An attempt at introducing the concept of exposure age

Eric Deleersnijder, July 24 - December 3, 2012

1. The concept of exposure age

So far, in all of the studies involving CART's ages[1], the clock attached to every constituent particle kept ticking at any location of the domain of interest. It was only at some of the domain boundaries that the clock would be reset to zero. With the noticeable exception of Liu et al. (2012), no one seems to have explored the possibility of having the clock function in a subdomain and be at a standstill in the rest of the domain of interest. It is likely that this alternative approach will be conducive to interesting results — provided it is used in a way that is inspired by the features of the flow under consideration.

In the present working note, one will show that a simple modification of the age concentration equation will lead to the abovementioned age. For obvious reasons, such an age might be termed "exposure age", for it would be related to the time spent by the particles in the subdomain in which the clock is active.

Let Ω , Ω_a and Ω_i denote the domain of interest, the sub-domain where the clock used for estimating the age is active and the sub-domain where the clock is inactive, respectively, implying that $\Omega = \Omega_a \cup \Omega_i$. The volumes of the aforementioned (sub-)domains are denoted

$$V_a = \int_{\Omega_a} d\mathbf{x} \quad , \quad V_i = \int_{\Omega_i} d\mathbf{x} \quad , \quad V = V_a + V_i \ . \tag{1}$$

As will be seen, in some circumstances, it is convenient to introduce the following dimensionless parameter

$$\mu \ = \ \frac{V_a}{V} \ = \ \frac{V_a}{V_a + V_i} \ . \tag{2}$$

Without any significant loss of generality, one may restrict the present discussion to passive constituents. Indeed, taking into account reactive (*i.e.* source/sink or production/destruction) terms would have no crucial impact on the issues addressed hereinafter. In the classical version of CART (e.g. Delhez et al. 1999, Deleersnijder et al. 2001), the concentration and age concentration of a passive constituent, $C(t,\mathbf{x})$ and $\alpha(t,\mathbf{x})$, obey the following equations

$$\frac{\partial C}{\partial t} \ = \ - \ \nabla \bullet (\mathbf{v}C - \mathbf{K} \bullet \nabla C) \ , \tag{3}$$

$$\frac{\partial \alpha}{\partial t} \ = \ C \ - \ \nabla \bullet (\mathbf{v}\alpha - \mathbf{K} \bullet \nabla \alpha) \ . \tag{4}$$

The age is obtained as the ratio of the age concentration to the concentration,

$$a(t,\mathbf{x}) \ = \ \frac{\alpha(t,\mathbf{x})}{C(t,\mathbf{x})} \ . \tag{5}$$

The age concentration equation (4) is valid where the clock of every particle is ticking.

[1] CART: Constituent-oriented Age and Residence time Theory (www.climate.be/cart)

To compute the exposure age, the only equation to be modified is (4), which then transforms to

$$\begin{cases} \mathbf{x} \in \Omega_a: & \dfrac{\partial \alpha^*}{\partial t} = C - \nabla \bullet (\mathbf{v}\alpha^* - \mathbf{K} \bullet \nabla \alpha^*) \\ \mathbf{x} \in \Omega_i: & \dfrac{\partial \alpha^*}{\partial t} = -\nabla \bullet (\mathbf{v}\alpha^* - \mathbf{K} \bullet \nabla \alpha^*) \end{cases} \tag{6}$$

The exposure age is

$$a^*(t,\mathbf{x}) = \frac{\alpha^*(t,\mathbf{x})}{C(t,\mathbf{x})} . \tag{7}$$

Clearly, the asterisk identifies variables related to the exposure age.

The ageing term, C, of the age concentration equation is to be omitted in the sub-domain Ω_i, thereby reflecting that the clock is not ticking in this subdomain — and, hence, that the particles are not ageing whenever they are present in this subdomain. To obtain the exposure age rather than the classical one, replacing (4) by (6) is the only modification needed. Though the modifications of the mathematical and numerical models are almost trivial, their impact of the results and the interpretations thereof are unlikely to be equally straightforward.

A few problems admitting analytical solutions are tackled below, hoping they will help gain insight into the functioning and potential usefulness of the alternative age suggested above. In the first two illustrations, the size of the domain of interest is infinite, making it impossible to evaluate and use the dimensionless parameter μ.

2. Illustration in a semi-infinite, one-dimensional domain

Consider a one-dimensional flow in which the velocity and diffusivity are the positive constants U and K. The domain of interest is semi-infinite, i.e. $x \in [0,\infty[$, where x is the relevant space coordinate. The subdomains in which the clock is active or inactive are defined as follows:

$$\Omega_a: \ 0 \le x < \Lambda \ , \quad \Omega_i: \ \Lambda \le x < \infty \ . \tag{8}$$

In other words, the clock is ticking in a segment whose length is Λ that is adjacent to the upstream boundary of the domain. In the remainder of the domain, the clock is inactive. This highly idealised problem is inspired by Liu et al. (2012).

It is convenient to introduce dimensionless variables that are based on the timescale Λ / U and length scale Λ. The related Peclet number is $Pe = U\Lambda / K$. In the present section, from here on, only dimensionless variables will be used.

The boundary condition for the tracer concentration is

$$C(t,0) = 1 \ . \tag{9}$$

Therefore, at a steady state, the tracer concentration is equal to unity in the whole domain of interest. Then, it is readily seen that the classical steady-state age $a(x)$ obeys the following equation:

$$\frac{d}{dx}\left(\frac{1}{Pe}\frac{da}{dx} - a\right) = -1 \ . \tag{10}$$

Then, prescribing the age to be zero at x=0, this age is

$$a(x) \;=\; x \;. \tag{11}$$

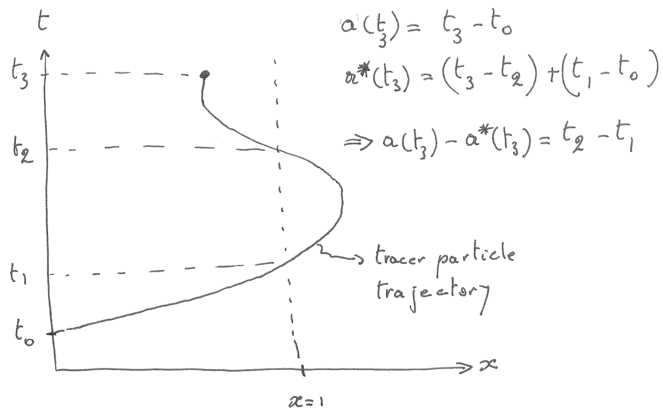


Figure 1. Illustration of the behaviour of the age, a, and exposure age, a^*, of a single tracer particle in the one-dimensional flow considered in Section 2. The domain of interest is semi-infinite ($0 \le x < \infty$) while the subdomain in which the clock is functioning is defined by the inequalities $0 \le x \le 1$. Dimensionless variables are used.

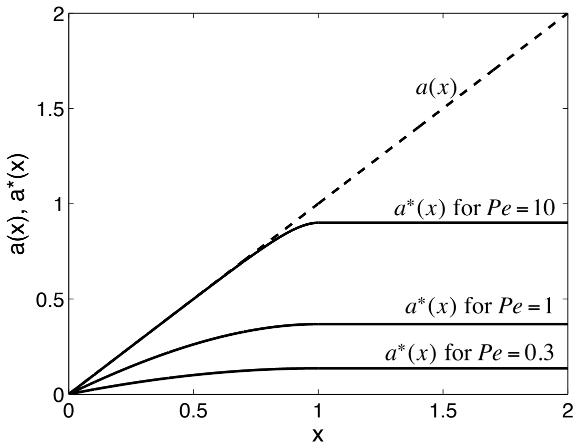


Figure 2. Illustration for various values of the Peclet number of the age $a(x)$ and the exposure age $a^*(x)$ as obtained from expressions (11) and (13), respectively. The age does not depend on the Peclet number, as opposed to the exposure age. The domain of interest is semi-infinite ($0 \le x < \infty$) while the subdomain in which the clock is functioning is defined by the inequalities $0 \le x \le 1$. Dimensionless variables are used.

Let $a^*(x)$ denote the steady-state exposure age. The latter obeys the equation

$$\frac{d}{dx}\left(\frac{1}{Pe}\frac{da^*}{dx} - a^*\right) = \begin{cases} -1\ , & 0 \le x < 1 \\ 0\ , & 1 < x < \infty \end{cases} \tag{12}$$

If the exposure age is also prescribed to be zero at x=0, the solution to (12) reads

$$a^*(x) = \begin{cases} x + \dfrac{1 - e^{Pe\,x}}{Pe\,e^{Pe}}\ , & 0 \le x < 1 \\[2ex] 1 + \dfrac{1 - e^{Pe}}{Pe\,e^{Pe}}\ , & 1 < x < \infty \end{cases} \tag{13}$$

The behaviour of the age and exposure age of a single tracer particle is illustrated in Figure 1, while expressions (11) and (13) are illustrated in Figure 2 for various values of the Peclet number. In addition, at x=1, the asymptotic behaviour of the exposure age is a follows:

$$a^*(1) \sim \frac{Pe}{2} - \frac{Pe^2}{6}\ , \quad Pe \to 0 \tag{14}$$

and

$$a^*(1) \sim 1 - \frac{1}{Pe}\ , \quad Pe \to \infty\ . \tag{15}$$

3. Illustration in an infinite, one-dimensional domain

The exposure age may also be estimated in an infinite domain. If the subdomain in which the clock is active is the same as in the previous example, one must solve the differential equation

$$\frac{d}{dx}\left(\frac{1}{Pe}\frac{da^*}{dx} - a^*\right) = \begin{cases} 0\ , & -\infty < x < 0 \\ -1\ , & 0 \le x < 1 \\ 0\ , & 1 < x < \infty \end{cases} \tag{16}$$

under the following boundary and matching conditions

$$\begin{cases} a^*(-\infty) = 0,\ \ a^*(0^-) = a^*(0^+),\ \left[\dfrac{da^*}{dx}\right]_{x=0^-} = \left[\dfrac{da^*}{dx}\right]_{x=0^+} \\[2ex] a^*(1^-) = a^*(1^+),\ \left[\dfrac{da^*}{dx}\right]_{x=1^-} = \left[\dfrac{da^*}{dx}\right]_{x=1^+},\ \ a^*(+\infty) < \infty \end{cases} \tag{17}$$

The solution of this problem is

$$a^*(x) = \begin{cases} \dfrac{e^{Pe} - 1}{Pe\,e^{Pe}} e^{Pe\,x}\ , & -\infty < x < 0 \\[2ex] \dfrac{1}{Pe} + x - \dfrac{e^{Pe\,x}}{Pe\,e^{Pe}}\ , & 0 \le x < 1 \\[2ex] 1\ , \ \ 1 < x < \infty \end{cases} \tag{18}$$

In addition, at x=0, the asymptotic behaviour of the exposure age is a follows:

$$a^*(0) \sim 1 - \frac{Pe}{2}\ , \quad Pe \to 0 \tag{19}$$

and

$$a^*(1) \sim \frac{1}{Pe}\ , \quad Pe \to \infty\ . \tag{20}$$

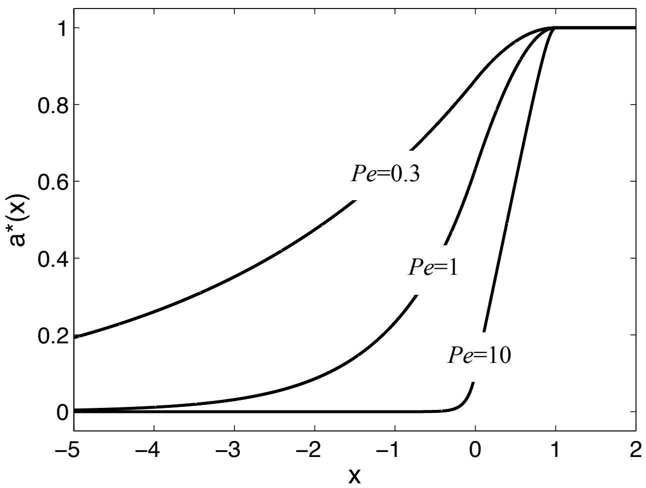


Figure 3. Illustration for various values of the Peclet number of the exposure age $a^*(x)$ as obtained from formula (18). The domain of interest is infinite ($-\infty < x < \infty$) while the subdomain in which the clock is functioning is defined by the inequalities $0 \le x \le 1$. Dimensionless variables are used.

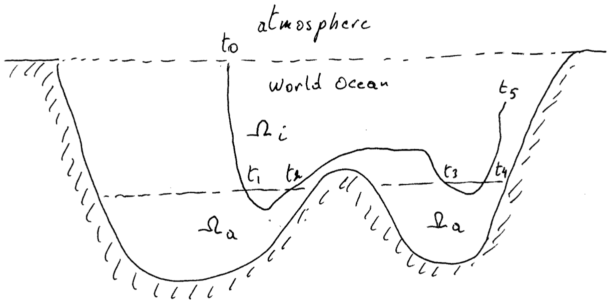
Solution (18) is illustrated in Figure 3. For $1 < x$, the exposure age is equal to unity irrespective of the value of the Peclet number. The physical explanation thereof has yet to be found.

4. Idealised models of the World Ocean ventilation

The age has been widely used for the last decades to diagnose ventilation processes in the World Ocean (e.g. Thiele and Sarmiento 1990, England 1995, Haine and Hall 2002). In this context, the age is defined as the time elapsed since leaving the ocean surface, implying that the age must be prescribed to be zero at the ocean-atmosphere interface. The exposure age might be used to focus on the deepest part of the ocean. Accordingly, the subdomain Ω_a might be defined as the fraction of the ocean lying below a given depth. Any point located above this depth would then belong to Ω_i (Figure 4).

This idea may be first tested in several highly idealised, one-dimensional models of oceanic ventilation, namely a purely diffusive ocean, the Munk loop (Munk 1966) and the leaky funnel (Mouchet and Deleersnijder 2008, Mouchet et al. 2012) (Figure 5). Those three types of models are illustrated in Figure 4 and the associated steady-state ages are listed in Table 1. It is worth stressing that the exposure age has to be continuously differentiable at the boundary between Ω_a and Ω_i.

Analytical solutions were arrived at, except for the Munk loop (Table 1, Figures 6-8). For the latter the related calculations presumably are too intricate.



$$a(t_5) = t_5 - t_0$$

$$a^*(t_5) = (t_2 - t_1) + (t_4 - t_3)$$

$$a(t_5) - a^*(t_5) = (t_1 - t_0) + (t_3 - t_2) + (t_5 - t_4)$$

Figure 4. Illustration of the behaviour of the age, a, and exposure age, a^*, of a single tracer particle in the World Ocean, assuming that the subdomain in which the clock is active (Ω_a) is the fraction of the ocean lying under a given depth. In other words, the boundary separating Ω_a from Ω_i is a horizontal surface.

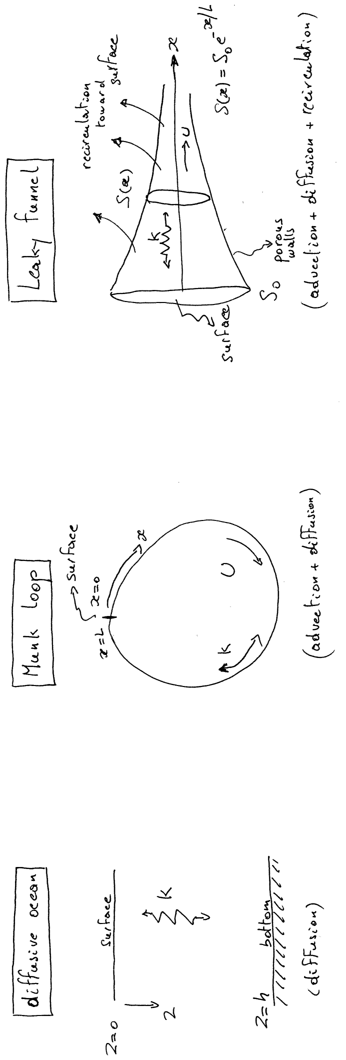


Figure 5. Illustration of the geometry and transport features of three highly-idealised, one-dimensional models of the World Ocean ventilation, namely the purely diffusive ocean, the Munk loop (Munk 1966) and the leaky funnel (Mouchet and Deleersnijder 2008, Mouchet et al. 2012).

Table 1. Analytical expressions of the age and exposure age at a steady state in three highly-idealised, one-dimensional models of the ventilation of the World Ocean (purely diffusive ocean, Munk loop and leaky funnel). The Peclet number is defined as $Pe = UL/K$, where the positive constants U, L and K are the relevant velocity, length and diffusivity scales.

		diffusive ocean	Munk loop	leaky funnel
	Ω	$0 \le z \le h$	$0 \le x \le L$	$0 \le x < \infty$, $S(x) = S_0\, e^{-x/L}$
	Ω_a	$(1-\mu)h \le z \le h$	$(1-\mu)/(2L) \le x \le (1+\mu)/(2L)$	$-L \log\mu \le x < \infty$, $S(x) = S_0\, e^{-x/L}$
	dimensionless variables	$\tilde{z} = \frac{z}{h}$, $(\tilde{a}, \tilde{a}^*) = \frac{(a, a^*)}{h^2/K}$	$\tilde{x} = \frac{x}{L}$, $(\tilde{a}, \tilde{a}^*) = \frac{(a, a^*)}{L/U}$	$\tilde{x} = \frac{x}{L}$, $(\tilde{a}, \tilde{a}^*) = \frac{(a, a^*)}{L/U}$
dimensionless variables (after dropping the tildes)	age problem	$\frac{d^2 a}{dz^2} = -1$ $a(0) = 0$, $[da/dz]_{z=1} = 0$	$\frac{1}{Pe}\frac{d^2 a}{dx^2} - \frac{da}{dx} = -1$ $a(0) = 0 = a(1)$	$\frac{1}{Pe}\frac{d^2 a}{dx^2} - \frac{1+Pe}{Pe}\frac{da}{dx} = -1$ $a(0) = 0$, $\lim\limits_{x\to\infty}\frac{a(x)}{x} < \infty$
	age	$a(z) = z - \frac{z^2}{2}$	$a(x) = x - \frac{e^{Pe\,x} - 1}{e^{Pe} - 1}$	$a(x) = \frac{Pe\,x}{1+Pe}$
	exposure age problem	$\frac{d^2 a^*}{dz^2} = \begin{cases} 0 , & 0 \le z < 1-\mu \\ -1 , & 1-\mu < z \le 1 \end{cases}$ $a^*(0) = 0$, $[da^*/dz]_{z=1} = 0$	$\frac{1}{Pe}\frac{d^2 a^*}{dx^2} - \frac{da^*}{dx} = \begin{cases} 0 , & 0 \le x < (1-\mu)/2 \\ -1 , & (1-\mu)/2 < x < (1+\mu)/2 \\ 0 , & (1+\mu)/2 < x \le 1 \end{cases}$ $a^*(0) = 0 = a^*(1)$	$\frac{1}{Pe}\frac{d^2 a^*}{dx^2} - \frac{1+Pe}{Pe}\frac{da^*}{dx} = \begin{cases} 0 , & 0 < x < -\log\mu \\ -1 , & -\log\mu < x < \infty \end{cases}$ $a^*(0) = 0$, $\lim\limits_{x\to\infty}\frac{a^*(x)}{x} < \infty$
	exposure age	$a^*(z) = \begin{cases} \mu z , & 0 \le z < 1-\mu \\ z - \frac{z^2 + (1-\mu)^2}{2} , & 1-\mu < z \le 1 \end{cases}$	**?**	$a^*(x) = \begin{cases} \frac{Pe\,\mu^{1+Pe}[e^{(1+Pe)x} - 1]}{(1+Pe)^2} , & 0 \le x < -\log\mu \\ \frac{Pe(x + \log\mu)}{1+Pe} + \frac{Pe(1-\mu^{1+Pe})}{(1+Pe)^2} , & -\log\mu < x < \infty \end{cases}$
	mean ages	$\left(\overline{a}, \overline{a^*}\right) = \int_0^1 [a(z), a^*(z)]\, dz$ $\overline{a} = \frac{1}{3}$, $\overline{a^*} = \frac{\mu(3-\mu^2)}{6}$	$\left(\overline{a}, \overline{a^*}\right) = \int_0^1 [a(x), a^*(x)]\, dx$ $\overline{a} = \frac{1}{2} + \frac{1}{e^{Pe} - 1} - \frac{1}{Pe}$, $\overline{a^*} = ?$	$\left(\overline{a}, \overline{a^*}\right) = \int_0^\infty [a(x), a^*(x)] e^{-x}\, dx$ $\overline{a} = \frac{Pe}{1+Pe}$, $\overline{a^*} = \mu - \frac{\mu^{1+Pe}}{1+Pe}$
	mean age ratio	$\theta = \frac{\overline{a^*}}{\overline{a}} = \frac{\mu(3-\mu^2)}{2}$	**?**	$\theta = \frac{\overline{a^*}}{\overline{a}} = \frac{(1+Pe)\mu}{Pe} - \frac{\mu^{1+Pe}}{Pe}$

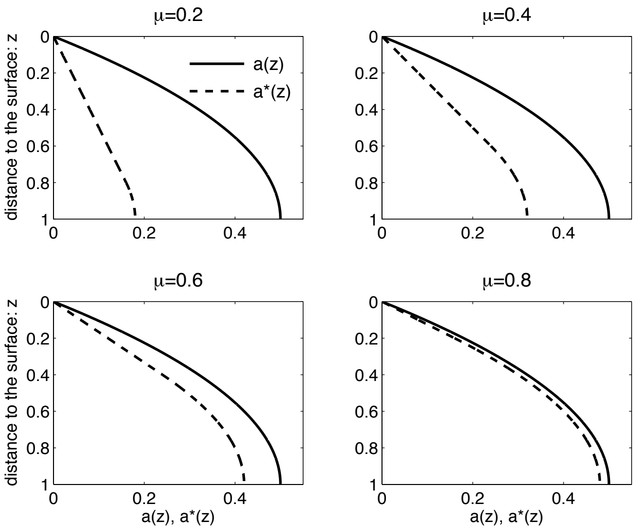


Figure 6. Profiles of the steady-state age $a(z)$ and exposure age $a^*(z)$ in the purely diffusive, one-dimensional ocean model for various values of the parameter μ. Dimensionless variables are used.

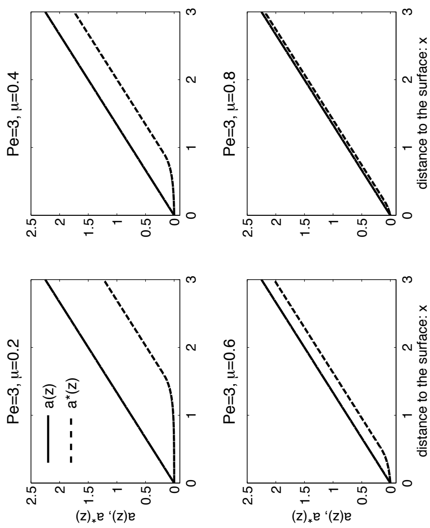
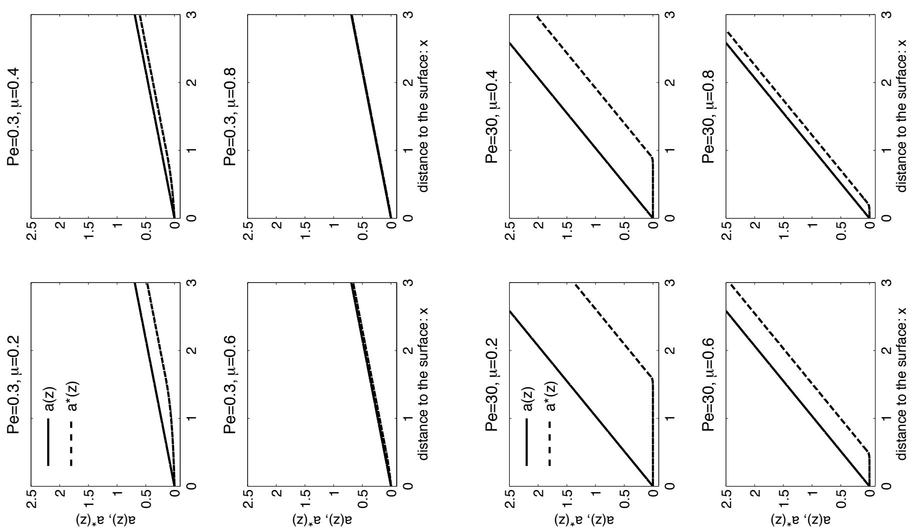


Figure 7. Distribution of the steady-state age $a(x)$ and exposure age $a^*(x)$ in the leaky funnel for various values of the parameter μ and the Peclet number $Pe = UL/K$. Dimensionless variables are used.



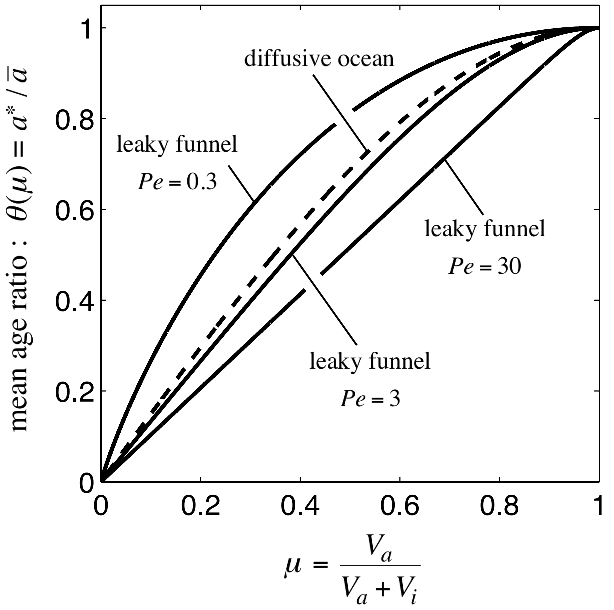


Figure 8. Ratio of the domain-averaged exposure age to the domain-averaged age in the diffusive ocean and the leaky funnel as a function of the parameter μ.

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Toward a generalisation of the concept of exposure age

Eric Deleersnijder, December 11, 2012

Anne Mouchet's conundrum and its solution

On December 11, 2012, Dr Anne Mouchet sent me an e-mail containing the following sentences:

> *Si on divise l'océan en M régions bien distinctes physiquement et telles que tout l'océan est représenté par ces M régions, alors si on définit un âge d'exposition dans chacune de ces M régions on a age_eau(x)=Somme(ages expo de chaque région, x), x étant une position quelconque dans océan.*
> *Oui?*

To see whether or not Anne's statement is true, one first divides the World Ocean Ω into M sub-domains Ω_m , m=1,2...M, with $\Omega = \Omega_1 \cup \Omega_2 \cup ....\Omega_M$ and $\Omega_m \cap \Omega_n = \varnothing$ if $m \neq n$. Then, it is convenient to define the following functions of the position vector $\mathbf{x}$:

$$\omega_m(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x} \in \Omega_m \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

These functions satisfy

$$\sum_{m=1}^{M} \omega_m(\mathbf{x}) \ = \ 1 \ . \tag{2}$$

Next, one introduces the tracer concentrations $C_m^*(t,\mathbf{x})$ and their age concentrations $\alpha_m^*(t,\mathbf{x})$, which are the solutions of the differential problems

$$\begin{cases} \dfrac{\partial C_m^*}{\partial t} \ = \ - \ \nabla \bullet (\mathbf{v} C_m^* - \mathbf{K} \bullet \nabla C_m^*) \\ C_m^*(0,\mathbf{x}) = C_{m,0}^*(\mathbf{x}) \ , \ \ [C_m^*(t,\mathbf{x})]_{\mathbf{x} \in \Gamma^s} = 1 \ , \ \ \left[(\mathbf{K} \bullet \nabla C_m^*) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^i} = 0 \end{cases} \tag{3}$$

and

$$\begin{cases} \dfrac{\partial \alpha_m^*}{\partial t} \ = \ \omega_m C_m^* \ - \ \nabla \bullet (\mathbf{v} \alpha_m^* - \mathbf{K} \bullet \nabla \alpha_m^*) \\ \alpha_m^*(0,\mathbf{x}) = \alpha_{m,0}^*(\mathbf{x}) \ , \ \ [\alpha_m^*(t,\mathbf{x})]_{\mathbf{x} \in \Gamma^s} = 0 \ , \ \ \left[(\mathbf{K} \bullet \nabla \alpha_m^*) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^i} = 0 \end{cases} \tag{4}$$

where Γ^s is the ocean surface, while Γ^i is the rest of the boundary of Ω, which is impermeable; $\mathbf{n}$ is the outward unit normal vector to the domain boundary; $\mathbf{v}(\mathbf{x})$ is the velocity, which is time-independent and divergenceless ($\nabla \bullet \mathbf{v} = 0$); $\mathbf{K}(\mathbf{x})$ is the diffusivity tensor, which is symmetric and positive-definite. Finally, the exposure ages of the tracers defined above are

$$a_m^*(t,\mathbf{x}) \ = \ \frac{\alpha_m^*(t,\mathbf{x})}{C_m^*(t,\mathbf{x})} \ . \tag{5}$$

In the limit $t \to \infty$, the tracer concentrations tend to unity. Therefore, the steady-state exposure ages $a_m^*(\infty,\mathbf{x})$ are the solutions of

$$\begin{cases} \nabla \bullet (\mathbf{v} a_m^* - \mathbf{K} \bullet \nabla a_m^*) \;=\; \omega_m \\ [a_m^*(\infty,\mathbf{x})]_{\mathbf{x}\in\Gamma^s} = 0 \;,\; \left[(\mathbf{K} \bullet \nabla a_m^*) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \end{cases} \tag{6}$$

Using property (2), it may be seen that the sum of these ages,

$$a(\mathbf{x}) \;=\; \sum_{m=1}^{M} a_m^*(\infty,\mathbf{x}) \tag{7}$$

satisfies a partial differential problem that is equivalent to that from which the steady-state water age is obtained, i.e.

$$\begin{cases} \nabla \bullet (\mathbf{K} \bullet \nabla a - \mathbf{v} a) \;=\; -1 \\ [a(\mathbf{x})]_{\mathbf{x}\in\Gamma^s} = 0 \;,\; \left[(\mathbf{K} \bullet \nabla a) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \end{cases} \tag{8}$$

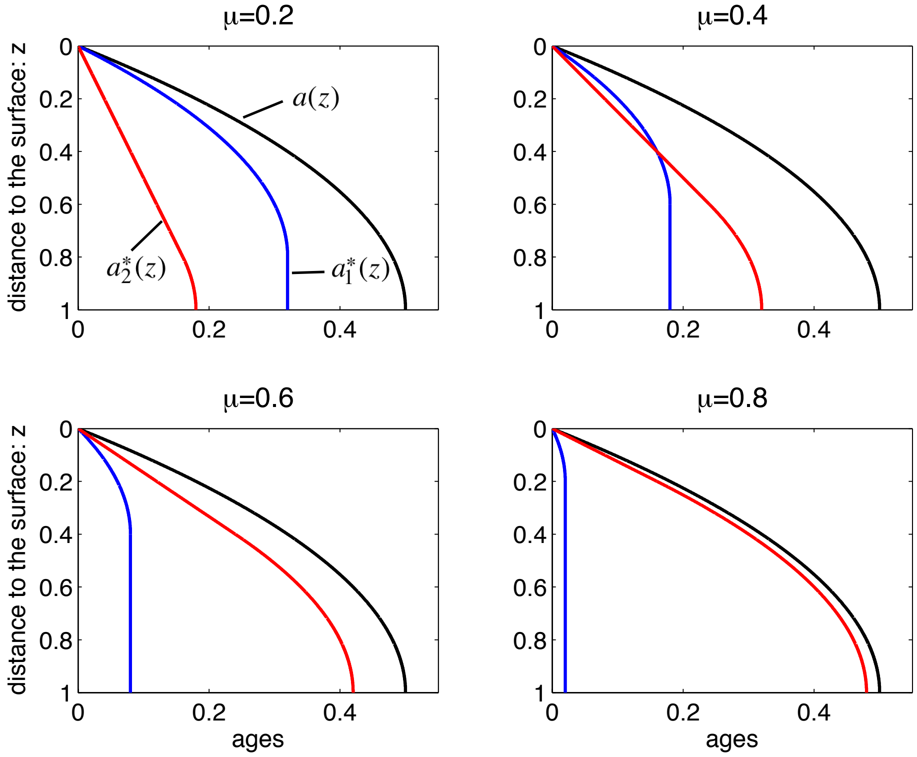
Unfortunately, I could not find a transient-state counterpart to this remarkable property. This does not imply, however, that it does not exist. One should not stop racking the brains...

Idealised illustrations

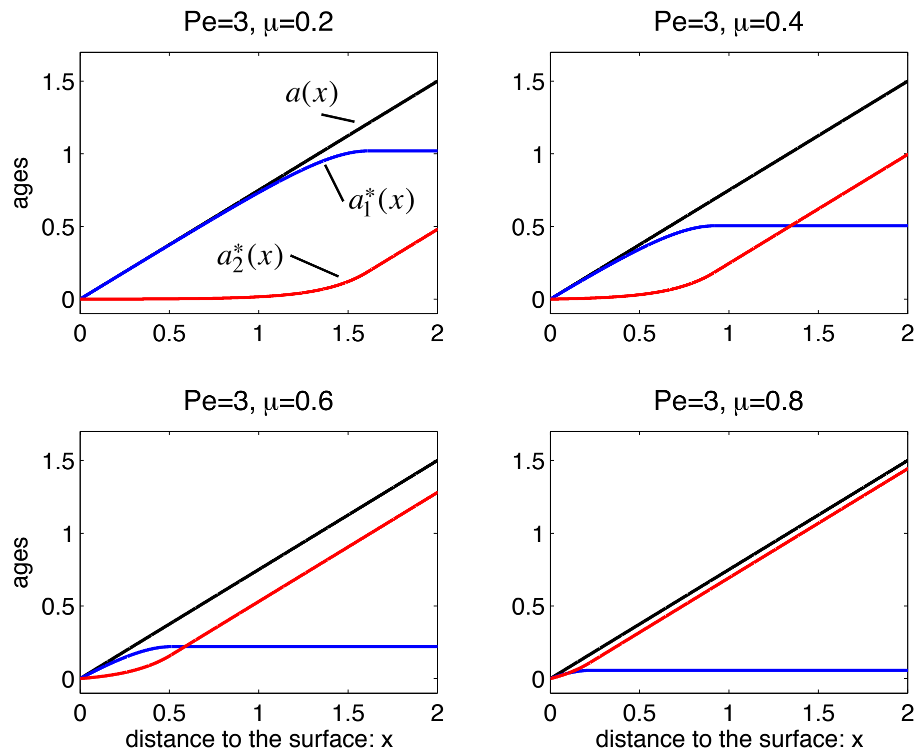
To illustrate the discussion above, two idealised models of the World Ocean ventilation dealt with in the previous working note are revisited. The domain of interest is divided into two subdomains. The subscript "2" refers to the lower part of the ocean, while "1" is associated with the subdomain adjacent to the ocean surface. The analytical solutions (in dimensionless variables) are listed in the table below and plotted in the figures below.

		diffusive ocean	leaky funnel
dimensionless variables	age problem	$\frac{d^2a}{dz^2} = -1$ $a(0)=0 \;,\; [da/dz]_{z=1} = 0$	$\frac{1}{Pe}\frac{d^2a}{dx^2} - \frac{1+Pe}{Pe}\frac{da}{dx} = -1$ $a(0)=0 \;,\; \lim_{x\to\infty} \frac{a(x)}{x} < \infty$
	age	$a(z) = z - \frac{z^2}{2}$	$a(x) = \frac{Pe\,x}{1+Pe}$
	exposure age problems	$\frac{d^2a_1^*}{dz^2} = \begin{cases} -1 \;,\; 0 \le z < 1-\mu \\ 0 \;,\; 1-\mu < z \le 1 \end{cases}$ $\frac{d^2a_2^*}{dz^2} = \begin{cases} 0 \;,\; 0 \le z < 1-\mu \\ -1 \;,\; 1-\mu < z \le 1 \end{cases}$ $a_m^*(0)=0 \;,\; [da_m^*/dz]_{z=1} = 0 \quad (m=1,2)$	$\frac{1}{Pe}\frac{d^2a_1^*}{dx^2} - \frac{1+Pe}{Pe}\frac{da_1^*}{dx} = \begin{cases} -1 \;,\; 0 < x < -\log\mu \\ 0 \;,\; -\log\mu < x < \infty \end{cases}$ $\frac{1}{Pe}\frac{d^2a_2^*}{dx^2} - \frac{1+Pe}{Pe}\frac{da_2^*}{dx} = \begin{cases} 0 \;,\; 0 < x < -\log\mu \\ -1 \;,\; -\log\mu < x < \infty \end{cases}$ $a_m^*(0)=0 \;,\; \lim_{x\to\infty} \frac{a_m^*(x)}{x} < \infty \quad (m=1,2)$
	exposure age	$a_1^*(z) = \begin{cases} (1-\mu)z - \frac{z^2}{2} \;,\; 0 \le z < 1-\mu \\ \frac{(1-\mu)^2}{2} \;,\; 1-\mu < z \le 1 \end{cases}$ $a_2^*(z) = \begin{cases} \mu z \;,\; 0 \le z < 1-\mu \\ z - \frac{z^2+(1-\mu)^2}{2} \;,\; 1-\mu < z \le 1 \end{cases}$	$a_1^*(x) = \begin{cases} \frac{Pe\,x}{1+Pe} - \frac{Pe\,\mu^{1+Pe}[e^{(1+Pe)x}-1]}{(1+Pe)^2} \;,\; 0 \le x < -\log\mu \\ -\frac{Pe\log\mu}{1+Pe} - \frac{Pe(1-\mu^{1+Pe})}{(1+Pe)^2} \;,\; -\log\mu < x < \infty \end{cases}$ $a_2^*(x) = \begin{cases} \frac{Pe\,\mu^{1+Pe}[e^{(1+Pe)x}-1]}{(1+Pe)^2} \;,\; 0 \le x < -\log\mu \\ \frac{Pe(x+\log\mu)}{1+Pe} + \frac{Pe(1-\mu^{1+Pe})}{(1+Pe)^2} \;,\; -\log\mu < x < \infty \end{cases}$

Diffusive ocean



Leaky funnel



Remarque sur l'addendum d'Eric Delhez du 12 décembre 2012

Eric Deleersnijder, le 12 décembre 2012

Ce que propose Eric Delhez dans la brève note de travail reproduite ci-après tranche résolument avec les habitudes prises quant à l'utilisation de l'*age-averaging hypothesis* de CART et avec les interprétations lagrangiennes réalisées dans ce cadre. S'il faut s'en rendre compte, il n'est pas question d'affirmer que la proposition d'Eric Delhez est malvenue. Au contraire, elle pourrait ouvrir des perspectives qui n'ont pas encore été explorées dans le cadre de CART.

L'*age-averaging hypothesis* a été formulée explicitement par Deleersnijder et al. (2001), mais elle était déjà utilisée implicitement dans les publications précédentes, notamment celle de Delhez et al. (1999). Cette hypothèse (purement arbitraire) indique comment calculer l'âge moyen d'un ensemble de particules: il s'agit d'effectuer une moyenne pondérée par les masses des différents âges. Cette méthode s'applique à des particules de même nature ou à des particules de natures différentes, ce qui permet alors de calculer l'âge d'un aggrégat.

On a toujours supposé qu'une particule n'avait qu'une seule horloge. En introduisant M concentrations d'âge mais un seul traceur, Eric Delhez suggère que chaque particule possède M horloges — et non plus une seule horloge. A tout moment et en tout lieu, une seule de ces horloges est active, les autres étant à l'arrêt. Ainsi l'horloge portant le numéro m est active dans le sous-domaine Ω_m et inactive dans tous les autres. Toutes les horloges d'une particule sont remises à zéro à l'instant où la particule qui les porte touche la surface de l'océan.

L'âge d'une particule suggéré par Eric Delhez n'est pas la moyenne des temps indiqués par les différentes horloges mais la somme de ceux-ci.

Deleersnijder E., J.-M. Campin and E.J.M. Delhez, 2001, The concept of age in marine modelling: I. Theory and preliminary model results, *Journal of Marine Systems*, 28, 229-267

Delhez E.J.M., J.-M. Campin, A.C. Hirst and E. Deleersnijder, 1999, Toward a general theory of the age in ocean modelling, *Ocean Modelling*, 1, 17-27

Addendum d'Eric Delhez du 12 décembre 2012

Notons d'abord que la propriété est évidente si on s'attache à la signification physique des variables.

Dans le contexte envisagé, l'introduction d'une équation pour la concentration $C_m^\star$ permet de suivre les particules qui ont touché la surface et pour lesquelles l'âge ou l'âge d'exposition peut être défini. Les conditions initiales de l'équation sont donc identiquement nulles dans tout le domaine. Dès lors, $C_m^\star$ est indépendant de m. Un traceur unique, celui qui est normalement utilisé pour calculer l'âge normal, doit donc être utilisé. Notons C la concentration de ce traceur qui vérifie

$$\begin{cases} \dfrac{\partial C}{\partial t} = -\nabla \cdot (\mathbf{v}C - \mathbf{K} \cdot \nabla C) \\ C(0,\mathbf{x}) = 0, \quad [C(t,\mathbf{x})]_{x\in\Gamma^s} = 1, \quad [(\mathbf{K} \cdot \nabla C) \cdot \mathbf{n}]_{x\in\Gamma^i} = 0 \end{cases}$$

À ce traceur unique, on attache M concentrations d'âge d'exposition $\alpha_m \quad (m = 1,2,\ldots,M)$ telles que, en reprenant les notations d'Eric,

$$\begin{cases} \dfrac{\partial \alpha_m^\star}{\partial t} = \omega_m C - \nabla \cdot (\mathbf{v}\alpha_m^\star - \mathbf{K} \cdot \nabla \alpha_m^\star) \\ \alpha_m^\star(0,\mathbf{x}) = 0, \quad [\alpha_m^\star(t,\mathbf{x})]_{x\in\Gamma^s} = 0, \quad [(\mathbf{K} \cdot \nabla \alpha_m^\star) \cdot \mathbf{n}]_{x\in\Gamma^i} = 0 \end{cases}$$

Remarquons que l'on ne peut rien dire à l'instant initial et que les concentrations d'âge d'exposition sont toutes nulles.

Si on somme les équations pour les concentrations d'âge d'exposition en tenant compte de

$$\sum_{m=1}^{M} \omega_m C = C \sum_{m=1}^{M} \omega_m = C$$

il vient

$$\begin{cases} \dfrac{\partial \alpha^\star}{\partial t} = C - \nabla \cdot (\mathbf{v}\alpha^\star - \mathbf{K} \cdot \nabla \alpha^\star) \\ \alpha^\star(0,\mathbf{x}) = 0, \quad [\alpha^\star(t,\mathbf{x})]_{x\in\Gamma^s} = 0, \quad [(\mathbf{K} \cdot \nabla \alpha^\star) \cdot \mathbf{n}]_{x\in\Gamma^i} = 0 \end{cases}$$

où

$$\alpha^\star = \sum_{m=1}^{M} \alpha_m^\star$$

Cette concentration d'âge d'exposition totale $\alpha^\star$ est bien identique à la concentration d'âge habituelle de sorte que

$$a = \frac{\alpha^\star}{C}$$

où a est l'âge de l'eau de surface habituel.

La propriété est donc démontrée dans le cas instationnaire.

Exposure ages and the age symmetry

Eric Deleersnijder, December 15, 2012

standard tracer and age

$$\frac{d^2C}{dx^2} - \frac{dC}{dx} = \delta(x-0) \qquad \Rightarrow \qquad C(x) = \begin{cases} e^{-|x|} \ , & x<0 \\ 1 \ , & 0<x \end{cases}$$

$$\frac{d^2\alpha}{dx^2} - \frac{d\alpha}{dx} = C \qquad \Rightarrow \qquad \alpha(x) = \begin{cases} (2+|x|)e^{-|x|} \ , & x<0 \\ 2+x \ , & 0<x \end{cases}$$

$$a(x) = 2+|x|$$

first exposure age

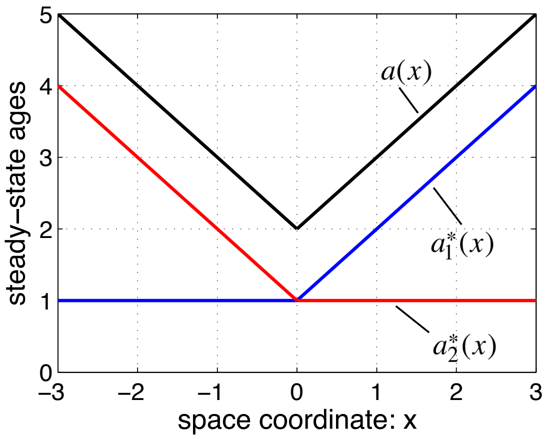
$$\frac{d^2\alpha_1^*}{dx^2} - \frac{d\alpha_1^*}{dx} = \begin{cases} 0 \ , & x<0 \\ -1 \ , & 0<x \end{cases} \qquad \Rightarrow \qquad \alpha_1^*(x) = \begin{cases} e^{-|x|} \ , & x<0 \\ 1+x \ , & 0<x \end{cases}$$

$$a_1^*(x) = \begin{cases} 1 \ , & x<0 \\ 1+x \ , & 0<x \end{cases}$$

second exposure age

$$\frac{d^2\alpha_2^*}{dx^2} - \frac{d\alpha_2^*}{dx} = \begin{cases} -e^{-|x|} \ , & x<0 \\ 0 \ , & 0<x \end{cases} \qquad \Rightarrow \qquad \alpha_2^*(x) = \begin{cases} (1+|x|)e^{-|x|} \ , & x<0 \\ 1 \ , & 0<x \end{cases}$$

$$a_2^*(x) = \begin{cases} 1+|x| \ , & x<0 \\ 1 \ , & 0<x \end{cases}$$

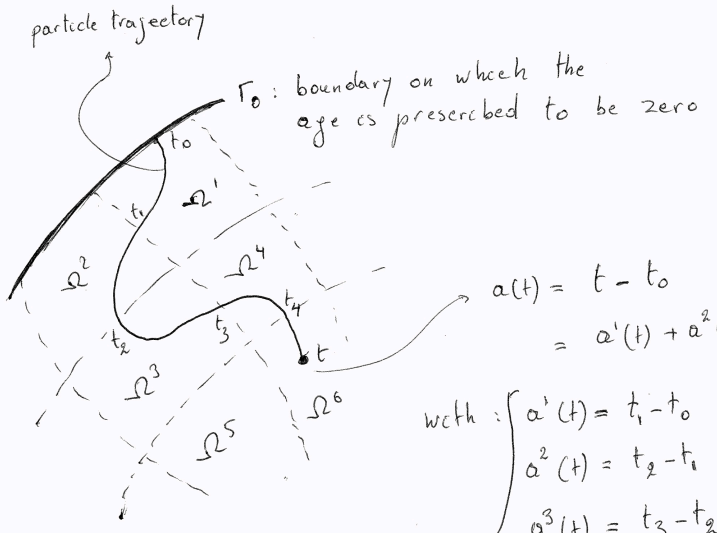


The equations leading to the exposure age

Eric Deleersnijder & Eric J.M. Delhez, September 4, 2013

So far, in all of the studies involving CART's ages[1], every constituent particle had one clock tied to it that kept ticking at any location of the domain of interest and was reset to zero upon hitting the boundary where the age was prescribed to be zero. Accordingly, the age was a measure of the time elapsed since leaving the aforementioned boundary. Over the past year, Anne Mouchet, Eric J.M. Delhez and Eric Deleersnijder devised another age concept, which was tentatively termed the "exposure age". Several working notes, authored by Eric J.M. Delhez or Eric Deleersnijder, were devoted to this topic. The definitions and equations ensuing from this theoretical research work are summarised hereinafter.

In the framework of the new approach, the domain of interest Ω is split into several subdomains, Ω^m (m=1,2,...M), with $\Omega = \Omega^1 \cup \Omega^2 \cup ... \Omega^M$. Instead of being equipped with a single clock, every particle has M clocks, i.e. a number of clocks that is equal to the number of subdomains. The m-th clock is ticking in the subdomain Ω^m and is at a standstill elsewhere. In other words, of all the clocks of a given particle, there is only one that is active at any time and location. All of the clocks of the particle under consideration are reset to zero at the moment it hits the boundary on which the age is prescribed to be zero, which is denoted Γ_0 hereinafter. This is illustrated in the figure below.



$$a(t) = t - t_0$$
$$= a^1(t) + a^2(t) + a^3(t) + a^4(t) + a^5(t) + a^6(t)$$

with :
$$\begin{cases} a^1(t) = t_1 - t_0 \\ a^2(t) = t_2 - t_1 \\ a^3(t) = t_3 - t_2 \\ a^4(t) = t_4 - t_3 \\ a^5(t) = 0 \\ a^6(t) = t - t_4 \end{cases}$$

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[1] CART: Constituent-oriented Age and Residence time Theory (www.climate.be/cart)

According to the rules set out above, the time indicated by the m-th clock is the time spent by the particle in the m-th subdomain — or the time this particle has been exposed to this subdomain — since it has hit for the last time the boundary Γ_0. Then, the sum of the times indicated by the clocks is the classical CART age, i.e. the time elapsed since leaving Γ_0.

In Lagrangian simulations, the approach proposed herein is rather easy to implement. If, on the other hand, the Eulerian description is adopted, the classical CART equations (e.g. Delhez et al. 1999, Deleersnijder et al. 2001) must be modified. This is seen below.

So far we have only considered a passive — or inert — tracer, which may be the seawater itself. If t and $\mathbf{x}$ denote the time and the position vector, the concentration $C(t,\mathbf{x})$ of such a tracer obeys the following partial differential equation

$$\frac{\partial C}{\partial t} = -\nabla\bullet(\mathbf{v}C - \mathbf{K}\bullet\nabla C) \;, \tag{1}$$

where $\mathbf{v}$ is the velocity, which is divergence-free ($\nabla\bullet\mathbf{v}=0$), and $\mathbf{K}$ is the diffusivity tensor, which must be symmetric and positive-definite (e.g. Deleersnijder et al. 2001). The key difference with the classical CART approach is that several age concentration equations are now needed, i.e. one equation per clock. For the m-th clock, the age concentration $\alpha^m(t,\mathbf{x})$ obeys

$$\frac{\partial \alpha^m}{\partial t} = \omega^m C - \nabla\bullet(\mathbf{v}\alpha^m - \mathbf{K}\bullet\nabla\alpha^m) \;, \quad m = 1,2...M \;, \tag{2}$$

where $\omega^m(\mathbf{x})$ is equal to unity in the m-th subdomain and is zero elsewhere, i.e.

$$\omega^m(\mathbf{x}) = \begin{cases} 1 \;, & \text{if } \mathbf{x}\in\Omega^m \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

All the clocks are reset to zero on Γ_0, leading to the boundary condition

$$\left[\alpha^1(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma_0} = \left[\alpha^2(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma_0} = \ldots \left[\alpha^M(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma_0} = 0 \;. \tag{4}$$

Finally, the m-th age, i.e. the time indicated by the m-th clock, is

$$a^m(t,\mathbf{x}) = \frac{\alpha^m(t,\mathbf{x})}{C(t,\mathbf{x})} \;. \tag{5}$$

It must be underscored that the m-th exposure age is defined in all of the subdomains — though the m-th clock is only ticking in Ω^m.

It is readily seen that the function defined by (3) satisfies at any location in the domain of interest the following property

$$\sum_{m=1}^{M} \omega^m(\mathbf{x}) = 1 \;. \tag{6}$$

Next, introduce the following variable

$$\alpha(t,\mathbf{x}) = \sum_{m=1}^{M} \alpha^m(t,\mathbf{x}) \;. \tag{7}$$

Combining (2)-(7) yields

$$\frac{\partial \alpha}{\partial t} = C - \nabla\bullet(\mathbf{v}\alpha - \mathbf{K}\bullet\nabla\alpha) \;, \tag{8}$$

suggesting that $\alpha(t,\mathbf{x})$ actually is the classical CART age concentration so that the usual age is

$$a(t,\mathbf{x}) = \frac{\alpha(t,\mathbf{x})}{C(t,\mathbf{x})} = \frac{\sum_{m=1}^{M} \alpha^m(t,\mathbf{x})}{C(t,\mathbf{x})} = \sum_{m=1}^{M} \frac{\alpha^m(t,\mathbf{x})}{C(t,\mathbf{x})} = \sum_{m=1}^{M} a^m(t,\mathbf{x}) \tag{9}$$

It is likely that the new type of age introduced above will be of use in connectivity studies and, perhaps, for several other purposes. The concept of exposure age can also be applied to non-passive constituents, but the relevant theory has yet to be developed. It is also possible that the method outline herein may be applied to the residence or exposure time; this has yet to be seen.

References

Delhez E.J.M., J.-M. Campin, A.C. Hirst and E. Deleersnijder, 1999, Toward a general theory of the age in ocean modelling, *Ocean Modelling*, 1, 17-27

Deleersnijder E., J.-M. Campin and E.J.M. Delhez, 2001, The concept of age in marine modelling: I. Theory and preliminary model results, *Journal of Marine Systems*, 28, 229-267

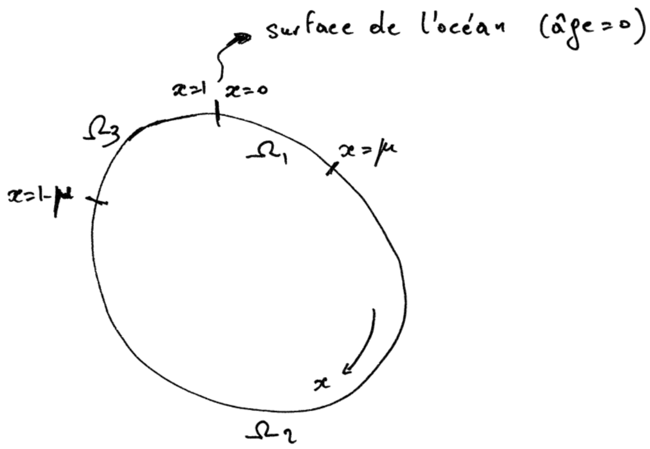
Les âges partiels dans la boucle de Munk

Eric Deleersnijder, 29 septembre 2013

On va estimer l'âge et des âges partiels dans la boucle de Munk dans un état stationnaire. On utilisera uniquement des variables rendues adimensionnelles de la manière habituelle. Le domaine d'intérêt, Ω, est défini par les inégalités $0 < x < 1$. On définit les sous-domaines suivants:

(1) $\Omega_1 : \; x \in]0, 1-\mu[\quad , \quad \Omega_2 : \; x \in]\mu, 1-\mu[\quad , \quad \Omega_1 : \; x \in]1-\mu, 1[$.

L'océan profond correspond au sous-domaine Ω_2 , tandis que les couches superficielles de l'océan sont associées à $\Omega_1 \cup \Omega_3$. Voir figure ci-dessous.



L'âge étant utilisé pour mesurer le taux de ventilation, tous les âges doivent être nuls à la surface de l'océan, c'est-à-dire en x=0 et x=1.

Dans la configuration définie plus haut, l'âge (adimensionnel) est la solution de l'équation

(2) $$\frac{1}{Pe}\frac{d^2 a}{dx^2} - \frac{da}{dx} = -1$$

Il vient alors aisément

(3) $$a(x) = x - \frac{e^{Pe\,x} - 1}{e^{Pe} - 1}$$

Les équations régissant les âges partiels et l'expression de ceux-ci sont

(4) $$\frac{1}{Pe}\frac{d^2 a_1}{dx^2} - \frac{da_1}{dx} = \begin{cases} -1 \;,\; 0 < x < \mu \\ 0 \;,\; \mu < x < 1 \end{cases}$$

(5) $$a_1(x) = \begin{cases} x + \dfrac{1 - \mu Pe - e^{Pe(1-\mu)}}{Pe(e^{Pe} - 1)}(e^{Pe\,x} - 1) \;,\; 0 < x < \mu \\ \dfrac{-1 + \mu Pe + e^{-Pe\,\mu}}{Pe(e^{Pe} - 1)}(e^{Pe} - e^{Pe\,x}) \;,\; \mu < x < 1 \end{cases}$$

(6) $$\frac{1}{Pe}\frac{d^2 a_2}{dx^2} - \frac{da_2}{dx} = \begin{cases} 0 \;,\; 0 < x < \mu \\ -1 \;,\; \mu < x < 1 - \mu \\ 0 \;,\; 1 - \mu < x < 1 \end{cases}$$

(7) $$a_2(x) = \begin{cases} \dfrac{-Pe + 2\mu Pe - e^{Pe\,\mu} + e^{Pe(1-\mu)}}{Pe(e^{Pe} - 1)}(e^{Pe\,x} - 1) \;,\; 0 < x < \mu \\ x + \dfrac{1 - Pe + \mu Pe - e^{Pe\,\mu}}{Pe(e^{Pe} - 1)}(e^{Pe\,x} - 1) + \dfrac{1 - \mu Pe - e^{-Pe\,\mu}}{Pe(e^{Pe} - 1)}(e^{Pe} - e^{Pe\,x}) \;,\; \mu < x < 1 - \mu \\ \dfrac{Pe - 2\mu Pe - e^{-Pe\,\mu} + e^{-Pe(1-\mu)}}{Pe(e^{Pe} - 1)}(e^{Pe} - e^{Pe\,x}) \;,\; 1 - \mu < x < 1 \end{cases}$$

(8) $$\frac{1}{Pe}\frac{d^2 a_3}{dx^2} - \frac{da_3}{dx} = \begin{cases} 0 \;,\; 0 < x < 1 - \mu \\ -1 \;,\; 1 - \mu < x < 1 \end{cases}$$

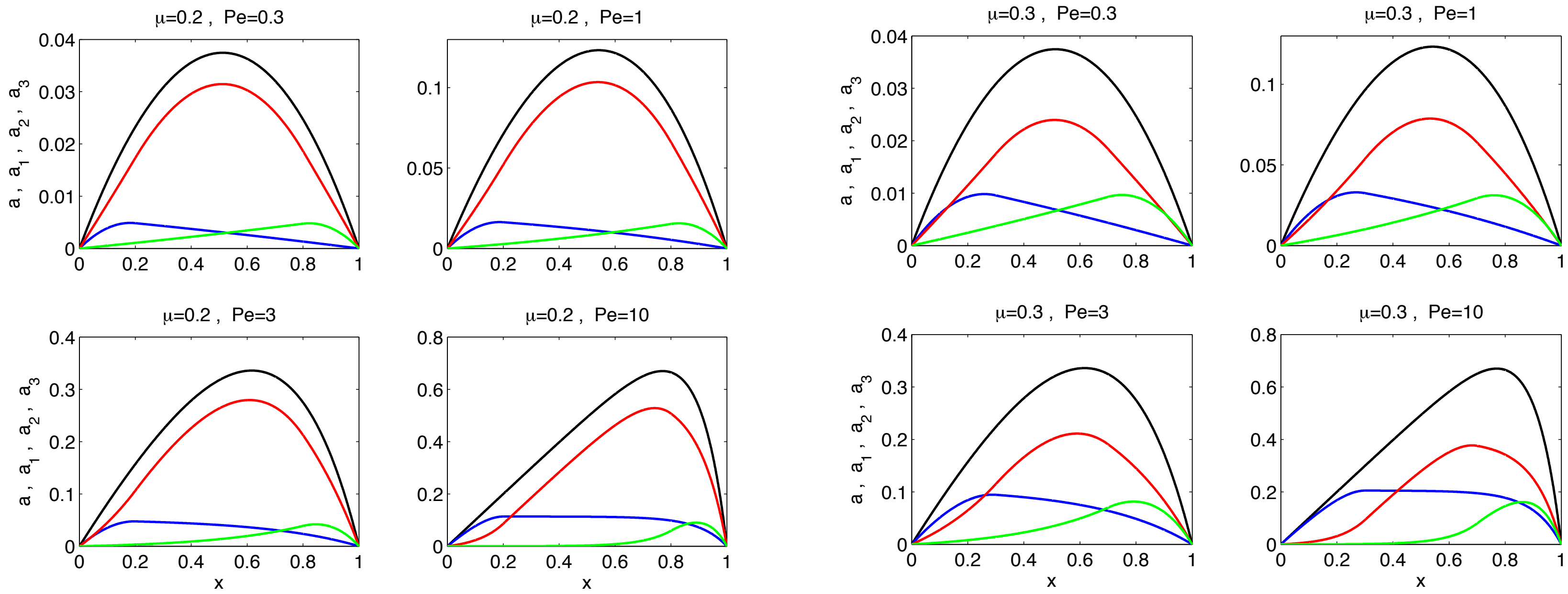
$$(9) \qquad a_3(x) = \begin{cases} \dfrac{-1-\mu\, Pe + e^{Pe\,\mu}}{Pe\,(e^{Pe}-1)}\,(e^{Pe\,x}-1)\ , \ 0 < x < 1-\mu \\[2ex] -1 + x + \dfrac{1+\mu\, Pe - e^{-Pe(1-\mu)}}{Pe\,(e^{Pe}-1)}\,(e^{Pe} - e^{Pe\,x})\ , \ 1-\mu < x < 1 \end{cases}$$

Comme il se doit, ces âges satisfont la relation suivante

$$(10) \qquad a(x) = a_1(x) + a_2(x) + a_3(x)$$

Les figures ci-dessous illustrent les solutions analytiques que l'on vient d'obtenir. Les couleurs correspondent aux âges suivants:

noir: $a(x)$, bleu: $a_1(x)$, rouge: $a_2(x)$, vert: $a_3(x)$



Détermination du maximum de l'âge partiel ou l'art d'enfoncer à grand'peine des portes ouvertes...

Eric Deleersnijder, 2-4 octobre 2013

On décompose le domaine d'intérêt Ω en M sous-domaines Ω^m (m=1,2... M), avec

$$\Omega = \Omega^1 \cup \Omega^2 \cup ... \Omega^M \qquad (1)$$

et

$$\Omega^i \cap \Omega^j = \varnothing \ , \quad i \neq j \ . \qquad (2)$$

La frontière du domaine, Γ, se compose de deux parties, Γ_i et Γ_0, avec $\Gamma = \Gamma_i \cup \Gamma_0$ et $\Gamma_i \cap \Gamma_0 = \varnothing$. La surface Γ_i est imperméable. Sur Γ_0, on impose une valeur nulle de l'âge, qui mesure donc le temps écoulé depuis qu'une particule d'eau a quitté cette frontière. On note $\mathbf{n}$ la normale extérieure unitaire à la frontière du domaine.

On considère une situation stationnaire. Comme la m-ème horloge d'une particule ne fonctionne que dans le m-ème sous-domaine et comme cette même horloge est remise à zéro dès que la particule touche un point de la surface Γ_0, la valeur maximale de l'âge associé à cette horloge est atteinte quand la particule se trouve dans le sous-domaine Ω^m ; toutefois, il est possible que la valeur maximale soit aussi atteinte, **sans être dépassée**, en dehors de ce sous-domaine.

On peut penser que ce raisonnement Lagrangien possède une contrepartie Eulérienne: le maximum absolu de $a^m(\mathbf{x})$ est égal au maximum de cette fonction dans le sous-domaine Ω^m. Cette propriété est en accord avec l'intuition physique la plus élémentaire. Il est toutefois souhaitable d'en produire une démonstration aussi rigoureuse que possible. Atteindre cet objectif est le but de cette note de travail.

La vitesse $\mathbf{v}(\mathbf{x})$ est indivergentielle,

$$\nabla \bullet \mathbf{v} = 0 \ . \qquad (3)$$

L'imperméabilité de la frontière Γ_i conduit à imposer la condition de bord

$$\left[\mathbf{v} \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma_i} = 0 \ . \qquad (4)$$

Par ailleurs, le tenseur des diffusivités $\mathbf{K}(\mathbf{x})$ est symétrique ($\mathbf{K} = \mathbf{K}^T$) et défini positif. Cette second caractéristique implique que, pour tout vecteur $\mathbf{y}$ non-nul ($\mathbf{y} \neq 0$), le tenseur des diffusivités est tel que

$$\mathbf{y} \bullet \mathbf{K} \bullet \mathbf{y} > 0 \ . \qquad (5)$$

L'âge partiel $a^m(\mathbf{x})$ satisfait l'équation différentielle aux dérivées partielles

$$\nabla \bullet (\mathbf{K} \bullet \nabla a^m - a^m \mathbf{v}) = \begin{cases} -1 \ , & \mathbf{x} \in \Omega^m \\ 0 \ , & \mathbf{x} \in \Omega \setminus \Omega^m \end{cases} \qquad (6)$$

et les conditions aux limites

$$\left[a^m(\mathbf{x})\right]_{\mathbf{x} \in \Gamma_0} = 0 \ , \qquad (7)$$

$$\left[(\mathbf{K} \bullet \nabla a^m) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma_i} = 0 \ . \qquad (8)$$

On pose

$$\hat{a}^m(\mathbf{x}) = a^m(\mathbf{x}) - a^m_{\max} , \tag{9}$$

où $a^m_{\max}$ représente la valeur maximale de l'âge $a^m(\mathbf{x})$ dans le sous-domaine Ω^m, c'est-à-dire

$$a^m_{\max} = \max_{\mathbf{x}\in\Omega^m} a^m(\mathbf{x}) \tag{10}$$

Le dépassement d'âge $\hat{a}^m(\mathbf{x})$ satisfait le problème différentiel suivant:

$$\nabla\bullet(\mathbf{K}\bullet\nabla\hat{a}^m - \mathbf{v}\hat{a}^m) = \begin{cases} -1 , & \mathbf{x}\in\Omega^m \\ 0 , & \mathbf{x}\in\Omega\setminus\Omega^m \end{cases} \tag{11}$$

$$\left[\hat{a}^m(\mathbf{x})\right]_{\mathbf{x}\in\Gamma_0} = -a^m_{\max} , \tag{12}$$

$$\left[(\mathbf{K}\bullet\nabla\hat{a}^m)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma_i} = 0 . \tag{13}$$

A l'évidence, la partie positive de la variable $\hat{a}^m(\mathbf{x})$,

$$\hat{a}^m_+(\mathbf{x}) = \frac{\hat{a}^m(\mathbf{x}) + \left|\hat{a}^m(\mathbf{x})\right|}{2} , \tag{14}$$

doit être nulle dans le sous-domaine Ω^m,

$$\left[\hat{a}^m_+(\mathbf{x})\right]_{\mathbf{x}\in\Omega^m} = 0 . \tag{15}$$

Il reste à démontrer qu'elle est également nulle dans le reste du domaine d'intérêt. On multiplie (11) par $\hat{a}^m(\mathbf{x})$, on intègre sur le domaine d'intérêt et on utilise le théorème de la divergence. On obtient ainsi

$$\begin{aligned} &\overbrace{\int_{\Gamma_i} (\hat{a}^m_+\,\mathbf{K}\bullet\nabla\hat{a}^m_+)\bullet\mathbf{n}\,d\Gamma_i}^{=0,\text{ voir (13)}} - \frac{1}{2}\overbrace{\int_{\Gamma_i} (\hat{a}^m_+)^2\,\mathbf{v}\bullet\mathbf{n}\,d\Gamma_i}^{=0,\text{ voir (4)}} + \frac{1}{2}\overbrace{\int_{\Gamma_0} [2\hat{a}^m_+\,\mathbf{K}\bullet\nabla\hat{a}^m_+ - (\hat{a}^m_+)^2\,\mathbf{v}]\bullet\mathbf{n}\,d\Gamma_0}^{=0,\text{ voir (12)}} \\ &\quad - \underbrace{\int_{\Omega^m} \nabla\hat{a}^m_+\bullet\mathbf{K}\bullet\nabla\hat{a}^m_+\,d\Omega}_{=0,\text{ voir (15)}} - \int_{\Omega\setminus\Omega^m} \nabla\hat{a}^m_+\bullet\mathbf{K}\bullet\nabla\hat{a}^m_+\,d\Omega = -\underbrace{\int_{\Omega^m} \hat{a}^m_+\,d\Omega^m}_{=0,\text{ voir (15)}} \end{aligned} \tag{16}$$

Cette relation se simplifie en

$$\int_{\Omega\setminus\Omega^m} \nabla\hat{a}^m_+\bullet\mathbf{K}\bullet\nabla\hat{a}^m_+\,d\Omega = 0 . \tag{17}$$

Le tenseur $\mathbf{K}$ étant défini positif, la relation (17) implique que la partie positive du dépassement d'âge est nulle dans $\Omega\setminus\Omega^m$,

$$\left[\hat{a}^m_+(\mathbf{x})\right]_{\mathbf{x}\in\Omega\setminus\Omega^m} = 0 . \tag{18}$$

En combinant (15) et (18), on aboutit à la conclusion que la partie positive du dépassement d'âge est nulle dans tout le domaine d'intérêt. Par conséquent, l'âge partiel $a^m(\mathbf{x})$ est partout inférieur ou égal à $a^m_{\max}$, ce qui signifie que le maximum absolu de $a^m(\mathbf{x})$ est égal au maximum de cette fonction dans le sous-domaine Ω^m.

Le maximum du m-ème âge est atteint dans le m-ème sous-domaine. Toutefois, cet âge partiel peut aussi atteindre la même valeur en-dehors du m-ème sous-domaine. On va illustrer ceci par un exemple unidimensionnel purement diffusif (pas d'advection), que l'on traitera en utilisant des variables rendues adimensionnelles selon la méthode habituelle. L'échelle de longueur est L, la longueur du domaine, et l'échelle de temps vaut L^2/K, où la constante positive K est la diffusivité.

Le domaine d'intérêt est défini par les inégalités $0 \le x \le 1$. L'âge est nul sur une frontière du domaine (x=0); l'autre frontière est imperméable (x=1). L'âge (adimensionnel) est donc la solution du problème différentiel suivant:

$$\frac{d^2 a}{dx^2} = -1 \ , \tag{19}$$

$$a(0) = 0 \ , \quad \left[\frac{da}{dx}\right]_{x=1} = 0 \ . \tag{20}$$

L'âge vaut donc (Figure 1)

$$a(x) = \frac{x(2-x)}{2} \ . \tag{21}$$

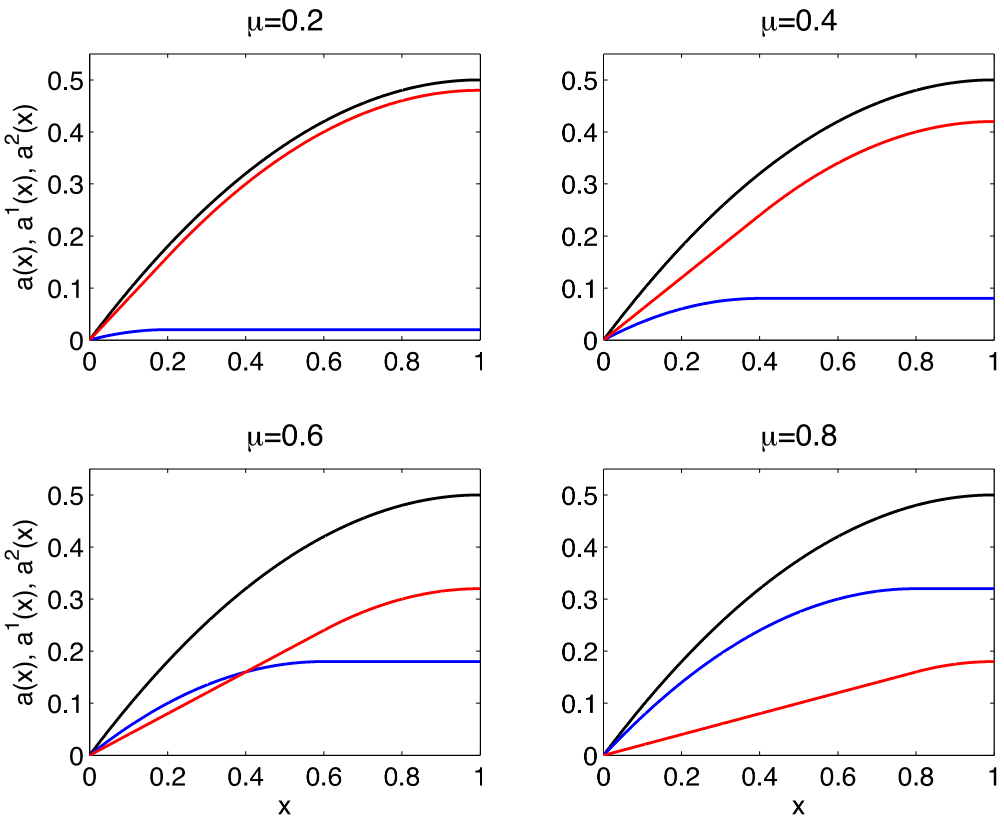


Figure 1. Représentation pour différentes valeurs du paramètre μ de l'âge de l'eau $a(x)$ (courbe noire) ainsi que des âges partiels $a^1(x)$ (courbe bleue) et $a^2(x)$ (courbe rouge). Toutes les variables sont adimensionnelles.

On divise maintenant le domaine d'intérêt en deux parties:

$$\Omega^1 : \ 0 \le x < \mu \ , \quad \Omega^2 : \mu < x \le 1 \tag{22}$$

Le premier âge partiel est la solution du problème différentiel

$$\frac{d^2a^1}{dx^2} = \begin{cases} -1 \ , & 0 \le x < \mu \\ 0 \ , & \mu < x \le 1 \end{cases} \quad , \quad a^1(0) = 0 \ , \quad \left[\frac{da^1}{dx}\right]_{x=1} = 0 \tag{23}$$

et il vaut

$$a^1(x) \ = \ \begin{cases} \dfrac{x(2\mu - x)}{2} \ , & 0 \le x < \mu \\ \dfrac{\mu^2}{2} \ , & \mu < x \le 1 \end{cases} \tag{24}$$

Le second âge partiel est la solution du problème différentiel

$$\frac{d^2a^2}{dx^2} = \begin{cases} 0 \ , & 0 \le x < \mu \\ -1 \ , & \mu < x \le 1 \end{cases} \quad , \quad a^2(0) = 0 \ , \quad \left[\frac{da^2}{dx}\right]_{x=1} = 0 \tag{25}$$

et il vaut

$$a^2(x) \ = \ \begin{cases} (1-\mu)x \ , & 0 \le x < \mu \\ \dfrac{-x^2 + 2x - \mu^2}{2} \ , & \mu < x \le 1 \end{cases} \tag{26}$$

Dans le second sous-domaine, $a^1(x)$ est constant et égal à la valeur maximale de cet âge. Par contre, pour l'autre âge partiel, la valeur maximale est atteinte en x=1 et la valeur de cet âge dans le premier sous-domaine est inférieure à ce maximum.

La porte ouverte est finalement enfoncée...

Sur le maximum de l'âge partiel d'un traceur radioactif

Eric Deleersnijder, 13 février 2014

Deleersnijder (2013b) a montré qu'à l'état stationnaire le maximum du m-ème âge partiel de l'eau[1] n'est pas supérieur au maximum de cet âge dans le m-ème sous-domaine. Cette propriété pourrait sembler triviale: en effet, la m-ème horloge de la particule étudiée ne fonctionnant que dans le m-ème sous-domaine, le temps maximal qu'elle indique est atteint au moment où la particule quitte le m-ème sous-domaine. Ce raisonnement ne peut pas être étendu aux particules qui interviennent pour calculer l'âge partiel en un endroit donné, car ces particules ont des histoires différentes. D'ailleurs, Deleersnijder (2013a) a produit une solution analytique stationnaire dans laquelle le maximum du m-ème âge partiel se situe à l'intérieur du m-ème sous domaine et non sur sa frontière.

Dans la présente note de travail, on va montrer qu'à l'état stationnaire le maximum du m-ème âge partiel d'un traceur radioactif n'est pas supérieur au maximum de cet âge dans le m-ème sous-domaine. Il s'agit donc d'une généralisation du résultat obtenu dans Deleersnijder (2013b).

Concentration du traceur radioactif

Soient Ω et Γ le domaine d'intérêt et la surface limitant celui-ci. On note $\mathbf{n}$ la normale extérieure à cette surface, qui se compose de deux parties, Γ_i et Γ_0, avec $\Gamma = \Gamma_i \cup \Gamma_0$ et $\Gamma_i \cap \Gamma_0 = \varnothing$. La surface Γ_i est imperméable. Sur Γ_0, on impose la concentration du traceur, qui est égale à la fonction non-négative connue $C_0(\mathbf{x})$, où $\mathbf{x}$ représente le vecteur position.

La vitesse $\mathbf{v}(\mathbf{x})$ est indivergentielle,

$$\nabla \bullet \mathbf{v} = 0 \ . \tag{1}$$

L'imperméabilité de la frontière Γ_i conduit à imposer la condition de bord

$$[\mathbf{v} \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma_i} = 0 \ . \tag{2}$$

Par ailleurs, le tenseur des diffusivités $\mathbf{K}(\mathbf{x})$ est symétrique ($\mathbf{K} = \mathbf{K}^T$) et défini positif. Cette seconde caractéristique implique que, pour tout vecteur $\mathbf{y}$ non-nul ($\mathbf{y} \neq 0$), le tenseur des diffusivités est tel que

$$\mathbf{y} \bullet \mathbf{K} \bullet \mathbf{y} > 0 \ . \tag{3}$$

Si la constante positive γ symbolise le taux de décroissance du traceur radioactif[2], alors sa concentration stationnaire $C(\mathbf{x})$ est la solution de l'équation suivante:

$$\nabla \bullet (\mathbf{K} \bullet \nabla C - C\,\mathbf{v}) - \gamma C = 0 \, . \tag{4}$$

Comme indiqué ci-dessus, on impose sur Γ_0 la condition de Dirichlet

$$[C(\mathbf{x})]_{\mathbf{x} \in \Gamma_0} = C_0(\mathbf{x}) \geq 0 \ . \tag{5}$$

[1] L'eau est traitée comme un traceur dont la concentration est égale à l'unité en tout point et à tout instant.

[2] La vie moyenne du traceur considéré est égale γ^{-1} tandis que sa demi-vie vaut $\gamma^{-1} \log 2 \approx 0.7\gamma^{-1}$.

L'imperméabilité de la surface Γ_i conduit à

$$\left[(-\mathbf{K}\bullet\nabla C + C\,\mathbf{v})\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma_i} = 0 \ . \qquad (6)$$

En combinant (2) et (6), on obtient la condition de Neumann

$$\left[(\mathbf{K}\bullet\nabla C)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma_i} = 0 \ . \qquad (7)$$

L'intuition physique suggère que la concentration qui vérifie les relations (4)-(5) et (7) est positive, notamment parce que la concentration imposée $C_0(\mathbf{x})$ est positive. Toutefois, il est souhaitable de produire une démonstration rigoureuse, dont résultat sera utile plus loin.

On note $C^-(\mathbf{x})$ l'hypothétique partie négative de la concentration, c'est-à-dire

$$C^-(\mathbf{x}) = \frac{C(\mathbf{x}) - |C(\mathbf{x})|}{2} \ . \qquad (8)$$

On multiplie (4) par $C^-(\mathbf{x})$, on intègre sur le domaine d'intérêt et, en utilisant le théorème de la divergence et la contrainte (1), on obtient la relation intégrale

$$\overbrace{\int_{\Gamma_i} C^-(\mathbf{K}\bullet\nabla C)\bullet\mathbf{n}\, d\Gamma_i}^{=0,\text{ voir }(7)} - \frac{1}{2}\overbrace{\int_{\Gamma_i} (C^-)^2\mathbf{v}\bullet\mathbf{n}\, d\Gamma_i}^{=0,\text{ voir }(2)} + \frac{1}{2}\overbrace{\int_{\Gamma_0} [2C^-(\mathbf{K}\bullet\nabla C) - (C^-)^2\mathbf{v}]\bullet\mathbf{n}\, d\Gamma_i}^{=0,\text{ voir }(5)}$$
$$- \int_{\Omega} [\nabla C^-\bullet\mathbf{K}\bullet\nabla C^- + \gamma(C^-)^2]\, d\Omega = 0 \qquad (9)$$

qui se simplifie en

$$\int_{\Omega} [\nabla C^-\bullet\mathbf{K}\bullet\nabla C^- + \gamma(C^-)^2]\, d\Omega = 0 \qquad (10)$$

Le tenseur des diffusivités étant défini positif, l'intégrande est positive ou nulle en tout point du domaine d'intégration. Comme l'intégrale est nulle, l'intégrande doit être nulle en tout point, ce qui implique que la partie négative de la concentration, C^-, est nulle partout. Par conséquent, la concentration ne peut être négative en aucun point du domaine d'intérêt:

$$C(\mathbf{x}) \geq 0 \qquad (11)$$

Concentrations d'âge partiel

On décompose le domaine d'intérêt Ω en M sous-domaines Ω_m (m=1,2... M), avec

$$\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \Omega_M \qquad (12)$$

et

$$\Omega_i \cap \Omega_j = \varnothing \ , \quad i \neq j \ . \qquad (13)$$

La m-ème concentration d'âge partiel, $\alpha_m(\mathbf{x})$, est la solution de l'équation différentielle aux dérivées partielles suivantes

$$\nabla\bullet(\mathbf{K}\bullet\nabla\alpha_m - \alpha_m\mathbf{v}) - \gamma\alpha_m = \begin{cases} -C \ , & \mathbf{x}\in\Omega_m \\ 0 \ , & \mathbf{x}\in\Omega\setminus\Omega_m \end{cases} \qquad (14)$$

Sur la frontière Γ_0, on impose que tous les âges partiels soient nuls:

$$\left[\alpha_m(\mathbf{x})\right]_{\mathbf{x}\in\Gamma_0} = 0 \ . \tag{15}$$

Par ailleurs, l'imperméabilité de l'autre partie de la frontière du domaine conduit à la condition de Neumann suivante

$$\left[(\mathbf{K}\bullet\nabla\alpha_m)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma_i} = 0 \ . \tag{16}$$

Pour démontrer que chaque concentration d'âge partiel est positive ou nulle, on emploie une méthode semblable à celle utilisée ci-dessus pour la concentration du traceur radioactif. Ainsi, on établit que la partie négative de la m-ème concentration d'âge partiel,

$$\alpha_m^-(\mathbf{x}) = \frac{\alpha_m(\mathbf{x})-\left|\alpha_m(\mathbf{x})\right|}{2} \tag{17}$$

satisfait la relation intégrale suivante:

$$\overbrace{\int_{\Gamma_i}\alpha_m^-(\mathbf{K}\bullet\nabla\alpha_m)\bullet\mathbf{n}\,d\Gamma_i}^{=0,\text{ voir (16)}} - \frac{1}{2}\overbrace{\int_{\Gamma_i}(\alpha_m^-)^2\mathbf{v}\bullet\mathbf{n}\,d\Gamma_i}^{=0,\text{ voir (2)}} + \frac{1}{2}\overbrace{\int_{\Gamma_0}[2\alpha_m^-(\mathbf{K}\bullet\nabla\alpha_m)-(\alpha_m^-)^2\mathbf{v}]\bullet\mathbf{n}\,d\Gamma_i}^{=0,\text{ voir (15)}}$$
$$-\int_{\Omega}[\nabla\alpha_m^-\bullet\mathbf{K}\bullet\nabla\alpha_m^-+\gamma(\alpha_m^-)^2]\,d\Omega = -\int_{\Omega_m}C\alpha_m^-\,d\Omega \tag{18}$$

En éliminant les termes nuls, cette relation se réduit à

$$\int_{\Omega}[\nabla\alpha_m^-\bullet\mathbf{K}\bullet\nabla\alpha_m^-+\gamma(\alpha_m^-)^2]\,d\Omega = \int_{\Omega_m}C\alpha_m^-\,d\Omega \ . \tag{19}$$

Le membre de gauche de (19) ne peut être négatif, alors que le membre de droite ne peut être positif, car $C \geq 0$ et $\alpha_m^- \leq 0$. Par conséquent, la partie négative de la concentration d'âge doit être nulle partout. En d'autres termes, toutes les concentrations d'âge partiel sont positives ou nulles en tout point du domaine d'intérêt:

$$\alpha_m(\mathbf{x}) \geq 0 \tag{20}$$

Ages partiels

Le m-ème âge partiel est défini comme suit:

$$a_m(\mathbf{x}) = \frac{\alpha_m(\mathbf{x})}{C(\mathbf{x})} \ . \tag{21}$$

La concentration du traceur et toutes les concentrations d'âge partiel étant positives ou nulles, les âges partiels ne peuvent être négatifs:

$$a_m(\mathbf{x}) \geq 0 \ . \tag{22}$$

Cette propriété est en accord avec l'intuition physique la plus élémentaire. Encore fallait-il démontrer rigoureusement que la solution du modèle employé ici obéissait à cette contrainte.

Soit A_m la valeur maximale du m-ème âge partiel dans le m-ème sous-domaine:

$$A_m = \max_{\mathbf{x}\in\Omega_m} a_m(\mathbf{x}) \tag{23}$$

On souhaite montrer que le m-ème âge partiel ne peut dépasser cette valeur, ce qui revient à prouver que la variable auxiliaire

$$\xi_m = C(A_m - a_m) \tag{24}$$

ne peut être négative. Cette variable est la solution du problème différentiel suivant:

$$\nabla\bullet(\mathbf{K}\bullet\nabla\xi_m - \xi_m\,\mathbf{v}) - \gamma\xi_m = \begin{cases} C\ , & \mathbf{x}\in\Omega_m \\ 0\ , & \mathbf{x}\in\Omega\setminus\Omega_m \end{cases} \tag{25}$$

$$\left[\xi_m(\mathbf{x})\right]_{\mathbf{x}\in\Gamma_0} = \left[C(\mathbf{x})A_m\right]_{\mathbf{x}\in\Gamma_0} = C_0(\mathbf{x})A_m \geq 0 \tag{26}$$

$$\left[(\mathbf{K}\bullet\nabla\xi_m)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma_i} = 0 \tag{27}$$

La partie négative de ξ_m,

$$\xi_m^-(\mathbf{x}) = \frac{\xi_m(\mathbf{x}) - |\xi_m(\mathbf{x})|}{2}\ , \tag{28}$$

satisfait la relation intégrale suivante

$$\begin{aligned} &\overbrace{\int_{\Gamma_i} \xi_m^-(\mathbf{K}\bullet\nabla\xi_m)\bullet\mathbf{n}\,d\Gamma_i}^{=0,\text{ voir (27)}} - \frac{1}{2}\overbrace{\int_{\Gamma_i}(\xi_m^-)^2\mathbf{v}\bullet\mathbf{n}\,d\Gamma_i}^{=0,\text{ voir (2)}} + \frac{1}{2}\overbrace{\int_{\Gamma_0}[2\xi_m^-(\mathbf{K}\bullet\nabla\xi_m) - (\xi_m^-)^2\mathbf{v}]\bullet\mathbf{n}\,d\Gamma_i}^{=0,\text{ voir (26)}} \\ &- \int_{\Omega\setminus\Omega_m}[\nabla\xi_m^-\bullet\mathbf{K}\bullet\nabla\xi_m^- + \gamma(\xi_m^-)^2]\,d\Omega = \underbrace{\int_{\Omega_m} C\,\xi_m^-\,d\Omega}_{=0,\text{ voir (23)-(24)}} \end{aligned} \tag{29}$$

Cette expression simplifie en

$$\int_{\Omega}[\nabla\xi_m^-\bullet\mathbf{K}\bullet\nabla\xi_m^- + \gamma(\xi_m^-)^2]\,d\Omega = 0 \tag{30}$$

Le tenseur des diffusivité étant défini positif, les deux termes de l'intégrande ne peuvent être négatifs. Par conséquent, la fonction ξ_m^- doit être nulle dans tout le domaine d'intérêt, ce qui implique

$$\boxed{0 \leq a_m(\mathbf{x}) \leq A_m} \tag{31}$$

A l'état stationnaire, le maximum du *m*-ème âge partiel d'un traceur radioactif n'est pas supérieur au maximum de cet âge dans le *m*-ème sous-domaine. QED.

Remarques

La démonstration réalisée ici peut sembler quelque peu complexe. Toutefois, je suis persuadé que toutes les étapes sont nécessaires pour aboutir à un résultat convaincant.

On voit sans peine que cette démonstration peut être appliquée à l'âge de l'eau. Pour ce faire, il suffit de poser $\gamma = 0$ et $C_0(\mathbf{x}) = 1$. En d'autres termes, la démonstration de Deleersnijder (2013b) n'est qu'un cas particulier des développements réalisés dans la présente note de travail.

La configuration étudiée ici est inspirée des études de ventilation dans l'océan mondial. D'autres types de configuration mériteraient d'être abordées.

Je suis persuadé que l'on ne pourrait pas obtenir un résultat similaire dans un problème instationnaire.

Références

Deleersnijder E., 2013a, *Les âges partiels dans la boucle de Munk*, Note6.pdf

Deleersnijder E., 2013b, *Détermination du maximum de l'âge partiel ou l'art d'enfoncer à grand'peine des portes ouvertes...,* Note7.pdf

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Partial ages for an abrupt release in a one-dimensional flow

Eric Deleersnijder, 26 February 2014

The age of tracer particles released abruptly into a fluid flow is equal to the time elapsed since the instant at which the tracer injection occurred. Therefore, in this case, the age is an uninteresting diagnosis. However, as was suggested by Emmanuel Hanert, partial ages are likely to be useful — though the sum of all partial ages will be equal to the elapsed time at any time and position. This is why Anouk de Brauwere is simulating numerically a hypothetical accidental tracer release into the Scheldt Estuary and computing the partial ages related to previously-defined subdomains (e.g. Soetaert and Herman 1995, de Brauwere et al. 2011). In addition, as will be seen in the present working note, for a highly-idealised set up (one-dimensional flow, constant velocity and diffusivity) an analytical solution may be obtained. The partial ages are seen to be independent of the flow velocity, which is presumably why they exhibit a symmetry property that has yet to be explained by means of physical arguments.

As usual, upon resorting to a modern tracer and timescale method (e.g. CART, www. climate.be/cart), there is no shortage of surprising results...

Tracer concentration

Let t and x denote the time and the space coordinate. In a one-dimensional flow for which the velocity and the diffusivity are the positive constants U and K, the concentration $C(t,x)$ of a passive tracer whose injection rate is $q(t,x)$ is the solution of the equation

$$K\frac{\partial^2 C}{\partial x^2} - U\frac{\partial C}{\partial x} - \frac{\partial C}{\partial t} = q \ . \tag{1}$$

The domain of interest, Ω, is assumed to be infinite ($-\infty < x < \infty$). In addition, there is no tracer in the domain of interest at the initial time

$$C(0^-,x) = 0 \ . \tag{2}$$

Further assume that the source function corresponds to the injection of a unit amount of tracer at the initial time at the point x=0. The corresponding release rate is

$$q(t,x) = \delta(t-0)\,\delta(x-0) \ , \tag{3}$$

where δ is the Dirac function. In this case, the physical dimension of the solution $C(t,x)$ is length^{-1}. For this forcing, $C(t,x)$ actually is the Green's function of the problem. However, for the sake of simplicity, $C(t,x)$ will still be termed "concentration".

It is convenient to introduce the following dimensionless variables:

$$t' = \frac{t}{K/U^2} \ , \quad x' = \frac{x}{K/U} \ , \quad C' = \frac{C}{U/K} \ . \tag{4}$$

From here on, unless otherwise stated, only dimensionless variables will be dealt with. Therefore, for notational simplicity, the primes may be dropped. Accordingly, the (dimensionless) concentration is governed by the equation

$$\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} - \frac{\partial C}{\partial t} = -\delta(t-0)\,\delta(x-0) \ . \tag{5}$$

It is readily seen that the Laplace transform of the concentration,

$$\tilde{C}(s,x) = \int_0^\infty C(t,x) e^{-st}\, dt \ , \tag{6}$$

satisfies

$$\frac{\partial^2 \tilde{C}}{\partial x^2} - \frac{\partial \tilde{C}}{\partial x} - s\tilde{C} = -\delta(x-0) \tag{7}$$

In the vicinity of the release point ($x = 0$), the Laplace transform must satisfy the matching conditions

$$\left[\tilde{C}(s,x)\right]_{x=0^-}^{x=0^+} = 0 \ , \tag{8}$$

and

$$\left[\frac{\partial}{\partial x}\tilde{C}(s,x)\right]_{x=0^-}^{x=0^+} = -1 \ . \tag{9}$$

The solution of (7)-(9) reads

$$\tilde{C}(s,x) = \frac{1}{2\lambda} \exp\left(\frac{x}{2} - \lambda |x|\right) \ , \tag{10}$$

with

$$\lambda = \sqrt{s + \frac{1}{4}} \ . \tag{11}$$

The original thereof is the well known expression (e.g. Abramowitz and Stegun 1972) (Figure 1)

$$C(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{(x-t)^2}{4t}\right] \tag{12}$$

The classical age

The (dimensionless) age of the tracer under consideration may be obtained from the formula

$$a(t,x) = \frac{\alpha(t,x)}{C(t,x)} \ , \tag{13}$$

where $\alpha(t,x)$ is the (dimensionless) age concentration. Assuming that the age of the tracer particles is zero at the instant they are released into the flow, the age concentration obeys the equation

$$\frac{\partial^2 \alpha}{\partial x^2} - \frac{\partial \alpha}{\partial x} - \frac{\partial \alpha}{\partial t} = -C \ , \tag{14}$$

that must be solved under the initial condition

$$\alpha(0,x) = 0 \tag{15}$$

and the matching conditions

$$\left[\alpha(t,x)\right]_{x=0^-}^{x=0^+} \;=\; 0 \;=\; \left[\frac{\partial}{\partial x}\alpha(t,x)\right]_{x=0^-}^{x=0^+} \;. \tag{16}$$

The Laplace transform of the age concentration,

$$\tilde{\alpha}(s,x) \;=\; \int_0^\infty \alpha(t,x)e^{-st}\,dt \;, \tag{17}$$

satisfies

$$\begin{cases} \dfrac{\partial^2\tilde{\alpha}}{\partial x^2} - \dfrac{\partial\tilde{\alpha}}{\partial x} - s\tilde{\alpha} \;=\; -\tilde{C} \\ \left[\tilde{\alpha}(s,x)\right]_{x=0^-}^{x=0^+} \;=\; 0 \;=\; \left[\dfrac{\partial}{\partial x}\tilde{\alpha}(s,x)\right]_{x=0^-}^{x=0^+} \end{cases} \tag{18}$$

The solution thereof is

$$\tilde{\alpha}(s,x) \;=\; \frac{1+\lambda|x|}{4\lambda^3}\exp\left(\frac{x}{2}-\lambda|x|\right) \;, \tag{19}$$

and the original of this Laplace transform reads (e.g. Abramowitz and Stegun 1972)

$$\alpha(t,x) \;=\; \frac{t}{\sqrt{4\pi t}}\exp\left[-\frac{(x-t)^2}{4t}\right] \tag{20}$$

By combining (12), (13) and (19), the age is arrived at

$$a(t,x) \;=\; t \tag{21}$$

Thus, as expected, the age is equal to the elapsed time.

The partial ages

The domain of interest Ω is split into two subdomains, Ω^1 and Ω^2, with $\Omega = \Omega^1 \cup \Omega^2$ and

$$\Omega^1 : \; -\infty < x < 0 \;, \quad \Omega^2 \;:\; 0 < x < \infty \;. \tag{22}$$

Then, the partial differential problems obeyed by the related partial age concentrations, $\alpha^1(t,x)$ and $\alpha^2(t,x)$, read

$$\begin{cases} \dfrac{\partial^2\alpha^1}{\partial x^2} - \dfrac{\partial\alpha^1}{\partial x} - \dfrac{\partial\alpha^1}{\partial t} \;=\; \begin{cases} 0 \;,\; 0 < x < \infty \\ -C \;,\; -\infty < x < 0 \end{cases} \\ \alpha^1(0,x) = 0 \;,\; \left[\alpha^1(t,x)\right]_{x=0^-}^{x=0^+} = 0 = \left[\dfrac{\partial}{\partial x}\alpha^1(t,x)\right]_{x=0^-}^{x=0^+} \end{cases} \tag{23}$$

and

$$\begin{cases} \dfrac{\partial^2\alpha^2}{\partial x^2} - \dfrac{\partial\alpha^2}{\partial x} - \dfrac{\partial\alpha^2}{\partial t} \;=\; \begin{cases} -C \;,\; 0 < x < \infty \\ 0 \;,\; -\infty < x < 0 \end{cases} \\ \alpha^2(0,x) = 0 \;,\; \left[\alpha^2(t,x)\right]_{x=0^-}^{x=0^+} = 0 = \left[\dfrac{\partial}{\partial x}\alpha^2(t,x)\right]_{x=0^-}^{x=0^+} \end{cases} \tag{24}$$

The Laplace transforms of the partial age concentrations, $\tilde{\alpha}^1(s,x)$ and $\tilde{\alpha}^2(s,x)$, are the solutions of

$$\begin{cases} \dfrac{\partial^2 \tilde{\alpha}^1}{\partial x^2} - \dfrac{\partial \tilde{\alpha}^1}{\partial x} - s\tilde{\alpha}^1 = \begin{cases} 0 \ , \ 0 < x < \infty \\ -\tilde{C} \ , \ -\infty < x < 0 \end{cases} \\ \left[\tilde{\alpha}^1(s,x)\right]_{x=0^-}^{x=0^+} = 0 = \left[\dfrac{\partial}{\partial x}\tilde{\alpha}^1(s,x)\right]_{x=0^-}^{x=0^+} \end{cases} \tag{25}$$

and

$$\begin{cases} \dfrac{\partial^2 \tilde{\alpha}^2}{\partial x^2} - \dfrac{\partial \tilde{\alpha}^2}{\partial x} - s\tilde{\alpha}^2 = \begin{cases} -\tilde{C} \ , \ 0 < x < \infty \\ 0 \ , \ -\infty < x < 0 \end{cases} \\ \left[\tilde{\alpha}^2(s,x)\right]_{x=0^-}^{x=0^+} = 0 = \left[\dfrac{\partial}{\partial x}\tilde{\alpha}^2(s,x)\right]_{x=0^-}^{x=0^+} \end{cases} \tag{26}$$

After some calculations, one obtains

$$\tilde{\alpha}^1(s,x) = \frac{1+\lambda(|x|-x)}{4\lambda^3}\exp\left(\frac{x}{2}-\lambda|x|\right) \tag{27}$$

and

$$\tilde{\alpha}^2(s,x) = \frac{1+\lambda(|x|+x)}{4\lambda^3}\exp\left(\frac{x}{2}-\lambda|x|\right) \tag{28}$$

As expected, the Laplace transforms of the partial age concentrations satisfy

$$\tilde{\alpha}^1(s,x) + \tilde{\alpha}^2(s,x) = \tilde{\alpha}(s,x) \ . \tag{29}$$

The originals of (27) and (28) are (e.g. Abramowitz and Stegun 1972)

$$\alpha^1(t,x) = \left[\frac{t}{2} - \frac{x\sqrt{\pi t}\, e^{x^2/(4t)}}{4}\operatorname{erfc}\left(\frac{|x|}{\sqrt{4t}}\right)\right] C(t,x) \tag{30}$$

and

$$\alpha^2(t,x) = \left[\frac{t}{2} + \frac{x\sqrt{\pi t}\, e^{x^2/(4t)}}{4}\operatorname{erfc}\left(\frac{|x|}{\sqrt{4t}}\right)\right] C(t,x) \tag{31}$$

where erfc denote the complementary error function, which is defined by the integral

$$\operatorname{erfc}(\xi) = \frac{2}{\sqrt{\pi}}\int_{\xi}^{\infty} e^{-\upsilon^2}\, d\upsilon \ . \tag{32}$$

Naturally, the partial age concentrations satisfy the constraint

$$\alpha^1(t,x) + \alpha^2(t,x) = \alpha(t,x) \ .$$

Now the partial ages are readily obtained (Figure 1)

$$a^1(t,x) \equiv \frac{\alpha^1(t,x)}{C(t,x)} = \frac{t}{2} - \frac{x\sqrt{\pi t}\, e^{x^2/(4t)}}{4}\operatorname{erfc}\left(\frac{|x|}{\sqrt{4t}}\right) \tag{33}$$

and

$$a^2(t,x) \equiv \frac{\alpha^2(t,x)}{C(t,x)} = \frac{t}{2} + \frac{x\sqrt{\pi t}\, e^{x^2/(4t)}}{4} \operatorname{erfc}\left(\frac{|x|}{\sqrt{4t}}\right) \tag{34}$$

It is easily seen that sum of the partial ages is equal to the classical age:

$$a^1(t,x) + a^2(t,x) = a(t,x) \ . \tag{35}$$

In addition, the partial ages satisfy

$$a^1(t,x) = a^2(t,-x) \tag{36}$$

The physical interpretation of this somewhat surprising symmetry property has yet to be provided. The partial ages admit the following asymptotic expressions

$$a^1(t,x) = a^2(t,-x) \sim \begin{cases} t - \dfrac{t^2}{x^2} \ , & \dfrac{x}{\sqrt{t}} \to -\infty \\ \dfrac{t^2}{x^2} \ , & \dfrac{x}{\sqrt{t}} \to \infty \end{cases} \tag{37}$$

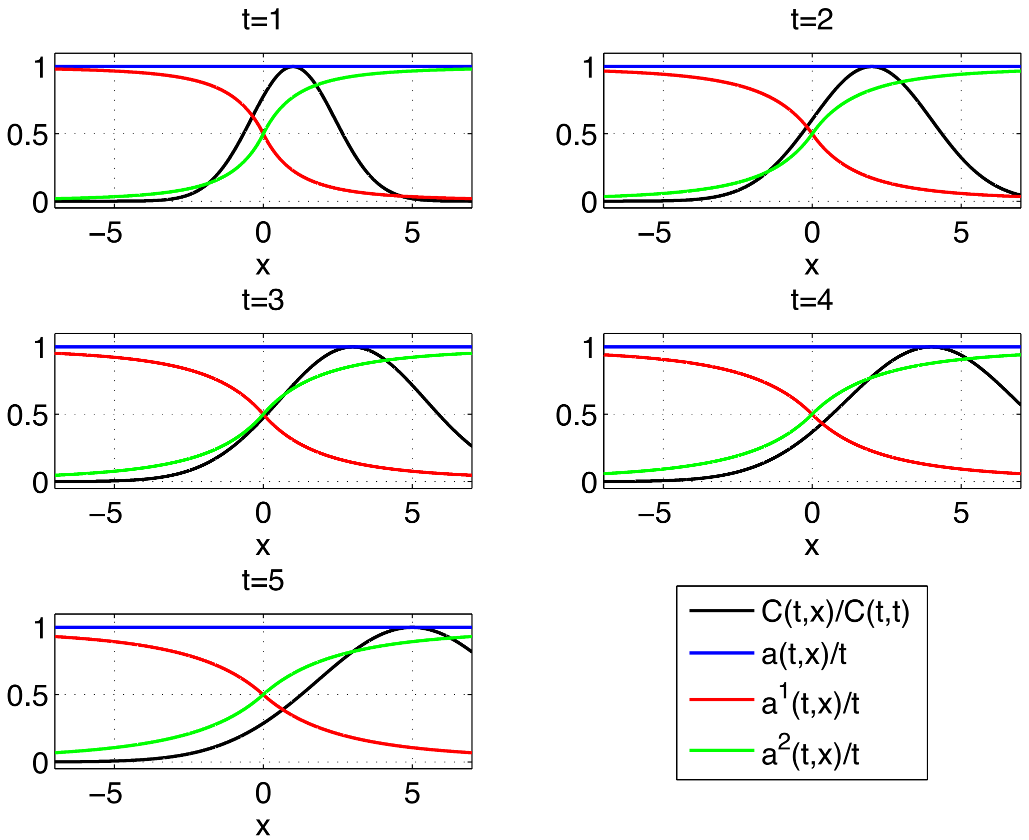

Figure 1. Illustration of the space distribution of the concentration, and the classical and partial ages at various times. The normalised concentration $C(t,x)/C(t,t)$ is plotted, where $C(t,t)=\sqrt{4\pi t}$ is the maximum over the domain of interest of the concentration at time t. All the ages represented in the graph are normalised by the elapsed time t. This figure neatly illustrates the relations (35) and (36)

Suggestions for further developments

The scaling (4) is such that the Peclet number is equal to unity. I believe this can be done without any loss of generality. Would it be worth checking this thoroughly?

In the past, the physical explanation of some surprising results about tracer and timescales has been built by resorting to a Lagrangian approach (Hall and Haine 2004, Delhez and Deleersnijder 2012). A similar strategy might prove to be successful for gaining insight into the symmetry property (36).

The age distribution function of the tracer under consideration above is

$$c(t,x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \exp\left[-\frac{(x-\tau)^2}{4\tau}\right] \delta(\tau - t) \ . \tag{38}$$

Would it be possible to derive similar distribution functions for the partial ages? The latter would be denoted $c^1(t,x,\tau)$ and $c^2(t,x,\tau)$, and would presumably obey the partial differential equations

$$\frac{\partial^2 c^1}{\partial x^2} - \frac{\partial c^1}{\partial x} - \frac{\partial c^1}{\partial t} = \begin{cases} 0 \ , & 0 < x < \infty \\ \dfrac{\partial c}{\partial \tau} \ , & -\infty < x < 0 \end{cases} \tag{39}$$

$$\frac{\partial^2 c^2}{\partial x^2} - \frac{\partial c^2}{\partial x} - \frac{\partial c^2}{\partial t} = \begin{cases} \dfrac{\partial c}{\partial \tau} \ , & 0 < x < \infty \\ 0 \ , & -\infty < x < 0 \end{cases} \tag{40}$$

and satisfy the constraint

$$c^1(t,x,\tau) + c^2(t,x,\tau) = c(t,x,\tau) \ . \tag{41}$$

Are (39)-(41) relevant and correct? If so, would it be possible to solve (39)-(40) analytically? Next, the question arises as to whether it is possible or not to derive the equations governing the partial distribution functions in any type of flow?

The partial ages may be rewritten using dimensional variables, yielding

$$a^1(t,x) = \frac{t}{2} - \frac{x\sqrt{\pi t}\, e^{x^2/(4Kt)}}{4\sqrt{K}} \operatorname{erfc}\left(\frac{|x|}{\sqrt{4Kt}}\right) \tag{42}$$

$$a^2(t,x) = \frac{t}{2} + \frac{x\sqrt{\pi t}\, e^{x^2/(4Kt)}}{4\sqrt{K}} \operatorname{erfc}\left(\frac{|x|}{\sqrt{4Kt}}\right) \tag{43}$$

The partial ages are independent of the fluid velocity U! There is little doubt that this surprising property is consistent with the symmetry property discovered above. Presumably, there is a single cause for these two peculiar features. On the other hand, this naturally leads one to wonder whether the partial ages that one would obtain under the assumption that the fluid velocity is zero (U=0) are equal or not to the expressions (42)-(43).

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