equation (13)). After some manipulations it yields:

$$(1+Z)\overline{P}_e\left[\left(\frac{1}{\rho}-\frac{1}{\gamma+\rho}\right)\exp[(\gamma+\rho)\overline{T}]+\frac{1}{\rho}\exp(\gamma\overline{T})+\frac{1}{1+\rho}\right]=1.$$

When Z rises, firms compensate the higher tax by dropping the scrapping age for a fixed level of energy price.

However, according to the numerical solution, the negative direct effect is overcome by an **indirect effect** of the energy expenditure tax on the scrapping age through the variation of the level of energy price. From equation (22), this effect is described by the following expression:

$$-\left(\frac{1}{\overline{T}}-\frac{\gamma\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1}+\frac{\gamma}{1-\alpha}\right)^{-1}\frac{1}{(1-\alpha)}\frac{1}{\overline{P}_e}\frac{\partial\overline{P}_e}{\partial Z}.$$

By solving numerically the non-linear system given by the equations (13), (14) and (16) for different values of Z, we get that $\partial\overline{P}_e/\partial Z<0$ (see Appendix B, Tables 1 and 2, second column)[7]. The explanation of this inverse relationship between the level of energy price and the energy expenditure tax is provided by the assumption of an exogenous long run energy supply. Considering the economy along the BGP, if Z rises the energy supply is not affected because it is exogenous and always decreasing with t. However the energy demand decreases since energy is now more expensive, for a fixed level of production and scrapping age. Then, the level of energy price decreases. Therefore, since the term between brackets is positive (see Appendix A), the indirect effect of Z on the scrapping age is positive.

However, as we just noticed, the net effect of Z on $\overline{T}$ results from the combination of both direct and indirect effects. As it was observed at the beginning of this section, we can not establish an analytical result measuring the strength of each effect. Nevertheless, by solving numerically the system previously described, we can conclude the sign of

[7]Notice that we checked the robustness of this result to changes on the parameters of the economy. Moreover, in the next sections, we did the same for $\partial\overline{P}_e/\partial A$, $\partial\overline{P}_e/\partial\overline{e}_s$ and $\partial\overline{P}_e/\partial\gamma$.

$\partial\overline{T}/\partial Z$. Following that, we get that for a large enough Z (according to our parametrization, greater than 84%), the sign of this derivative is positive (otherwise, it is negative). Then, our result establishes that an increase of the energy tax does not necessarily induce earlier replacement of machines (*i.e.*, lower $\overline{T}$) if Z is large enough (see Appendix B, Tables 1 and 2, second column). The interpretation of this result is the following. When the energy tax increases, firms can compensate the higher tax by an earlier replacement of their equipment. However, if this tax is large enough, they should replace the machines very often. But this replacement is too costly because it should be done frequently. Then, firms prefer to keep their old equipment, more energy demanding.

Optimal investment ($\bar{i}$)

Let us study the effect of Z on $\bar{i}$, *i.e.*, $\partial\bar{i}/\partial Z$. By differentiating equation (14) with respect to Z, and rearranging terms, we get

$$\frac{\partial\bar{i}}{\partial Z} = -\bar{i}\left[\frac{1}{1-\alpha}\frac{1}{1+Z} + \left(\frac{1}{\overline{T}} + \frac{\gamma}{1-\alpha}\right)\frac{\partial\overline{T}}{\partial Z} + \frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial\overline{P}_e}{\partial Z}\right]. \qquad (18)$$

From equation (18), we can distinguish both a direct and indirect effect of Z on investment. For a fixed scrapping age and level of energy price, we get the **direct effect**:

$$\frac{\partial\bar{i}}{\partial Z}_{|\overline{P}_e \text{ and } \overline{T} \text{ fixed}} = -\bar{i}\left(\frac{1}{1-\alpha}\frac{1}{1+Z}\right) < 0.$$

Since the term between brackets is positive, the direct effect is negative. If the tax increases, the operation cost of machines rises too. Then, firms decide to reduce their investment for a fixed scrapping age and level of energy price.

However, there are two additional **indirect effects** from the variation of both scrapping age and level of energy price. This effect is given by the following expression:

$$-\bar{i}\left[\left(\frac{1}{\overline{T}} + \frac{\gamma}{1-\alpha}\right)\frac{\partial\overline{T}}{\partial Z} + \frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial\overline{P}_e}{\partial Z}\right].$$

The sign of the indirect effect is undetermined because $\partial\overline{T}/\partial Z > 0$ (see the previous section) and $\partial\overline{P}_e/\partial Z < 0$. Nevertheless, considering both effects together (direct and indirect effect), the net result is provided by the numerical solution (see Appendix B): $\partial\bar{i}/\partial Z < 0$. The explanation is the following. The direct effect reduces investment because an increase of the tax raises the operation cost. However, this effect is not offset by the indirect effect of the scrapping age and level of energy price. Indeed, according to the numerical solution, the behavior of the scrapping age reinforces the direct effect.

Final good output ($\overline{y}$)

The comparative static of the final good output is directly provided by the comparative static of both optimal scrapping age and investment considered before. The $\partial\overline{y}/\partial Z$ is given by the equation of the final good output along the BGP, *i.e.*, $\overline{y} = A(\bar{i}\overline{T})^{\alpha}$. Differentiating that expression with respect to Z, we get

$$\frac{\partial\overline{y}}{\partial Z} = \alpha\overline{y}\left(\frac{1}{\bar{i}}\frac{\partial\bar{i}}{\partial Z} + \frac{1}{\overline{T}}\frac{\partial\overline{T}}{\partial Z}\right).$$

From the previous sections, we know that $\partial\overline{T}/\partial Z > 0$ and $\partial\bar{i}/\partial Z < 0$. According to the numerical solution, the effect of increasing an already high energy expenditure tax on the final good output is negative. Indeed, the decrease of the optimal investment overcomes the later replacement of machines.

3.4.2 Disembodied technical progress (A)

This section studies the effect of an increase of A, which is the technical progress that affects all generations of machines of the economy, *i.e.*, disembodied technical progress. We apply a similar strategy as in the previous section with Z. Taking logs and differentiating equations (14) and (16) with respect to A, we get

$$\frac{\partial\bar{i}}{\partial A} = \bar{i}\left[-\left(\frac{1}{\overline{T}} + \frac{\gamma}{1-\alpha}\right)\frac{\partial\overline{T}}{\partial A} - \frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial\overline{P}_e}{\partial A} + \frac{1}{A}\frac{1}{1-\alpha}\right], \tag{19}$$

$$\frac{\partial\bar{i}}{\partial A} = -\bar{i}\left(\frac{\gamma\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1}\right)\frac{\partial\overline{T}}{\partial A}. \tag{20}$$

Optimal scrapping age

Combining equations (19) and (20), it yields

$$\frac{\partial \overline{T}}{\partial A} = \left(\frac{1}{\overline{\overline{T}}} + \frac{\gamma}{1-\alpha} - \frac{\gamma \exp(\gamma \overline{T})}{\exp(\gamma \overline{T}) - 1}\right)^{-1} \left(\frac{1}{A}\frac{1}{1-\alpha} - \frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial \overline{P}_e}{\partial A}\right). \quad (21)$$

From equation (21), we identify the **direct effect** of an increase of the disembodied technical progress (A) on the scrapping age along the BGP ($\overline{T}$):

$$\frac{\partial \overline{T}}{\partial A}_{|\overline{P}_e fixed} = \left(\frac{1}{\overline{\overline{T}}} + \frac{\gamma}{1-\alpha} - \frac{\gamma \exp(\gamma \overline{T})}{\exp(\gamma \overline{T}) - 1}\right)^{-1} \frac{1}{A}\frac{1}{1-\alpha}.$$

This effect is positive (see Appendix A), *i.e.*, $\partial \overline{T}/\partial A_{|\overline{P}_e fixed} > 0$. When A increases, the marginal productivity of all machines rises too, and firms decide to replace later their equipment. Therefore, $\overline{T}$ increases.

However, the net effect of A is a combination of the direct effect described above and the following **indirect effect** of A on $\overline{T}$ thought the level of energy price ($\overline{P}_e$). This indirect effect is given by the expression

$$-\left(\frac{1}{\overline{\overline{T}}} + \frac{\gamma}{1-\alpha} - \frac{\gamma \exp(\gamma \overline{T})}{\exp(\gamma \overline{T}) - 1}\right)^{-1} \frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial \overline{P}_e}{\partial A}.$$

The indirect effect is negative because $\partial \overline{P}_e/\partial A > 0$ (see Appendix B, Tables 1 and 2, third column). The explanation is the following. When A rises, the marginal productivity of all machines increases too. Therefore, firms replace later their equipment. The older a machine the greater its energy requirements. Then, the demand of energy rises for a fixed level of energy supply. As a consequence, the level of energy price ($\overline{P}_e$) increases too.

According to the numerical solution (see Appendix B), the net effect of an increase of the disembodied technical progress is a rise of replacement age. Indeed, one can observe that, on the one hand, the higher A the greater marginal productivity of all machines. Therefore, firms scrap later their equipment (direct effect). On the other, the level of energy price increases, reducing the scrapping age (indirect effect). However, the former effect is stronger than the direct effect. Hence, firms replace earlier their equipment, *i.e.*, $\partial\overline{T}/\partial A < 0$.

Optimal investment

The effect of a rise of the disembodied technical progress on the optimal investment is given by equation (19)[8]. We can distinguish a **direct effect** of A on the optimal investment:

$$\frac{\partial\bar{i}}{\partial A}_{|\overline{P}_e\ and\ \overline{T}\ fixed} = \bar{i}\frac{1}{1-\alpha}\frac{1}{A} > 0.$$

The greater A the higher marginal productivity of all machines. Consequently, firms invest more (see the investment rule along the BGP, *i.e.*, equation (13)) for a given scrapping age and level of energy price.

Moreover, the direct effect is reinforced by a decline in the scrapping age for a given level of energy price (**indirect effect through the scrapping age**):

$$-\bar{i}\left(\frac{1}{\overline{T}} + \frac{\gamma}{1-\alpha}\right)\frac{\partial\overline{T}}{\partial A} > 0.$$

Indeed, when A increases firms replace earlier their equipment, then investment rises.

But there is an **indirect effect through the level of energy prices**, which is given by the following expression:

$$-\bar{i}\left(\frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\right)\frac{\partial\overline{P}_e}{\partial A} < 0.$$

This effect is negative because the greater A the higher the level of energy price. However, according to the numerical solution (see Appendix B), this effect is not strong enough to offset the positive effects. Hence, the net result is that an increase in the disembodied technical progress raises the optimal investment, *i.e.*, $\partial\bar{i}/\partial A > 0$.

[8]Notice that equation (20) directly provides the $\partial\bar{i}/\partial A$ as a function of $\partial\overline{T}/\partial A$. However the variation on $\overline{T}$ contains the effect of the energy price level. For that reason we take equation (27), which considers both effects separately.

Final good output

The final good output is given by $\overline{y} = A(\overline{i}\overline{T})^{\alpha}$. Then, taking logs and differentiating with respect to A, we get

$$\frac{\partial \overline{y}}{\partial A} = \overline{y}\left[\frac{1}{A} + \alpha\left(\frac{1}{\overline{i}}\frac{\partial \overline{i}}{\partial A} + \frac{1}{\overline{T}}\frac{\partial \overline{T}}{\partial A}\right)\right].$$

As we can observe, an increase in the disembodied technical progress rises the final good output since the marginal productivity of all machines grows (**direct effect**), for fixed $\overline{i}$ and $\overline{T}$:

$$\frac{\partial \overline{y}}{\partial A}_{|\overline{i}\ and\ \overline{T}\ fixed} = \overline{y}\frac{1}{A} > 0.$$

Moreover, this effect is reinforced by a positive effect of A on $\overline{i}$ (**indirect effect though the optimal investment**):

$$\alpha\frac{1}{\overline{i}}\frac{\partial \overline{i}}{\partial A} > 0.$$

However, the negative **indirect effect of A through the scrapping age**,

$$\alpha\overline{y}\frac{1}{\overline{T}}\frac{\partial \overline{T}}{\partial A} < 0,$$

does not offset the previous positive effects (see Appendix B). Hence, the net result of an increase in the disembodied technical progress is a rise of the final good output, *i.e.*, $\partial\overline{y}/\partial A > 0$.

3.4.3 Energy supply level ($\overline{e}_s$)

In this section we study the effect of an increase of the level of (exogenous) energy supply ($\overline{e}_s$) on the optimal replacement, investment and output of our economy. As in the previous section, taking logs and differentiating equations (14) and (16) with respect to $\overline{e}_s$, we get

$$\frac{\partial \overline{i}}{\partial \overline{e}_s} = \overline{i}\left[-\left(\frac{1}{\overline{T}} + \frac{\gamma}{1-\alpha}\right)\frac{\partial \overline{T}}{\partial \overline{e}_s} - \frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial \overline{P}_e}{\partial \overline{e}_s}\right], \tag{22}$$

$$\frac{\partial \overline{i}}{\partial \overline{e}_s} = \overline{i}\left(\frac{1}{\overline{e}_s} - \frac{\gamma\exp(\gamma\overline{T})}{\exp(\gamma\overline{T}) - 1}\frac{\partial \overline{T}}{\partial \overline{e}_s}\right). \tag{23}$$

Optimal scrapping age

The effect of an increase in the level of energy supply on the replacement age is given by combining equations (22) and (23):

$$\frac{\partial \overline{T}}{\partial \overline{e}_s} = \left(\frac{\gamma \exp(\gamma \overline{T})}{\exp(\gamma \overline{T}) - 1} - \frac{1}{\overline{\overline{T}}} - \frac{\gamma}{1-\alpha}\right)^{-1} \left(\frac{1}{1-\alpha}\frac{1}{\overline{\overline{P}}_e}\frac{\partial \overline{P}_e}{\partial \overline{e}_s} + \frac{1}{\overline{e}_s}\right). \quad (24)$$

From equation (24), we get the **direct effect** of $\overline{e}_s$ on $\overline{T}$:

$$\frac{\partial \overline{T}}{\partial \overline{e}_{s\,|\overline{P}_e\ fixed}} = \left(\frac{\gamma \exp(\gamma \overline{T})}{\exp(\gamma \overline{T}) - 1} - \frac{1}{\overline{\overline{T}}} - \frac{\gamma}{1-\alpha}\right)^{-1} \frac{1}{\overline{e}_s} < 0.$$

This is a negative effect (see Appendix A). If we fix the level of energy price, firms have to replace earlier their equipment. However, it is not possible to interpret the effect of $\overline{e}_s$ without considering the variation of $\overline{P}_e$. For that reason, we have to study the **indirect effect** through the level of energy price. This effect is described by the expression

$$\left(\frac{\gamma \exp(\gamma \overline{T})}{\exp(\gamma \overline{T}) - 1} - \frac{1}{\overline{\overline{T}}} - \frac{\gamma}{1-\alpha}\right)^{-1} \left(\frac{1}{1-\alpha}\frac{1}{\overline{\overline{P}}_e}\right)\frac{\partial \overline{P}_e}{\partial \overline{e}_s} > 0.$$

The numerical solution shows that $\partial \overline{P}_e/\partial \overline{e}_s < 0$ (see Appendix B, Tables 1 and 2, fourth column). The interpretation of this result is that the level of energy price decreases when the energy supply level increases. Then, the indirect effect is negative.

Finally, following the numerical solution, the net effect is positive. Indeed, a rise in the energy supply level reduces the level of energy price. Hence, firms decide to replace later their equipment, *i.e.*, $\partial \overline{T}/\partial \overline{e}_s > 0$.

Optimal investment

Equation (22) describes the effect of an increase of the energy supply level on the optimal investment. In this case, this increase affects the optimal investment through two indirect effects.

If we fix the level of energy price, the rise of the energy supply level rises the optimal investment because the scrapping age is lower:

$$\frac{\partial \overline{T}}{\partial \overline{e}_{s\,|\overline{P}_e\ fixed}} < 0.$$

Then, the **indirect effect through the scrapping age**, which is given by the expression

$$\frac{\partial \bar{i}}{\partial \overline{e}_s}_{|\overline{P}_e\ fixed} = -\bar{i}\left(\frac{1}{\overline{\overline{T}}} + \frac{\gamma}{1-\alpha}\right)\frac{\partial \overline{T}}{\partial \overline{e}_s}_{|\overline{P}_e\ fixed} > 0,$$

is positive. As we have shown in the previous section, the higher energy supply level the lower scrapping age for a fixed level of energy price. Then, for a fixed $\overline{P}_e$, when the energy supply level increases, firms invest more because they decide to replace earlier their equipment.

Moreover, there is an additional effect from the variation of the level of energy price. This is a positive **indirect effect through the level of energy price**:

$$\frac{\partial \bar{i}}{\partial \overline{e}_s}_{|\overline{T}\ fixed} = -\bar{i}\frac{1}{1-\alpha}\frac{1}{\overline{\overline{P}}_e}\frac{\partial \overline{P}_e}{\partial \overline{e}_s} > 0.$$

When the energy supply level increases, the level of energy price decreases for a fixed scrapping age. As a consequence, firms invest more because the operation cost it is reduced.

Taking both effects together, the numerical solution concludes that the net result is positive, *i.e.*, $\partial\bar{i}/\partial\overline{e}_s > 0$. Hence, firms invest more when the energy supply level increases.

Final good output

The effect on the final good output is straightforward from the behavior of the optimal scrapping age and investment. As for the case Z, but differentiating with respect to the level of energy supply, we get

$$\frac{\partial \overline{y}}{\partial \overline{e}_s} = \alpha\left(\frac{1}{\bar{i}}\frac{\partial \bar{i}}{\partial \overline{e}_s} + \frac{1}{\overline{\overline{T}}}\frac{\partial \overline{T}}{\partial \overline{e}_s}\right) > 0.$$

Since both effects are positive, the greater energy supply level the higher final good output.

3.4.4 Embodied technical progress (γ)

This section studies the effect of an increase of the growth rate of energy-saving technical progress (γ), which is embodied in new equipment. Taking logs and differentiating equations (14) and (16) with respect to γ, we get

$$\frac{\partial \bar{i}}{\partial \gamma} = \bar{i}\left(\frac{1}{\gamma} - \frac{\overline{T}\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1} - \frac{\gamma\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1}\frac{\partial\overline{T}}{\partial\gamma}\right), \tag{25}$$

$$\frac{\partial \bar{i}}{\partial \gamma} = \bar{i}\left[-\left(\frac{1}{\overline{T}} + \frac{\gamma}{1-\alpha}\right)\frac{\partial\overline{T}}{\partial\gamma} - \frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial\overline{P}_e}{\partial\gamma} - \frac{\overline{T}}{1-\alpha}\right]. \tag{26}$$

Optimal scrapping age

The effect of the embodied energy-saving technical progress on the replacement decision of firms is described by combining equations (25) and (26):

$$\frac{\partial\overline{T}}{\partial\gamma} =$$

$$\left(\frac{1}{\overline{T}} + \frac{\gamma}{1-\alpha} - \frac{\gamma\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1}\right)^{-1}\left(\frac{\overline{T}\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1} - \frac{1}{\gamma} - \frac{\overline{T}}{1-\alpha} - \frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial\overline{P}_e}{\partial\gamma}\right).$$

For a fixed level of energy price, an increase of γ reduces the scrapping age, *i.e.*, **direct effect**:

$$\frac{\partial\overline{T}}{\partial\gamma}_{|\overline{P}_e fixed} =$$

$$\left(\frac{1}{\overline{T}} + \frac{\gamma}{1-\alpha} - \frac{\gamma\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1}\right)^{-1}\left(\frac{\overline{T}\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1} - \frac{1}{\gamma} - \frac{\overline{T}}{1-\alpha}\right) < 0.$$

The interpretation is the following. The greater γ the lower energy requirements of new equipment. Then, for a fixed level of energy price, firms decide to replace their equipment earlier.

However the replacement decision is also determined by the variation of the level of energy price, which is $\partial\overline{P}_e/\partial\gamma < 0$ (see Appendix B, Tables 1 and 2, fifth column). When γ increases, the new equipment requires less energy than the old machines. Then, for a given scrapping age, the demand of energy decreases. As a consequence, the level of energy price decreases too. This is the **indirect effect** of γ through the level of energy price:

$$-\left(\frac{1}{\overline{\overline{T}}}+\frac{\gamma}{1-\alpha}-\frac{\gamma\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1}\right)^{-1}\frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial\overline{P}_e}{\partial\gamma}>0,$$

which is a positive effect[9]. Indeed, since the level of energy price decreases, the operation cost decreases too and firms decide to replace their equipment later.

The combination of both effects gives us the net effect of γ on the scrapping age. From the numerical solution (see Appendix B), the effect of a decrease of the level of energy price (positive indirect effect) is not strong enough to overcome the negative direct effect. Hence, the higher growth rate of embodied energy-saving technical progress the lower replacement age, *i.e.*, $\partial\overline{T}/\partial\gamma<0$.

Optimal investment

Equation (26) gives us the behavior of the optimal investment ($\bar{i}$) when there is a variation of the growth rate embodied energy-saving technical progress (γ). In this case, we can distinguish a **direct effect** of γ on $\bar{i}$:

$$\frac{\partial\bar{i}}{\partial\gamma}_{|\overline{T}\ and\ \overline{P}_e\ fixed}=-\bar{i}\frac{\overline{T}}{1-\alpha}<0.$$

For a fixed scrapping age and level of energy price, when the growth rate of embodied energy-saving technical progress increases, firms rise investment because the new equipment requires less energy. This is clear from equation (13) (optimal investment rule along the BGP). Indeed, for fixed $\overline{T}$ and $\overline{P}_e$, it is straightforward that the RHS rises when γ increases[10]. To keep the equality, investment has to increase in order to rise the LHS.

However, two positive **indirect effects** overcome the negative direct effect. Indeed, following the previous section, a variation of γ changes

[9] From Appendix A, changing $\overline{T}$ by γ and viceversa, it is straightforward that $\left(\frac{1}{\overline{\overline{T}}}+\frac{\gamma}{1-\alpha}-\frac{\gamma\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1}\right)^{-1}>0$.

[10] By differentiating RHS with respect to γ, for $\overline{T}$ and $\overline{P}_e$ fixed.

the replacement age ($\partial\overline{T}/\partial\gamma < 0$). This **indirect effect through the scrapping age** is described by the expression

$$-\bar{i}\left(\frac{1}{\overline{T}} + \frac{\gamma}{1-\alpha}\right)\frac{\partial\overline{T}}{\partial\gamma} > 0.$$

The greater γ the lower $\overline{T}$. Then, firms decide to invest more.

This effect is reinforced by a second positive **indirect effect through the level of energy prices** ($\partial\overline{P}_e/\partial\gamma < 0$), which is given by

$$-\bar{i}\frac{1}{1-\alpha}\frac{1}{\overline{P}_e}\frac{\partial\overline{P}_e}{\partial\gamma} > 0.$$

When γ rises, the level of energy price decreases. As a consequence, the operation cost of machines falls, and firms decide to invest more.

Finally, from the numerical solution (see Appendix B), the net result of an increase in the growth rate of embodied energy-saving technical progress is a rise of the optimal investment, *i.e.*, $\partial\bar{i}/\partial\gamma > 0$.

Final good output

Taking the final good output along the BGP, we get

$$\frac{\partial\overline{y}}{\partial\gamma} = \overline{y}\alpha\left(\frac{1}{\bar{i}}\frac{\partial\bar{i}}{\partial\gamma} + \frac{1}{\overline{T}}\frac{\partial\overline{T}}{\partial\gamma}\right).$$

From the previous sections, we know that $\partial\bar{i}/\partial\gamma > 0$ and $\partial\overline{T}/\partial\gamma < 0$. Hence, according to numerical solution, the net effect of an increase of the growth rate of embodied energy-saving technical progress is negative, *i.e.*, $\partial\overline{y}/\partial\gamma < 0$. Indeed, the negative effect of the replacement age overcomes the positive effect of the optimal investment. As a consequence, the level of final good output falls[11]. This result is consistent with Boucekkine *et al.* (1997) and Boucekkine *et al.* (1998). These authors get decreasing level of final good output after a positive shock of the (exogenous) embodied technical progress. Indeed, the economy

[11] Observe that a fall in the level of final good output does not necessary implies a decrease in welfare. Indeed, gains in welfare could be found during the transition to the BGP

considered by these authors shows long-run growth. However, the levels (series without the trend) decrease. In our model, this fall can be compensated by including additional elements to the investment, such as human capital investment. Furthermore, if we increase the embodied energy-saving technical progress (γ) together with disembodied technical progress (A), the final good output rises (see Appendix B, Tables 1 and 2, last column).

3.5 Concluding remarks

Along the same lines as Chapter 2, this chapter studied the hypothesis of the trade-off between energy reduction and growth, which is less severe if energy conservation is raised by means of energy-saving technologies. Our model is an extension, to the general equilibrium case, of the work of Boucekkine and Pommeret (2004). We found that our model is very rich to capture a wide range of different aspects that affect the long-run behavior of our economy. In particular, we pointed out the frequently ignored issue of equipment replacement, which plays important role in energy-saving technical change. Indeed, an important consequence of considering such a replacement was that the increase of a tax on the energy expenditure of firms does not necessarily induce earlier replacement of machines if this tax is high enough. The reason is the following. To offset the effect of a higher energy expenditure tax, firms should replace earlier their equipment. Since replacement is costly, firms prefer to take higher energy expenditure than a faster (and too costly) equipment replacement.

This chapter entails several limitations. In the following, we point out three main boundaries of our analysis. First, as Section 2 observes, due to the difficult analytical tractability of vintage capital models, we do not explicitly study the extraction sector. However, it is widely accepted that the extraction sector plays important role to determine the evolution of energy prices, with particular significance of market power issues. Therefore, a valuable extension of this chapter is to include the extraction sector into our framework.

Second, this chapter shows that a tax on the energy expenditure of firms has important implications on both investment and replacement. However, it is clear that this tax will also affect the investment in technical progress. Indeed, this is an important decision for firms, which is assumed to be exogenous in this chapter. Then, another extension is to consider energy-saving technical progress as an endogenous variable. Some studies point out the importance of assuming endogenous technical progress in this kind of models (see for instance, Carraro *et al.* (2003) and Buonnano *et al.* (2003)). Since it is considered a general equilibrium set-up, we could implement this idea by means of a R&D sector (see Löschel (2002) for a survey about technical change in economic models of environmental policy).

Finally, let us consider some comments about the robustness of our numerical results. As Section 3.4.1 observes, the strategy followed in this chapter allows us to identify the different effects of an increase of Z, A, $\overline{e}_s$ and γ on the economy, which is one of the main objectives of this chapter. However, we had to solve numerically a static system of non-linear equations to measure the strength of the different effects (*i.e.*, the net effect). As we observed in Section 3.4.1 (footnote 7), the effects on the level of energy price are robust. However, the net effect depends on the value of the parameters. Therefore, we chose a parametrization which is empirically supported by the economic literature. Nevertheless, this calls for an accurate econometric calibration of our model, considering the particularities of each country.

Appendix A

Proposition 1. *If* $0 < \alpha < 1$, $0 < Z < 1$, $\gamma > 0$, $\rho > 0$ *and* $\overline{T} > 0$ *then*

$$\left(\frac{1}{\overline{T}} - \frac{\gamma \exp(\gamma \overline{T})}{\exp(\gamma \overline{T}) - 1} + \frac{\gamma}{1-\alpha}\right)^{-1} > 0.$$

Proof. The term inside the brackets is equal to

$$\frac{(\exp(\gamma\overline{T}) - 1)(1-\alpha) - \gamma\overline{T}\exp(\gamma\overline{T})(1-\alpha) + \gamma\overline{T}(e^{\gamma\overline{T}} - 1)}{\overline{T}(\exp(\gamma\overline{T}) - 1)(1-\alpha)}$$

If $0 < \alpha < 1$, $\gamma > 0$ and $\overline{T} > 0$, the denominator is greater than zero[12]. About the numerator, rearranging terms, we obtain that it is equal to

$$\exp(\gamma\overline{T})(1-\alpha+\alpha\gamma\overline{T}) - (1-\alpha+\overline{T}\gamma).$$

If we rename $\gamma\overline{T} = x$, the numerator is a function $f(x) = \exp(x)(1-\alpha+\alpha x) - (1-\alpha+x)$. It is easy to see that $f(0) = 0$ and $f'(x) > 0$ because $x > 0$. Then, the numerator is greater than zero.

Since numerator and denominator are greater than zero, then

$$\left(\frac{1}{\overline{\overline{T}}} - \frac{\gamma\exp(\gamma\overline{T})}{\exp(\gamma\overline{T})-1} + \frac{\gamma}{1-\alpha}\right)^{-1} > 0$$

■

Appendix B

In this chapter we write a program for Gauss to solve the static system of non-linear equations given by the equations (13), (14) and (16). The structure of the program is simple. First, we assign values to the parameters of our model from the economic literature. Since we want to study the effect of the change of some parameters on the optimal scrapping age, investment, energy price and output, we generate a sequence of them. After that, we solve the static system of non-linear equations by a standard Newton-Raphson algorithm. Finally, we plot the optimal scrapping age ($\overline{T}$), investment ($\bar{i}$), level of energy price ($\overline{P}_e$) and final good output ($\overline{y}$) against the different values of the parameters, concluding the corresponding sign of the derivatives.

[12]Observation: if $\gamma > 0$, $\rho > 0$ and $\overline{T} > 0$ then $\exp(\gamma\overline{T}) > 1$ and $\exp(\rho\overline{T}) > 1$.

Table 1: Parametrization

Parameter	ΔZ	ΔA	$\Delta \overline{e}_s$	$\Delta \gamma$
A	15	[15,16]	15	15
α	1/3	1/3	1/3	1/3
ρ	0.03	0.03	0.03	0.03
γ	0.02	0.02	0.02	[0.02,0.04]
$\overline{e}_e$	1300	1300	[1300,1310]	1300
Z	[0.85,0.95]	0.85	0.85	0.85

Table 2: Results

Variable	ΔZ	ΔA	$\Delta \overline{e}_s$	$\Delta \gamma$	$\Delta \gamma \Delta A$
$\overline{P}_e$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$
$\overline{T}$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$
$\bar{i}$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
$\overline{y}$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$

Chapter 4

The problem of non-renewable energy resources in the production of physical capital

(Pérez-Barahona, 2007a. The problem of non-renewable energy resource in the production of physical capital. CORE Discussion Papers. 2007/8)

4.1 Introduction

Standard papers on non-renewable energy resources (see for instance, Dasgupta and Heal (1974, 1979), Stiglitz (1974), Solow (1974a), Hartwick (1989), and Smulders and Nooij (2003)) assume the same technology for both physical capital production and consumption (which implies that the energy intensity of both sectors is the same), as well as the existence of reasonable substitution possibilities between energy and physical capital. Moreover, this literature also claims that if energy is produced by means of non-renewable resources, physical capital accumulation could offset the constraints on production possibilities due to non-renewable energy resources (Dasguptan and Heal (1979)). Nevertheless, this chapter argues that this explanation about physical capital accumulation as a solution to the problem of non-renewable energy resources is not totally satisfactory since there is empirical evidence showing that physical capital production is relatively more energy-intensive than consumption (see Chapter 5 or Pérez-Barahona (2007b)). Indeed, under this hypothesis, if energy were produced by means of non-renewable energy resources, this would limit growth through physical capital accumulation. Then, all solution based on rising physical capital accumulation can have dubious effectiveness to solve the trade-off between economic growth and the usage of non-renewable energy resources if one does not analyze the role of energy in physical capital accumulation, *i.e.*, equipment good production. Following that idea, this chapter shows that, if physical capital production is relatively more energy-intensive than consumption, technical progress (in particular, energy-saving technical progress) plays a central role to guarantee long-run growth. Indeed, we establish that if the growth rate of technical progress is high enough there is balanced growth path (BGP) with positive growth of all the endogenous variables. Otherwise, the economy decreases at constant rate, with all the endogenous variables converging to zero asymptotically. This result contrasts with the claim of Stiglitz (1974) and Dasgupta and Heal (1979). These authors argue that, even without technical progress, capital accumulation can offset the constraint on production possibilities due to non-renewable energy resources. However, this chapter shows that the economy decreases (converging to zero asymptotically) if there is no growth of technical progress.

This chapter emphasizes the role of energy-saving technical progress as a solution to solve the trade-off between economic growth and the usage of non-renewable energy resources, such as fossil fuels. Boucekkine and Pommeret (2004) point out the importance of energy-saving technologies. The idea of this kind of technologies is energy efficiency. According to various studies (EU (2005)), the European Union (EU) could save at least 20% of its present energy consumption (EUR 60 billion per year, or the present combined energy consumption of Germany and Finland) by improving energy-efficiency.

In order to illustrate the ideas previously presented, this chapter builds a general equilibrium model, based on Hartwick (1989), with three sectors: final good, equipment good, and extraction sector, where physical capital production is relatively more energy-intensive than consumption. In this model, the equipment good sector produces physical capital by means of a technology defined over two inputs: energy and investment, where the energy is directly obtained from a given stock of a non-renewable energy resource. This work presents the conditions under which technical progress (in particular, energy-saving technical progress) can guarantee positive long-run growth.

The chapter is organized as follows. Section 4.2 introduces the model, with a description of each sector. The optimal solution of this economy is presented in Section 4.3, together with the interpretation of the main results of the chapter. Finally, Section 4.4 concludes.

4.2 The model

Following Hartwick (1989), let us consider an economy with three sectors where the energy is obtained by means of non-renewable energy resources, and the population is constant [1]. The final good sector produces a non-durable good with AK technology, where physical capital is the only input. As usual, the final good is assigned to consumption or investment. The equipment sector produces physical capital (equipment good) be means of a technology defined over two inputs: energy and investment. Finally, the extraction sector directly produces energy by extracting a non-renewable resource from a given stock.

4.2.1 Final good sector

The final good sector produces a non-durable good $Y(t)$ by means of the following AK technology:

$$Y(t) = A(t)K(t), \tag{1}$$

[1] See Stigliz (1974) for an analysis of the growth possibilities open to an economy with non-renewable energy resources and an exponential growing labour force.

where $A(t)$ is the (exogenous) disembodied technical progress, and $K(t)$ represents the equipment good, which is the only input to produce the final good. Moreover, the final good is used either to consume, $C(t)$, or to invest in physical capital, $I(t)$, verifying the budget constraint of the economy

$$Y(t) = C(t) + I(t). \tag{2}$$

As the introduction points out, the standard literature on non-renewable energy resources considers energy as an additional input in the production of the final good[2], assuming the same technology for both physical capital production and consumption. Then, if energy is produced by means of non-renewable energy resources, there is (*a priori*) a limit to economic growth due to the stock restrictions of this kind of natural resources. In general, the solution to this problem (without replacement of non-renewable resources by renewable ones) is based on greater physical capital accumulation. However, this literature does not study the case of having different technologies for physical capital production and consumption, which is empirically supported (see Chapter 5). Indeed, if physical capital production is relatively more energy-intensive than consumption, non-renewable resources can limit growth through the equipment production. In fact, the aim of this chapter is to study the effects of this hypothesis on the long-run growth of the economy. For simplicity reasons, an AK technology is assumed, considering energy as input for equipment production, *i.e.*, physical capital production is relatively more energy-intensive than consumption.

4.2.2 Equipment good sector

The equipment good sector produces physical capital by means of a technology with two inputs. On the one hand, the equipment good sector

[2]In general, $Y(t) = F[K(t), R(t)]$. Examples:

$$Y(t) = [\alpha_1 K(t)^{(\sigma-1)/\sigma} + \alpha_2 R(t)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)} \text{ CES},$$

$$Y(t) = K(t)^{\alpha_1} R(t)^{\alpha_2} \text{ Cobb–Douglas}.$$

takes the fraction of final good devoted to equipment production, *i.e.*, $I(t)$. And on the other, the equipment production needs energy, $R(t)$, which is directly produced by the extraction sector. The technology for equipment production is represented by the following Cobb-Douglas function:

$$K(t) = [\theta(t)R(t)]^{\alpha} I(t)^{1-\alpha}, \text{ with } 0 < \alpha < 0, \tag{3}$$

where α is the share of non-renewable energy resource in equipment production, and $\theta(t)$ denotes the embodied energy-saving technical progress. In this chapter, technical progress is considered as exogenous variable.

At this point, two observations are needed. First, this model considers physical capital as an intermediate good (equipment good). Indeed, it is assumed total depreciation of physical capital for each period. This allows us to regard $K(t)$ as a flow variable instead of a stock, getting an AK configuration, which has no dynamical transition. However, Chapter 5 (2006) eliminates this simplification assuming $K(t)$ as a durable good (stock variable). This entails technical difficulties[3] and dynamical transition. Nevertheless, both models essentially have the same behavior along the BGP. Then, since the aim of this chapter is to study the conditions under which technical progress can sustain long-run growth, $K(t)$ is considered here as a flow variable for simplicity reasons.

Second, equation (3) assumes substitutability between energy and investment. However, there is a very well known debate about substitutability *vs.* complementarity. Indeed, if one considers the idea of a minimum energy requirement to use a machine, the assumption of complementarity should be chosen (see Chapter 2 and 3). Nevertheless, taking the argument of Dasgupta and Heal (1979), if $I(t)$ is interpreted as final good service providing a certain (minimum) energy flow, one can keep the substitutability assumption.

[3]Chapter 5 deals with an optimal control problem with mixed constraints and two states variables: capital accumulation and stock of non-renewable resource. He provides a full analytical characterization of both short and long-run dynamics applying the technique of Special Functions representation.

4.2.3 Extraction sector

The energy is directly produced by extracting a non-renewable energy resource $R(t)$ from a given homogeneous stock $S(t)$. As Dasgupta and Heal (1974), and Hartwick (1989), it is assumed costless extraction[4]. The evolution of this resource stock is described by the expression

$$R(t) = -\dot{S}(t), \text{ where } S(0) \text{ is given.} \tag{4}$$

Since one can not extract more than the available stock, the following restriction should be included

$$S(t) \geq 0. \tag{5}$$

4.2.4 Central planner solution

The central planner (optimal solution) maximizes the instantaneous utility function of the representative household:

$$\max W = \int_0^\infty ln[C(t)] \exp(-\rho t) dt, \text{with } \rho > 0,$$

subject to equations (1)-(5), where ρ is the time preference parameter (it is assumed to be a positive discount factor).

4.3 Equilibrium

This section presents the central planner solution, providing a full analytical characterization of the equilibrium of this economy. In the following, we solve the optimal control problem involved by this model. Moreover, the corresponding dynamical system is analyzed, providing the interpretation of the results.

[4] According to Dasgupta and Heal (1974), the extraction cost do not introduce any great problem if one assumes any non-convexities. Indeed, one can easily introduce extraction cost, $EC(t)$, by modifying the budget constraint of the economy:

$$Y(t) = C(t) + I(t) + EC(R, S), \text{ where } \partial EC/\partial R > 0 \text{ and } \partial EC/\partial S \leq 0.$$

4.3.1 Optimal control problem

We can easily rewrite the central planner problem as an optimal control problem with mixed constraints:

$$\max \int_0^\infty ln[A(t)\theta(t)^\alpha R(t)^\alpha I(t)^{1-\alpha} - I(t)] \exp(-\rho t) dt$$

subject to:

$$\dot{S}(t) = -R(t),$$

$$S(t) \geq 0,$$

$$\text{with } 0 < \alpha < 1,\ \rho > 0, \text{ and } S(0) \text{ given},$$

where $S(t)$ is the state variable, $I(t)$ and $R(t)$ are the control variables, and $A(t)$ and $\theta(t)$ are exogenous functions.

Following Sydsæter *et al.* (1999, pages 109-110), the Lagrangian associated with this problem is

$$\mathcal{L}(\bullet) = ln[A(t)\theta(t)^\alpha R(t)^\alpha I(t)^{1-\alpha} - I(t)] \exp(-\rho t) - \lambda(t)R(t) + q(t)S(t),$$

where $\lambda(t)$ and $q(t)$ are the Lagrangian multipliers. The corresponding first order conditions (FOC) are:

$$\frac{\partial \mathcal{L}}{\partial R(t)} = 0; \frac{\partial \mathcal{L}}{\partial I(t)} = 0;$$

$$\frac{\partial \mathcal{L}}{\partial S(t)} = -\dot{\lambda}(t);$$

$$q(t) \geq 0 (= 0 \text{ if } S(t) > 0);$$

$$\lim_{t \to \infty} \lambda(t) S(t) = 0 \text{ (Transversality condition)}.$$

The FOC for $R(t)$, $I(t)$, and $S(t)$, yield, respectively,

$$\frac{A(t)\theta(t)^\alpha \alpha \frac{1}{R(t)} \left(\frac{R(t)}{I(t)}\right)^\alpha}{A(t)\theta(t)^\alpha \left(\frac{R(t)}{I(t)}\right)^\alpha - 1} \exp(-\rho t) = \lambda(t) + q(t), \tag{6}$$

$$\frac{R(t)}{I(t)} = [A(t)\theta(t)^\alpha (1-\alpha)]^{-\frac{1}{\alpha}}, \tag{7}$$

$$q(t) = -\dot{\lambda}(t). \tag{8}$$

Since the non-renewable energy resource is assumed to be essential input, it is not optimal to completely deplete the stock $S(t)$ in a finite t (see Hartwick (1989)). Therefore, $S(t) > 0$ for all t, which implies that $q(t) = 0$ for all t. Then, from equation (8), one concludes that the shadow price of the non-renewable resource is constant for all t, *i.e.*, $\lambda(t) = \lambda$.

Notice that, since $\lambda(t) = \lambda$ for all t, the transversality condition implies that the stock of non-renewable energy resource should be depleted asymptotically, *i.e.*,

$$\lim_{t\to\infty} S(t) = 0. \tag{9}$$

4.3.2 Dynamical system

Taking the FOC, the equilibrium of this economy is characterized by the following dynamical system:

Proposition 1. *The optimal solution of this economy is a path*

$$\{R(t), I(t), K(t), Y(t), C(t), S(t)\},$$

and a constant λ that satisfy the following conditions for all t:

1)

$$\frac{A(t)\theta(t)^{\alpha}\alpha\frac{1}{R(t)}\left(\frac{R(t)}{I(t)}\right)^{\alpha}}{A(t)\theta(t)^{\alpha}\left(\frac{R(t)}{I(t)}\right)^{\alpha}-1}\exp(-\rho t) = \lambda; \tag{10}$$

2) equations (1)-(4), (7) and (9);

where $A(t)$ and $\theta(t)$ are exogenous functions.

Proposition 1 provides a dynamical system of 7 equations and 7 unknowns, which describes the equilibrium of this economy (optimal solution) for all t. Notice that λ is the shadow price of the non-renewable resource.

4.3.3 Optimal solution

Solving the dynamical system of Proposition 1, one obtains the optimal solution paths for $\{R(t), I(t), K(t), Y(t), C(t), S(t)\}$, and the constant λ. Following Solow (1974a), it is assumed $A(t) = \overline{A} \cdot \exp(\gamma_a t)$ and $\theta(t) = \overline{\theta} \cdot \exp(\gamma_\theta t)$, where $\overline{A} > 0$, $\overline{\theta} > 0$, $\gamma_A \geq 0$, and $\gamma_\theta \geq 0$. Then, Proposition 2 establishes the optimal solution of this economy:

Proposition 2. *For a given $S(0) > 0$, $\rho > 0$, $0 < \alpha < 1$, $\overline{A} > 0$, $\overline{\theta} > 0$, $\gamma_A \geq 0$, and $\gamma_\theta \geq 0$, the equilibrium of the economy in every time t is given by*

$$R(t) = \rho S(0) \exp(-\rho t), \tag{11}$$

$$S(t) = S(0) \exp(-\rho t), \tag{12}$$

$$I(t) = \rho S(0)\overline{\theta}[\overline{A}(1-\alpha)]^{\frac{1}{\alpha}} \exp\left\{\left(\frac{1}{\alpha}\gamma_A + \gamma_\theta - \rho\right)t\right\}, \tag{13}$$

$$K(t) = \rho S(0)\overline{\theta}[\overline{A}(1-\alpha)]^{\frac{1-\alpha}{\alpha}} \exp\left\{\left(\frac{1-\alpha}{\alpha}\gamma_A + \gamma_\theta - \rho\right)t\right\}, \tag{14}$$

$$Y(t) = \rho S(0)\overline{\theta}\overline{A}^{\frac{1}{\alpha}}(1-\alpha)^{\frac{1-\alpha}{\alpha}} \exp\left\{\left(\frac{1}{\alpha}\gamma_A + \gamma_\theta - \rho\right)t\right\}, \tag{15}$$

$$C(t) = \rho S(0)\overline{\theta}\frac{\alpha}{1-\alpha}[\overline{A}(1-\alpha)]^{\frac{1}{\alpha}} \exp\left\{\left(\frac{1}{\alpha}\gamma_A + \gamma_\theta - \rho\right)t\right\}, \tag{16}$$

and the shadow price of the stock of non-renewable resource is

$$\lambda(t) = \lambda = \frac{1}{\rho S(0)}. \tag{17}$$

Proof. The proof is provided in Appendix ■

4.3.4 Interpretation of the results

As general comment, one can observe that this economy is always in the BGP, which is defined as the situation where all the endogenous variables grow at constant rate, *i.e.*, $x(t) = \overline{x} \cdot \exp(\gamma_x t)$, where γ_x is the growth rate of the variable x. Indeed, Proposition 2 implies that

$$\gamma_S = \gamma_R = -\rho, \tag{18}$$

$$\gamma_Y = \gamma_C = \gamma_I = \frac{1}{\alpha}\gamma_A + \gamma_\theta - \rho, \tag{19}$$

$$\gamma_K = \frac{1-\alpha}{\alpha}\gamma_A + \gamma_\theta - \rho. \tag{20}$$

The reason is the following. Section 4.2.2 points out the implications of assuming physical capital as a flow variable (total depreciation). Indeed, the assumption of physical capital as a non-durable good implies that the economy described in this chapter has an AK representation. Then, this economy is always in the BGP, without dynamical transition. Furthermore, taking a similar set-up to this chapter, Chapter 5 proves that the assumption of physical capital as a stock variable (durable good) adds dynamical transition with essentially identical long-run behavior as the economy with physical capital as flow variable[5]. Since this chapter focuses on long-run growth, one can avoid some technical difficulties assuming physical capital as a non-durable good.

Extraction sector

Equation (18) implies that both the stock of non-renewable energy resource $S(t)$ and the extraction flow $R(t)$ decrease at the same constant rate. The explanation is the following. $\gamma_S = \gamma_R$ because energy is directly produced by extracting $R(t)$ from the stock $S(t)$ (see equation (4)). Moreover, the reason why both variables decrease at constant rate is because energy is produced by means of a non-renewable energy resource. Then, since the non-renewable energy resource is assumed to be essential input, the stock of the resource should be depleted asymptotically. The depletion rate is ρ, which is the time preference parameter of the household. Indeed, the greater ρ, the less important the future for the household. Then, if ρ rises, the non-renewable resource is depleted faster, reducing the growth rates of the economy (see equations (19) and (20)):

$$\frac{\partial \gamma_i}{\partial \rho} < 0, \text{ for all } i = Y, C, I, K, R, S. \tag{21}$$

[5] As Chapter 5 observes, capital accumulation also implies that the growth rate of the technical progress affects the levels of our endogenous variables.

This effect is also reflected in the shadow price of the non-renewable energy resource (see equation (17)). The greater ρ, the lower shadow price of the resource. Then, the resource extraction increases[6].

In addition, regarding to the levels, *i.e.*, the initial values of the endogenous variables, Proposition 2 implies that the greater ρ, the greater the initial values, but with lower growth rates.

Final and equipment good sectors

From Proposition 2, one observes that technical progress (in particular, energy-saving technical progress) is a key element to guarantee long-run growth. Indeed, if there is no growth of technical progress, *i.e.*, $\gamma_A = \gamma_\theta = 0$, the economy decreases due to the negative effect of ρ (see equations (19) and (20)). Furthermore, one can establish the conditions for technical progress to preserve positive growth:

Proposition 3. $\gamma_K > 0$ *if and only if* $(1-\alpha)\gamma_A + \alpha\gamma_\theta > \alpha\rho$*;* $\gamma_Y (= \gamma_C = \gamma_I) > 0$ *if and only if* $\gamma_A + \alpha\gamma_\theta > \alpha\rho$.

Notice that, according to this model, the growth rate of physical capital γ_K is lower than the growth rate of output $\gamma_Y (= \gamma_C = \gamma_I)$. Indeed, one can easily obtain that $\gamma_K = \gamma_Y - \gamma_A$. This is because the output incorporates the effect of the disembodied technical progress $A(t)$ (see equations (1) and (3))[7]. Then, the condition for positive growth of all the endogenous variables of the model is presented in the following proposition:

Proposition 4. $\gamma_K > 0$ *and* $\gamma_Y (= \gamma_C = \gamma_I) > 0$ *if and only if* $(1-\alpha)\gamma_A + \alpha\gamma_\theta > \alpha\rho$.

Proposition 4 establishes that the growth rate of technical progress

[6]Notice that a higher endowment of non-renewable energy resource ($S(0)$) also reduces the shadow price of the resource. Both ρ and $S(0)$ increase the extraction level. However, only ρ affects the depleting growth rate. This is because ρ represents the time preference of the household.

[7]If $\gamma_A = 0$ then $\gamma_K = \gamma_Y (= \gamma_C = \gamma_I)$.

should be high enough to maintain positive growth. Otherwise, the economy decreases at constant rate, with all the endogenous variables converging to zero asymptotically. Moreover, from equations (19) and (20), it is clear that the higher the growth rate of technical progress, the higher the long-run growth rates of the economy[8].

A simple quantitative evaluation can be done by applying equations (18)-(20) and the previous condition. Taking an explicit target of GDP annual growth rate (γ_Y), one can determine the minimum annual growth rate of energy-saving technical progress (γ_θ) compatible with that target. The empirical literature widely accepts a value for the time preference of household (ρ) of 3%, as well as an annual growth rate of disembodied technical progress (γ_A) about 1%. According to this model, for a target of $\gamma_Y = 2\%$ (which is a common target for developed countries), the annual depleting rate of the resource should be 3%, *i.e.*, $\gamma_S = \gamma_R = -3\%$. Moreover, for given values of the share of energy resource in the production of equipment goods (α), the second row of Table 1 provides the compatible values of the annual growth rate of energy-saving technical progress (γ_θ).

Table 1

α	0.25	0.5	0.75
γ_θ	1%	3%	3.7%
γ_Y with $\gamma_\theta = 0$	1%	-1%	-1.7%
γ_K with $\gamma_\theta = 0$	0%	-2%	-2.7%

The values of the second row can be compared with empirical estimations of γ_θ. Azomahou *et al.* (2004) estimates an annual growth rate of energy-saving technical progress for the USA around 1.9%, and 0.9% for France. According to this exercise, problems could arise if the share of non-renewable energy resource in equipment production (α) is high. Indeed, one can observe (see Table 1, second row) that the higher α the higher growth rate of energy-saving technical progress (γ_θ) is required. This effect can be illustrated by the following two figures.

[8]Notice that the levels of the endogenous variables are positively affected by the levels of technical progress (see Proposition 2).

Figure 1

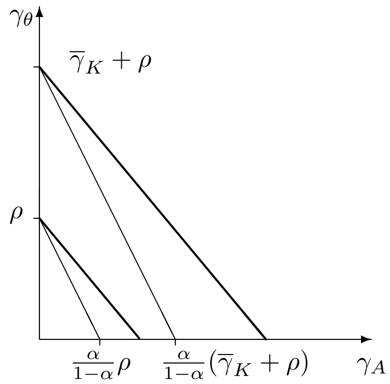


Figure 2

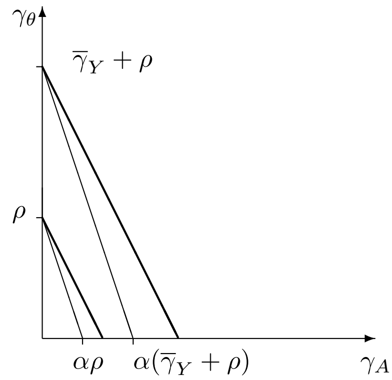


Figure 1 represents the set of pairs $(\gamma_\theta, \gamma_A)$ that yields the same growth rate of physical capital[9], *i.e.*, level curves. In this model, the level curves for the growth rate of physical capital are parallel straight lines with negative slope $-\frac{1-\alpha}{\alpha}$, where the growth rate of physical capital $(\overline{\gamma}_K)$ increases as one moves to the top right-hand corner of the figure. If the share of non-renewable resources in equipment good production rises, the absolute value of the slope decreases. Then, the level curves become flatter (bold line). Hence, for a fixed growth rate of disembodied technical progress (γ_A), the growth rate of energy-saving technical progress (γ_θ) should be higher in order to maintain a target of growth rate of physical capital. The reason is unambiguous. The greater the share of non-renewable resources in equipment good production, the greater required growth rate of energy-saving technical progress to ensure a target of growth rate of physical capital. Figure 2 replicates the same logic as Figure 1, but for the case of output growth rate. Indeed, the greater the share of non-renewable resources in equipment good production, the greater required growth rate of energy-saving technical progress to ensure a target of output growth rate.

Finally, Table 1 also illustrates how important is energy-saving tech-

[9] From equation (20), for a fixed value of $\overline{\gamma}_K$, we can easily obtain the level curve $\gamma_\theta = (\overline{\gamma}_K + \rho) - \frac{1-\alpha}{\alpha}\gamma_A$. Notice that $\gamma_\theta = \rho - \frac{1-\alpha}{\alpha}\gamma_A$ is the level curve corresponding to $\overline{\gamma}_K = 0$.

nical progress to guarantee positive growth. Rows third and fourth provide, respectively, the growth rate of output and physical capital, when there is no growth of energy-saving technical progress (*i.e.*, $\gamma_\theta = 0$), but there is the disembodied one ($\gamma_A = 1\%$). When $\alpha = 0.25$, the growth rate of output is 1%. However, there is no growth of physical capital ($\gamma_K = 0\%$). Moreover, as α increases, both the growth rate of output and physical capital are negative. Nevertheless, as the second row illustrates, the economy can maintain a growth rate target (in this case, $\gamma_Y = 2\%$) opting for energy-saving technologies.

Energy efficiency

The previous section points out the role of technical progress to guarantee long-run growth. The mechanism behind that is energy efficiency: technical progress allows for a BGP with positive growth because it reduces energy intensity. Let us define energy intensity as the quantity of energy per unit of output, *i.e.*, the ratio $R(t)/Y(t)$ [10]. Taking equations (12) and (15), one easily obtains the following proposition:

Proposition 5. *For a given $\rho > 0$, $0 < \alpha < 1$, $\overline{A} > 0$, $\overline{\theta} > 0$, $\gamma_A \geq 0$, and $\gamma_\theta \geq 0$, the energy intensity of the economy in every time t is given by:*

$$\frac{R(t)}{Y(t)} = \frac{1}{\overline{\theta}\overline{A}^{\frac{1}{\alpha}}(1-\alpha)^{\frac{1-\alpha}{\alpha}}} \exp\left\{-\left(\frac{1}{\alpha}\gamma_A + \gamma_\theta\right)t\right\}. \tag{22}$$

From this proposition, we can conclude that the energy intensity decreases at the constant rate[11]

$$\gamma_{\frac{R(t)}{Y(t)}} = -\left(\frac{1}{\alpha}\gamma_A + \gamma_\theta\right). \tag{23}$$

[10] There are two alternative definitions of energy intensity: $\frac{R(t)}{K(t)}$ and $\frac{R(t)}{I(t)}$.

[11] Notice that, following the alternative definitions, energy intensity decreases at constant rate too. Indeed,

$$\gamma_{\frac{R(t)}{K(t)}} = -\left(\frac{1-\alpha}{\alpha}\gamma_A + \gamma_\theta\right); \quad \gamma_{\frac{R(t)}{I(t)}} = -\left(\frac{1}{\alpha}\gamma_A + \gamma_\theta\right).$$

Then, technical progress decreases energy intensity, which is consistent with empirical works, such as Azomahou *et al.* (2004). Indeed, the greater the growth rate of technical progress, the lower the growth rate of energy intensity. Furthermore, if technical progress does not grow, the energy intensity is constant and the economy does not overcome the growth limit problem of non-renewable energy resources (indeed, in this case, output decreases at the rate ρ).

4.4 Concluding remarks

This chapter studied the possibilities of technical progress to deal with the growth limit problem imposed by the usage of non-renewable energy resources, when physical capital production is relatively more energy-intensive than consumption. In order to illustrate the problem, we considered a general equilibrium model with three sectors: final good, equipment good, and extraction sector. In this model, the equipment good sector produces physical capital by means of a technology defined over two inputs: energy and investment, where the energy is directly obtained from a given stock of a non-renewable energy resource. This work presented the conditions under which technical progress (in particular, energy-saving technical progress) guarantees positive long-run growth. Indeed, Proposition 4 established that the growth rate of technical progress should be high enough to maintain a positive growth of all the endogenous variables. Otherwise, the economy decreases at constant rate, with all the endogenous variables converging to zero asymptotically. The mechanism behind that was energy efficiency (Proposition 5): technical progress reduces energy intensity, allowing for a BGP with positive growth. The optimal solution (central planner solution) was considered. However, one can easily obtain the same results for the decentralized economy, which is a typical outcome when no externality is considered in the economy.

This chapter entails several limitations. The main one is pointed out in Section 4.2.2. This model considers physical capital as a non-durable good, *i.e.*, flow variable. However, physical capital is usually considered as a durable good, *i.e.*, stock variable. The assumption of physical capital as a flow variable allow us to deal with an optimal control problem with only one state variable (the stock of non-renewable energy resources) instead of two state variables (the previous one, and the stock of physical capital). Indeed, this model has AK representation, which has no dynamical transition (see Proposition 2). The simplification of physical capital as a flow variable is equivalent to consider physical capital as a stock variable, with total depreciation for each period. However, this is not a very realistic assumption because one important characteristic of physical capital is its partial depreciation, *i.e.*, physical capital is a durable good. Then, one should examine the consistency of the results presented in this chapter considering energy as an input for physical capital accumulation, *i.e.*, physical capital as a stock variable. As Section 4.2.2 also observes, this kind of analysis is done in Chapter 5, where physical capital is a durable good. This chapter entails technical difficulties and dynamical transition. However, the behavior of this model along the BGP is essentially identical to our model with physical capital as a non-durable good. If one focuses on the behavior along the BGP, the simplification of physical capital as a non-durable good provides a simpler set-up, without dynamical transition.

Two possible extensions of the model could be done. First, the quantitative evaluation presented in Section 4.3.4 points out the convenience for an accurate estimate of the share of energy resources in the equipment good sector (α). As Table 1 notices, the share of energy resources in the equipment good production has important role to determine the growth rate of technical progress to guarantee a target of long-run growth. Since the standard literature on non-renewable energy resources assumes the same technology for both physical capital production and consumption, there are only estimations of this share for final good production. Finally, as second extension, one should observe that technical progress is assumed to be exogenous variable. However, technical progress (in particular energy-saving technical progress) is a variable decided endogenously by the economy. Due to the tractability of the set-up considered in this chapter, one can incorporate endogenous technical progress by adding a R&D sector for the energy-saving technical progress. Following Aghion and Howitt (1992) and Grossman and Helpman (1991a,b), one can model energy-saving technical progress with improvements in the "quality" of investment goods. Doing that, one can study the capacity of different policies (such as fossil fuel taxes or R&D subsidies) to induce greater investment in energy-saving technologies, and their effects on the short and long-run growth of the economy.

Appendix

The strategy to solve the dynamical system of Proposition 1 is the following:

Step 1: Replacing equation (7) into equation (10), it yields $R(t)$.

Step 2: Replacing $R(t)$ into equation (4), $S(t)$ is obtained by solving a linear first-order differential equation.

Step 3: Knowing $S(t)$, the shadow price of the non-renewable resource λ is determined by equation (9).

Step 4: Taking $R(t)$ into equation (7), one obtains $I(t)$.

Step 5: Applying $R(t)$ and $I(t)$ into equation (3), $K(t)$ is determined.

Step 4: Taking $K(t)$ into the final good technology (equation (1)), the output $Y(t)$ is easily obtained.

Step 6: Since $Y(t)$ and $I(t)$ have been determined, one gets $C(t)$ from the budget constraint of the economy (equation (2)).

In the following, we apply the previous strategy to calculate the equilibrium.

Step 1 yields

$$R(t) = \frac{1}{\lambda} \exp(-\rho t). \tag{$A.1$}$$

The linear first-order differential equation involved in Step 2 is

$$R(t) = -\dot{S}(t), \text{ where } S(0) \text{ is given.}$$

The solution of this equation is given by the following expression:

$$S(t) = S(0) - \int_0^t R(\tau)d\tau.$$

Solving the integral, one easily obtains

$$S(t) = S(0) + \frac{1}{\lambda}\frac{1}{\rho}[\exp(-\rho t) - 1]. \qquad (A.2)$$

Taking the preceding expression for $S(t)$ into the equation (9), it yields equation (17) of Proposition 2

$$\lambda = \frac{1}{\rho S(0)}.$$

Replacing equation (17) into equations (A.1) and (A.2), one gets, respectively, the equations (11) and (12) of Proposition 2. Following Step 4 - 6, one can easily complete the proof ■

Chapter 5

Capital accumulation and non-renewable energy resources: a Special Functions case

(Pérez-Barahona, 2007b. Capital accumulation and non-renewable energy resources: a Special Functions case. CORE Discussion Papers. 2007/9)

5.1 Introduction

A central problem in resources economics is the production of energy by means of non-renewable energy resources, such as fossil fuels. Indeed, authors as Dasgupta and Heal (1974), Hartwick (1989), and Smulders and Nooij (2003) point out that the usage of non-renewable energy resources implies a limit to the economic growth of modern economies.

The standard literature on non-renewable energy resources (see for instance, Dasgupta and Heal (1974, 1979), Solow (1974a,b), and Stiglitz (1974)) gives central position to physical capital accumulation to offset the constraint on production possibilities due to non-renewable energy resources. Indeed, as Stigliz (1974) and Dasgupta and Heal (1979) claim, physical capital accumulation can compensate the negative effect of non-renewable energy resources on economic growth, even without technical progress (the economy can achieve, at least, steady state). However, this literature assumes the same technology for both physical capital accumulation and consumption. Nevertheless, since physical capital is a crucial element to solve the problem imposed by non-renewable energy resources, the assumptions on the technology for physical capital accumulation play an important role too. Indeed, the assumption of same technology for both physical capital accumulation and consumption implies (among other things) that the energy intensity of both sectors is the same. However, data does not to support this implication. There are not available data to construct an energy-intensity measurement of the theoretical physical capital and consumption sectors considered in this kind of literature. However, we can use databases at sector level as proxy for our energy-intensity data. Physical capital accumulation, *i.e.*, equipment good production, usually involves the transformation of raw material such as iron, steel, non-ferrous metals or non-metallic minerals. Moreover, transport and storage also plays important role on these kinds of activities. However, consumption good sectors are more related with other activities, such as food, clothes or construction. Azomahou *et al.* (2006), taking the Structural Analysis Database of the OECD and the energy databases of the IEA, build an energy intensity measurement (ratio between energy consumption and value added) of 14 sectors of the economy. They find that this ratio (mean) is particularly high for iron and steel (0.809) and transport and storage (0.85). Moreover, non-ferrous metals (0.599) and non-metallic minerals (0.507) also present considerably high energy intensity. However, the activities more related with consumption goods present lower energy intensity. Indeed, food and tobacco accounts for an energy intensity of 0.134. Furthermore, the energy intensity of textile and leather (0.082) and construction (0.018) is lower. Then, we can conclude that data seems to support that physical capital accumulation is more energy-intensive than consumption.

In this chapter, we study the implications of assuming different technologies for physical capital accumulation and consumption. More precisely, we assume that physical capital accumulation is relatively more energy-intensive than consumption. We use a general equilibrium model with three sectors (non-durable good, physical capital and extraction sector), based on Dasgupta and Heal (1974), Hartwick (1989) and Chapter 4. We point out four main contributions of this chapter. First, in contrast with the previously presented claim of Stiglitz (1974) and Dasgupta and Heal (1979), technical progress plays a crucial role to sustain long-run growth in this chapter. Indeed, Chapter 4, assuming that physical capital accumulation is more energy-intensive than consumption, and total capital depreciation (*i.e.*, physical capital as a flow variable), shows that the economy decreases at constant rate (with all the endogenous variables converging to zero asymptotically) if there is no growth of technical progress. Moreover, considering physical capital as a stock variable, Chapter 5 shows that neither balanced growth path (BGP) nor steady state equilibrium do not exist if there is no growth of technical progress.

The second contribution of this chapter is mainly methodological. This model entails some technical difficulties. In particular, we have to deal with an optimal control problem with mixed constraints and two states variables: capital accumulation and non-renewable energy resource stock. Moreover, the capital accumulation law is a non-linear function with respect to investment. However, we provide a close-form solution of the optimal solution paths of our variables in levels for every time t by applying the technique of Special Functions representation. This is very well known method in the field of mathematical physics to describe the behavior of dynamical systems. This tool provides a full analytical characterization of both short and long-run dynamics, without using qualitative techniques, such as phase diagrams. In this chapter, we use a family of Special Functions called Gauss Hypergeometric functions (see Boucekkine and Ruiz-Tamarit (2007) for an application to the Lucas-Uzawa model).

Applying the previously presented technique of Special Functions representation, we provide the necessary and sufficient conditions for existence and uniqueness of BGP, which is the third contribution of this chapter. Moreover, we prove that the economy asymptotically converges to the BGP. This result contrast with Chapter 4, where the economy does not present dynamical transition. The reason is that Chapter 4 assumes physical capital as a flow variable. Here, we eliminate that simplification considering physical capital as a durable good.

Finally, since the Special Functions representation provides a close-form solution of the optimal solution paths of our variables in levels, the fourth contribution of this chapter is that we can explicitly study the monotonicity properties of the optimal trajectories in levels, which is typically unallowed in standard approaches. We obtained that the assumption of physical capital as relatively more energy-intensive sector than consumption, and the technical progress (in particular, energy-saving technical progress), introduce new results of non-monotonicity of the optimal trajectories in levels, which contrast with Dasgupta and Heal (1974)'s results about the monotonicity of consumption and extraction of non-renewable energy resources.

In this chapter, we give special attention to energy-saving technologies. The idea behind energy-saving technical progress is energy efficiency, which is define as the inverse of the energy intensity (quantity of energy per unit of output). The importance of this kind of technologies is pointed out in Boucekkine and Pommeret (2004). The interest for energy efficiency is shared by developed countries, such as the EU, regarding to the Environment and Energy Policy. In particular, the European Commission, through the Green Paper "Doing more with less" (EU (2005)), opened the debate about the opportunities of energy-efficiency savings in the EU. Indeed, according to EU (2005), the European Union (EU) could save at least 20% of its present energy consumption improving energy-efficiency.

The chapter is organized as follows. Section 5.2 describes our economy, providing the dynamical system corresponding to the optimal solution. In Section 5.3, we study the BGP equilibrium. Section 5.4 solves the dynamical system presented in Section 5.2 using Gauss Hypergeometric functions. In Section 5.5, we provide the analytical expressions of the optimal solution paths. Section 5.6 studies the monotonicity properties of the optimal trajectories in levels. Finally, some concluding remarks are considered in Section 5.7.

5.2 The model

Let us consider a three sector economy with exhaustible energy resources, where the population is constant[1], based on Dasgupta and Heal (1974), Hartwick (1989) and Chapter 4. The final good sector only produces a non-durable good, which can be assigned to consumption or investment. The final good is produced by means of AK technology, where the physical capital (durable good) is the only input. The durable good sector accumulates physical capital by means of a technology defined over two inputs: investment and energy. Finally, the extraction sector directly produces energy by extracting non-renewable energy resources from a given stock.

5.2.1 Final good sector

The final good sector produces a non-durable good $Y(t)$ by means of the following AK technology:

$$Y(t) = A(t)K(t), \tag{1}$$

where $A(t)$ is the disembodied technical progress (or scale parameter), and $K(t)$ represents the physical capital, which is the only input to produce the final good. Here, we consider AK technology because it is the simplest set-up[2]. Final good is used either to consume, $C(t)$, or to invest in physical capital, $I(t)$, verifying the budget constraint of the economy:

$$Y(t) = C(t) + I(t). \tag{2}$$

The standard literature on non-renewable energy resources (see for instance Dasgupta and Heal (1974, 1979), Stiglitz (1974), Solow (1974),

[1] As Chapter 4 observes, Stigliz (1974) studies an economy with non-renewable resources and exponential growing labor force.

[2] Nevertheless, one could include more inputs such as energy (for final good production), or labor.

Hartwick (1989), or Smulders and Nooij (2003)) assumes the same technology for physical capital accumulation, $\dot{K}(t)$, and consumption[3]. However, as we will see in Section 5.2.2, this chapter assumes that physical capital accumulation is a function of investment and non-renewable energy resources, $R(t)$, (*i.e.*, $\dot{K}(t) = g[I(t), R(t)]$). This implies that physical capital accumulation is relatively more energy-intensive than consumption.

5.2.2 Durable good sector

The durable good sector accumulates physical capital, $\dot{K}(t)$, by using two inputs. On the one hand, the durable good sector takes the fraction of final good devoted to investment. And on the other, this sector uses the energy, $R(t)$, produced by the extraction sector. The technology for physical capital accumulation is represented by the following Cobb-Douglas function:

$$\dot{K}(t) = [\theta(t)R(t)]^{\alpha} I(t)^{1-\alpha}, \text{ with } 0 < \alpha < 0, \tag{3}$$

where $\theta(t)$ denotes the embodied energy-saving technical progress, and $K(0)$ is given. In this chapter, technical progress is considered as an exogenous variable.

Two observations should be made at this point. First, notice that we assume substitutability between energy and investment. Indeed, there is a very well known debate about substitutability *vs.* complementarity. If one considers the idea of minimum energy requiring to use a machine, the assumption of complementarity should be chosen. However, following the argument of Dasgupta (1979), one can assume substitutability if $I(t)$ is interpreted as final good service. Following that, we assume that the provision of a flow of final good $I(t)$ implies the provision of a certain energy flow.

As second observation, one can notice that the main different between Chapter 4 and our model is that Chapter 4 considers physical capital as a flow variable instead of a stock. As it is pointed out in that

[3]They consider $Y(t) = F[K(t), R(t)]$, where $Y(t) = C(t) + I(t)$ and $\dot{K}(t) = I(t)$.

chapter, this is a simplification which assumes total capital depreciation for each period. However, our model eliminates that simplification, considering physical capital as a stock variable. The main consequence of our assumption is the following. The model of Chapter 4 has not dynamical transition, *i.e.*, the economy is always in the BGP. However, as we will prove later, considering physical capital as a stock generates dynamical transition towards the BGP. Moreover, despite of having the same long-run growth rate as Chapter 4, our BGP levels are different because they are affected by the growth rates of technical progress.

5.2.3 Extraction sector

Energy is directly produced by extracting non-renewable energy resources, $R(t)$, from a given homogeneous stock $S(t)$. As Dasgupta and Heal (1974), and Hartwick (1989), we assume costless extraction[4]. The law of motion of the stock of non-renewable energy resources is described by the expression

$$R(t) = -\dot{S}(t), \text{ where } S(0) \text{ is given.} \tag{4}$$

Since one can not extract more than the available stock, the following restriction should be included

$$S(t) \geq 0. \tag{5}$$

5.2.4 Optimal solution

The central planner (optimal solution) maximizes the instantaneous utility function of the representative household:

$$\max W = \int_0^\infty ln[C(t)] \exp(-\rho t) dt, \text{ with } \rho > 0,$$

[4]Dasgupta and Heal (1974): "Extraction cost do not appear to introduce any great problem, provided that we assume any non-convexities". Indeed, one can easily introduce extraction cost $EC(t)$ by modifying the budget constraint of the economy:

$$Y(t) = C(t) + I(t) + EC(R, S), \text{ where } \partial EC/\partial R > 0 \text{ and } \partial EC/\partial S \leq 0.$$

subject to the equations (1)-(5), where ρ is the time preference parameter (it is assumed to be a positive discount factor).

One can easily rewrite the problem as optimal control problem with mixed constraints:

$$\max \int_0^\infty ln[A(t)K(t) - I(t)] \exp(-\rho t) dt, \text{ with } \rho > 0$$

subject to:

$$\dot{K}(t) = [\theta(t)R(t)]^\alpha I(t)^{1-\alpha}, \text{ with } 0 < \alpha < 0,$$

$$R(t) = -\dot{S}(t),$$

$$S(t) \geq 0,$$

$$\text{with } K(0) \text{ and } S(0) \text{ given,}$$

where $K(t)$ and $S(t)$ are the state variables, and $I(t)$ and $R(t)$ are the control variables. Following Sydsæter *et al.* (1999), the sufficient condition for this problem is a dynamical system described by the next proposition:

Proposition 1. *Given the initial conditions $K(0)$ and $S(0)$, the optimal solution of the optimal control problem described above is a path $\{C(t), R(t), I(t), S(t), K(t)\}$ that satisfies the following conditions:*

$$\dot{K}(t) = \theta(t)^\alpha R(t)^\alpha I(t)^{1-\alpha}, \tag{6}$$

$$\frac{\dot{C}(t)}{C(t)} = (1-\alpha)A(t)\theta(t)^\alpha \left(\frac{R(t)}{I(t)}\right)^\alpha - \alpha \left(\frac{\dot{\theta}(t)}{\theta(t)} - \frac{\dot{I}(t)}{I(t)} + \frac{\dot{R}(t)}{R(t)}\right) - \rho, \tag{7}$$

$$(1-\alpha)A(t)\theta(t)^\alpha \left(\frac{R(t)}{I(t)}\right)^\alpha = \alpha \frac{\dot{\theta}(t)}{\theta(t)} + (1-\alpha)\left(\frac{\dot{I}(t)}{I(t)} - \frac{\dot{R}(t)}{R(t)}\right), \tag{8}$$

$$A(t)K(t) = C(t) + I(t), \tag{9}$$

$$\dot{S}(t) = -R(t). \tag{10}$$

The proof is provided in Appendix A ■

This system has five equations and five unknowns [5]. Equation (6) is the accumulation law of physical capital. Notice that, in this model, energy is an input to accumulate physical capital. If one denotes $\dot{K}(t) = g[\theta(t), I(t), R(t)]$, for a general utility function $U[C(t)]$, equation (7) could be rewritten as

$$\frac{U_{cc}\dot{C}}{U_c} - \rho = \frac{\dot{g}_I}{g_I} - Ag_I.$$

This is the Ramsey optimal savings relation. Similarly, equation (8) is equivalent to the following expression:

$$A(t)g_I = \frac{\dot{g}_R}{g_R},$$

which is the efficient condition for using up the non-renewable resource, *i.e.*, a variant of the Hotelling rule. Finally, equation (9) is the budget constraint of the economy, and equation (10) is the law of motion of the stock of non-renewable energy resources.

5.3 Balanced growth path equilibrium

Let us define balanced growth path equilibrium (BGP) as the situation where all the endogenous variables grow at constant rate, *i.e.*, $x(t) = \overline{x}\exp(\gamma_x t)$. Following Solow (1974a), we assume that $A(t) = \overline{A}\exp(\gamma_A t)$ and $\theta(t) = \overline{\theta}\exp(\gamma_\theta t)$ for all t, where $\overline{A}, \overline{\theta}, \gamma_A, \gamma_\theta$ are strictly positive and exogenous parameters. Indeed, we will prove later that both kind of (exogenous) technical progress should grow at a high enough rate in order to have BGP with positive growth rates.

[5] Indeed, there are two more equations: one transversality condition (TC) for each state variable:

$$\lim_{t\to\infty} \psi(t)K(t) = 0 \text{ and} \lim_{t\to\infty} \lambda(t)S(t) = 0,$$

where $\psi(t)$ and $\lambda(t)$ are the corresponding lagrangian multipliers. We will use these two TC conditions in order to obtain the two constants involved in the calculation of the endogenous variables of the dynamical system. Actually, one could write a system of seven equations (the five previous equations and these two TC) and seven unknowns (the preceding unknowns and the two constants related with the state variables).

The following proposition summarizes the growth rates of the endogenous variables of our model, along the BGP (necessary condition):

Proposition 2. *Along the BGP, $Y(t)$, $C(t)$, and $I(t)$ grow at rate*

$$\gamma_Y = \gamma_C = \gamma_I = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho,$$

the growth rate of the stock of physical capital is

$$\gamma_K = \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho,$$

and extraction, $R(t)$, and stock of non-renewable energy resources, $S(t)$, grow at rate

$$\gamma_R = \gamma_S = -\rho.$$

The proof is provided in Appendix B ■

Parameter ρ is the (positive) discount rate, which represents the degree of impatience of the household. Indeed, the greater ρ the more impatient the household. Then, household prefer a higher level of present consumption than to postpone it. As one can observe from the previous proposition, the greater ρ the greater decreasing rate of the stock of non-renewable energy resources. This is because, since household does not want to postpone consumption, he prefers to extract a higher amount of energy resource in order to have greater present consumption.

Furthermore, since the stock of non-renewable energy resources is lower, this reduces the long-run growth rates of the economy for a given growth rate of technical progress. However, this negative effect of impatience could be compensated by greater growth rates of both disembodied technical progress (γ_A) and energy-saving technical progress (γ_θ). Indeed, from Proposition 2, if $(1-\alpha)\gamma_A + \alpha\gamma_\theta > \alpha\rho$, then $\gamma_Y (= \gamma_C = \gamma_I) > 0$ and $\gamma_K > 0$ (if BGP exists). The reason is that technical progress (in particular, energy-saving technical progress) reduces the energy intensity of the economy (*i.e.,* increases energy efficiency). If one defines energy intensity as the ratio $R(t)/Y(t)$, it is easy to prove that, in our model, the growth rate of energy intensity along the BGP is[6]

$$\gamma_{R/Y} = -\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A\right),$$

which is negative for $\gamma_A > 0$ and $\gamma_\theta > 0$. This is consistent with empirical works, such as Azomatou *et al.* (2004).

However, the following observation should be done regarding to the cases less or equal. If $(1-\alpha)\gamma_A + \alpha\gamma_\theta = \alpha\rho$, then $\gamma_Y (= \gamma_C = \gamma_I) = \gamma_A$ and $\gamma_K = 0$. An interesting example of this case is $\gamma_A = 0$ and $\gamma_\theta = \rho$, which implies (see Proposition 2) $\gamma_Y (= \gamma_C = \gamma_I) = 0$ and $\gamma_K = 0$, *i.e.*, steady state equilibrium[7]. This result contrasts with Dasgupta and Heal (1974). As we observe in the Introduction, these authors achieve steady state equilibrium without technical progress. In our case, where physical capital accumulation is relatively more energy-intensive than consumption, we need technical progress (in particular, energy-saving technical progress) to reach, at least, steady state equilibrium. Indeed, let us consider the case $(1-\alpha)\gamma_A + \alpha\gamma_\theta < \alpha\rho$. If $\gamma_A > 0$ and/or $\gamma_\theta > 0$, then $\gamma_K < 0$. However, this is not possible because we do not consider physical capital depreciation. A particular example of our last case is $\gamma_A = \gamma_\theta = 0$. Rewriting the dynamical system (6)-(10), one can easily obtain that, along the BGP (if it exists), all variables decrease at the constant rate ρ. However, as for the previous case, it is not possible that the stock of physical capital decreases because we do not consider physical capital depretation. Then, BGP (neither steady state) does not exist if there is no growth of technical progress.

5.4 Analytical solution using Gauss Hypergeometric Functions

In Section 4, we will fully characterize the dynamics of our economy. As we explain in Section 1, the equilibrium of our economy is described by

[6]Notice that we can consider two alternatives definitions of energy intensity, $R(t)/K(t)$ or $R(t)/I(t)$. For these definitions, the corresponding BGP growth rates are $\gamma_{R/K} = -\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right)$ and $\gamma_{R/I} = -\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A\right)$.

[7]As one will verify later, our analysis is also valid for $\gamma_A = 0$ or $\gamma_\theta = 0$ (we do not consider negative growth rates of technical progress). The case $\gamma_A = \gamma_\theta = 0$ is discussed at the end of this paragraph.

the dynamical system (6)-(10). Here, we will use a new technique in economics called Special Functions[8]. This is a very well known technique in the field of mathematical physics. The tool of Special Functions provides us a full analytical characterization of the dynamics of our system, without using qualitative techniques such as phase diagrams. However, this methodology has not been sufficiently explored in economic theory and, in particular, in growth models[9].

Since we are interested in the dynamics towards the BGP, it is useful to detrend our dynamical system[10]. Let us define the following change of variable $\tilde{x}(t) = x(t)/\exp(\gamma_x t)$. Then, the transformed (detrended) variables will be constant along the BGP (if it exists). Applying this change of variable, one obtains the following detrended dynamical system with five equations and five unknowns $(\tilde{K}, \tilde{C}, \tilde{R}, \tilde{I}, S)$:

$$\dot{\tilde{K}}(t) + \left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho\right)\tilde{K}(t) = \overline{\theta}^{\alpha}\tilde{R}(t)^{\alpha}\tilde{I}(t)^{1-\alpha}, \tag{11}$$

$$\frac{\dot{\tilde{C}}(t)}{\tilde{C}(t)} = (1-\alpha)\overline{A\theta}^{\alpha}\left(\frac{\tilde{R}(t)}{\tilde{I}(t)}\right)^{\alpha} - \alpha\left(\frac{\dot{\tilde{R}}(t)}{\tilde{R}(t)} - \frac{\dot{\tilde{I}}(t)}{\tilde{I}(t)}\right) - \gamma_\theta - \frac{1-\alpha}{\alpha}\gamma_A, \tag{12}$$

$$(1-\alpha)\overline{A\theta}^{\alpha}\left(\frac{\tilde{R}(t)}{\tilde{I}(t)}\right)^{\alpha} = (1-\alpha)\left(\frac{\dot{\tilde{I}}(t)}{\tilde{I}(t)} - \frac{\dot{\tilde{R}}(t)}{\tilde{R}(t)}\right) + \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A, \tag{13}$$

$$\tilde{C}(t) + \tilde{I}(t) = \overline{A}\tilde{K}(t), \tag{14}$$

$$\dot{S}(t) = \tilde{R}(t)\exp(-\rho t). \tag{15}$$

Achieving the analytical solution of this dynamical system, one fully characterizes the dynamics of the economy described by in this chapter. Moreover, if there is BGP, the variables involved in this (detrended) dynamical system will be constant along the BGP. Furthermore, since γ_A and γ_θ are exogenous parameters, one can easily recovers the original variables by retrieving the change of variable.

[8]See Abramowitz and Stegun (1970).

[9]See Boucekkine and Ruiz-Tamarit (2007) for an application to the Lucas-Uzawa model.

[10]Notice that this is not a compulsory step to apply Special Functions. However, eliminating the long-run trends allows us to have less messy expressions.

In order to obtain the analytical solution of the dynamical system (11)-(15), we will apply the following strategy:

Step 1. Let us define the ratio $X(t) = \frac{\tilde{R}(t)}{\tilde{I}(t)}$.

Step 2. One can easily observes that equation (13) is a Bernoulli's equation (see Section 5.4.2). Then, we can obtain the analytical solution for the ratio $X(t)$.

Step 3. Taking $X(t)$ into equation (12), one obtains a linear first-order differential equation. Solving this differential equation, the analytical expression for $\tilde{C}(t)$ is straightforward.

Step 4. Rewriting equation (11), one gets

$$\dot{\tilde{K}}(t) + \left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - r\right)\tilde{K}(t) = \overline{\theta}^\alpha X(t)^\alpha[\overline{A}\tilde{K}(t) - \tilde{C}(t)].$$

Since we have the analytical solution for $\tilde{C}(t)$ (Step 3), the previous expression is a linear first-order differential equation. By solving it, one gets the analytical solution for $\tilde{K}(t)$. Notice that, this step will involve the application of Special Functions. In particular, we will use a family of Special Functions called Gauss Hypergeometric functions. Later, we will provide a briefly introduction to these fuctions.

Step 5. From equation (14),

$$\tilde{I}(t) = \overline{A}\tilde{K}(t) - \tilde{C}(t).$$

Since we know both $\tilde{C}(t)$ (Step 3) and $\tilde{K}(t)$ (Step 4), the previous expression gives us the analytical solution for $\tilde{I}(t)$.

Step 6. Since we know $\tilde{I}(t)$ and $X(t)$, one can easly gets $\tilde{R}(t)$.

Step 7. Finally, from equation (15), $S(t)$ is obtained by solving a linear first-order differential equation.

5.4.1 Step 1.

Let us define the ratio $X(t) = \frac{\tilde{R}(t)}{\tilde{I}(t)}$. Taking logs and differentiating with respect to t, one gets

$$\frac{\dot{X}(t)}{X(t)} = \frac{\dot{\tilde{R}}(t)}{\tilde{R}(t)} - \frac{\dot{\tilde{I}}(t)}{\tilde{I}(t)}. \tag{16}$$

5.4.2 Step 2.

Applying equation (16) into equation (13), one finds

$$\dot{X}(t) = \frac{\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A}{1-\alpha} X(t) + \overline{A\theta}^{\alpha} X(t)^{1+\alpha}. \tag{17}$$

One can observe that equation (17) is a Bernoulli's equation. Following Sydsæter *et al.* (1999), a Bernoulli's equation is defined as

$$\dot{x}(t) = q(t)x(t) + r(t)x(t)^n,$$

which has the following solution:

$$x(t) = \exp\left(\frac{p(t)}{1-n}\right)\left(\varphi + (1-n)\int r(t)\exp(-p(t))dt\right)^{\frac{1}{1-n}},$$

where $p(t) = (1-n)\int q(t)dt$, φ is a constant, and $n \neq 1$.

Then, the solution of the Bernouilli's equation (17) is

$$X(t) = \exp\left(\frac{1}{\alpha}Zt\right)\left(\varphi + \frac{\alpha\overline{A\theta}^{\alpha}}{Z}\exp(Zt)\right)^{-\frac{1}{\alpha}}, \tag{18}$$

where $Z = \frac{\alpha}{1-\alpha}\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right)$, and φ is a constant which should be calculated.

5.4.3 Step 3.

Applying equations (16) and (18) into equation (12), one finds that

$$\frac{\dot{\tilde{C}}(t)}{\tilde{C}(t)} = \frac{\overline{A\theta}^{\alpha}\exp(Zt)}{\varphi + \alpha\overline{A\theta}^{\alpha}\exp(Zt)} - \frac{Z}{\alpha}, \tag{19}$$

which is a linear first-order differential equation.

A linear first-order differential equation is defined as

$$\dot{x}(t) + a(t)x(t) = b(t),$$

whose solution is given by the expression (Sydsæter *et al.* (1999))

$$x(t) = x(0)\exp\left(-\int_0^t a(\xi)d\xi\right) + \int_0^t b(\tau)\exp\left(-\int_\tau^t a(\xi)d\xi\right)d\tau.$$

Applying this result into equation (19), we find that

$$\tilde{C}(t) =$$

$$\tilde{C}(0)\exp\left[\frac{1}{\alpha}\Big(ln(\varphi Z+\alpha\overline{A\theta}^{\alpha}\exp(Zt))-ln(\varphi Z+\alpha\overline{A\theta}^{\alpha})-Zt\Big)\right], \qquad (20)$$

where $\tilde{C}(0)(=C(0))$ is the initial condition for consumption, which should be determined.

5.4.4 Step 4.

Rewriting equation (11), one gets

$$\dot{\tilde{K}}(t)+\left(\frac{1-\alpha}{\alpha}Z-\rho-\overline{A\theta}^{\alpha}X(t)^{\alpha}\right)\tilde{K}(t)=-\overline{\theta}^{\alpha}X(t)^{\alpha}\tilde{C}(t) \qquad (21)$$

Since from the previous steps we know the analytical solution for $X(t)$ (equation (18)) and $\tilde{C}(t)$ (equation (20)), $\tilde{K}(t)$ is the solution of the linear first-order differential equation (21).

Applying the previous result about the solution of a linear first-order differential equation, from equation (21) one finds that

$$\tilde{K}(t)=\exp\left[-\frac{1}{\alpha}ln(\varphi Z+\alpha\overline{A\theta}^{\alpha})\right]$$

$$\cdot\exp\left\{\frac{1}{\alpha}\Big[ln(\varphi Z+\alpha\overline{A\theta}^{\alpha}\exp(Zt))-((1-\alpha)Z-\alpha\rho)t\Big]\right\}$$

$$\cdot\left\{K(0)+\frac{Z}{Z+\rho}\frac{\tilde{C}(0)}{\alpha\overline{A}}\left[\frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)}{\exp((Z+\rho)t)}-{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)\right]\right\}, \qquad (22)$$

where

$${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)={}_2F_1\left(1,\frac{Z+\rho}{Z},1+\frac{Z+\rho}{Z};-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right),$$

$${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)={}_2F_1\left(1,\frac{Z+\rho}{Z},1+\frac{Z+\rho}{Z};-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right),$$

and $K(0)(=\tilde{K}(0))$ is the (given) initial condition for capital.

As one can observe, equation (22) involves Special Functions (${}_2F_1$). In this chapter, we use a family of Special Functions called Gauss Hypergeometric functions[11], which are defined as

$$ {}_pF_q(a;b;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \times (a_2)_n \times \dots (a_p)_n}{(b_1)_n \times (b_2)_n \times \dots (b_q)_n} \frac{z^n}{n!}, $$

where $a = (a_1, a_2, \dots, a_p)$, $b = (b_1, b_2, \dots, b_q)$, and $(x)_n$ is the Pochhammer symbol, defined by:

$$ (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, $$

where $\Gamma(x)$ is the *Gamma function.*

The proof of equation (22) is provided in Appendix C ■

5.4.5 Step 5.

Since we have the solution of $\tilde{C}(t)$ (equation (20)) and $\tilde{K}(t)$ (equation (22)), from equation (14) one finds that

$$ \tilde{I}(t) = \exp\left\{\frac{1}{\alpha}\left[ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(zt)) - ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}) - Zt\right]\right\} \cdot \left\{\overline{A}\exp((Z+\rho)t)\zeta(t) - \tilde{C}(0)\right\}, \quad (23) $$

where

$$ \zeta(t) = K(0) + \frac{Z}{Z+\rho}\frac{\tilde{C}(0)}{\alpha\overline{A}}\left[\frac{1}{\exp((Z+\rho)t)}{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right) - {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)\right]. $$

5.4.6 Step 6.

Since $X(t)$ is defined as $X(t) = \tilde{R}(t)/\tilde{I}(t)$, taking the solution of $X(t)$ (equation (18)) and $\tilde{I}(t)$ (equation (23)), one finds the solution for the energy flow $\tilde{R}(t)$:

$$ \tilde{R}(t) = \exp\left[-\frac{1}{\alpha}ln(\varphi Z + \alpha\overline{A\theta}^{\alpha})\right] Z^{\frac{1}{\alpha}} \exp\left(\frac{1}{\alpha}Zt\right) \cdot \left\{\overline{A}\exp\left\{\left(\frac{1-\alpha}{\alpha}Z - \rho\right)t\right\}\zeta(t) - \tilde{C}(0)\exp\left(-\frac{1}{\alpha}Zt\right)\right\}. \quad (24) $$

[11]See Boucekkine and Ruiz-Tamarit (2007) for a quick overview of Gauss Hypergeometric functions.

5.4.7 Step 7.

Since we have the solution for $\tilde{R}(t)$ (equation (24)), one can find the solution for the stock of non-renewable energy resources, $S(t)$, by solving equation (15), which is a Linear first-order differential equation. The solution of this equation is

$$S(t) = S(0) + \int_0^t \tilde{R}(\tau)\exp(-\rho\tau)d\tau, \tag{25}$$

where $S(0)$ is given. The corresponding integral is equal to

$$\int_0^t \tilde{R}(\tau)\exp(-\rho\tau)d\tau = [1]+[2]+[3]+[4],$$

where

$$[1] = \frac{(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} \overline{A}K(0)}{2(\frac{1}{\alpha}Z-\rho)-Z}\left\{\exp\left[\left(2\left(\frac{1}{\alpha}Z-\rho\right)-Z\right)t\right]-1\right\},$$

$$[2] = A_2\frac{1}{Z+\Omega}\frac{Z}{Z+\rho}$$

$$\cdot\left[{}_3F_2\left(a;b;-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)-\exp\left(-(Z+\Omega)\frac{Z+\rho}{Z}t\right){}_3F_2\left(a;b;-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)\right],$$

$$[3] = -A_2\frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}{2(\frac{1}{\alpha}Z-\rho)-Z}\left\{\exp\left[\left(2\left(\frac{1}{\alpha}Z-\rho\right)-Z\right)t\right]-1\right\},$$

$$[4] = \frac{(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0)}{\rho}\left(\exp(-\rho t)-1\right),$$

with

$$A_2 = (\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}}\frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho}\frac{C(0)}{\alpha},$$

$$\Omega = \left(2\rho-\frac{2-\alpha}{\alpha}Z\right)\left(\frac{Z}{Z+\rho}\right),$$

$$a = \left(1,\frac{Z+\rho}{Z},\frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}\right),$$

$$b = \left(1+\frac{Z+\rho}{Z},1+\frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}\right),$$

and

$$\left|-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right|<1.$$

The proof is provided in Appendix D ■

5.4.8 Computation of the constants φ and $C(0)$

Equations (20), and (22)-(25) are the analytical solutions for, respectively, $\tilde{C}(t)$, $\tilde{K}(t)$, $\tilde{I}(t)$, $\tilde{R}(t)$, and $S(t)$. However, we still need to determine the value of the constants φ and $C(0)(=\tilde{C}(0))$, in order to finish the calculations. To do this, we use the two transversality conditions (TC) involved by the states variables of our model:

$$\lim_{t\to\infty} \psi(t)K(t) = 0, \tag{26}$$

$$\lim_{t\to\infty} \lambda(t)S(t) = 0, \tag{27}$$

where $\psi(t)$ and $\lambda(t)$ are the corresponding Lagrangian multipliers. In addition, the TCs will also imply restrictions on the parameters, which allow us to complete the full characterization of the optimal trajectories.

TC for the stock of physical capital ($K(t)$)

From equation (A.1) (see Appendix A), we obtain the shadow price of physical capital, $\psi(t)$:

$$\psi(t) = \frac{(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{\alpha}}{(1-\alpha)\overline{\theta}^{\alpha} C(0)} (\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha} \exp(Zt))^{-\frac{1-\alpha}{\alpha}}. \tag{28}$$

Taking equation (22), one gets[12]

$$\psi(t)K(t) = \frac{(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha} \exp(Zt))}{(1-\alpha)\overline{\theta}^{\alpha} C(0)}$$

$$\cdot \left\{ K(0) + \frac{Z}{Z+\rho}\frac{C(0)}{\alpha\overline{A}} \left[\frac{1}{\exp((Z+\rho)t)} {}_2F_1\left(\frac{-\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}\right) - {}_2F_1\left(\frac{-\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)\right]\right\}.$$

Then, the TC for $K(t)$ (equation (26)) implies that

$$C(0) = \frac{\alpha\overline{A}}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)} \frac{Z+\rho}{Z} K(0), \tag{29}$$

where $K(0)$ is given.

Equation (29) establishes a first relationship between φ and $C(0)$. In order to have a system of two equations and two unknowns, we can obtain the second equation from the TC for the non-renewable energy resources (equation (27)).

[12] Notice that $K(t) = \tilde{K}(t)\exp\left[\left(\frac{1-\alpha}{\alpha}Z - \rho\right)t\right]$.

TC for the stock of non-renewable energy resource ($S(t)$)

Since $\lambda(t) = \lambda$ for all t (see Appendix A)[13], equation (27) implies that

$$\lim_{t\to\infty} S(t) = 0. \tag{30}$$

The transversality condition for $S(t)$ implies that the stock of non-renewable energy resources can not infinitely increase (*i.e.*, as Boucekkine and Ruiz-Tamarit (2007), non-explosive solution) and it should be depleted in the infinite (this is because we assume that this kind of energy resources are essential and there are not alternative resources, such as renewable energy resources).

Taking equation (29) into equation (25), we get that

$$S(t) = S(0) - \frac{(\varphi Z + \alpha \overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0)}{\rho} (\exp(-\rho t) - 1) - A_2 \frac{1}{Z+\Omega} \frac{Z}{Z+\rho}$$
$$\cdot \left[{}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) - \exp\left(-(Z+\Omega)\frac{Z+\rho}{Z} t\right) {}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}\exp(Zt)}\right)\right], \tag{31}$$

where

$${}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) = {}_3F_2\left(a; b; -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right),$$
$${}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}\exp(Zt)}\right) = {}_3F_2\left(a; b; -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}\exp(Zt)}\right).$$

From equation (31), if the term $(\Omega + Z)\frac{Z+\rho}{Z}$ is negative[14], then $S(t)$ will infinitely increases. Hence, $(\Omega + Z)\frac{Z+\rho}{Z}$ should be greater than zero in order to avoid explosive solutions. Since ρ and Z are positive, we have to ensure that $\Omega + Z$ is positive too[15]. Then, from the definition of Ω (equation (25)), we can establish the following proposition:

Proposition 3. *Any particular non-explosive solution to the dynamical system (6)-(10) has to satisfy the following condition:*

$$\rho > \frac{2}{3}\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right).$$

[13]Since $q(t) = 0$, λ is easily obtained taking equation (28) into equation (A.2) in Appendix A.

[14]${}_3F_2(a; b; 0) = 1$ for a and b different than zero.

[15]Notice that, from Proposition 2 and the definition of Ω (equation (25)), Ω should be less or equal to 0 in order to have BGP.

Moreover, Proposition 2 establishes that the growth rate of technical progress should be high enough to ensure BGP equilibrium. Indeed, we can consider both conditions in the following proposition:

Proposition 4. *Any particular BGP solution to the dynamical system (6)-(10) has to satisfy the following condition:*

$$\frac{2}{3}\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right) < \rho < \left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right). \tag{32}$$

The interpretation of Proposition 4 is the following. As we observe in Section 3, ρ represents the household's impatience degree. Indeed, the greater is ρ the less important is future for the household (*i.e.*, the more impatient is the household). Proposition 2 establishes that technical progress should be high enough ($(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A) > \rho$) to guarantee BGP, which is the right hand side (RHS) of condition (32). However, the left hand side (LHS) of condition (32) establishes that the growth rate of technical progress should be not too high ($\frac{2}{3}(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A) < \rho$). If this is not the case, an explosive solution will appear (see Proposition 3). Nevertheless, an explosive solution is not possible because of the transversality condition for $S(t)$.

After eliminating the possibility of explosive solution, we can establish a condition on the parameters to deplete the stock of non-renewable energy resources in the infinite. From equation (31) and condition (32), it is clear that

$$\lim_{t\to\infty} S(t) =$$

$$S(0) + \frac{C(0)}{\rho}\left(\frac{Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}} - A_2 \frac{1}{Z+\Omega}\frac{Z}{\Omega+\rho}\, {}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right). \tag{33}$$

From equation (30), $\lim_{t\to\infty} S(t) = 0$. Then, we obtain the following condition replacing A_2 in equation (33) and equalizing to zero:

$$S(0) =$$

$$C(0)\left(\frac{Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}\left[\left(\frac{Z}{Z+\rho}\right)^2 \frac{1}{\alpha}\frac{1}{Z+\Omega}\, {}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) - \frac{1}{\rho}\right], \tag{34}$$

where $S(0)$ is given.

Equation (34) is the second relationship between φ and $C(0)$. Then, equations (29) and (34) is a non-linear system of two equations and two unknowns. Solving this system, φ and $C(0)$ will be determined.

5.5 The optimal solution paths

Section 4 provided the analytical solution of the dynamical system (11)-(15) using Gauss Hypergeometric Functions. But this is the solution of the detrended dynamical system. However, one can easily recover the solution of the original dynamical system (6)-(10) by multiplying each detrended variable by its corresponding BGP growth rate, *i.e*, $x(t) = \tilde{x}(t)\exp(\gamma_x t)$. Moreover, Section 4 also established the parameters constraints to have BGP solution. In Section 5, we put together all the previous conditions, providing the optimal solution paths of our economy.

5.5.1 Computation of the constant φ

Taking equation (29) into equation (34), we reduce the previous non-linear system of two equations ((29) and (34)) and two unknowns (φ and $C(0)$) to the following single equation for φ, where $C(0)$ is directly found (equation (29)) once φ is determined:

$$\frac{S(0)}{K(0)} = \alpha\overline{A}\left(\frac{Z}{\varphi Z + \alpha\overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+R}\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)} - \frac{Z+\rho}{Z}\frac{1}{\rho}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)}\right], \tag{35}$$

where $K(0)$ and $S(0)$ are given.

Since equation (25) requires $\left|-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right| < 1$, we only consider parameterizations with solution for equation (35) such that[16]

$$\varphi \in \left(-\frac{\alpha\overline{A\theta}^{\alpha}}{Z}, \frac{\alpha\overline{A\theta}^{\alpha}}{Z}\right). \tag{36}$$

5.5.2 Optimal solution paths for the ratio $X(t)$

From equation (18), rearranging terms, we get the following proposition for the optimal solution path for $X(t)$:

Proposition 5. *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

(i) *it does exist a unique path for $X(t)$, starting from $X(0) = \left(\frac{Z}{\varphi Z+\alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}$, such that*

$$X(t) = \left(\frac{Z}{\varphi Z\exp(-Zt) + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}; \tag{37}$$

(ii) *this equilibrium path shows transitional dynamics, approaching asymptotically to the constant*

$$X_{BGP} = \left(\frac{Z}{\alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}.$$

Moreover, it is easy to see that condition (36) implies that the starting point, the transitional dynamics, and the BGP of $X(t)$ are always positive:

Corollary 1. *Under the conditions (32) and (36), the optimal solution path for $X(t)$ is positive.*

5.5.3 Optimal solution paths for the consumption $C(t)$

From equation (20), we recover $C(t)$ by multiplying $\tilde{C}(t)$ by $\exp(\gamma_C t)$ (see Proposition 2). Then, rearranging terms, we establish the following

[16] If some particular parametrization generates a solution for the equation (35) out the interval (36), we should study the continuation formulas of the Gauss Hypergeometric representation for $S(t)$. However, this is beyond of the objective of this chapter.

proposition:

Proposition 6. *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

(i) *it does exist a unique path for $C(t)$, starting from*

$$C(0) = \frac{\alpha \overline{A}}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right)} \frac{Z+\rho}{Z} K(0),$$

such that

$$C(t) = C(0)\left[\left(\frac{\varphi Z}{\varphi Z + \alpha \overline{A\theta}^{\alpha}}\right)\exp(-Zt) + \frac{\alpha \overline{A\theta}^{\alpha}}{\varphi Z + \alpha \overline{A\theta}^{\alpha}}\right]^{\frac{1}{\alpha}}$$
$$\cdot \exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\}; \tag{38}$$

(ii) *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$C_{BGP}(t) = C(0)\left(\frac{\alpha \overline{A\theta}^{\alpha}}{\varphi Z + \alpha \overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}} \exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\};$$

where $K(0)$ is given.

Regarding to the sigh of the optimal solution path, for $C(0) > 0$, condition (36) implies positive starting point, transitional dynamics and BGP. From Proposition 6, the condition for $C(0) > 0$ is that ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 0$. Then, we can establish the following corollary:

Corollary 2. *Under the conditions (32), (36), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 0$, the optimal solution path for $C(t)$ is positive.*

5.5.4 Optimal solution paths for the physical capital $K(t)$

Taking equation (29) into equation (22) and rearranging terms , we obtain the optimal solution path for the detrended physical capital. Then, multiplying $\tilde{K}(t)$ by $\exp(\gamma_K t)$, one can establish the Proposition 7:

Proposition 7. *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

(i) *it does exist a unique path for $K(t)$, starting from $K(0)$, such that*

$$K(t) = K(0)(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)} \left(\frac{\varphi Z}{\exp(Zt)} + \alpha\overline{A\theta}^{\alpha}\right)^{\frac{1}{\alpha}}$$
$$\cdot \exp\left\{\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho\right)t\right\} \quad (39);$$

(ii) *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$K_{BGP}(t) =$$
$$K(0)\left(\frac{\alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}} \frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)} \exp\left\{\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho\right)t\right\};$$

where $K(0)$ is given.

Furthermore, as in the previous case, we can establish the following corollary for positive optimal solution path:

Corollary 3. *Under the conditions (32), (36), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 0$, the optimal solution path for $K(t)$ is positive, for a given $K(0) > 0$.*

The proof is provided in Appendix E ■

5.5.5 Optimal solution paths for the investment $I(t)$

Applying equation (29) into equation (23), one gets the optimal solution path for the detrended investment. After multiplying $\tilde{I}(t)$ by $\exp(\gamma_I t)$ and rearranging terms, we find the following proposition:

Proposition 8. *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

(i) *it does exist a unique path for $I(t)$, starting from*

$$I(0) = \overline{A}K(0)\left(1 - \alpha\frac{Z+\rho}{Z}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)}\right),$$

such that

$$I(t) = \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)}\left(\frac{\varphi Z + \alpha\overline{A}\theta^{\alpha}\exp(Zt)}{(\varphi Z + \alpha\overline{A}\theta^{\alpha})\exp(Zt)}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}\exp(Zt)}\right) - \alpha\frac{Z+\rho}{Z}\right]\exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\}; \quad (40)$$

(ii) *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$I_{BGP}(t) = \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)}\left(\frac{\alpha\overline{A}\theta^{\alpha}}{\varphi Z + \alpha\overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}\left(1 - \frac{Z+\rho}{Z}\alpha\right)$$

$$.\exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\};$$

where $K(0)$ is given.

It is easy to prove that condition (32) implies $1 > \alpha\frac{Z+\rho}{Z}$. Indeed, if ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right) > 1$, then $I(0) > 0$. Since ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}\exp(Zt)}\right) > {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right) > 1$ (see Appendix E), equation (40) will be positive too. Then, we establish the following corollary:

Corollary 4. *Under the conditions (32), (36), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right) > 1$, the optimal solution path for $I(t)$ is positive, for a given $K(0) > 0$.*

5.5.6 Optimal solution paths for the output $Y(t)$

Since $Y(t) = C(t) + I(t)$, Propositions 6 and 8 provides the optimal solution path for the output:

Proposition 9. *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

(i) *it does exist a unique path for $Y(t)$, starting from $Y(0) = \overline{A}K(0)$, such that*

$$Y(t) = \overline{A}K(0) \left(\frac{\varphi Z + \alpha \overline{A}\theta^{\alpha} \exp(Zt)}{(\varphi Z + \alpha \overline{A}\theta^{\alpha}) \exp(Zt)} \right)^{\frac{1}{\alpha}} \frac{{}_2F_1 \left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha} \exp(Zt)} \right)}{{}_2F_1 \left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}} \right)}$$

$$\cdot \exp \left\{ \left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho \right) t \right\}; \tag{41}$$

(ii) *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$Y_{BGP}(t) = Y(t) = \overline{A}K(0) \left(\frac{\alpha \overline{A}\theta^{\alpha}}{\varphi Z + \alpha \overline{A}\theta^{\alpha}} \right)^{\frac{1}{\alpha}} \frac{1}{{}_2F_1 \left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}} \right)}$$

$$\cdot \exp \left\{ \left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho \right) t \right\};$$

where $K(0)$ is given.

Concerning to the sign of the optimal solution path, we proceed as in Corollary 3:

Corollary 5. *Under the conditions (32), (36), and ${}_2F_1 \left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}} \right) > 0$, the optimal solution path for $Y(t)$ is positive, for a given $K(0) > 0$.*

5.5.7 Optimal solution paths for the extraction of non-renewable resources $R(t)$

Since $\tilde{R}(t) = X(t)\tilde{I}(t)$, we get the optimal solution path for the detrended non-renewable resources from Proposition 5 and equations (23) and (29) (notice that $\tilde{C}(0) = C(0)$). Considering $\gamma_R = -\rho$, we recover the optimal solution path for $R(t)$. Then, we establish the following proposition:

Proposition 10. *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

(i) *it does exist a unique path for $R(t)$, starting from*

$$R(0) = \overline{A}K(0)\left(\frac{Z}{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}\left(1 - \alpha\frac{Z+\rho}{Z}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)}\right),$$

such that

$$R(t) =$$

$$\left(\frac{Z}{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}\frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)}\left[{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}\right) - \alpha\frac{Z+\rho}{Z}\right]$$

$$\cdot\exp(-\rho t); \tag{42}$$

(ii) *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$R_{BGP}(t) = \left(\frac{Z}{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}\frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)}\left[1 - \alpha\frac{Z+\rho}{Z}\right]$$

$$\cdot\exp(-\rho t);$$

where $K(0)$ is given.

Following the same reasoning as in Corollary 4, we can establish the following condition for positive optimal solution path for $R(t)$:

Corollary 6. *Under the conditions (32), (36), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right) > 1$, the optimal solution path for $R(t)$ is positive, for a given $K(0) > 0$.*

5.5.8 Optimal solution paths for the stock of non-renewable resources $S(t)$

Taking equation (31) into equation (25), rearranging terms, we get the optimal solution path for the stock of non-renewable energy resources:

Proposition 11. *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

(i) *it does exist a unique path for $S(t)$, starting from $S(0)$, such that*

$$S(t) = C(0)\left(\frac{Z}{\varphi Z + \alpha \overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\cdot \left\{ \frac{\left(\frac{Z}{Z+\rho}\right)^2 \frac{1}{\alpha}\frac{1}{Z+\Omega}}{\exp\left((Z+\Omega)\frac{Z+\rho}{Z}t\right)} {}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}\exp(Zt)}\right) - \frac{1}{\rho}\frac{1}{\exp(\rho t)} \right\}$$

$$\cdot \exp(-\rho t); \tag{43}$$

(ii) *this equilibrium path shows transitional dynamics, approaching asymptotically to zero;*

where $K(0)$ and $S(0)$ are given.

Regarding to the sigh of the optimal solution path for the stock of non-renewable resources, for $C(0) > 0$, it is enough to require that $\exp(\rho t)$ increases faster than $\exp\left((Z+\Omega)\frac{Z+\rho}{Z}t\right)$:

Corollary 7. *Under the conditions (32), (36), ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 0$, and $\rho > (Z+\Omega)\frac{Z+\rho}{Z}$, the optimal solution path for $S(t)$ is positive, for $K(0) > 0$ and $S(0) > 0$ given*[17].

Finally, we conclude this section with a corollary of all the conditions to guarantee positive optimal solution paths of our economy:

Corollary 8. *Under the conditions (32), (36), ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 1$, $\rho > (Z+\Omega)\frac{Z+\rho}{Z}$, ${}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 1$ and , the optimal solution paths for $X(t)$, $C(t)$, $K(t)$, $I(t)$, $Y(t)$, $R(t)$, and $S(t)$, are positive, for $K(0) > 0$ and $S(0) > 0$ given.*

[17]Notice that $(Z+\Omega)\frac{Z+\rho}{Z} > 0$ (see Section 5.8.2). Moreover, the condition ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 0$ is required to guarantee $C(0) > 0$.

5.6 On the monotonicity of the optimal solution paths

Applying Gauss Hypergeometric representation, Section 5.5 provided the closed-form solution of the optimal paths of our variables in level. An important outcome of these closed-form solutions is that we can explicitly study the monotonicity of the optimal trajectories of the variables in level. These kinds of studies are typically unallowed in standard approaches[18].

5.6.1 The relationship between φ and $S(0)/K(0)$

We will see later that the sign of φ is important to determine the monotonicity properties of our optimal solutions paths. Therefore, we have to study the sign of φ before dealing with the monotonicity issue. From condition (35), taking all parameters as given, one observes that the ratio $S(0)/K(0)$ decides φ (and, in particular, the sign of φ)[19].

Let us define $(S(0)/K(0))^*$ as the ratio corresponding to $\varphi = 0$. Then,

$$\left(\frac{S(0)}{K(0)}\right)^* = \alpha\overline{A}\left(\frac{Z}{\alpha\overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}\left(\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+\Omega} - \frac{Z+\rho}{Z}\frac{1}{\rho}\right). \tag{44}$$

From (35), applying the implicit function theorem, one obtains that

$$\frac{\partial\varphi}{\partial(S(0)/K(0))} < 0.$$

The proof is provided in Appendix F ■

Hence, for $S(0)/K(0) > (<)(S(0)/K(0))^*$, φ is negative (positive).

[18] An exception is Dasgupta and Heal (1974), where only consumption and extraction can be analyzed in level.

[19] Notice that Corollary 8 implies that both left and right hand side of equation (35) are positive.

5.6.2 Monotonicity of the ratio $X(t)$

From equation (37), we easily obtains that

$$\frac{dX(t)}{dt} = \varphi \left[\frac{Z}{\alpha} \frac{X(t)}{\varphi + \frac{\alpha \overline{A\theta}^{\alpha}}{Z} \exp(Zt)} \right].$$

Taking Corollary 8, one can observe that the term between the square brackets is positive. Then, the sign of the derivative is determined by the sign of φ. If $\varphi < (>)0$, then $dX(t)/dt < (>)0$ for all t. Moreover, if $\varphi = 0$, then $X(t)$ is constant for all t. Since the sign of φ is determined by the ratio $S(0)/K(0)$ (see Section 6.1), we can establish the following proposition:

Proposition 12.

(1) *If $S(0)/K(0) > (<)(S(0)/K(0))^*$, then $X(t)$ decreases (increases) monotonically for all t.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then*

$$X(t) = \left(\frac{Z}{\alpha \overline{A\theta}^{\alpha}} \right)^{\frac{1}{\alpha}}$$

for all t.

See Appendix H, Figure 1, for an illustration.

5.6.3 Monotonicity of consumption

Equation (38) is the optimal solution path for consumption. First, let us study the detrended consumption

$$\tilde{C}(t) = C(0) \left[\left(\frac{\varphi Z}{\varphi Z + \alpha \overline{A\theta}^{\alpha}} \right) \exp(-Zt) + \frac{\alpha \overline{A\theta}^{\alpha}}{\varphi Z + \alpha \overline{A\theta}^{\alpha}} \right]^{\frac{1}{\alpha}}. \tag{45}$$

From equation (45), we get

$$\frac{d\tilde{C}(t)}{dt} =$$

$$-\varphi\left\{\tilde{C}(t)\frac{Z\exp(-Zt)}{\alpha}\left(\frac{\varphi Z\exp(-Zt)+\alpha\overline{A\theta}^{\alpha}}{\varphi Z+\alpha\overline{A\theta}^{\alpha}}\right)^{-1}\frac{1}{\varphi+\frac{\alpha\overline{A\theta}^{\alpha}}{Z}}\right\} \tag{46}$$

As for $X(t)$, the expression between braces is positive. Then, the sign of φ determines the sign of the derivative. Hence, we can establish the following proposition:

Proposition 13.

(1) *If $S(0)/K(0) > (<)(S(0)/K(0))^*$, then $\tilde{C}(t)$ increases (decreases) monotonically for all t.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then*

$$\tilde{C}(t) = \alpha\overline{A}\frac{Z+\rho}{Z}K(0)$$

for all t.

See Appendix H, Figure 2, for an illustration.

Knowing the monotonicity properties of $\tilde{C}(t)$, let us study $C(t)$. As we established before, $C(t) = \tilde{C}(t)\cdot\exp(\gamma_C t)$, where $\gamma_C = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho$. Then, it is straightforward that

$$\frac{dC(t)}{dt} = \exp(\gamma_C t)\left(\frac{d\tilde{C}(t)}{dt} + \tilde{C}(t)\gamma_C\right). \tag{47}$$

Taking Corollary 8, one can observe that all terms in equation (47) are positive, except $d\tilde{C}(t)/dt$. However, we can study this derivative from the monotonicity properties of $\tilde{C}(t)$. If $S(0)/K(0) > (S(0)/K(0))^*$, then $d\tilde{C}(t)/dt > 0$ for all t. Consequently, from equation (47), $dC(t)/dt > 0$ for all t. If $S(0)/K(0) = (S(0)/K(0))^*$, then $d\tilde{C}(t)/dt = 0$ for all t. Nevertheless, from equation (47), we conclude that $dC(t)/dt > 0$ for all t.

However, the difficult case arises for $S(0)/K(0) < (S(0)/K(0))^*$ because $d\tilde{C}(t)/dt < 0$ for all t. Replacing equation (46) into equation (47), and rearranging terms, it yields

$$\frac{dC(t)}{dt} = \exp(\gamma_C t)\tilde{C}(t)\left[\gamma_C - \left(\frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z+\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)\right]. \tag{48}$$

As in the previous section, all terms of equation (48) are positive. Since $\exp(Zt)$ increases with time, the term between brackets achieves its maximum value when $t = 0$. Thus, if

$$\gamma_C \geq \frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha \overline{A\theta}^{\alpha}} (\equiv \gamma_C^*),$$

then, $dC(t)/dt > 0$ for all $t > 0$[20]. However, if $\gamma_C < \gamma_C^*$, there exist a strictly positive t^* such that $dC(t)/dt < 0$ for all $0 < t < t^*$, $dC(t)/dt = 0$ for $t = t^*$, and $dC(t)/dt > 0$ for all $t > t^*$. It is easy to prove that

$$t^* = \frac{ln\left[\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right)\right]}{Z}.$$

The proof is provided in Appendix G ■

Moreover, it is easy to see that

$$\frac{dt^*}{d\rho} = \frac{1}{\gamma_C(Z - \alpha\gamma_C)}.$$

From Proposition 2, taking the definition of Z (equation (18)), one concludes that $dt^*/d\rho > 0$. This result implies that the more impatient is the household (*i.e.*, the greater is ρ) the greater is the number of periods where consumption decreases (t^*)[21].

We summarize the monotonicity properties of $C(t)$ in the following proposition:

Proposition 14.

(1) *If $S(0)/K(0) \geq (S(0)/K(0))^*$, then $C(t)$ increases monotonically for all t.*

(2) *If $S(0)/K(0) < (S(0)/K(0))^*$, then*

(2.1) *if $\gamma_C \geq \gamma_C^*$ then $C(t)$ increases monotonically for all $t > 0$;*

(2.2) *if $\gamma_C < \gamma_C^*$ then $C(t)$ decreases monotonically for all $0 < t < t^*$, and increases monotonically for all $t > t^*$.*

[20] If $\gamma_C = \gamma_C^*$, then $dC(t)/dt = 0$ for $t = 0$, and $dC(t)/dt > 0$ for all $t > 0$.

[21] Unfortunately, we can not provide a general conclusion of the effect of technical progress on t^*.

See Appendix H, Figure 3, for an illustration. Moreover, Figure 4 considers an example of initially decreasing consumption, but after $t^* = 99.85$ periods, consumption increases monotonically.

The economic interpretation of Propositions 13 and 14 is the following. If the proportion of non-renewable energy resources with respect to physical capital is high enough (*i.e.*, $S(0)/K(0) > (S(0)/K(0))^*$), then the detrended consumption ($\tilde{C}(t)$) increases monotonically until its BGP level. Moreover, this effect is reinforced by the growth rates of technical progress through the long-run growth rate of consumption. Therefore, the consumption ($C(t)$) increases monotonically for all $t > 0$. If $S(0)/K(0) = (S(0)/K(0))^*$, the detrended consumption is constant (its BGP level) for all $t > 0$. However, as in the previous case, technical progress allows for a consumption increasing monotonically for all $t > 0$. Finally, if the proportion of non-renewable energy resources is too low (*i.e.*, $S(0)/K(0) < (S(0)/K(0))^*$), the detrended consumption decreases monotonically for all $t > 0$. However, if the growth rates of technical progress are high enough to achieve a high enough long-run growth rate of consumption (*i.e.*, $\gamma_c \geq \gamma_c^*$) , then the consumption increases monotonically for all $t > 0$. Nevertheless, if the growth rates of technical progress are not so high (*i.e.*, $\gamma_c < \gamma_c^*$), then the consumption decreases monotonically until t^*. And then, consumption increases monotonically for all $t > t^*$ because the technical progress effect is now high enough (notice that technical progress increases with time).

An observation should be made at this moment. Taking a standard model of non-renewable energy resources with same technology for both physical capital and consumption, Dasgupta and Heal (1974) also described a non-monotonic behavior. However, we find two main differences between Dasgupta and Heal's result and our case. First, notice that what matters for Dasgupta and Heal's result is the size of both $S(0)$ and $K(0)$. However, since our model assumes that physical capital accumulation is more energy intensive that consumption, the key element for our result is the proportion of $S(0)$ with respect to $K(0)$. Second, according to Dasgupta and Heal (1974), if the initial stocks of $S(0)$ and

$K(0)$ are high enough, the consumption initially rises and, after some periods, it will decreases. However, in our model, the non-monotonic behavior appears when the proportion of $S(0)$ with respect to $K(0)$ is not high enough. Indeed, as we explained before, if the growth rates of technical progress are high enough ($\gamma_c \geq \gamma_c^*$) the negative effect of a low proportion of $S(0)$ with respect to $K(0)$ is compensated. Otherwise, if the growth rates of technical progress are not so high ($\gamma_c < \gamma_c^*$), the consumption decreases during an initial period. But, since technical progress rises with time, consumption increases after some periods.

5.6.4 Monotonicity of physical capital, investment and output

Equation (39) provides the optimal solution path for $K(t)$. Let us study the detrended capital:

$$\tilde{K}(t) =$$

$$K(0)(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{\frac{1}{\alpha}} \frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)} \left(\frac{\varphi Z}{\exp(Zt) + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}. \quad (49)$$

From equation (49), taking the formula (E.1) from Appendix E, we get

$$\frac{d\tilde{K}(t)}{dt} = \tilde{K}(t)\frac{\varphi Z^2}{\alpha\exp(Zt)}$$

$$\cdot \left[\frac{Z+\rho}{2Z+\rho}\frac{1}{\overline{A\theta}^{\alpha}}\frac{{}_2F_1\left(a,b,c;-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)} - \frac{1}{\alpha\overline{A\theta}^{\alpha} + \varphi Z\exp(Zt)}\right], \quad (50)$$

where

$$a = 2; b = 1 + \frac{Z+\rho}{Z}; c = b+1.$$

Since ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right) > 0$, applying the Euler integral representation as in Appendix D, one verifies that ${}_2F_1\left(a,b,c;-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right) > 0$. Therefore, following Corollary 8, all terms in equation (50) are positive. However, we can neither establish the sign of $d\tilde{K}(t)/dt$ nor conditions on the parameters to find out that sign. Nevertheless, we can conclude that $\tilde{K}(t)$ is not necessarily monotonic. Moreover, since $K(t) = \tilde{K}(t)\exp(\gamma_K t)$, where $\gamma_K = \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho$, $K(t)$ is not necessarily monotonic neither:

Proposition 15.

(1) *$\tilde{K}(t)$ is not necessarily monotonic.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $\tilde{K}(t) = K(0)$ for all t.*

(3) *$K(t)$ is not necessarily monotonic.*

(4) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $K(t)$ increases monotonically for all t.*

See Appendix H, Figure 5, for an illustration.

Let us study the monotonicity of investment. Since $\tilde{I}(t) = \overline{A}\tilde{K}(t) - \tilde{C}(t)$, then $d\tilde{I}/dt = \overline{A}d\tilde{K}(t)/dt - d\tilde{C}(t)/dt$. Hence, from Sections 5.6.3 and 5.6.4, we conclude that the detrended investment is not necessarily monotonic[22]. Moreover, since $I(t) = \tilde{I}(t)\exp(\gamma_I t)$, where $\gamma_I = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho$, the investment is not necessarily monotonic neither:

Proposition 16.

(1) *$\tilde{I}(t)$ is not necessarily monotonic.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then*

$$\tilde{I}(t) = \overline{A}K(0)\left(1 - \alpha\frac{Z+\rho}{Z}\right)$$

for all t.

(3) *$I(t)$ is not necessarily monotonic.*

(4) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $I(t)$ increases monotonically for all t.*

[22]We achieve the same conclusion by differentiating equation (40) with respect to t.

See Appendix H, Figure 6, for an illustration.

Since the detrended output $\tilde{Y}(t)$ is equal to $\overline{A}\tilde{K}(t)$, then $d\tilde{Y}(t)/dt = \overline{A}d\tilde{K}(t)/dt$. From section Proposition 15, we know that $\tilde{K}(t)$ is not necessarily monotonic, but we can not establish a general condition for this non-monotonicity. Hence, we can only conclude that $\tilde{Y}$ is not necessarily monotonic[23]. Since $Y(t) = \tilde{Y}(t)\exp(\gamma_Y t)$, where $\gamma_Y = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho$, we conclude that $Y(t)$ is not necessarily monotonic neither:

Proposition 17.

(1) *$\tilde{Y}(t)$ is not necessarily monotonic.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $\tilde{Y}(t) = \overline{A}K(0)$ for all t.*

(3) *$Y(t)$ is not necessarily monotonic.*

(4) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $Y(t)$ increases monotonically for all t.*

See Appendix H, Figure 7, for an illustration.

5.6.5 Monotonicity of extraction and stock of non-renewable energy resources

Let us conclude Section 5.6 studying the monotonicity properties of the extraction ($R(t)$) and stock ($S(t)$) of non-renewable energy resources. From Equation (42), we obtain the detrended $R(t)$:

$$\tilde{R}(t) =$$

$$\cdot \left(\frac{Z}{\varphi Z + \alpha\overline{A}\theta^\alpha}\right)^{\frac{1}{\alpha}} \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)} \left[{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha\exp(Zt)}\right) - \alpha\frac{Z+r}{Z}\right] \quad (51)$$

Taking logs and differentiating with respect to t, it yields

$$\frac{d\tilde{R}(t)}{dt} =$$

[23] As for investment, we get the same conclusion by differentiating equation (41) with respect to t.

$$\left[\tilde{R}(t)\frac{Z+\rho}{2Z+\rho}\frac{Z^2}{\alpha\overline{A}\theta^{\alpha}\exp(Zt)}\frac{{}_2F_1\left(a,b,c;-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}\exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}\exp(Zt)}\right)-\alpha\frac{Z+\rho}{Z}}\right]\varphi, \tag{52}$$

where

$$a=2; b=1+\frac{Z+\rho}{Z}; c=b+1.$$

Following Corollary 8, we observe that the terms between square brackets in equation (52) are positive. Then, the sign of $d\tilde{R}(t)/dt$ is determined by the sigh of φ. Hence, we establish the following proposition:

Proposition 18.

(1) *If $S(0)/K(0) > (S(0)/K(0))^*$, then $\tilde{R}(t)$ decreases monotonically for all t.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then*[24]

$$\tilde{R}(t)=\overline{A}K(0)\left(\frac{Z}{\alpha\overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}\left(1-\alpha\frac{Z+\rho}{Z}\right).$$

(3) *If $S(0)/K(0) < (S(0)/K(0))^*$, then $\tilde{R}(t)$ increases monotonically for all t.*

See Appendix H, Figure 8, for an illustration.

Since $R(t)=\tilde{R}(t)\exp(-\rho t)$, then $dR(t)/dt=\exp(-\rho t)(d\tilde{R}(t)/dt-\rho\tilde{R}(t))$. Hence, applying equation (52), we obtain

$$\frac{dR(t)}{dt}=\tilde{R}(t)\exp(-\rho t)$$

$$\cdot\left\{\left[\frac{Z+\rho}{2Z+\rho}\frac{Z^2}{\alpha\overline{A}\theta^{\alpha}\exp(Zt)}\frac{{}_2F_1\left(a,b,c;-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}\exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}\exp(Zt)}\right)-\alpha\frac{Z+\rho}{Z}}\right]\varphi-\rho\right\}. \tag{53}$$

Since the terms between square brackets are positives, we obtain the monotonicity properties of $R(t)$:

[24] Notice that condition (32) implies that $1>\alpha\frac{Z+\rho}{Z}$.

Proposition 19.

(1) *If $S(0)/K(0) \geq (S(0)/K(0))^*$, then $R(t)$ decreases monotonically for all t.*

(2) *If $S(0)/K(0) < (S(0)/K(0))^*$, then $R(t)$ is not necessarily monotonic.*

See Appendix H, Figure 9, for an illustration. As Proposition 19 establishes, if the proportion of non-renewable energy resources with respect to physical capital is too low, the extraction of non-renewable energy resources is not necessarily monotonic. Indeed, it is possible to generate an example of non-monotonic behavior of $R(t)$ (see Figure 11). But unfortunately, we can not establish an analytical condition for this case. Figure 10 shows a non-monotonic case of the extraction of non-renewable resources, where $R(t)$ increases during an initial period. And then, $R(t)$ decreases monotonically until zero. The economic interpretation of this case is the following. As we observe in Section 5.2.4, the Hotelling rule is the efficient condition for using up the non-renewable resources. Indeed, equation (8) is the Hotelling rule for our economy. For this particular case, Figure 10 shows that $I(t)$ always increases. Then, for a fixed $R(t)$, the RHS of equation (8) (*i.e.,* the growth rate of the marginal productivity of extraction of non-renewable energy resources) rises. Since the RHS of equation (8) (*i.e.,* the marginal productivity of investment) equals the LHS, then LHS has to increase too. Hence, $R(t)$ should rise, increasing the LHS and reducing the RHS. However, technical progress increases with t. Then, after some periods, technical progress can increase the LHS even if $R(t)$ decreases[25]. Finally, we point out that this result contrasts with Dasgupta and Heal (1974). These authors obtained that $R(t)$ decreases monotonically for all t. However, our model allows for a non-monotonic behavior of $R(t)$. Indeed, this possibility rises the importance of considering physical capital production as

[25]Notice that energy-saving technical progress also affects the RHS, through its growth rate γ_θ. However, we assumed that γ_θ is constant. Then, the effect of energy-saving technical progress on the RHS is constant with t. Nevertheless, the effect of technical progress (in particular, energy-saving technical progress) on the LHS increases with t.

more energy-intensive than consumption sector, and the role of technical progress (in particular, energy-saving technical progress).

To conclude, we study the monotonic properties of the stock of non-renewable energy resources. Equation (10) establishes that $\dot{S}(t) = -R(t)$. Then, $dS(t)/dt = -R(t)$. Hence, since $R(t) > 0$ for all t, $dS(t)/dt < 0$ for all t:

Proposition 20. *$S(t)$ decreases monotonically for all t.*

See Appendix H, Figure 11, for an illustration.

5.7 Concluding remarks

In this chapter, we studied the implications of assuming that physical capital accumulation is relatively more energy-intensive than consumption. We pointed out four main contributions of this chapter. First, we concluded that technical progress (in particular, energy-saving technical progress) is a key factor to sustain long-run growth when physical capital accumulation is more energy-intensive than consumption. As we noticed, this result contrasts with Stiglitz (1974) and Dasgupta and Heal (1979), where the economy achieves steady state even without technical progress. Indeed, we showed that, in our model, neither BGP nor steady state are possible if there is not technical progress. The second contribution was the application of Special Functions representation to provide a close-form solution of the optimal solution paths of our variables in levels for every time t. This new technique in economic theory allows us to have a full characterization of both short and long-run dynamics. Indeed, the third and fourth contributions emerge thanks to this technique. Taking the close-form solutions, we provided the necessary and sufficient conditions for existence and uniqueness of BGP, which was the third contribution of this chapter. Moreover, we also proved that there is asymptotic convergence to the BGP. Finally, the fourth contribution was the study of the monotonicity properties of the optimal solution paths of our variables in levels. As we pointed out, this kind of studies are typically unallowed in standard approaches. We found that our assumptions on physical capital accumulation (relatively more energy-intensive than consumption), and technical progress, implies new non-monotonic behavior of the optimal trajectories. In particular, we showed that consumption and extraction of non-renewable energy resources can depict non-monotonic behavior. Moreover, we noticed that this non-monotonic behavior of consumption and extraction contrast with the non-monotonicity result of Dasgupta and Heal (1974).

This model has several limitations. Among of them, we point out the following three limits of our analysis. First, following Sollow (1974a), we assumed unlimited (exogenous) technical progress. Regarding to our disembodied technical progress ($A(t)$), this assumption models the idea of unlimited growth of knowledge. However, unlimited energy-saving technical progress ($\theta(t)$) could arise physical incompatibility problems related to the thermodynamic laws. In the literature of exogenous energy-saving technical progress, the assumption of unlimited growth of energy efficiency is widely used (see for instance Boucekkine and Pommeret (2004), Azomahou *et al.* (2004), and Pérez-Barahona and Zou (2006a,b)). Indeed, these authors assume that the limit of improving energy efficiency is very far away. In our model, we take that point of view. However, from Section 5.5, one should observe that the effect of technical progress on our economy is a combination of both disembodied and energy-saving technical progress (see constant Z and BGP growth rates). Then, for high enough growth rates of disembodied technical progress (γ_A), it is possible to have positive BGP without growth of energy-saving technical progress (see Proposition 2). Nevertheless, if γ_A is not high enough, the economy could achieve positive BGP by adopting renewable energy resources. This case is not analyzed here, but it is a very interesting extension to our model since it could rise additional questions such as the depletion of non-renewable energy resources in a finite period.

A second limitation of our analysis is that we assumed exogenous technical progress. The reason of taking that assumption is the following. The aim of this chapter was to study the conditions under which technical progress can offset the trade-off between economic growth and the usage of non-renewable energy resources. Then, we did not analyze how the economy can achieve these conditions. However, a very interesting extension to our model is to assume technical progress (in particular, energy-saving technical progress) as an endogenous variable. Indeed, as we observed in Proposition 2, our BGP growth rates are not affected by the stock of non-renewable energy resources. However, the scarcity of energy resources could rise new investments in energy-saving technologies, establishing an effect of the stock of non-renewable energy resource on the BGP growth rates. As it is suggested in Chapter 4, a possibility to endogenize energy-saving technical progress could be by means of an R&D sector (for energy-saving technologies) following Aghion and Howitt (1992) and Grossman and Helpman (1991a,b).

Finally, as third limitation, we notice that it was assumed no physical capital depreciation. Indeed, our model is based on Dasgupta and Heal (1974), Hartwick (1989) and Chapter 4, which do not consider capital depreciation neither. This simplification is frequently considered in models of resources economics (see for instance Sollow (1974a,b), Pezzey and Withagen (1998), and Stokey (1998)). However, for an accurate understanding of the relationship between non-renewable energy resources and physical capital accumulation, capital depreciation should be considered.

Appendix A

Following Sydsæter *et al.* (1999, pages 109-110), the Lagrangian associated with our optimal control problem with mixed constraints is

$$\mathcal{L}(\bullet) = U[A(t) - I(t)]\exp(-\rho t) + \psi(t)\{g[\theta(t), I(t), R(t)]\} - \lambda(t)R(t) + q(t)S(t),$$

where $\psi(t)$, $\lambda(t)$ and $q(t)$ are the Lagrangian multipliers, $U[\bullet]$ is a general utility function, and $g[\bullet]$ is a general production function for physical capital accumulation.

The corresponding first order conditions (FOC) are:

$$\frac{\partial \mathcal{L}}{\partial I(t)} = 0; \frac{\partial \mathcal{L}}{\partial R(t)} = 0;$$

$$\frac{\partial \mathcal{L}}{\partial K(t)} = -\dot{\psi}(t); \frac{\partial \mathcal{L}}{\partial S(t)} = -\dot{\lambda}(t);$$

$$q(t) \geq 0 (= 0 \text{ if } S(t) > 0);$$

$\lim_{t\to\infty} \psi(t)K(t) = 0$ and $\lim_{t\to\infty} \lambda(t)S(t) = 0$ (Transvarsality conditions).

From the FOC, one obtains

$$\psi(t)g_I = U_c \exp(-\rho t), \tag{A.1}$$

$$\psi(t)g_R = \lambda(t) + q(t), \tag{A.2}$$

$$\dot{\psi}(t) = -U_c A(t) \exp(-\rho t), \tag{A.3}$$

$$q(t) = -\dot{\lambda}(t). \tag{A.4}$$

Since it is not optimal to completely deplete the stock of non-renewable energy resource in a finite t[26], *i.e.*, $S(t) > 0$ for all t, then $q(t) = 0$. Taking equation (A.4), this implies $\lambda(t) = \lambda$ for all t.

Applying equation (A.1) into equation (A.3), one gets

$$\frac{\dot{\psi}(t)}{\psi(t)} = -A(t)g_I. \tag{A.5}$$

Taking logs in equation (A.2) and applying equation (A.5), one obtains the Hotelling rule

$$A(t)g_I = \frac{\dot{g}_R}{g_R}.$$

Taking the functional forms of my model, equation (8) is easily obtained from the Hotelling rule.

Taking logs in equation (A.1) and applying equation (A.5), one finds the Ramsey (saving) rule

$$\frac{U_{cc}\dot{C}}{U_c} - \rho = \frac{\dot{g}_I}{g_I} - Ag_I.$$

[26]Notice that non-renewable energy resources is an essential input for physical capital accumulation. Moreover, energy-saving technical progress is embodied in the new equipment goods.

Applying the functional forms of my model, one gets equation (7).

Finally, equation (6) is the technology for physical capital accumulation, equation (9) is the budget constraint of the economy, and equation (10) is the law of motion of the stock of non-renewable energy resource. ■

Appendix B

Along the BGP, all the endogenous variables grow at constant rate, *i.e.*, $x(t) = \overline{x}\exp(\gamma_x t)$. Moreover, we assume that $A(t) = \overline{A}\exp(\gamma_A t)$ and $\theta(t) = \overline{\theta}\exp(\gamma_\theta t)$ for all t, where $\overline{A}, \overline{\theta}, \gamma_A, \gamma_\theta$ are strictly positive and exogenous parameters.

By writing equation (7) along the BGP, one obtains

$$\gamma_A + \alpha\gamma_\theta = \alpha(\gamma_I - \gamma_R), \tag{B.1}$$

which implies that

$$\gamma_C = (1-\alpha)\overline{A\theta}^{\alpha}\left(\frac{\overline{R}}{\overline{I}}\right)^{\alpha} + \gamma_A - \rho. \tag{B.2}$$

Since $Y(t) = C(t) + I(t)$, $\gamma_Y = \gamma_I = \gamma_C$. Moreover, from the final good technology ($Y(t) = A(t)K(t)$) one gets that $\gamma_K = \gamma_Y - \gamma_A$. Then,

$$\gamma_K = (1-\alpha)\overline{A\theta}^{\alpha}\left(\frac{\overline{R}}{\overline{I}}\right)^{\alpha}. \tag{B.3}$$

From equation (8) along the BGP, it is easy to obtain that

$$\left(\frac{\overline{R}}{\overline{I}}\right)^{\alpha} = \frac{\alpha\gamma_\theta + (1-\alpha)(\gamma_I - \gamma_R)}{(1-\alpha)\overline{A\theta}^{\alpha}}. \tag{B.4}$$

Applying equation (B.4) into equation (B.3), one gets the value of the ratio

$$\left(\frac{\overline{R}}{\overline{I}}\right)^{\alpha} = \frac{\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A}{(1-\alpha)\overline{A\theta}^{\alpha}}. \tag{B.5}$$

Replacing this ratio into equations (B.2) and (B.3), one obtains

$$\gamma_Y = \gamma_C = \gamma_I = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho$$

and

$$\gamma_K = \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho.$$

From equation (B.1) and the two previous expressions, one finds that the long-run growth rate of the energy is $\gamma_R = -\rho$. Finally, since $\dot{S}(t) = -R(t)$,

$$\gamma_S = \gamma_R (= -\rho) \blacksquare$$

Appendix C

Equation (21) can be rewritten as a standard linear first-order differential equation:

$$\dot{\tilde{K}}(t) + a(t)\tilde{K}(t) = b(t), \qquad (C.1)$$

where

$$a(t) = \frac{1-\alpha}{\alpha}Z - \rho - \overline{A\theta}^{\alpha}X(t)^{\alpha},$$
$$b(t) = -\overline{\theta}^{\alpha}X(t)^{\alpha}\tilde{C}(t),$$

and $K(0)(=\tilde{K}(0))$ is given.

Since the previous steps provide the solutions for $X(t)$ (equation (18)) and $\tilde{C}(t)$ (equation (20)), one can solve the differential equation (C.1) applying the following result (Sydsæter *et al.* (1999)):

$$\tilde{K}(t) =$$
$$\tilde{K}(0)\exp\left(-\int_0^t a(\xi)d\xi\right) + \int_0^t b(\tau)\exp\left(-\int_\tau^t a(\xi)d\xi\right)d\tau. \qquad (C.2)$$

Taking equations (18) and (20), one obtains that

$$a(t) = \frac{1-\alpha}{\alpha}Z - \rho - \frac{Z\overline{A\theta}^{\alpha}}{\frac{\varphi Z}{\exp(Zt)} + \alpha\overline{A\theta}^{\alpha}}. \qquad (C.3)$$

After some calculations, one finds

$$\int a(\xi)d\xi = \left(\frac{1-\alpha}{\alpha}Z - \rho\right)\xi + \frac{1}{\alpha}ln\left(\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(Z\xi)}\right). \qquad (C.4)$$

Evaluating the integral (C.4) in t and 0, and after rearranging terms, one obtains the first part of equation (C.2):

$$\tilde{K}(0)\exp\left(-\int_0^t a(\xi)d\xi\right) = \tilde{K}(0)\exp\left(-\frac{1}{\alpha}ln(\varphi Z + \alpha\overline{A\theta}^{\alpha})\right)$$

$$\cdot \exp\left(\frac{1}{\alpha}(ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}) - ((1-\alpha)Z - \alpha\rho)t)\right).$$

In order to calculate the second part of equation (C.2), we first obtain

$$b(\tau)\exp\left(-\int_{\tau}^{t} a(\xi)d\xi\right) =$$

$$-Z\overline{\theta}^{\alpha}C(0)\exp\left\{\frac{1}{\alpha}[ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)) - ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})]\right\}$$

$$\cdot \exp\left\{\left(-\frac{1-\alpha}{\alpha}Z + \rho\right)t\right\}\frac{\exp(-\rho\tau)}{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Z\tau)}, \qquad (C.5)$$

by evaluating the integral (C.4) in τ and t. Then,

$$\int_0^t b(\tau)\exp\left(-\int_{\tau}^{t} a(\xi)d\xi\right)d\tau =$$

$$-Z\overline{\theta}^{\alpha}C(0)\exp\left\{\frac{1}{\alpha}[ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)) - ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})]\right\}$$

$$\cdot \exp\left\{\left(-\frac{1-\alpha}{\alpha}Z + \rho\right)t\right\}$$

$$\cdot \int_0^t \frac{\exp(-\rho\tau)}{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Z\tau)}d\tau. \qquad (C.6)$$

In order to calculate equation (C.6), we need to obtain

$$\int_0^t \frac{\exp(-\rho\tau)}{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Z\tau)}d\tau. \qquad (C.7)$$

To do this, we follow Boucekkine and Ruiz-Tamarit (2004). Equation (C.7) is equal to

$$\int_0^t \left(\varphi Z\exp(\rho\tau) + \alpha\overline{A}\overline{\theta}^{\alpha}\exp((Z+\rho)\tau)\right)^{-1}d\tau.$$

Taking the following change of variable $V = \exp(-(\rho+Z)\tau)$,

$$\int_0^t \left(\varphi Z\exp(\rho\tau) + \alpha\overline{A}\overline{\theta}^{\alpha}\exp((Z+\rho)\tau)\right)^{-1}d\tau =$$

$$\int_1^{\exp(-(\rho+Z)t)}\left(1 + \frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}V^{\frac{Z}{Z+\rho}}\right)^{-1}dV. \qquad (C.8)$$

Recalling $y = \exp(-(\rho+Z)t)$, $A_0 = \frac{\varphi Z}{\alpha \overline{A}\theta^\alpha}$, and $\beta = -1$, equation (C.8) can be rewritten as

$$\int_1^y \left(1 + A_0 V^{\frac{Z}{Z+\rho}}\right)^\beta dV. \tag{C.9}$$

Applying the binomial theorem:

$$(1+ax^b)^c = \sum_{n=0}^{\infty} (-c)_n \frac{(-ax^b)^n}{n!},$$

equation (C.9) equals

$$\sum_{n=0}^{\infty} \frac{(-\beta)_n}{n!} (-A_0)^n \int_1^y V^{\frac{Z}{Z+\rho}n} dV,$$

which is equal to

$$\sum_{n=0}^{\infty} (-\beta)_n \frac{-(A_0)^n}{n!} \frac{1}{\frac{Z}{Z+\rho}n+1} \left(y^{\frac{Z}{Z+\rho}n} y - 1\right). \tag{C.10}$$

Applying the property $\frac{X}{X+n} = \frac{(X)_n}{(1+X)_n}$, equation (C.10) equals

$$y \sum_{n=0}^{\infty} \frac{(a)_n(-\beta)_n}{(1+a)_n} \frac{(-A_0 y^{\frac{Z}{Z+\rho}})^n}{n!}$$

$$-\sum_{n=0}^{\infty} \frac{(a)_n(-\beta)_n}{(1+a)_n} \frac{(-A_0)^n}{n!}. \tag{C.11}$$

Taking the definition of Gauss hypergeometric function

$$_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

one finds that equation (C.8) equals

$$\exp(-(\rho+Z)t)\,_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha \exp(Zt)}\right) - {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right), \tag{C.12}$$

where

$$_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha \exp(Zt)}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1+\frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha \exp(Zt)}\right),$$

$$_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1+\frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)$$

Hence, retrieving the change of variables,

$$\int_0^t \frac{\exp(-\rho\tau)}{\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(Z\tau)}d\tau = \frac{1}{\alpha\overline{A\theta}^{\alpha}(Z+\rho)}$$

$$\cdot \left[{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) - \frac{1}{\exp[(\rho+Z)t]}{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)\right]. \qquad (C.13)$$

Finally, taking together equations (C.3), (C.5), and (C.13), and rearranging terms, one can easily obtain equation (22) ■

Appendix D

Equation (24) can be rewritten as follows:

$$\tilde{R}(t) = [a] + [b] + [c] + [d],$$

where

$$[a] = (\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} \overline{A} K(0) \exp\left\{\left(\frac{2-\alpha}{\alpha}Z - \rho\right)t\right\},$$

$$[b] =$$

$$(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho}\frac{C(0)}{\alpha}{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)\exp\left\{2\left(\frac{1-\alpha}{\alpha}Z - \rho\right)t\right\},$$

$$[c] =$$

$$-(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho}\frac{C(0)}{\alpha}{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)\exp\left\{\left(\frac{2-\alpha}{\alpha}Z - r\right)t\right\},$$

$$[d] = -(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0).$$

Then,

$$\int_0^t \tilde{R}(\tau)\exp(-\rho\tau)d\tau = \int_0^t [a]\exp(-\rho\tau)d\tau + \int_0^t [b]\exp(-\rho\tau)d\tau$$

$$+ \int_0^t [c]\exp(-\rho\tau)d\tau + \int_0^t [d]\exp(-\rho\tau)d\tau.$$

It is easy to prove that

$$\int_0^t [a]\exp(-\rho\tau)d\tau =$$

$$\frac{(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} \overline{A} K(0)}{2(\frac{1}{\alpha}Z - \rho) - Z}\left\{\exp\left[\left(2\left(\frac{1}{\alpha}Z - \rho\right) - Z\right)t\right] - 1\right\} = [1],$$

$$\int_0^t [c]\exp(-\rho\tau)d\tau =$$

$$-A_2 \frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^\alpha}\right)}{2(\frac{1}{\alpha}Z-\rho)-Z}\left\{\exp\left[\left(2\left(\frac{1}{\alpha}Z-\rho\right)-Z\right)t\right]-1\right\} = [3],$$

$$\int_0^t [d]\exp(-\rho\tau)d\tau = \frac{(\varphi Z+\alpha\overline{A\theta}^\alpha)^{-\frac{1}{\alpha}}Z^{\frac{1}{\alpha}}C(0)}{\rho}(\exp(-\rho t)-1) = [4],$$

where

$$A_2 = (\varphi Z+\alpha\overline{A\theta}^\alpha)^{-\frac{1}{\alpha}}\frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho}\frac{C(0)}{\alpha}.$$

The integral [b] involves ${}_3F_2$ Hypergeometric functions. Indeed, applying the Euler integral representation (Abramowitz and Stegun (1970), page 558, formula 15.3.1), we obtain that

$$\int_0^t [b]\exp(-\rho\tau)d\tau = A_2\frac{Z+\rho}{Z}(\alpha\overline{A\theta}^\alpha)^{\frac{Z+\rho}{Z}}$$

$$\cdot\int_0^1 (1-\xi)^{\frac{\rho}{Z}}\left(\int_0^t\left(\alpha\overline{A\theta}^\alpha\exp(Z\tau)+\varphi Z\xi\right)^{-\frac{Z+\rho}{Z}}\exp\left\{\left(\frac{2-\alpha}{\alpha}Z-2\rho\right)\tau\right\}d\tau\right)d\xi,$$

The integral $\int_0^t(\cdot)d\tau$ can be rewritten as

$$\int_0^t\left\{\varphi Z\xi\exp(R\tau)+\alpha\overline{A\theta}^\alpha\exp[(Z+\Omega)\tau]\right\}^{-\frac{Z+\rho}{Z}}d\tau, \qquad (D.1)$$

where

$$\Omega = -\left(\frac{2-\alpha}{\alpha}Z-2\rho\right)\frac{Z}{Z+\rho}.$$

Following a similar strategy as in Appendix C (equation (C.7)), we can apply the change of variable $V = \exp\{(\Omega+\gamma)\tau\}$ to equation (D.1). Taking $\gamma = -(Z+\Omega)[(Z+\rho)/Z]-\Omega$, one can prove that equation (D.1) equals

$$-\left(\alpha\overline{A\theta}^\alpha\right)^{-\frac{Z+\rho}{Z}}\frac{Z}{Z+\rho}\frac{1}{Z+\Omega}$$

$$\cdot\left[\exp\left\{-(Z+\Omega)\frac{Z+\rho}{Z}t\right\}{}_2F_1\left(\hat{a},\hat{b},\hat{c};\frac{-\varphi Z\xi}{\alpha\overline{A\theta}^\alpha\exp(Zt)}\right)-{}_2F_1\left(\hat{a},\hat{b},\hat{c};\frac{-\varphi Z\xi}{\alpha\overline{A\theta}^\alpha}\right)\right],$$

where

$$\hat{a} = \frac{Z+\rho}{Z}\frac{Z+\Omega}{Z};\hat{b} = \frac{Z+\rho}{Z};\hat{c} = 1+\frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}.$$

Then,

$$\int_0^t [b]\exp(-\rho\tau)d\tau = A_2\frac{1}{Z+\Omega}$$

$$\cdot \left\{ [INT1] - \exp\left(-(Z+\Omega)\frac{Z+\rho}{Z} t \right) [INT2] \right\}, \qquad (D.2)$$

where

$$[INT1] = \int_0^1 (1-\xi)^{\frac{\rho}{Z}} {}_2F_1 \left(\hat{a}, \hat{b}, \hat{c}; -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha}} \right) d\xi,$$

$$[INT2] = \int_0^1 (1-\xi)^{\frac{\rho}{Z}} {}_2F_1 \left(\hat{a}, \hat{b}, \hat{c}; -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha} \exp(Zt)} \right) d\xi.$$

Assuming that $\left| -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right| < 1$, we can apply the following property (see Luke (1975), Vol.1, page 58, equation (10)):

$$_{p+1}F_{q+1} \left\{ (\beta, \alpha_p); (\beta+\sigma, \rho_p); Z \right\} =$$

$$\frac{\Gamma(\beta+\sigma)}{\Gamma(\beta)\Gamma(\sigma)} \int_0^1 t^{\beta-1}(1-t)^{\sigma-1} {}_{p+1}F_{q+1} \left\{ (\alpha_p); (\rho_p\ Zt) \right\} dt$$

for $|Z| < 1$, $Re(\beta) > 0$, and $Re(\sigma) > 0$.

In our case, $t \equiv \xi$, $p \equiv 2$, $q \equiv 1$, $\beta \equiv 1$, $\sigma \equiv \frac{r}{Z} + 1$,

$$\alpha_p \equiv \left(\frac{Z+\rho}{Z} \frac{Z+\Omega}{Z}, \frac{Z+\rho}{Z} \right),$$

$$\rho_p \equiv \left(1 + \frac{Z+\rho}{Z} \frac{Z+\Omega}{Z} \right),$$

and $Z \equiv -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha}}$ for [INT1], and $Z \equiv -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha} \exp(Zt)}$ for [INT2]. Then,

$$[INT1] = \frac{Z}{Z+\rho} {}_3F_2 \left(a; b; -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha}} \right),$$

$$[INT1] = \frac{Z}{Z+\rho} {}_3F_2 \left(a; b; -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha} \exp(Zt)} \right),$$

where

$$a = \left(1, \frac{Z+\rho}{Z}, \frac{Z+\rho}{Z} \frac{Z+\Omega}{Z} \right),$$

$$b = \left(1 + \frac{Z+\rho}{Z}, 1 + \frac{Z+\rho}{Z} \frac{Z+\Omega}{Z} \right).$$

Applying this result into equation (D.2), we obtain [2] ■

Appendix E

Under the conditions (32), (36), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right) > 0$, it is clear that the only problem is the term

$$\frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha \exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)}$$

in equation (39). Since we know that the denominator is positive, we have to verify that the numerator is positive too. To do that, we use the following property (see Abramowitz and Stegun, 15.2.1, page 557):

$$\frac{d}{dz}{}_2F_1(a,b,c;z) = \frac{ab}{c}{}_2F_1(a+1,b+1,c+1;z) \qquad (E.1).$$

Since our a, b, c terms and ${}_2F_1(a,b,c;z)$ are positive, we conclude that ${}_2F_1(a+1,b+1,c+1;z)$ and the derivative are positive too. Then, ${}_2F_1(a,b,c;z)$ increases as z raises. Since $\exp(Zt)$ increases our z term, then

$$ {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha \exp(Zt)}\right) > {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right) > 0$$

■

Appendix F

Let us define

$$F\left(\frac{S(0)}{K(0)}\right) = \frac{S(0)}{K(0)} - \alpha\overline{A}\left(\frac{Z}{\varphi Z + \alpha\overline{A}\theta^\alpha}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+R}\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)} - \frac{Z+\rho}{Z}\frac{1}{\rho}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)}\right]. \qquad (F.1)$$

Condition (35) implies

$$F\left(\frac{S(0)}{K(0)}\right) = 0.$$

Then, applying the implicit function theorem, one obtains

$$\frac{\partial\varphi}{\partial(S(0)/K(0))} = -\frac{\partial F/\partial(S(0)/K(0))}{\partial F/\partial\varphi}. \qquad (F.2)$$

From (F.1), $\partial F/\partial(S(0)/K(0)) = 1$. Moreover, $\partial F/\partial\varphi = -\partial g(\varphi)/\partial\varphi$, where

$$g(\varphi) = \alpha\overline{A}\left(\frac{Z}{\varphi Z + \alpha\overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+R}\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)} - \frac{Z+\rho}{Z}\frac{1}{\rho}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)}\right]. \qquad (F.3)$$

Applying the following property (Luke (1975), Vol.1, page 111 (Property 38))

$${}_3F_2\left\{(a,b,c);(d+1,c+1);Z\right\} = \frac{c}{c-d}{}_2F_1\left(a,b,d+1;Z\right)$$

$$-\frac{d}{c-d}{}_3F_2\left\{(a,b,c);(d,c+1);Z\right\},$$

one obtains that

$$\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)} = \frac{Z+\Omega}{\Omega} - \frac{Z}{\Omega}\frac{{}_2F_1\left(a',b',c',-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}}\right)},$$

where

$$a' = 1; b' = \frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}; c' = 1+b'.$$

Then, we can rewrite $g(\varphi)$ with ${}_2F_1$ functions. Applying the formula (E.1) for the derivative of ${}_2F_1$, one gets $\partial g(\varphi)/\partial\varphi$. Rearranging terms and taking Corollary 6, we obtain that $\partial g(\varphi)/\partial\varphi < 0$. Hence, from (F.2), one concludes that

$$\frac{\partial\varphi}{\partial(S(0)/K(0))} < 0$$

■

Appendix G

Let us consider the case

$$\gamma_C < \frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A}\theta^{\alpha}}.$$

Taking

$$\gamma_C = \frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(Zt^*)}, \tag{G.1}$$

we obtain t^*. From (G.1), one gets

$$\exp(Zt^*) = \frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right). \tag{G.2}$$

Since, under Corollary 8, the term $Z - \alpha\gamma_C$ is strictly positive, we can apply logs in (G.2). Then,

$$t^* = \frac{ln\left[\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right)\right]}{Z}.$$

If the term between square brackets is > 1, then $t^* > 0$. We can prove it by contradiction. Let us assume that

$$\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right) \leq 1.$$

Then, it easy to prove that

$$\gamma_C \geq \frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}},$$

which contradicts our initial statement. Then $t^* > 0$ ■

Appendix H

Notice that this is not a numerical simulation. We proceed as follows. First, for given values of the parameters, we obtain the corresponding constant φ by solving numerically the non-linear equation (35)[27]. Once φ is known, we plot the optimal solution paths provided in Section 5.5.

[27]Since our problem is concave, the solution paths are unique. Then, equation (35) has a unique solution for φ. A formal proof of uniqueness of the solution of equation (35) could be done. However, this is beyond the objective of this chapter. The uniqueness can be tested by plotting equation (35), for given values of the parameters.

Parametrization

α	γ_θ	γ_A	$\overline{A}$	$\overline{\theta}$	$(S(0)/K(0))^*$	ρ
0.32	0.019	0.01	1	2	0.0056	0.03

Note for the Figures

- All Figures share the same parametrization except Figure 4, where we specify the parameters value.
- φ is the constant corresponding to the ratio $S(0)/K(0)$. In particular, $\varphi = 0$ is the constant corresponding to the ratio $(S(0)/K(0))^*$.
- $x(t)$ represents the variable and $detrx(t)$ is the corresponding detrended variable.

Figures

Figure 1: $X(t)$

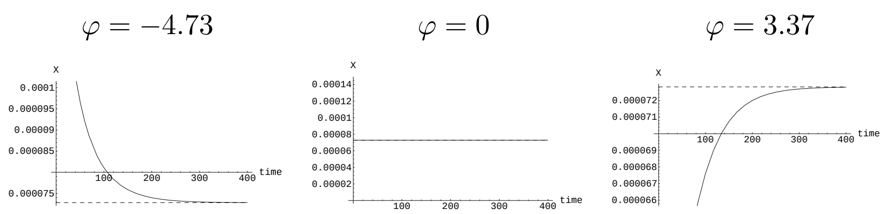


Figure 2: $\tilde{C}(t)$

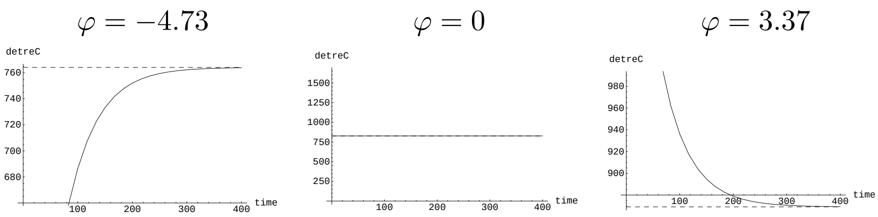


Figure 3: $C(t)$

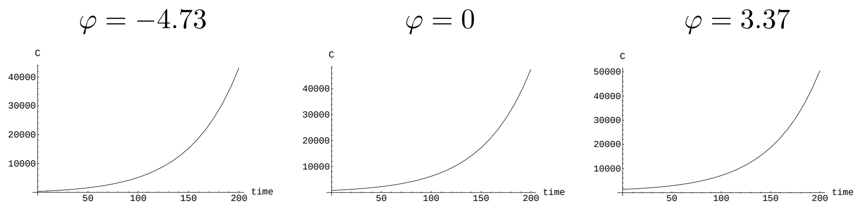


Figure 4: special case of $C(t)$

α	γ_θ	γ_A	$\overline{A}$	$\overline{\theta}$	$S(0)/K(0)$	ρ
0.32	0.01	0.001	1	2	0.00015	0.01

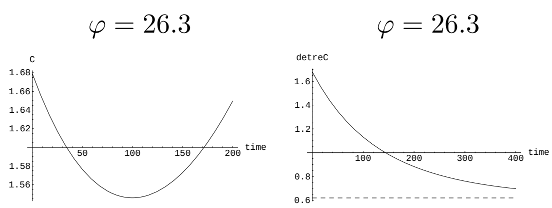


Figure 5: $\tilde{K}(t)$ and $K(t)$

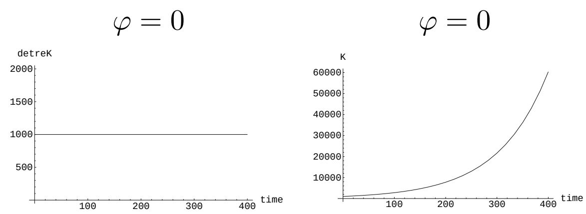


Figure 6: $\tilde{I}(t)$ and $I(t)$

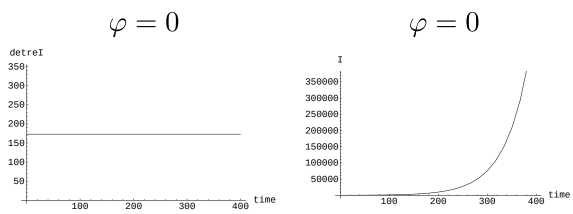


Figure 7: $\tilde{Y}(t)$ and $Y(t)$

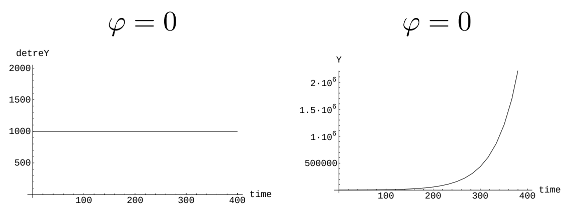


Figure 8: $\tilde{R}(t)$

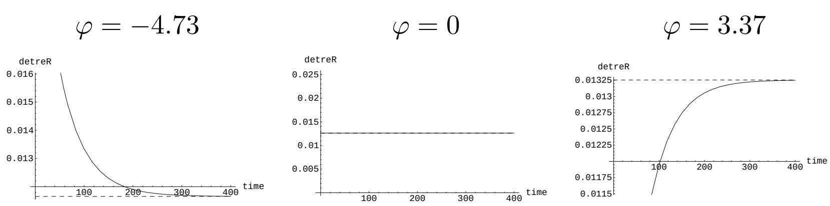


Figure 9: $R(t)$

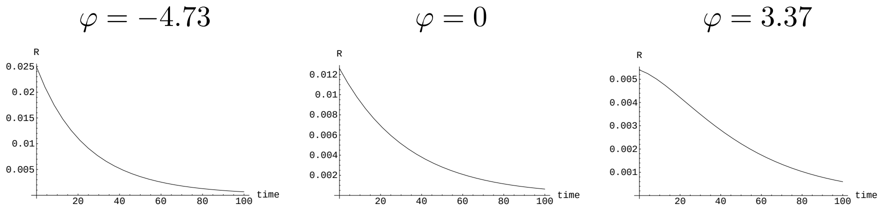


Figure 10: special case of $R(t)$

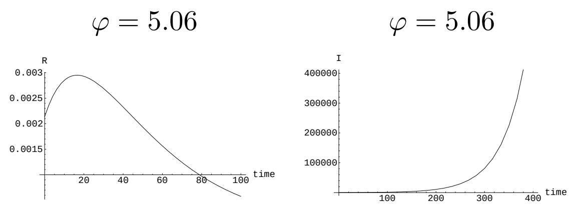
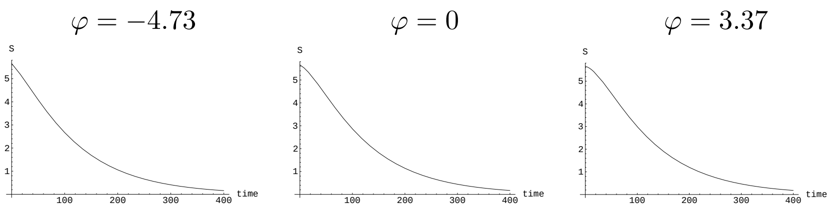


Figure 11: $S(t)$



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