2014/55

Single-period cutting planes for inventory routing problems

Pasquale Avella, Maurizio Boccia
and Laurence A. Wolsey

DISCUSSION PAPER

Center for Operations Research
and Econometrics

Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium
http://www.uclouvain.be/core

CORE
Voie du Roman Pays 34, L1.03.01
B-1348 Louvain-la-Neuve, Belgium.
Tel (32 10) 47 43 04
Fax (32 10) 47 43 01
E-mail: corestat-library@uclouvain.be
http://www.uclouvain.be/en-44508.html

CORE DISCUSSION PAPER
2014/55

Single-period cutting planes for inventory routing problems

Pasquale AVELLA[1], Maurizio BOCCIA[1] and Laurence A. WOLSEY[2]

November 2014

Abstract

IRP involves the distribution of one or more products from a supplier to a set of clients over a discrete planning horizon. Each client has a known demand to be met in each period and can only hold a limited amount of stock. The product is shipped through a distribution network by one or more vehicles of limited capacity. The objective is to find replenishment decisions minimizing the sum of the storage and distribution costs.

In this paper we present reformulations of IRP, under the Maximum Level replenishment policy, derived from a single-period substructure. We define a generic family of valid inequalities, and then introduce two specific subclasses for which the separation problem of generating violated inequalities can be solved effectively. A basic Branch-and-Cut algorithm has been implemented to demonstrate the strength of the single-period reformulations. Computational results are presented for the benchmark instances with 50 clients and three periods and 30 clients and six periods.

Keywords: inventory routing, valid inequalities, cutting planes.

Mathematics Subject Classification: 90C11, 90C26.

This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming, contract no. P7/36, Combinatorial Optimization: Metaheuristics and Exact Methods. The scientific responsibility is assumed by the authors.

[1] Dipartimento di Ingegneria, Universita del Sannio, Italy.
[2] CORE, Université catholique de Louvain, Voie du Roman Pays 34, B-1348 Louvain-la-Neuve, Belgium.

1 Introduction

The Inventory Routing Problem (IRP) arises from the integration of two basic components of the logistic supply chain, namely Inventory Management and Vehicle Routing. IRP involves the distribution of one or more products from a supplier to a set of clients over a discrete planning horizon. Each client has a known demand to be met in each period and can only hold a limited amount of stock. Replenishment policies are defined by the supplier, the most common being known as order-up (OU) and maximum level (ML). Under the order-up policy, if a client is visited in a period then the amount shipped to the client must bring the stock level up to the upper bound. The maximum level policy only requires that the maximum stock level in each period cannot be exceeded.
The product is shipped through a distribution network by one or more vehicles of limited capacity. IRP has found applications in several contexts, such as maritime logistics and the distribution of gas, perishable items, groceries, etc. The objective is to find replenishment decisions minimizing the sum of the storage and distribution costs.

1.1 Literature review

Archetti et al. [2] were the first to propose a Branch-and-Cut algorithm for the IRP. They addressed the single-vehicle case and both the variants with OU and ML replenishment policies. Then Solyalı and Süral [13] proposed a shortest-path network reformulation of the single-vehicle problem for the order-up replenishment policy. Avella et al. [4] presented tight reformulations for the lot-sizing subproblems that arise for each client designed to strengthen the dual lower bounds, showing computational results for the single vehicle problem.
Exact algorithms for multi-vehicle IRP have been proposed by Adulyasak et al. [1], Coelho and Laporte [6, 7] and Desaulniers et al. [9]. Adulyasak et al.[1] proposed reformulations for the OU and the ML policies using vehicle indices and extended some of the inequalities introduced by Archetti et al. [2] to the multi-vehicle case. Coelho and Laporte [6] presented a Branch-and-Cut algorithm able to address several variants of IRP, including the case with multiple vehicles, under the ML policy. Then in a subsequent paper [7] the same authors presented new valid inequalities based on the minimum number of visits each client must receive and showed that modifying the order of input data and symmetry breaking constraints can be effective in tightening the formulation. Desaulniers et al. [9] have introduced an extended formulation for the ML replenishment policy, based on route delivery patterns, presenting a branch-and-price algorithm which outperforms the branch-and-cut algorithm of Coelho and Laporte [7] on the instances with 4 and 5 vehicles. All the algorithms proposed in the above papers have been tested on the same set of benchmark instances. For a more general review on algorithms for IRP, we refer to Coelho et al. [5].

1.2 Outline of the paper

IRP naturally decomposes into two families of highly structured subproblems: a lot-sizing problem for each client, discussed by the authors in detail in [4] and a single-period inventory routing problem for each time period, denoted SIRP. In this paper we present reformulations of IRP, under the ML replenishment policy, derived essentially from this single-period substructure. We define a generic family of valid inequalities for SIRP, and then introduce two specific subclasses for which the separation problem of generating violated inequalities can be solved effectively.

A basic Branch-and-Cut algorithm, without any special-purpose primal heuristics, has been implemented to demonstrate the strength of the single-period reformulations. Computational results are presented for the benchmark instances with 50 clients and three periods and 30 clients and six periods.

The remainder of the paper is organized as follows. In Section 2 we give a more formal definition of IRP and an initial mixed integer programming formulation. In section 3 we introduce a simple version of the generic family of valid inequalities for the single-period IRP. In 4 we describe in detail the family of “disjoint route inequalities" as well as the two subfamilies that we use as cutting planes. In Section 5 we also describe a separation procedure for another family of inequalities obtained by aggregating demands over several time periods. In Section 6 we describe the components of our branch-and-cut algorithm. Finally in Section 7 we report on detailed results for instances with 50 clients and three periods and 30 clients and six periods, showing the effectiveness of the proposed inequalities. Detailed results for the instances with $15, 20, 25$ clients and six periods are given in the Appendix 9

2 Problem definition and formulation

Let $T = \{1, 2, ..., T_{max}\}$ be a discrete time horizon. In each period $t \in T$, D^0 units of a single item are delivered to the supplier 0, and the supplier then uses a fleet of K vehicles of capacity C to supply a set of clients $I = \{1, 2, \ldots, n\}$. The deliveries must be planned so that the demand D^i of each client $i \in I$ in each period is satisfied and his stock capacity is not exceeded. In addition, in each period t, the vehicle delivering to the clients leaves before the arrival of the D^0 units at the supplier.

Let $T_0 = T \cup \{0\}$, let $sinit^0$ be the initial stock of the supplier and let $sinit^i$ be the initial stock of client i. Let h^0 and h^i denote the unit storage costs for the supplier and for the clients $i \in I$, respectively.

Under the ML replenishment policy, there is an upper bound $\bar{S}^i$ on the stock held by customer $i \in I$ at the end of a period. Under the assumption that a delivery to client i (if any) in a period occurs before the demand D^i is met, it follows that the maximum value W^i of a shipment to client i in a period is $W^i = \bar{S}^i + D^i$, and with this notation $\bar{S}^i = W^i - D^i$.

Besides defining stock levels, the vehicle routes in each period $t \in T$ must be determined. The distribution network in each period $t \in T$ is represented by a directed graph $G(I_0, A)$ where $I_0 = I \cup \{0\}$ denotes the set of the nodes corresponding to the supplier (depot) and to the clients and $A = \{ij \in I_0 \times I_0 : i \neq j\}$. A travel cost c^{ij} is associated with each arc $ij \in A$.

A solution of IRP involves a decision as to which clients are served by which vehicles in period t, the order in which they are served (i.e. the route at time t) and the delivery amounts in order to minimize the sum of the storage and routing costs over the time horizon.

The following variables are used:

> x_t^i is the amount shipped to client $i \in I$ in period $t \in T$;
>
> s_t^i is the stock of client $i \in I$ at the end of time period $t \in T_0$;
>
> s_t^0 is the stock level of the supplier at the end of time period $t \in T_0$;
>
> z_t^i is a binary variable which is 1 if the client $i \in I$ is visited at time $t \in T$, 0 otherwise;
>
> z_t^0 is an integer variable giving the number of vehicles delivering to one or more clients in period $t \in T$;
>
> y_t^{ij} is a binary variable which is 1 if the arc $(ij) \in A$ belongs to the route of some vehicle at time $t \in T$, 0 otherwise.

A basic IRP formulation is:

$$\min \quad \sum_{t \in T_0} \sum_{i \in I_0} h_t^i s_t^i + \sum_{t \in T} \sum_{ij \in A} c^{ij} y_t^{ij}$$

$$s_t^0 = s_{t-1}^0 + D^0 - \sum_{i \in I} x_t^i, \quad t \in T \tag{1}$$

$$s_t^i = s_{t-1}^i + x_t^i - D^i, \quad i \in I,\ t \in T \tag{2}$$

$$s_0^i = sinit^i, \quad i \in I_0 \tag{3}$$

$$x_t^i \leq W^i z_i^t \quad i \in I,\ t \in T \tag{4}$$

$$s_t^i \leq W^i - D^i, \quad i \in I,\ t \in T \tag{5}$$

$$z_t^i \leq z_t^0 \leq K, \quad i \in I, t \in T \tag{6}$$

$$\sum_{i \in I} x_t^i \leq C z_t^0, \quad t \in T \tag{7}$$

$$\sum_{j \in I_0} y_t^{ji} = \sum_{j \in I_0} y_t^{ij} = z_t^i, \quad i \in I_0,\ t \in T \tag{8}$$

$$\sum_{ij \in (I_0 \setminus S : S)} y_t^{ij} \geq z_t^\ell, \quad \ell \in S \subset I,\ t \in T \tag{9}$$

$$\sum_{ij \in (I_0 \setminus S : S)} C y_t^{ij} \geq \sum_{i \in S} x_t^i, \quad S \subseteq I, t \in T \tag{10}$$

$$\sum_{t \in T} z_t^0 \geq \left\lceil \frac{\sum_{i \in I} (T_{max} D^i - sinit^i)}{C} \right\rceil \tag{11}$$

$$z_t^i \in \{0, 1\}, \quad i \in I,\ t \in T \tag{12}$$

$$z_t^0 \in \mathbb{Z}^+, \quad t \in T \tag{13}$$

$$y_t^{ij} \in \{0, 1\}, \quad ij \in A,\ t \in T \tag{14}$$

$$s_t^i \geq 0, \quad i \in I_0,\ t \in T_0 \tag{15}$$

$$x_t^i \geq 0, \quad i \in I,\ t \in T, \tag{16}$$

where $(E:F)$ denotes the set of arcs $ij \in A$ with $i \in E$ and $j \in F$.

Constraints (1) and (2) are the balance constraints for the supplier and the clients respectively.
Constraints (3) give the initial stock levels for the supplier and the clients.
The variable upper bounds (4) bound the amount delivered to client i and enforce $z_t^i = 1$ if the client is served at time t. Constraints (5) give the stock upper bounds at each client.
Constraints (6) enforce $z_t^0 \geq 1$ if at least one client is served in period t.
Constraints (7) impose that the total amount shipped from the supplier to clients at t does not exceed the vehicle capacity C times the number of vehicles in use.
Constraints (8) ensure that the number of vehicles leaving the depot corresponds to the number of vehicles in use in each period and that a vehicle arrives at and leaves the client j in period t if and only if $z_t^j = 1$.
Constraints (9) are classical subtour elimination constraints that are added dynamically to the formulation. The separation algorithm consists of solving a min-cut problem between 0 and i, for each $i \in V$ and $t \in T$ on the graph $G(I_0, A)$, where the arcs A are weighted with the fractional values attained by the variables y_t^{ij} in the current LP relaxation, see for instance [11]
Constraints (10) are Capacitated Subtour Elimination constraints [1], which are added dynamically to the formulation. It again follows from Picard and Ratliff [12] that the separation algorithm is polynomially solvable as a min-cut problem.
Constraint (11) provides a lower bound on the number of vehicle tours needed to satisfy demand over the whole time horizon.

Formulation $(1)-(16)$ can be tightened by noting that $s_{t-1}^i \geq sinit^i - (t-1)D^i$ and so an upper bound on the delivery quantity in t is $W_t^i = W^i - (sinit^i - (t-1)D^i)^+$. In addition one can use the single-item reformulations of the lot-sizing problem with stock upper bounds discussed in [4].

3 The Single-Period Subproblem: Introducing Disjoint Route Inequalities

The Single-Period Inventory Routing Problem is defined by constraints $(1)-(10)$ of the formulation, for a fixed t. Specifically we consider the set (SIRP):

$$\sum_{i\in I_0} y_t^{ij} = \sum_{i\in I_0} y_t^{ji} = z_t^j, \quad j \in I_0 \tag{17}$$

$$\sum_{ij\in(I_0\setminus S:S)} y_t^{ij} \geq z_t^\ell, \quad \ell \in S \subseteq I \tag{18}$$

$$\sum_{ij\in(I_0\setminus S:S)} Cy_t^{ij} \geq \sum_{i\in S} x_t^i, \quad S \subseteq I \tag{19}$$

$$z_t^i \leq z_t^0 \leq K, \quad i \in I, \tag{20}$$

$$0 \leq x_t^i \leq W^i z_t^i, \quad i \in I \tag{21}$$

$$z_t^0 \in \mathbb{Z}_+^1, \; z_t^i \in \{0,1\} \; i \in I, \; y_t^{ij} \in \{0,1\} \; ij \in A \tag{22}$$

$$s_{\tau-1}^i + x_\tau^i = D^i + s_\tau^i, \; x_\tau^i \geq 0 \quad \tau \in \{t, t+1\}, \; i \in I \tag{23}$$

$$0 \leq s_\tau^i \leq W^i - D^i, \quad \tau \in \{t-1, t, t+1\}, \; i \in I. \tag{24}$$

Constraints (17) - (22) define the basic single period model (SIRP). The additional constraints (23) - (24) will allow us to generate inequalities involving the stock variables thereby linking together consecutive periods. Note that constraints (17) - (22) with the additional equations $x_t^i = D^i z_t^i$ describe the feasible region of the “Routing Problem with Profits" of Archetti et al. [3] and with $x_t^i = D^i z_t^i, z_t^i = 1$ it becomes that of the well-known capacitated vehicle routing problem (CVRP) [8, 10].

Now we introduce the Disjoint Route Inequalities for the basic model SIRP. The proof of validity is given for a more general case in the next section.

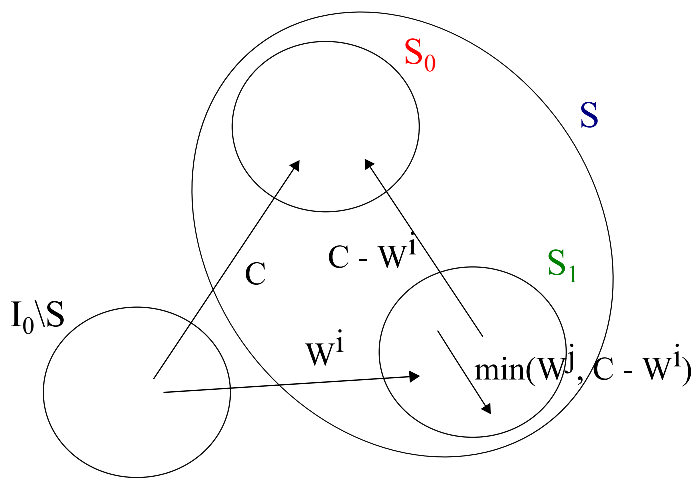

Figure 1: *A Disjoint Route Inequality*

Proposition 1. *Given $S \subseteq I$ and $\mu^{ij} \geq 0$ for $ij \in A$, the inequality*

$$\sum_{ij\in A} \mu^{ij} y_t^{ij} \geq \sum_{i\in S} x_t^i$$

is valid for SIRP $\big((17)$ *-* $(22)\big)$ *if*

$$\sum_{ij\in R}\mu^{ij} \geq \min(C, \sum_{i\in V(R)\cap S} W^i) \tag{25}$$

for every route (subtour) R *starting and ending at the depot, where* $V(R)$ *is the set of nodes in* I *of route* R*.*

In Figure 1 we demonstrate the μ^{ij} values leading to a disjoint route inequality. Specifically we have a partition S_0, S_1 of S. We set $\mu^{ij} = C$ for $ij \in (I_0 \setminus S : S_0)$, $\mu^{ij} = W^j$ for $ij \in (I_0 \setminus S : S_1)$, $\mu^{ij} = C - W^i$ for $ij \in (S_1 : S_0)$, $\mu^{ij} = \min(C - W^i, W^j)$ for $ij \in (S_1 : S_1)$ and $\mu^{ij} = 0$ otherwise. It is easily checked that the condition (25) holds for every route R and every $S \subseteq I$.
Note that with $\mu^{ij} = C$ for $ij \in (I_0 \setminus S : S)$ and $\mu^{ij} = 0$ for $ij \notin (I_0 \setminus S : S)$ one obtains the capacitated subtour constraint (19).

4 The Disjoint Route Inequalities

Now we present the more general family of Disjoint Route (DR) Inequalities.

Theorem 1. *Given a period* $t \in T$*, a subset* $S \subseteq I$*, a partition* (S_1, S_2, S_3, S_4) *of* S *and* $\mu^{ij} \geq 0$ *for* $ij \in A$*, the inequality*

$$\sum_{ij\in A}\mu^{ij}y_t^{ij} \geq \sum_{i\in S_1} x_t^i + \sum_{i\in S_2}(x_t^i - s_{t+1}^i) + \sum_{i\in S_3}(x_t^i - s_t^i) + \sum_{i\in S_4}(s_t^i - (W^i - 2D^i)) \tag{26}$$

is valid for SIRP if

$$\sum_{ij\in R}\mu^{ij} \geq \min\big(C, \sum_{i\in V(R)\cap S_1} W_t^i + \sum_{i\in V(R)\cap S_2} 2D^i + \sum_{i\in V(R)\cap(S_3\cup S_4)} D^i\big), \tag{27}$$

for every route R *starting and ending at the depot.*

Proof. Let (x_t, y_t, z_t, s) be a feasible solution of SIRP(17-24) for period t and $R_1, \ldots, R_Q$ with $Q \leq K$ be the corresponding routes. As the routes are arc-disjoint,

$$\sum_{ij\in A}\mu^{ij}y_t^{ij} = \sum_{q=1}^{Q}\sum_{ij\in R_q}\mu^{ij}.$$

Let R be one such route, $T_\ell = V(R) \cap S_\ell$ and $T = \cup_{\ell=1}^4 T_\ell$.
By hypothesis, $\sum_{ij\in R}\mu^{ij} \geq \min\big(C, \sum_{i\in T_1} W_t^i + \sum_{i\in T_2} 2D^i + \sum_{i\in T_3\cup T_4} D^i\big)$.
As each vehicle has capacity C, we have $C \geq \sum_{\ell=1}^4\sum_{i\in T_\ell} x_t^i$. Therefore

$$\sum_{ij\in R}\mu^{ij} \geq \sum_{i\in T_1}\min(x_t^i, W^i) + \sum_{i\in T_2}\min(x_t^i, 2D^i) + \sum_{i\in T_3\cup T_4}\min(x_t^i, D^i).$$

For $i \in T_1$, we have: $W^i \geq x_t^i$.
For $i \in T_2$, we have $x_t^i \geq x_t^i - s_{t+1}^i$ as $s_{t+1}^i \geq 0$. Also $2D^i \geq x_t^i - s_{t+1}^i$ as $x_t^i \leq 2D^i + s_{t+1}^i$.

For $i \in T_3$, we have $x_t^i \geq x_t^i - s_t^i$ as $s_t^i \geq 0$. Also $D^i \geq x_t^i - s_t^i$ as $x_t^i \leq D^i + s_t^i$.
For $i \in T_4$, we have $x_t^i = s_t^i + D^i - s_{t-1}^i$. As $s_{t-1}^i \leq W^i - D^i$, this gives $x_t^i \geq s_t^i - (W^i - 2D^i)$. Also $D^i \geq s_t^i - (W^i - 2D^i)$ as $s_t^i \leq W^i - D^i$.
It follows that:

$$\sum_{ij \in R} \mu_t^{ij} \geq \sum_{i \in T_1} x_t^i + \sum_{i \in T_2} (x_t^i - s_{t+1}^i) + \sum_{i \in T_3} (x_t^i - s_t^i) + \sum_{i \in T_4} (s_t^i - (W^i - 2D^i))$$

Summing over the Q disjoint routes gives the inequality (26).
□

Even for a given $S \subseteq I$ the separation problem for the DR Inequalities appears to be intractable, so we consider two subfamilies - namely Simple DR and h-DR Inequalities - whose separation problem can be efficiently solved in practice.

4.1 Simple Disjoint Route Inequalities

We assume for simplicity that $C \geq W^i \geq 2D^i \geq D^i$ for all $i \in I$. The following proposition introduces the *Simple Disjoint Route (Simple DR)* Inequalities.

Proposition 2. *Let $S \subseteq I$, $(S_0, S_1, S_2, S_3, S_4)$ be a partition of S. Let $v^i = C$ for $i \in S_0$, $v^i = W^i$ for $i \in S_1$, $v^i = 2D^i$ for $i \in S_2$, $v^i = D^i$ for $i \in S_3 \cup S_4$ and $v^i = 0$ for $i \in I_0 \setminus S$.*
Set $\mu^{ij} = \min(C - v^i, v^j)$ for all $ij \in A$.

Then the inequality

$$\sum_{ij \in A} \mu^{ij} y_t^{ij} \geq \sum_{j \in S_0 \cup S_1} x_t^j + \sum_{j \in S_2} (x_t^j - s_{t+1}^j) + \sum_{j \in S_3} (x_t^j - s_t^j) + \sum_{j \in S_4} (s_t^j - (W^j - 2D^j)) \qquad (28)$$

is a DR inequality valid for SIRP.

Proof. Consider any initial subpath P of a route R starting at the depot. We claim that

$$\sum_{ij \in P} \mu^{ij} \geq \min \big(C, \sum_{i \in V(P) \cap S} v^i\big). \qquad (29)$$

The claim clearly holds for paths of length $p = 1$. Now consider a subpath P of length p ending at node k and its extension $P' = P \cup (k\ell)$. It suffices to consider the case where $\sum_{ij \in P} \mu^{ij} < C$ and thus we assume $\sum_{ij \in P} \mu^{ij} \geq \sum_{i \in V(P) \cap S} v^i$.

i) If $\mu^{k\ell} = v^\ell$, then $\sum_{ij \in P'} \mu^{ij} \geq \sum_{i \in V(P) \cap S} v^i + v^\ell = \sum_{i \in V(P') \cap S} v^i$.
ii) If $\mu^{k\ell} = C - v^k$, then $\sum_{ij \in P'} \mu^{ij} = \sum_{ij \in P} \mu^{ij} + (C - v^k) \geq v^k + (C - v^k) = C$.

Thus the condition (29) holds for any subpath P' of length $p + 1$, and thus (27) holds for all routes. It follows that the (28) is a DR inequality.
□

Note that when $S_2 = S_3 = S_4 = \emptyset$, one obtains the inequality of Proposition 1 and Figure 1.

4.1.1 Separation of Simple DR Inequalities

The separation problem for the Simple DR Inequalities amounts to solving a 0-1 Quadratic Programming problem.
For simplicity of notation, set $S_5 = I_0 \setminus S$ with $v^5 = 0$. Let $(\bar{\mathbf{z}}, \bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{s}})$ be the current fractional solution. Let α_k^i be a binary variable which is 1 if $i \in S_k$ and 0 otherwise, and let v_k^i be the value of v^i when $i \in S_k$. Then the separation problem can be written as:

$$
\begin{aligned}
\min \quad & \sum_{ij \in A} \sum_{k=0}^{5} \sum_{l=0}^{5} \min(C - v_k^i, v_l^j) \bar{y}^{ij} \alpha_k^i \alpha_l^j \\
& - \sum_{j \in I_0} \bar{x}_t^j (\alpha_0^j + \alpha_1^j) - \sum_{j \in I_0} (\bar{x}_t^j - \bar{s}_{t+1}^j) \alpha_2^j \\
& - \sum_{j \in I_0} (\bar{x}_t^j - \bar{s}_t^j) \alpha_3^j - \sum_{j \in I_0} (\bar{s}_t^j - (W^j - 2D^j)) \alpha_4^j, \\
& \qquad \sum_{k=0}^{5} \alpha_k^j = 1, \quad j \in I_0 \\
& \alpha_k^j \in \{0,1\} \; j \in I_0, k \in 0, \ldots, 5, \; \alpha_5^0 = 1.
\end{aligned}
$$

Standard linearization gives an MIP that is easy in practice. In general the resulting linear program is not integral due to the constraints $\sum_{k=0}^{5} \alpha_k^j = 1$ for $j \in I_0$.

4.2 h-Disjoint Route Inequalities

By considering paths of maximum length $h \geq 2$, we get a different subfamily of DR Inequalities, the *h-Disjoint Route (h-DR) Inequalities*. As before $V(R)$ denotes the set of nodes in I of a route R and let i_R be the first node after leaving the depot.

Proposition 3. *Let $S \subseteq I$ and let (S_1, S_2, S_3, S_4) be a partition of S. Let Π_h be the set of routes R starting and ending at the depot with $|V(R)| = k \leq h$ The h-DR Inequality:*

$$
\sum_{ij \in (I_0 \setminus S:S) \cup ((S:S) \setminus (S_1:S_1))} \ell_j y_t^{ij} + \sum_{ij \in (S_1:S_1)} m_{ij} y_t^{ij} \geq \sum_{i \in S_1} x_t^i + \sum_{i \in S_2} (x_t^i - s_{t+1}^i) + \sum_{i \in S_3} (x_t^i - s_t^i) + \sum_{i \in S_4} (s_t^i - (W^i - 2D^i)) \tag{30}
$$

is a DR Inequality valid for SIRP if:

i) $W^j \leq \ell_j \leq C, \quad j \in S_1;$

ii) $2D^j \leq \ell_j \leq C, \quad j \in S_2;$

iii) $D^j \leq \ell_j \leq C, \quad j \in S_3 \cup S_4;$

iv) $\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq min(C, \sum_{i \in V(R)} W^i), \quad R \in \Pi_{h-1}$ *with* $V(R) \subseteq S_1;$

v) $\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq C, \quad R \in (\Pi_h \setminus \Pi_{h-1})$ *with* $V(R) \subseteq S_1$;

vi) $m_{ij} \geq 0, \quad ij \in (S_1 : S_1)$.

Proof. We show that inequality (30) is a DR inequality. Setting $\mu_{ij} = \ell_j$ for $ij \in (I_0 \setminus S : S) \cup ((S : S) \setminus (S_1 : S_1))$, $\mu_{ij} = m_{ij}$ for $ij \in (S_1 : S_1)$ and $\mu_{ij} = 0$ otherwise, we need to show that condition (27) holds on every route.

First we prove the proposition for the case $S_2 \cup S_3 \cup S_4 = \emptyset$ and then in the general case.

Case $S_2 \cup S_3 \cup S_4 = \emptyset$, $(S = S_1)$.

Let R be a route beginning and ending at the depot. We break up the route into subpaths with nodes $T_1, \ldots, T_L$ with $L \geq 1$ where each subpath T_ℓ consists of a path in $I_0 \setminus S$ followed by a path in S, except for the last subpath T_L which just consists of a subpath in $I_0 \setminus S$ ending at the depot. Let A_ℓ be the set of arcs of T_ℓ. We will show that

$$\sum_{ij \in A_\ell} \mu^{ij} \geq \min(C, \sum_{i \in T_\ell \cap S} W^i) \tag{31}$$

for all ℓ. Summing will then give

$$\sum_{ij \in R} \mu^{ij} = \sum_{\ell=1}^{L} \min(C, \sum_{i \in T_\ell \cap S} W^i) \geq \min(C, \sum_{i \in V(R) \cap S} W^i)$$

where the inequality follows as $\min(a,b) + \min(a,c) \geq \min(a, b+c)$ when $a, b, c \geq 0$ and the arcs linking subpaths all have heads in $I_0 \setminus S$ and $\mu^{ij} = 0$.
Thus it suffices to show that $lhs(31) \geq rhs(31)$ on each subpath T_ℓ.
But each subpath T_ℓ has the same μ-value as the route $(0, T_\ell, 0)$ of length $|T_\ell|$. So the conditions iv) and v) ensure that (31) holds for each T_ℓ.

Case $S_2 \cup S_3 \cup S_4 \neq \emptyset$.

$lhs(30)$ can be rewritten as the sum of three terms:

$$T1 \equiv \sum_{ij \in (I_0 \setminus S_1 : S_1)} \ell_j y_t^{ij} + \sum_{ij \in (S_1 : S_1)} m_{ij} y_t^{ij}$$

$$T2 \equiv \sum_{ij \in (I_0 : S_2)} \ell_j y_t^{ij}$$

$$T34 \equiv \sum_{ij \in (I_0 : S_3 \cup S_4)} \ell_j y_t^{ij}$$

For each route R, the μ-value of T_1 is the same as that of a route R' in which each visit to a node of $S_2 \cup S_3 \cup S_4$ is replaced by a node in $I_0 \setminus S$. The value of T_1 for this route R' was treated above and it was shown that

$$\sum_{ij \in (I_0 \setminus S_1 : S_1)} \ell_j y_t^{ij} + \sum_{ij \in (S_1 : S_1)} m_{ij} y_t^{ij} \geq min\big(C, \sum_{i \in V(R) \cap S_1} W^i\big) \tag{32}$$

Now we consider the term $T2$. By condition ii), $\ell_i \geq 2D^i$ for each $i \in S_2$. So, for each route starting and ending at the depot, we have:

$$\sum_{ij \in (I_0 : S_2)} \ell_j y_t^{ij} \geq \sum_{i \in V(R) \cap S_2} 2D^i \tag{33}$$

Finally we consider the term $T34$. By condition iii), $\ell_i \geq D^i$ for each $i \in S_3 \cup S_4$. So, for each route starting and ending at the depot, we have:

$$\sum_{ij \in (I_0 : S_3 \cup S_4)} \ell_j y_t^{ij} \geq \sum_{i \in V(R) \cap (S_3 \cup S_4)} D^i \tag{34}$$

Summing (32), (33) and (34), we have:

$$\sum_{ij \in R} \mu^{ij} \geq \min\big(C, \sum_{i \in V(R) \cap S_1} W^i + \sum_{i \in V(R) \cap S_2} 2D^i + \sum_{i \in V(R) \cap (S_3 \cup S_4)} D^i\big)$$

for each route. The last inequality uses $min(a, b) + c = min(a + c, b + c) \geq min(a, b + c)$. So condition (27) is satisfied and the claim follows.
$\square$

4.2.1 Separation of h-Disjoint Route Inequalities

Let $S \subset I$ and (S_1, S_2, S_3, S_4) be a partition of S and let $(\bar{\mathbf{z}}, \bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{s}})$ be the current fractional solution. The separation problem for a given set S and its partition amounts to solve the following LP:

$$v^* = \min \quad \sum_{ij \in A} \bar{y}_t^{ij} m_{ij} \tag{35}$$

$$\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq min(C, \sum_{i \in V(R)} W^i), \;\; R \in \Pi_{h-1} \tag{36}$$

$$\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq C, \;\; R \in \Pi_h \setminus \Pi_{h-1} \tag{37}$$

$$m_{ij} = \ell_j, \;\; ij \in (I_0 \setminus S : S) \tag{38}$$

$$m_{ij} = \ell_j, \;\; ij \in (S : S) \setminus (S_1 : S_1) \tag{39}$$

$$W^i \leq \ell_i \leq C, \;\; i \in S_1 \tag{40}$$

$$2D^i \leq \ell_i \leq C, \;\; i \in S_2 \tag{41}$$

$$D^i \leq \ell_i \leq C, \;\; i \in S_3 \cup S_4 \tag{42}$$

$$m_{ij} \geq 0, \;\; ij \in A.$$

The inequality (30) is violated if $v^* < \sum_{i \in S_1} \bar{x}_t^i + \sum_{i \in S_2} (\bar{x}_t^i - \bar{s}_{t+1}^i) + \sum_{i \in S_3} (\bar{x}_t^i - \bar{s}_t^i) + \sum_{i \in S_4} (\bar{s}_t^i - (W^i - 2D^i))$.

This separation LP becomes hard to solve when $|I| \geq 20$, because of the huge number of constraints (36) and (37). To overcome this problem, we adopt the following procedure:

- Solve the separation LP on a reduced network $(I_0, \bar{A})$ where $\bar{A} = \{ij : \bar{y}_t^{ij} > 0\}$. Let $\bar{\ell}_i$ for $i \in I$ and $\bar{m}_{ij}$ for $ij \in \bar{A}$ be its solution.
- Extend the solution by setting $\bar{m}_{ij} = 0$ for all $ij \in A \setminus \bar{A}$.
- Enumerate all the routes $R \in \Pi_h$. As long as a route R is encountered for which (36) or (37) is violated, let $uv \notin \bar{A}$ be an arc of R. Set

  $$\bar{m}_{uv} \leftarrow \min\big(C, \sum_{i \in V(R)\cap S_1} W^i + \sum_{i \in V(R)\cap S_2} 2D^i + \sum_{i \in V(R)\cap(S_3 \cup S_4)} D^i\big) - \sum_{ij \in R} \bar{m}_{ij} - \bar{\ell}_i$$

  .

  In case of ties, select uv corresponding to the variable y_t^{uv} with the maximum reduced cost in the IRP relaxation providing the current fractional solution $(\bar{\mathbf{z}}, \bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{s}})$.

Note that in the enumeration routes R are often truncated quickly because $\bar{\ell}_i + \sum_{ij \in R} \bar{m}_{ij} > C$.

We adopt two different strategies for the choice of S.

Approach 1: Select the set S and its partition obtained from the solution of the quadratic $0-1$ separation problem for the simple DR inequalities and then use the procedure outlined above.

Approach 2: With $S_2 = S_3 = S_4 = \emptyset$, the exact separation problem for the h-DR inequalities can be formulated as a (big-M) mixed integer program. Specifically let α^i be a binary variables such that $\alpha^i = 1$ if $i \in S$ with the formulation:

$$\min \quad \sum_{ij \in A} \bar{y}_t^{ij} m^{ij} - \sum_{i \in I} \bar{x}_t^i \alpha^i \tag{43}$$

$$\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq min(C, \sum_{i \in V(R)} W^i)\big(\sum_{j \in V(R)} \alpha^j - (|V(R)| - 1)\big), \; R \in \Pi_{h-1} \tag{44}$$

$$\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq C\big(\sum_{j \in V(R)} \alpha^i - (h-1)\big), \; R \in \Pi_h \setminus \Pi_{h-1} \tag{45}$$

$$m_{ij} \geq \ell_j - C\alpha^i, \; i \in I_0, j \in I \tag{46}$$

$$W^i \alpha^i \leq \ell_i \leq C\alpha^i, \; i \in I \tag{47}$$

$$m_{ij} \geq 0, \; ij \in A, \; \alpha^i \in \{0,1\}, \; i \in I.$$

Note that the inequalities (44) and (45) enforce (36) and (37) when the path lies in S. (46) and the non-negative coefficient of m_{ij} in the objective function (43) lead to satisfaction of (38). Finally (47) implies (40) when $i \in S$.
This MIP is then solved over the reduced network providing a set S and an initial solution $(\bar{\ell}, \bar{m})$. The conversion to a feasible solution again follows the procedure outlined above. Surprisingly the above MIP can be solved in a reasonable time on the benchmark instances.

5 Multi-period Capacitated Subtour Inequality

In [7] Coelho and Laporte introduced a family of multi-period inequalities that we call *Multiperiod Capacitated Subtours*.
Let $H = [t_{ini}, t_{end}]$ be a time interval within T. Let $ND_H^i = D^i(t_{end} - t_{ini} + 1) - (W^i - D^i)$ be the net total demand of client $i \in I$. Let $S \subset I$ be a subset of clients.

Proposition 4. *The Multiperiod Capacitated Subtour Inequality*

$$\sum_{t \in H} \sum_{ij \in (I_0 \setminus S : S)} y_t^{ij} \geq \left\lceil \frac{\sum_{i \in S} ND_H^i}{C} \right\rceil \tag{48}$$

is valid for IRP.

Here an exact separation algorithm based on an MIP formulation.

5.1 Separation of Multiperiod Capacitated Subtours

The separation of Multiperiod Capacitated Subtours for a time interval H amounts to solving a multiperiod $0-1$-quadratic program with a side constraint enforcing rounding. Let $(\bar{\mathbf{z}}, \bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{s}})$ be the current fractional solution, let $\alpha_j = 1$ if $j \in S$, 0 otherwise and let γ be an integer variable expressing the rhs:

$$\min \quad \sum_{t \in H} \sum_{ij \in A} \bar{y}_t^{ij}(1 - \alpha^i)\alpha^j - \gamma \tag{49}$$

$$C\gamma \leq \sum_{i \in I} ND_H^i \alpha^i + 1 - \varepsilon \tag{50}$$

$$\alpha^i \in \{0, 1\}, \ i \in I \tag{51}$$

$$\gamma \in \mathbb{Z}. \tag{52}$$

A standard linearization gives a block-structured MIP problem, having a block for each period. The linear problem is not integral because of the rounding constraint (50), which links all the blocks. However the problems is very efficiently solved in practice by a MIP solver.

6 Implementation details

The separation procedures for the Subtour, Disjoint Route and the Multiperiod Capacitated Subtour Inequalities have been embedded into the cutting planes generation callbacks of the commercial mixed integer programming solver FICO Xpress 7.6. The code is written in ANSI C using the Microsoft Visual C++ 2013 compiler. The resulting branch-and-cut algorithm is run in two versions: first as a heuristic to find an upper bound (upper bounding phase) and then we change the settings to get provably good solutions (optimality phase).
In the upper bounding phase we adopt a simple MIP-based “fixing" heuristic: after cutting plane generation at the root node, we *i*) fix to zero all the routing variables which are at zero in the

current fractional solution and then *ii*) solve the restricted MIP by a truncated version of our branch-and-cut algorithm with a branching strategy aimed at finding good feasible solutions. In both phases we set Xpress parameters to run the code with a single thread. We set a time limit of 1800 seconds for the upper bounding phase and of 3600 seconds for the optimality phase. The Xpress preprocessing, heuristics and cut generation procedures were disabled to allow the use of callbacks.

6.1 Preprocessing

Reduced cost fixing on the routing variables y_t^{ij} is performed at the end of the root node.

6.2 A Priori Reformulation

The upper bounds on deliveries W_t^i are tightened if the stock at the end of the previous period has a positive lower bound. The tight formulations for single item lot-sizing with stock upper bounds given in [4] are added for each item/client.

6.3 Cutting strategy

Separation procedures are performed in the following order:

i) Subtour Elimination Constraints (at every node of the search tree)

ii) Capacitated Subtour Constraints (at every node of the search tree)

ii) Multiperiod Capacitated Subtour Inequalities (only at the root node)

iv) Simple DR Inequalities (only at nodes with depth ≤ 5)

v) h-DR inequalities with S generated by the Simple Metric (only at nodes with depth ≤ 5)

vi) MIP separation of h-DR Inequalities with $h \leq 8$ if the number of clients does not exceed 30 (only at nodes with depth ≤ 5).

Each procedure is run repeatedly till the violation falls below a given tolerance. One then moves on to the next procedure. When all six procedures have terminated, one restarts with procedure i). However in the heuristic/upper bounding phase the separation procedures iii)-vi) are only performed at the top node.

6.4 Branching strategy

In the upper bounding phase, the node selection strategy is best-first for the first 100 nodes of the search tree, then depth-first. In the optimality phase, the strategy is best-first. The variable selection strategy is the XPRESS 7.6 default strategy.

7 Computational results

We report detailed computational results on a large set of IRP instances, comparing the results of our branch-and-cut algorithm - denoted ABW - with the reported results of the Branch-and-Cut algorithm of Coelho and Laporte [7] - denoted CL - based on a 3-index (arc ij on vehicle k) formulation and of the Branch-and-Cut-and-Price algorithm of Desaulniers, Rakke and Coelho [9] - denoted DRC - in which the columns correspond to route and delivery patterns for a given period.

Our computation was carried out on an Intel(R) Core(TM) i7-2620 processor clocked at 2.7 GHz with 8GB of RAM. The reported results for the DL and DRC algorithms were both obtained using an Intel(R) Core(TM) i7-2600 processor clocked at 3.40GHz with 8 cores and 16GB of RAM. Only a single core was used for both algorithms.

The test bed consists of the the largest and the most challenging instances in the dataset introduced in [2], namely those with 50 clients and three periods, and those with $15, 20, 25, 30$ clients and six periods. All the instances and the computational results for both the CL and DRC algorithms are available at L. Coelho's webpage *http://www.leandro-coelho.com/instances/*. The instances are partitioned in two main groups: those with "low "storage costs, namely $h_i \in [0.01, 0.05]$ and $h_0 = 0.03$, and those with "high "storage costs, namely $h_i \in [0.1, 0.5]$ and $h_0 = 0.3$. As usual in testing exact algorithms for IRP, travel costs are given by the euclidean distance between each pair of nodes, rounded to the nearest integer, and the vehicle capacity is obtained by dividing the original capacity by the number of vehicles, rounded to the nearest integer (x.5 is rounded to $x + 1$).

Each instance is labeled as nXX-TY-{low,high}-f, where XX is the number of clients, Y is the number of periods, the attribute {low, high} denotes the magnitude of the storage costs and f is a number identifying the instance.

We first report briefly on the behaviour of our cutting plane procedures. In Table 1 we present results for four variants of two instances with 20 clients and 6 periods. The initial formulation consists of (1)-(16) plus the single-item lot-sizing reformulations presented in [4]. Column 2 gives the number of vehicles K and column 3 the best upper bound obtained by our branch-and-cut algorithm. Columns 4-6 present the results obtained just using the subtour separation routines i) and ii) exhaustively at the top node, columns 4-6 using routines i), ii) and iii) (multi-period cuts) exhaustively at the top nodes, columns 7-9 using i)-iv) exhaustively and finally columns 10-12 using i)-vi) exhaustively. For each set of three columns we give the lower bound obtained, the time for separation and re-optimization and finally the %-gap relative to the best solution given in Column 3.

Next in Tables 2 and 3 we report results with the ABW algorithm for the instances with 50 clients and three periods and 30 clients and six periods instances as well as the reported results of CL and DRC. Detailed results for the instances with $15, 20, 25$ clients and six periods are reported in the Appendix 9. Each table is organized as follows. Column $Name$ shows the name of the instance, K is the number of vehicles. For each algorithm, LB, BLB and BUB denote the lower bound at the root node, the best lower bound and the best upper bound found after enumeration, respectively. Columns $Time$ show the CPU seconds spent to produce the final

			Subtours			Multi-period			S-DR			h-DR		
Name	K	*BUB*	LB	Time	%Gap	LB	Time	%Gap	LB	Time	%Gap	LB	Time	%Gap
low-1	2	7752.98	6902.48	3	10.97	7106.96	8	8.33	7112.12	23	8.27	7177.75	236	7.42
	3	9073.71	8099.41	6	10.74	8348.6	19	7.99	8395.32	90	7.48	8486.83	305	6.47
	4	10499.07	9546.5	9	9.07	10009.91	24	4.66	10105.56	178	3.75	10145.41	382	3.37
	5	12603.24	11049.65	10	12.33	11486.53	37	8.86	11570.32	265	8.20	11602.3	216	7.94
high-1	2	15677.13	15058.76	3	3.94	15214.86	7	2.95	15236.52	21	2.81	15325.86	156	2.24
	3	17107.01	16251.64	5	5.00	16476.5	15	3.69	16554.75	96	3.23	16669.22	285	2.56
	4	18558.81	17690.91	6	4.68	18105.31	18	2.44	18244.36	136	1.69	18289.07	200	1.45
	5	20715.52	19182.12	7	7.40	19580.9	24	5.48	19709.05	115	4.86	19757.54	211	4.62

Table 1: Lower bounds from cuts at the root node for two $n20$-$T6$- instances

BLB and BUB values.

For the instances $n50$-$T3$ we observe that:

- our lower bounds at the top node are generally slightly lower (less than 1% of difference) than those of DRC and always much higher than those of CL;

- our lower bounds after enumeration are generally slightly higher than those of DRC for $K = 2, 3$ and slightly worse for $K = 4, 5$, always much higher than those of CL

- the best upper bounds are significantly lower than those of the other two algorithms for $K = 4, 5$.

- the overall effect is that our algorithm returns gaps that are significantly smaller than those of the other two algorithms, particularly for $K = 4, 5$.

- we solved to optimality 9 instances that the DRC algorithm did not solve. Three of these instances were not solved by the CL algorithm either. For the instance n50-T3-low-4 with $K = 2$ we get an optimal value less than the optimal value reported for the CL algorithm on the L. Coelho webpage.

For the instances $n30$-$T6$ we observe that:

- our lower bounds at the top node as well as the lower bounds after enumeration are generally slightly lower (less than 2% of difference) than those of DRC and always much higher than those of CL;

- the best upper bounds are significantly lower than those of the other two algorithms, particularly for $K = 3, 4, 5$.

- the overall effect is that our algorithm returns gaps that are significantly smaller than those of the other two algorithms, particularly for $K = 3, 4, 5$.

- we solved to optimality two instances that were solved by CL, but not by DRC. For the instances n30-T6-high-2 and n30-T6-high-4, $K = 2$, we get an optimal value less than the lower bound reported for the CL algorithm on the L. Coelho webpage.

Furthermore we observe (see Appendix 9) that we solve to optimality four n15-T6, one n20-T6, and four n25-t6, K=2, instances which the DRC algorithm did not solve. Instances n25-T6-high-1 and n25-T6-high-4 were not solved by the CL algorithm either. For the instances n25-T6-high-1, n25-T6-high-2, n25-T6-high-4 and n25-T6-high-5, $K = 2$ we get a feasible solution whose value is smaller than the lower bound reported for the CL algorithm on the L. Coelho webpage.
Finally in the figures 2 and 3 we report, for each set of 10 instances, the average gap of the ABW algorithm, compared with the results of CL and DSR. The value of the average gap for each algorithm is computed without considering the instances for which the algorithm does not provide an upper bound. So the value of the average gap of the CL algorithm for the n50-T3 instances with $K = 5$ is not significant because the algorithm is able to find only two upper bounds on the 10 instances.

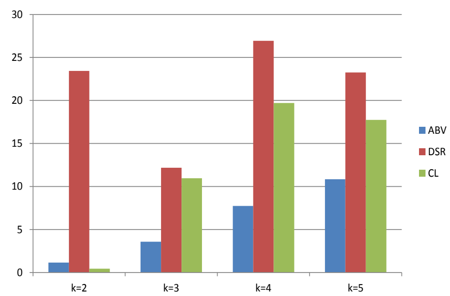

Figure 2: *Average gaps for ABW, CL and DRC on n50-T3 instances*

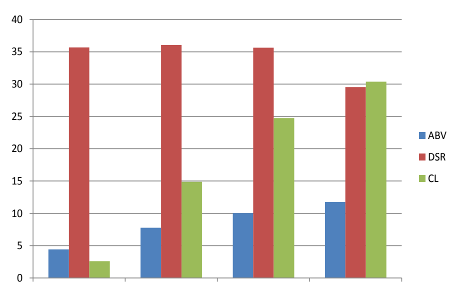

Figure 3: *Average gaps for ABW, CL and DRC on n30-T6 instances*

8 Final Remarks

Though the test sets for IRP are based on time-invariant demands and stock upper bounds, the disjoint route inequalities can easily be adapted to the case of time-varying demands and bounds D_t^i, W_t^i. However in any period it is necessary that the vehicle capacities are all the same.
As pointed out earlier, restrictions of the single period subproblem (SIRP) arise in the capacitated

Table 2: Computational results for $n50$-$T3$- instances

				ABW					DRC					CL				
Name	K	UB_{heu}	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time
low-1	2	4737.86	1800	4529.66	4554.42	4737.86	3.87	3600	4457.2	4457.2	7673.91	41.92	7200	4163.4	4567.16	4567.16	0.00	1536
low-2	2	4861.92	509	4741.71	4842.92	4842.92	0.00	382	4768.72	4770.25	4943.3	3.50	7200	4465.64	4842.92	4842.92	0.00	2707
low-3	2	4749.44	571	4629.22	4705.74	4705.74	0.00	1261	4636.45	4645.51	8054.85	42.33	7200	4463.88	4652.31	4730.31	1.65	7200
low-4	2	4648.42	103	4622	4648.42	4648.42	0.00	50	4636.68	4636.68	6287.6	26.26	7200	4480.91	4662.84	4662.84	0.00	118
low-5	2	4901.08	1800	4501.37	4541.65	4901.08	7.33	3600	4531.34	4531.34	7392.95	38.71	7200	4241.23	4507.91	4567.43	1.30	7200
high-1	2	15155.32	1029	15053.33	15097.1	15097.1	0.00	1246	14999.18	14999.18	18161.08	17.41	7200	14697.6	15097.1	15097.1	0.00	1763
high-2	2	15350.38	97	15239.22	15342.94	15342.94	0.00	641	15273.48	15273.48	18778.43	18.66	7200	14997.6	15186.5	15356.4	1.11	7200
high-3	2	15609.98	563	15447.61	15520.9	15520.9	0.00	1245	15455.64	15468.69	18388.44	15.88	7200	15294	15520.9	15520.9	0.00	3197
high-4	2	16775.12	215	16734.58	16775.12	16775.12	0.00	151	16747.42	16747.42	19641.16	14.73	7200	16593.7	16775.1	16775.1	0.00	801
high-5	2	16078.42	921	15950.13	16003.45	16067.87	0.40	3600	15978.09	15978.09	18790	14.96	7200	15689.6	15968.2	16027.7	0.37	7200
low-1	3	5202.18	1334	5186.06	5202.18	5202.18	0.00	236	5189.62	5189.62	5829.45	10.98	7200	4397.71	4668.74	5829.45	19.91	7200
low-2	3	5957.09	1800	5284.68	5385.02	5957.09	9.60	3600	5332.17	5351.07	6188.80	13.54	7200	4848.35	5015.45	6188.80	18.96	7200
low-3	3	5321.45	1109	4996.67	5047.37	5321.45	5.15	3600	5022.02	5041.01	5506.95	8.46	7200	4603.85	4744.26	5506.95	13.85	7200
low-4	3	5386.5	1800	5155.01	5216	5386.5	3.17	3600	5189.8	5213.18	5292.2	1.49	7200	4627.81	5053.54	5292.20	4.51	7200
low-5	3	5641.79	1800	5008.09	5041.32	5641.79	10.64	3600	5045.13	5054.91	5790.36	12.70	7200	4360.01	4537.52	5790.36	21.64	7200
high-1	3	15732.6	351	15709.43	15732.6	15732.6	0.00	293	15721.25	15721.25	15732.60	0.07	7200	14926.50	15069.50	16390.80	8.06	7200
high-2	3	16348.8	1800	15739.86	15858.75	16348.8	3.00	3600	15837.78	15854.55	21275.27	25.48	7200	15239.60	15299.70	16452.40	7.01	7200
high-3	3	16314.26	1262	15783.67	15866.64	16314.26	2.74	3600	15842.89	15854.46	19378.35	18.18	7200	15433.00	15569.70	16278.60	4.35	7200
high-4	3	17469.32	1093	17264.9	17364.51	17469.32	0.60	3600	17314.28	17326.98	19692.49	12.01	7200	16737.50	16968.80	17560.40	3.37	7200
high-5	3	16662.36	1174	16436.84	16512.84	16662.36	0.90	3600	16500.84	16510.20	20355.68	18.89	7200	15817.30	15998.30	17366.10	7.88	7200
low-1	4	6689.22	1800	5708.49	5754.65	6689.22	13.97	3600	5734.78	5760.4	8904.76	35.31	7200	4643.52	4750.38	7533.48	36.94	7200
low-2	4	7105.28	1800	5907.19	5968.52	7105.28	16.00	3600	6002.11	6035.23	10482.05	42.42	7200	4986.74	4989.10	-	-	7200
low-3	4	5589.17	1100	5398.38	5428.63	5589.17	2.87	3600	5419.76	5446.15	8693.82	37.36	7200	4824.93	4863.11	6516.87	25.38	7200
low-4	4	6986	1800	5790.53	5846.59	6986	16.31	3600	5902.73	5924.86	8725.63	32.10	7200	4784.96	4821.80	7119.92	32.28	7200
low-5	4	6456.78	1800	5641.01	5680.33	6456.78	12.03	3600	5684.83	5700.53	9251.72	38.38	7200	4526.44	4553.35	-	-	7200
high-1	4	16508.64	1561	16234.48	16302.13	16508.64	1.25	3600	16270.65	16281.23	19385.03	16.01	7200	15165.00	15190.70	17738.80	14.36	7200
high-2	4	17733.72	1799	16346.88	16460.04	17733.72	7.18	3600	16512.64	16538.71	20987.64	21.20	7200	15503.90	15540.90	17674.80	12.07	7200
high-3	4	16412.8	1018	16195.6	16265.77	16412.8	0.90	3600	16244.83	16266.05	19451.19	16.38	7200	15587.70	15682.10	17536.60	10.58	7200
high-4	4	18618.03	1800	17891.05	17994.82	18618.03	3.35	3600	18024.35	18043.89	20817.08	13.32	7200	16923.90	16975.00	19111.70	11.18	7200
high-5	4	17777.38	1800	17041.69	17135.24	17777.38	3.61	3600	17144.01	17156.75	20645.10	16.90	7200	15982.00	15992.50	18790.70	14.89	7200
low-1	5	7167.86	1800	6419.22	6446.11	7167.86	10.07	3600	6439.11	6452.71	10429.19	38.13	7200	4864.57	4881.49	-	-	7200
low-2	5	9134.85	1800	6518.19	6564.1	9134.85	28.14	3600	6577.23	6612.29	11967.18	44.75	7200	5246.68	-	-	-	7200
low-3	5	6889.06	1800	5823.12	5881.77	6889.06	14.62	3600	5913.07	5935.12	10045.05	40.91	7200	4930.35		-	-	7200
low-4	5	7594.7	1800	6390.44	6437.08	7594.7	15.24	3600	6570.07	6601.09	9741.39	32.24	7200	4993.16	-	-	-	7200
low-5	5	7035.78	1800	6197.02	6228.27	7035.78	11.48	3600	6225.53	6233.77	6233.77	0.00	7098	4746.76	-	-	-	7200
high-1	5	17954.77	1800	16944.35	16989.26	17954.77	5.38	3600	16972.92	16986.23	20925.28	18.82	7200	15387.40	-	-	-	7200
high-2	5	18233.75	1800	16986.48	17067.47	18233.75	6.40	3600	17093.89	17129.53	22496.79	23.86	7200	15769.70	15775.50	19517.90	19.17	7200
high-3	5	17391.73	1800	16608.09	16675.93	17391.73	4.12	3600	16739.11	16757.53	20840.11	19.59	7200	15752.10	-	-	-	7200
high-4	5	20111.17	1800	18524.5	18598.81	20111.17	7.52	3600	18689.82	18728.47	21847.94	14.28	7200	17140.90	-	-	-	7200
high-5	5	18688.59	1800	17589.85	17659.84	18688.59	5.50	3600	17678.44	17695.79	17695.79	0.00	5741	16207.30	16265.80	19438.00	16.32	7200

Table 3: Computational results for $n30$-$T6$- instances

				ABW					DRC					CL				
Name	K	UB_{heu}	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time
low-1	2	9385	1800	8792.7	8888.78	9385	5.29	3600	8820.3	8820.3	16251.45	45.73	7200	8008.29	8839.02	9123.99	3.12	7200
low-2	2	8763.46	1800	8074.9	8236.6	8763.46	6.01	3600	8193.31	8193.31	15729.43	47.91	7200	7783.33	8197.54	8565.34	4.29	7200
low-3	2	8478.43	1800	8419.55	8442.43	8442.43	0.00	487	8370.79	8378.61	14278.23	41.32	7200	8095.36	8442.43	8442.43	0.00	1888
low-4	2	8610.78	1800	7863.04	7947.95	8610.78	7.70	3600	8008.77	8039.12	15738.27	48.92	7200	7538.53	7735.56	8313.72	6.95	7200
low-5	2	8863.01	1800	7689.86	7770.25	8863.01	12.33	3600	7859.64	7859.64	14227.85	44.76	7200	7163.14	7664.56	8041.11	4.68	7200
high-1	2	24133.36	1800	23518.29	23659.08	24133.36	1.97	3600	23621.13	23621.13	31131.5	24.12	7200	22692.7	23441.7	23859.7	1.75	7200
high-2	2	20944.45	1800	20305.63	20500.1	20944.45	2.12	3600	20453.51	20504.19	28951.02	29.18	7200	22997.5	23416.8	23416.8	0.00	1340
high-3	2	23532.17	1800	23343.09	23416.8	23416.8	0.00	1280	23334.1	23395.26	29281.46	20.10	7200	23008.5	23416.8	23416.8	0.00	1373
high-4	2	18824.72	1800	17912.84	17979.61	18824.72	4.49	3600	18070.05	18094.02	25851.23	30.01	7200	18617.2	19155.3	19686.8	2.70	7200
high-5	2	20148.35	1800	19170.39	19274.1	20148.35	4.34	3600	19375.43	19396.77	25732.45	24.62	7200	18615.3	19092.5	19560.2	2.39	7200
low-1	3	11367.08	1800	9981.67	10020.79	11367.08	11.84	3600	10187.07	10201.41	19097.65	46.58	7200	8263.12	8607.01	11522.8	25.30	7200
low-2	3	9964.96	1800	9089.76	9178.73	9964.96	7.89	3600	9265.48	9291.15	17277.25	46.22	7200	8381.59	8654.56	10091.1	14.24	7200
low-3	3	9590.32	1800	9102.72	9151.21	9590.32	4.58	3600	9176.51	9198.47	16127.15	42.96	7200	8241.71	8410.07	9941.07	15.40	7200
low-4	3	10683.88	1800	8861.66	8898.2	10683.88	16.71	3600	9046.99	9069.01	17240.23	47.40	7200	7835.98	7898.9	10342.3	23.63	7200
low-5	3	10128.46	1800	8689.75	8732.31	10128.46	13.78	3600	8909.75	8928.70	16066.91	44.43	7200	7252.9	7397.05	10161.4	27.20	7200
high-1	3	25844.98	1800	24733.9	24777.11	25844.98	4.13	3600	24988.49	25011.69	34010.58	26.46	7200	23012	23452.6	26360.3	11.03	7200
high-2	3	22028.40	1800	21353.87	21458.7	22028.4	2.59	3600	21532.91	21588.27	30179.24	28.47	7200	20642.5	20994	21931.9	4.28	7200
high-3	3	24567.56	1800	24075.41	24131.96	24567.56	1.77	3600	24175.33	24201.97	31128.5	22.25	7200	23158.2	23428.4	24602.6	4.77	7200
high-4	3	20456.06	1800	18866.46	18923.09	20456.06	7.49	3600	19083.34	19124.46	27368.71	30.12	7200	17854.2	17976.8	19832.5	9.36	7200
high-5	3	21700.98	1800	20129.2	20199.77	21700.98	6.92	3600	20434.55	20462.29	27545.45	25.71	7200	18738.5	18985.2	21965.7	13.57	7200
low-1	4	13579.54	1800	11428.89	11455.41	13579.54	15.64	3600	11686.08	11697.35	18869.49	38.01	7200	8620.03	9016.31	14169.4	36.37	7200
low-2	4	11108.81	1800	10251.51	10311.45	11108.81	7.18	3600	10422.53	10436.33	18873.86	44.70	7200	9017.71	9143.11	12336.5	25.89	7200
low-3	4	11189.66	1800	9941.28	9965.94	11189.66	10.94	3600	10073.39	10108.28	20263.87	50.12	7200	8464.27	8591.04	11597	25.92	7200
low-4	4	11370.01	1800	9946.88	9983.02	11370.01	12.20	3600	10174.94	10238.87	17246.3	40.63	7200	8146.38	8149.34	-	-	7200
low-5	4	12295.51	1800	9715.36	9742.62	12295.51	20.76	3600	10009.37	10030.97	19655.39	48.97	7200	7512.67	7580.06	12085	37.28	7200
high-1	4	28602.54	1800	26167.46	26201.91	28602.54	8.39	3600	26440.68	26481.6	33783.58	21.61	7200	23392.9	23771.1	29016.7	18.08	7200
high-2	4	23500.25	1800	22428.43	22515.72	23500.25	4.19	3600	22681.27	22725.35	31166.8	27.08	7200	21269.4	21539.4	24880.5	13.43	7200
high-3	4	25988.86	1800	24912.45	24968.57	25988.86	3.93	3600	25080.1	25121.56	35239.26	28.71	7200	23417.9	23581.7	-	-	7200
high-4	4	22283.15	1800	19973.4	20035.34	22283.15	10.09	3600	20229.99	20302.14	27392.85	25.89	7200	18170.8	18182.9	-	-	7200
high-5	4	22829.25	1800	21147.16	21201.06	22829.25	7.13	3600	21533.92	21566.65	31105.4	30.67	7200	19013.3	19103.9	22787.6	16.17	7200
low-1	5	14824.81	1800	12878.96	12896.62	14824.81	13.01	3600	13194.27	13210.61	19001.15	30.47	7200	9110.97	-	-	-	7200
low-2	5	12614.88	1800	11326.29	11387.74	12614.88	9.73	3600	11559.06	11598.46	18565.88	37.53	7200	9595.19	9881.87	13206.4	25.17	7200
low-3	5	13150.13	1800	10878.07	10891	13150.13	17.18	3600	11066.22	11088.49	18458.13	39.93	7200	8725.88	8835.89	17774.5	50.29	7200
low-4	5	13673.77	1800	11134.64	11173.46	13673.77	18.29	3600	11418.24	11470.19	18885.37	39.26	7200	8459.84	-	-	-	7200
low-5	5	13261.28	1800	10813.85	10814.65	13261.28	18.45	3600	11141.07	11174.33	17450.56	35.97	7200		-	-	-	7200
high-1	5	31219.77	1800	27610.72	27661.09	31219.77	11.40	3600	27981.24	28007.94	33935.15	17.47	7200	23931	-	-	-	7200
high-2	5	25076.12	1800	23573.71	23658.8	25076.12	5.65	3600	23833.74	23881.11	30878.57	22.66	7200	21874.7	22086.3	26185.2	15.65	7200
high-3	5	27420.74	1800	25854.63	25903.65	27420.74	5.53	3600	26026.74	26101.72	33466.92	22.01	7200	23724.5	-	-	-	7200
high-4	5	23624.53	1800	21161.2	21209.33	23624.53	10.22	3600	21476.72	21515.16	30100.03	28.52	7200	18494.3	-	-	-	7200
high-5	5	24281.36	1800	22260.78	22324.9	24281.36	8.06	3600	22670.68	22703.38	28942.78	21.56	7200	19321.4	-	-	-	7200

vehicle routing problem and other single period variants. It would be interesting to see if and when the DR inequalities are useful for these problems. It would also be interesting to test whether the cutting planes complement the branch-and-price approach of Desaulniers et al.

References

[1] Y. Adulyasak, J-F. Cordeau, and R. Jans. Formulations and branch-and-cut algorithms for multi-vehicle production and inventory routing problems. *INFORMS Journal on Computing*, 26(1):103–120, 2014.

[2] C. Archetti, L. Bertazzi, G. Laporte, and M.G. Speranza. A branch-and-cut algorithm for a vendor-managed inventory-routing problem. *Transportation Science*, 41(3):382–391, 2007.

[3] C. Archetti, N. Bianchessi, and M.G. Speranza. Optimal solutions for routing problems with profits. *Discrete Applied Mathematics*, 161:547–557, 2013.

[4] P. Avella, M. Boccia, and L.A. Wolsey. Single-item reformulations for a vendor managed inventory routing problem: computational experience with benchmark instances. *Networks*, to appear, 2014.

[5] L.C. Coelho, J-F. Cordeau, and G. Laporte. Thirty years of inventory routing. *Transportation Science*, 48(1):1–19, 2014.

[6] L.C. Coelho and G. Laporte. The exact solution of several classes of inventory-routing problems. *Computers & Operations Research*, 40(2):558–565, 2013.

[7] L.C. Coelho and G. Laporte. Improved solutions for inventory-routing problems through valid inequalities and input ordering. *International Journal of Production Economics*, 155:391–397, 2014.

[8] G. Cornuéjols and F.Harche. Polyhedral study of the capacitated vehicle routing problem. *Mathematical Programming*, 60:21–52, 1993.

[9] G. Desaulniers, J.G. Rakke, and L.C. Coelho. A branch-price-and-cut algorithm for the inventory-routing problem. Working Paper G-2014-19, Gerad, Montreal, 2014.

[10] M. Fischetti, P. Toth, and D. Vogo. A branch and bound algorithm for the capacitated vehicle routing problem on directed graphs. *Operations Research*, 42:846–859, 1994.

[11] D. Naddef and G. Rinaldi. Branch-and-cut algorithms for the capacitated vrp. In P. Toth and D. Vigo, editors, *The Vehicle Routing Problem*, pages 53–84. Society for Industrial and Applied Mathematics, 2001.

[12] J.C. Picard and H.D. Ratliff. A graph-theoretic equivalence for integer programs. *Operations Research*, 21:261–269, 1973.

[13] O. Solyali and H. Süral. A branch-and-cut algorithm using a strong formulation and an a priori tour-based heuristic for an inventory-routing problem. *Transportation Science*, 45(3):335–345, 2011.

9 Appendix: Computational results for n15-T6, n20-T6 and n25-T6 instances

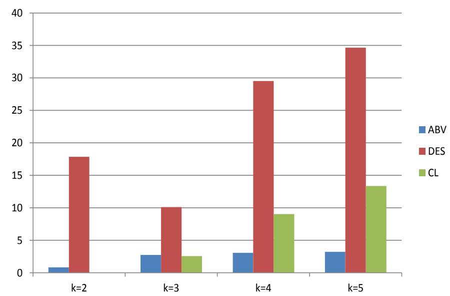

Figure 4: *Average gaps for ABW, CL and DRC on n15-T6 instances*

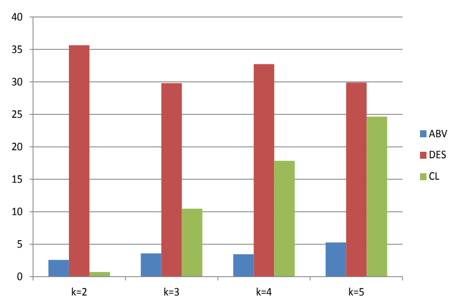

Figure 5: *Average gaps for ABW, CL and DRC on n20-T6 instances*

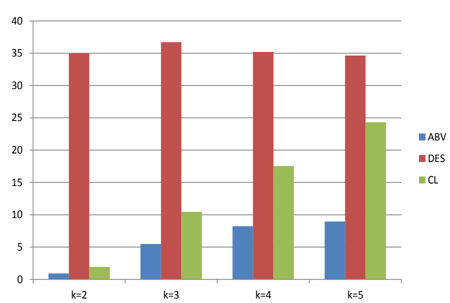

Figure 6: *Average gaps for ABW, CL and DRC on n25-T6 instances*

Table 4: Computational results for $n15$-$T6$- instances

				ABW					DRC					CL				
Name	k	UB_{heu}	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time
low-1	2	5987.4	258	5911.67	5987.4	5987.4	0.00	2903	5787.17	5891.59	6050.44	2.63	7200	5480.07	5987.4	5987.4	0.00	392
low-2	2	6152.89	83	6061.77	6152.89	6152.89	0.00	1664	6070.62	6120.67	6414.88	4.59	7200	5661.23	6152.89	6152.89	0.00	90
low-3	2	7057.4	571	6768.99	6879	7057.4	2.53	3600	6845.09	6928.5	12883.27	46.22	7200	6025.13	7011.2	7011.2	0.00	1418
low-4	2	6229.88	537	5962.25	6096.92	6229.88	2.13	3600	5997.48	6067.36	11609.15	47.74	7200	5647.19	6126.71	6126.71	0.00	208
low-5	2	6391.61	592	6237.02	6288.48	6391.61	1.61	3600	6187.58	6268.35	8803.74	28.80	7200	5301.53	6364.99	6364.99	0.00	311
high-1	2	12638.04	164	12537.76	12624.68	12624.68	0.00	1320	12445.94	12559.18	12653.3	0.74	7200	12108	12624.7	12624.7	0.00	179
high-2	2	12568.37	80	12480.36	12568.37	12568.37	0.00	712	12485.33	12540.84	12592.01	0.41	7200	12067.5	12568.4	12568.4	0.00	43
high-3	2	14519.78	587	14168.25	14310.93	14519.78	1.44	3600	14274.73	14392.5	14472.65	0.55	7200	13451	14450.1	14450.1	0.00	647
high-4	2	11267.7	501	11105.29	11249.31	11267.7	0.16	3600	11150.14	11235.72	15924.81	29.45	7200	10795.8	11267.7	11267.7	0.00	101
high-5	2	11374.4	259	11229.98	11315.53	11374.4	0.52	3600	11180.13	11290.06	13696.16	17.57	7200	10297.1	11374.4	11374.4	0.00	211
low-1	3	6966.03	766	6726.53	6772.3	6966.03	2.78	3600	6734.02	6840.66	6873.59	0.48	7200	5735.36	6609.01	6873.59	3.85	7200
low-2	3	7247.45	543	6991.54	7039.45	7247.45	2.87	3600	7010.53	7071.68	7107.77	0.51	7200	6132.77	6990.29	7107.77	1.65	7200
low-3	3	8175.42	914	7904.84	7955.4	8175.42	2.69	3600	7968.65	8046.21	8239.95	2.35	7200	6445.79	7640.68	8239.95	7.27	7200
low-4	3	7334.66	914	6916.15	7006.82	7334.66	4.47	3600	7027.42	7108.72	7157.6	0.68	7200	6210.94	7004.33	7157.6	2.14	7200
low-5	3	7798.09	760	7353.89	7410.98	7798.09	4.96	3600	7401.25	7512.69	7742.34	2.97	7200	5938.45	7340.01	7742.34	5.20	7200
high-1	3	13662.78	773	13364.19	13447.16	13662.78	1.58	3600	13398.43	13511.62	20211.22	33.15	7200	12380.8	13384	13517.6	0.99	7200
high-2	3	13608.81	505	13395.6	13455.72	13608.81	1.12	3600	13416.48	13494.36	13513.53	0.14	7200	12566.4	13478	13529.1	0.38	7200
high-3	3	15679.82	943	15322.66	15402.09	15679.82	1.77	3600	15409.34	15500.89	23175.85	33.12	7200	13883.9	15216.8	15626.3	2.62	7200
high-4	3	12548.94	230	12075.18	12168.8	12548.94	3.03	3600	12178.1	12265.27	12358.93	0.76	7200	11340.9	12240.5	12296	0.45	7200
high-5	3	12706.98	708	12344.66	12427.62	12706.98	2.20	3600	12387.19	12524.68	17150.1	26.97	7200	10940.7	12474.1	12610.7	1.08	7200
low-1	4	7860.32	852	7646.41	7682.3	7860.32	2.26	3600	7653.31	7710.16	7939.74	2.89	7200	5990.8	6774.36	8083.88	16.20	7200
low-2	4	8333.77	736	7968.22	8021.29	8333.77	3.75	3600	7974.1	8047.19	13725.37	41.37	7200	6647.61	7567.46	8194.73	7.65	7200
low-3	4	9465.35	1215	8949.58	9001.46	9465.35	4.90	3600	9039.63	9100.94	18045.19	49.57	7200	6925.93	7929.37	9481.35	16.37	7200
low-4	4	8618.09	1003	8123.33	8176.71	8618.09	5.12	3600	8062.45	8257.57	14793.88	44.18	7200	6805.63	7783.48	8467.29	8.08	7200
low-5	4	9087.01	1003	8643.58	8686.45	9087.01	4.41	3600	8680.32	8770.41	13626.02	35.63	7200	6706.16	8014.37	8987.54	10.83	7200
high-1	4	14441.12	790	14287.04	14345.88	14441.12	0.66	3600	14290.18	14373.83	14438.8	0.45	7200	12629.2	13364.4	14575	8.31	7200
high-2	4	14721.92	661	14377.35	14459.99	14721.92	1.78	3600	14387.48	14484.13	20193.38	28.27	7200	13084.1	14086	14629.6	3.72	7200
high-3	4	16812.23	1107	16385.86	16450.56	16812.23	2.15	3600	16489.22	16568.74	25418.7	34.82	7200	14360.1	15550.5	16867.3	7.81	7200
high-4	4	13722.95	859	13276.43	13338.68	13722.95	2.80	3600	13209.45	13415.3	19932.29	32.70	7200	11954.6	12887.7	13608.6	5.30	7200
high-5	4	14116.6	918	13630.3	13703.23	14116.6	2.93	3600	13692.74	13793.87	18497.1	25.43	7200	11720.4	13205.2	14064.4	6.11	7200
low-1	5	8890.26	890	8560.52	8616.96	8890.26	3.07	3600	8548.17	8626.66	14405.94	40.12	7200	6328.22	6970.21	9062.71	23.09	7200
low-2	5	9166.96	728	8899.53	8944.26	9166.96	2.43	3600	8950.14	9003.06	15107.48	40.41	7200	7245.98	8294.92	9196.5	9.80	7200
low-3	5	10803.11	1460	10207.2	10249.97	10803.11	5.12	3600	10187.14	10382.86	19639.92	47.13	7200	7555.87	8547.9	10769.3	20.63	7200
low-4	5	9581.77	956	9234.4	9265.02	9581.77	3.31	3600	9260.41	9325.03	16864.75	44.71	7200	7548.33	8522.49	9687.22	12.02	7200
low-5	5	10799.68	1302	10061.76	10100.68	10799.68	6.47	3600	10106.92	10227.14	14671.77	30.29	7200	7602.16	8650.96	10816.8	20.02	7200
high-1	5	15633.24	942	15216.54	15286.39	15633.24	2.22	3600	15216.23	15311.68	20391.78	24.91	7200	12966.2	13653.9	15605	12.50	7200
high-2	5	15625.91	622	15306.81	15373.99	15625.91	1.61	3600	15355.09	15430.59	21548.68	28.39	7200	13679.4	14818.3	15607.1	5.05	7200
high-3	5	18244.71	1204	17648.55	17696.11	18244.71	3.01	3600	17628.87	17826.88	26992.88	33.96	7200	14988.4	16031.6	17998.7	10.93	7200
high-4	5	14625.5	954	14376.94	14422.44	14625.5	1.39	3600	14410.89	14476.72	22110.72	34.53	7200	12693	13751	14707	6.50	7200
high-5	5	15669.7	1181	15049.1	15107.13	15669.7	3.59	3600	15114.91	15241.91	19560.88	22.08	7200	12618.4	13691	15769.5	13.18	7200

Table 5: Computational results for $n20$-$T6$- instances

				ABW					DRC					CL				
Name	k	UB_{heu}	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time
low-1	2	7752.98	994	7177.75	7262.01	7752.98	6.33	3600	7208.7	7252.39	12492.46	41.95	7200	6311.01	7388.8	7388.8	0.00	2188
low-2	2	6358.66	875	6281.04	6330.22	6358.66	0.45	3600	6306.18	6336.55	12473.08	49.20	7200	6034.15	6358.66	6358.66	0.00	2311
low-3	2	7570.72	1741	7333.94	7361.75	7570.72	2.76	3600	7310.82	7380.55	13593.15	45.70	7200	6969.95	7473.61	7473.61	0.00	1331
low-4	2	8513.63	1313	8091.06	8137.88	8513.63	4.41	3600	8111.16	8172.71	14706.25	44.43	7200	7187.48	8139.95	8389.51	2.97	7200
low-5	2	8735.87	978	8339.06	8414.41	8735.87	3.68	3600	8370.32	8417.77	14075.87	40.20	7200	7279.04	8231.85	8516.97	3.35	7200
high-1	2	15677.13	867	15325.86	15456.38	15677.13	1.41	3600	15396.03	15444.91	20683.47	25.33	7200	14465.9	15540.3	15540.3	0.00	1098
high-2	2	14908.15	637	14823.25	14908.15	14908.5	0.00	3600	14859.94	14900.87	21142.09	29.52	7200	14529	14940.3	14940.3	0.00	1735
high-3	2	15081.03	321	14843.92	14886.54	15081.03	1.29	3600	14844.91	14918.52	21117.71	29.36	7200	14469.3	14986.8	14986.8	0.00	1828
high-4	2	15534.74	872	15182.58	14908.15	15534.74	4.03	3600	15215.31	15285.29	20697.64	26.15	7200	14253.4	15383.2	15437.1	0.35	7200
high-5	2	17269.67	813	16912.28	17012.76	17269.67	1.49	3600	16971.05	17039.65	22562.64	24.48	7200	15873.2	17000.4	17101.9	0.59	7200
low-1	3	9073.71	1314	8486.83	8531.52	9073.71	5.98	3600	8629.23	8657.81	14350.86	39.67	7200	6720.31	7501.57	8898.51	15.70	7200
low-2	3	6983.1	1112	6834	6879.47	6983.1	1.48	3600	6834.47	6897.99	13754.59	49.85	7200	6205.48	6557.14	7001.49	6.35	7200
low-3	3	8562.53	1800	8219.38	8242.07	8562.53	3.74	3600	8260.6	8299.13	13537.88	38.70	7200	7253.51	7528.53	8672.4	13.19	7200
low-4	3	10627.34	1800	9683.43	9715.98	10627.34	8.58	3600	9789.05	9832.27	15362.74	36.00	7200	7555.61	8726.47	10343.5	15.63	7200
low-5	3	10878.71	1601	10156.08	10215.92	10878.71	6.09	3600	10240.24	10312.35	16226.63	36.45	7200	8128.58	8740.79	10751.2	18.70	7200
high-1	3	17107.01	1375	16669.22	16740.17	17107.01	2.14	3600	16815.77	16866.92	22622.08	25.44	7200	14890.3	15993.4	17178.7	6.90	7200
high-2	3	15504.99	852	15366.4	15440.89	15504.99	0.41	3600	15391.77	15452.24	15489.8	0.24	7200	14710.6	15254.8	15512.4	1.66	7200
high-3	3	16052.4	1644	15744.65	15781.32	16052.4	1.69	3600	15800.8	15846.52	21059.17	24.75	7200	14761.3	14996.6	16165.5	7.23	7200
high-4	3	17364.08	1800	16790.68	16833.49	17364.08	3.06	3600	16906.97	16956.62	22599.29	24.97	7200	14688.8	15664	17582.6	10.91	7200
high-5	3	19354.05	1642	18723.14	18811.47	19354.05	2.80	3600	18844.04	18917.08	24264.18	22.04	7200	16744.8	17490.3	19136.7	8.60	7200
low-1	4	10499.07	1380	10145.41	10175.04	10499.07	3.09	3600	10207.95	10242.87	16507.79	37.95	7200	7184.67	8185.77	10486.7	21.94	7200
low-2	4	7748.02	1506	7401.2	7452.17	7748.02	3.82	3600	7454.07	7527.69	15738.17	52.17	7200	6409.07	6756.96	7790.96	13.27	7200
low-3	4	9786.13	1800	9252.12	9266.36	9786.13	5.31	3600	9271.63	9298.8	15805.81	41.17	7200	7550.21	7789.81	9899.33	21.31	7200
low-4	4	11734.06	1741	11232.34	11271.05	11734.06	3.95	3600	11397.5	11441.88	19494.63	41.31	7200	8093.61	8707.1	12678.4	31.32	7200
low-5	4	12879.02	1552	12106.66	12148.1	12879.02	5.68	3600	12234.6	12304.28	19558.56	37.09	7200	9423.05	9876.82	13387.1	26.22	7200
high-1	4	18558.81	1388	18289.07	18346.18	18558.81	1.15	3600	18390.32	18444.6	18444.6	0.00	7077	15387.4	16531.8	19199.8	13.90	7200
high-2	4	16206.73	1306	15970.04	16031.73	16206.73	1.08	3600	16041.43	16118.55	24409.96	33.97	7200	14944.7	15300.1	16282.2	6.03	7200
high-3	4	17153.86	1800	16762.55	16798.13	17153.86	2.07	3600	16798.3	16845.41	23318.59	27.76	7200	15066.1	15409	17273.7	10.80	7200
high-4	4	19156.81	1797	18358.15	18401.99	19156.81	3.94	3600	18512.95	18568.32	26715.33	30.50	7200	15249.5	16069.4	19923.9	19.35	7200
high-5	4	21717.11	1528	20688.52	20746.6	21717.11	4.47	3600	20842.12	20921.61	28034.53	25.37	7200	18028.5	18557.5	21647.3	14.27	7200
low-1	5	12603.24	1800	11602.3	11627.74	12603.24	7.74	3600	11625.69	11674.24	17578.66	33.59	7200	7755.16	8576.9	12617.6	32.02	7200
low-2	5	8400.41	1800	8170.24	8196	8400.41	2.43	3600	8200.62	8240.46	16527.8	50.14	7200	6655.39	6997.62	8583.52	18.48	7200
low-3	5	10994.45	1800	10248.13	10271.64	10994.45	6.57	3600	10287.55	10340.97	17728.21	41.67	7200	7871.72	8130.26	11738.2	30.74	7200
low-4	5	14246.13	1800	12807.95	12834.93	14246.13	9.91	3600	12942.23	13006.65	19166.48	32.14	7200	8832.94	9456.66	14778.5	36.01	7200
low-5	5	15243.79	1800	13889.17	13935.77	15243.79	8.58	3600	14079.52	14118.4	21966.91	35.73	7200	10782.1	-	-	-	7200
high-1	5	20715.52	1653	19757.54	19796.02	20715.52	4.44	3600	19817.12	19870.7	27041.77	26.52	7200	15956.3	16855	21005.1	19.76	7200
high-2	5	16926.73	1464	16721.51	16778.69	16926.73	0.87	3600	16755.05	16825.03	16967.1	0.84	7200	15215.8	-	-	-	7200
high-3	5	18337.76	1633	17760.23	17799.36	18337.76	2.94	3600	17822.36	17879.72	25222.01	29.11	7200	15392.8	15721.7	18846.2	16.58	7200
high-4	5	20796.7	1800	19918.26	19966.47	20796.7	3.99	3600	20044.21	20125.46	26410.5	23.80	7200	15918.7	-	-	-	7200
high-5	5	23779.56	1800	22469.41	22530.98	23779.56	5.25	3600	22670.34	22732.62	30439.09	25.32	7200	19389.6	19822.8	24446.3	18.91	7200

Table 6: Computational results for $n25$-$T6$- instances

		ABW							DRC					CL				
Name	k	UB_{heu}	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time	LB	BLB	BUB	% Gap	Time
low-1	2	7469.85	235.12	7443.13	7461.55	7461.55	0.00	496.95	7432.35	7432.35	14775.93	49.70	7200	7427.71	7461.55	7461.55	0.00	199
low-2	2	8354.18	1800	8196.88	8223.58	8354.18	1.56	3600	8227.67	8238.51	13779.1	40.21	7200	7317.67	8116.95	8340.35	2.68	7200
low-3	2	8766.87	1800	8596.35	8652.41	8766.87	1.31	3600	8586.59	8602.01	15804.17	45.57	7200	7619.93	8134.35	8702.76	6.53	7200
low-4	2	7884.33	115.93	7872.31	7884.33	7884.33	0.00	373.72	7874.05	7874.05	13434.92	41.39	7200	7664.09	7884.33	7884.33	0.00	600
low-5	2	8838.13	1800	8389.36	8412.22	8838.13	4.82	3600	8375.65	8387.43	14361.46	41.60	7200	7746.24	8139.38	8540	4.69	7200
high-1	2	15954.81	217.8	15904.35	15954.81	15954.81	0.00	939.19	15901.77	15926.79	23351.5	31.80	7200	16579.2	17379.5	17596.9	1.24	7200
high-2	2	17601.63	1800	17450.86	17488.46	17601.63	0.64	3600	17487.73	17509.37	22904.38	23.55	7200	17866.5	19024.3	19024.3	0.00	5349
high-3	2	19056.26	1442.98	18876.71	18969.97	19056.26	0.45	3600	18909.22	18942.3	26209.71	27.73	7200	17867.4	18949.7	19024.3	0.39	7200
high-4	2	16679.9	1800	16591.45	16663.13	16663.13	0.00	2514.54	16601.14	16637.19	22325.72	25.48	7200	19213.7	19667	20077.4	2.04	7200
high-5	2	20003.28	1800	19845.81	19921.96	20003.28	0.41	3600	19892.4	19916.43	25803.9	22.82	7200	22691.8	23444.1	23875.4	1.81	7200
low-1	3	8697.26	1800	8146.1	8232.55	8697.26	5.34	3600	8215.11	8250.39	16656.45	50.47	7200	8011.27	8072.5	8554.74	5.64	7200
low-2	3	10288.61	1800	9391.96	9424.64	10288.61	8.40	3600	9464.08	9491.81	15716.37	39.61	7200	7531.38	7945.46	10127.3	21.54	7200
low-3	3	10814.81	1800	10077.41	10111.41	10814.81	6.50	3600	10142.59	10181.07	17497.89	41.82	7200	8014.88	8543.71	10802.5	20.91	7200
low-4	3	8841.54	1800	8648.66	8684.85	8841.54	1.77	3600	8700.47	8723.98	18316.08	52.37	7200	8021.66	8442.51	8932.45	5.48	7200
low-5	3	11826.54	1800	9976.97	9999.99	11826.54	15.44	3600	10048.90	10096.32	16378.71	38.36	7200	8627.34	8799.64	10793.8	18.48	7200
high-1	3	17046.29	1800	16616.93	16723.35	17046.29	1.89	3600	16702.14	16753.09	25955.12	35.45	7200	16458.6	16631.4	16972.7	2.01	7200
high-2	3	19771.15	1800	18649.12	18696.23	19771.15	5.44	3600	18740.12	18777.81	24942.14	24.71	7200	16798.9	17265.6	19325.1	10.66	7200
high-3	3	20863.6	1800	20414.95	20479.68	20863.6	1.84	3600	20510.60	20550.8	27906.74	26.36	7200	18350.9	19116	21371	10.55	7200
high-4	3	17780.79	1800	17395.79	17460.58	17780.79	1.80	3600	17468.98	17529.32	27265.69	35.71	7200	16789.6	17303.4	17644.4	1.93	7200
high-5	3	22930.42	1800	21458.98	21494.9	22930.42	6.26	3600	21557.39	21628.12	27794.83	22.19	7200	20094.4	20486.5	22084.5	7.24	7200
low-1	4	9480.53	1800	9116.12	9144.26	9480.53	3.55	3600	9119.46	9159.21	18719.37	51.07	7200	8607	8774.58	9628.95	8.87	7200
low-2	4	12441.13	1800	10722.65	10748.93	12441.13	13.60	3600	10828.83	10870.69	18445.86	41.07	7200	7786.4	8105.46	12595.9	35.65	7200
low-3	4	12782.16	1800	11769.06	11784	12782.16	7.81	3600	11848.62	11878.85	17830.45	33.38	7200	8677.67	9215.34	13476	31.62	7200
low-4	4	10205.82	1800	9572.86	9596.24	10205.82	5.97	3600	9662.51	9701.42	17830.68	45.59	7200	8440.22	9001.93	10095.8	10.83	7200
low-5	4	15139.16	1800	11857.09	11879.9	15139.16	21.53	3600	11988.42	12012.67	20241.26	40.65	7200	9520.15	9876.82	13387.1	26.22	7200
high-1	4	17873.43	1800	17551.37	17603.29	17873.5	1.51	3600	17600.03	17647.51	27283.89	35.32	7200	17051.4	17215.4	18185.1	5.33	7200
high-2	4	21240.67	1800	19975.2	20012.89	21240.67	5.78	3600	20108.78	20151.61	27684.54	27.21	7200	17062.6	17408.8	21110.8	17.54	7200
high-3	4	23636.04	1800	22076.83	22115.58	23636.04	6.43	3600	22208.88	22251.6	28205.36	21.11	7200	19059.3	19676.5	23148.1	15.00	7200
high-4	4	19255.82	1800	18327.14	18370.58	19255.82	4.60	3600	18451.43	18503.13	26769.36	30.88	7200	17238.4	17787.2	19172.6	7.23	7200
high-5	4	26457.14	1800	23355.12	23389.63	26457.14	11.59	3600	23497.44	23536.46	31667.32	25.68	7200	21005.4	21259.7	25658	17.14	7200
low-1	5	10745.98	1800	9994.46	10027.6	10745.98	6.69	3600	10086.89	10117.16	17732.85	42.95	7200	9230.58	9373.43	11511	18.57	7200
low-2	5	13807.81	1800	12201.5	12219.37	13807.81	11.50	3600	12306.58	12333.97	20533.47	39.93	7200	8082.59	8377.48	14214.7	41.06	7200
low-3	5	15388.58	1800	13550.9	13562.53	15388.58	11.87	3600	13633.23	13655.84	21241.57	35.71	7200	9467.29	10044.9	15306.8	34.38	7200
low-4	5	11919.28	1800	10455.94	10471.2	11919.28	12.15	3600	10548.29	10587.34	19863.33	46.70	7200	8997.14	9282.76	11687.4	20.57	7200
low-5	5	16882.52	1800	18468.83	18514.46	19752.14	6.27	3600	13878.21	13909.69	23911.19	41.83	7200	10448.5	-	-	-	7200
high-1	5	19752.14	1800	18468.83	18514.46	19752.14	6.27	3600	18573.35	18618.67	26333.27	29.30	7200	17679.6	-	-	-	7200
high-2	5	22986.29	1800	21469.83	21490.44	22986.29	6.51	3600	21582.51	21624.81	29760.62	27.34	7200	17357.1	-	-	-	7200
high-3	5	26192.77	1800	23830.26	23869.42	26192.77	8.87	3600	24000.27	24030.18	30819.29	22.03	7200	19832.1	-	-	-	7200
high-4	5	20714.22	1800	19220.06	19250.41	20714.22	7.07	3600	19338.8	19403.24	28804.02	32.64	7200	17789.6	18301.4	20256.3	9.65	7200
high-5	5	28743.58	1800	25143.82	25193.82	28743.58	12.35	3600	25379.25	25421.54	35333.55	28.05	7200	21944.7	22046.9	28157.9	21.70	7200

Recent titles

CORE Discussion Papers

2014/14 Jean-François CARPANTIER and Arnaud DUFAYS. Specific Markov-switching behaviour for ARMA parameters.
2014/15 Federico GRIGIS DE STEFANO. Strategic stability of equilibria: the missing paragraph.
2014/16 Claudio TELHA and Mathieu VAN VYVE. Efficient approximation algorithms for the economic lot-sizing in continuous time.
2014/17 Yukai YANG. Testing constancy of the error covariance matrix in vector models against parametric alternatives using a spectral decomposition.
2014/18 Koen DECANCQ, Marc FLEURBAEY and Erik SCHOKKAERT. Inequality, income, and well-being.
2014/19 Paul BELLEFLAMME and Martin PEITZ. Digital piracy: an update.
2014/20 Eva-Maria SCHOLZ. Licensing to vertically related markets.
2014/21 N. Baris VARDAR. Optimal energy transition and taxation of non-renewable resources.
2014/22 Benoît DECERF. Income poverty measures with relative poverty lines.
2014/23 Antoine DEDRY, Harun ONDER and Pierre PESTIEAU. Aging, social security design and capital accumulation.
2014/24 Biung-Ghi JU and Juan D. MORENO-TERNERO. Fair allocation of disputed properties.
2014/25 Nguyen Thang DAO. From agriculture to manufacture: How does geography matter ?
2014/26 Xavier Y. WAUTHY. From Bertrand to Cournot via Kreps and Scheinkman: a hazardous journey.
2014/27 Gustavo BERGANTIÑOS and Juan MORENO-TERNERO. The axiomatic approach to the problem of sharing the revenue from bundled pricing.
2014/28 Jean HINDRIKS and Yukihiro NISHIMURA. International tax leadership among asymmetric countries.
2014/29 Jean HINDRIKS and Yukihiro NISHIMURA. A note on equilibrium leadership in tax competition models.
2014/30 Olivier BOS and Tom TRUYTS. Auctions with prestige motives.
2014/31 Juan D. MORENO-TERNERO and Lars P. ØSTERDAL . Normative foundations for equity-sensitive population health evaluation functions.
2014/32 P. Jean-Jacques HERINGS, Ana MAULEON and Vincent VANNETELBOSCH. Stability of networks under Level-*K* farsightedness.
2014/33 Lionel ARTIGE, Laurent CAVENAILE and Pierre PESTIEAU. The macroeconomics of PAYG pension schemes in an aging society.
2014/34 Tanguy KEGELART and Mathieu VAN VYVE. A conic optimization approach for SKU rationalization.
2014/35 Ulrike KORNEK, Kei LESSMANN and Henry TULKENS. Transferable and non transferable utility implementations of coalitional stability in integrated assessment models.
2014/36 Ibrahim ABADA, Andreas EHRENMANN and Yves SMEERS. Endogenizing long-term contracts in gas market models.
2014/37 Julio DAVILA. Output externalities on total factor productivity.
2014/38 Diane PIERRET. Systemic risk and the solvency-liquidity nexus of banks.
2014/39 Paul BELLEFLAMME and Julien JACQMIN. An economic appraisal of MOOC platforms: business models and impacts on higher education.
2014/40 Marie-Louise LEROUX, Pierre PESTIEAU and Grégory PONTHIERE. Longévité différentielle et redistribution: enjeux théoriques et empiriques.
2014/41 Chiara CANTA, Pierre PESTIEAU and Emmanuel THIBAULT. Long term care and capital accumulation: the impact of the State, the market and the family.
2014/42 Gilles GRANDJEAN, Marco MANTOVANI, Ana MAULEON and Vincent VANNETELBOSCH. Whom are you talking with ? An experiment on credibility and communication structure.

Recent titles

CORE Discussion Papers - continued

2014/43 Julio DAVILA. The rationality of expectations formation.
2014/44 Florian MAYNERIS, Sandra PONCET and Tao ZHANG. The cleaning effect of minimum wages. Minimum wages, firm dynamics and aggregate productivity in China.
2014/45 Thierry BRECHET, Natali HRITONENKOVA and Yuri YATSENKO. Domestic environmental policy and international cooperation for global commons.
2014/46 Mathieu PARENTI, Philip USHCHEV and Jacques-François THISSE. Toward a theory of monopolistic competition.
2014/47 Takatoshi TABUCHI, Jacques-François THISSE and Xiwei ZHU. Does technological progress affect the location of economic activity?
2014/48 Paul CASTANEDA DOWER, Victor GINSBURGH and Shlomo WEBER. Colonial legacy, linguistic disenfranchisement and the civil conflict in Sri Lanka.
2014/49 Victor GINSBURGH, Jacques MELITZ and Farid TOUBAL. Foreign language learnings: An econometric analysis.
2014/50 Koen DECANCQ and Dirk NEUMANN. Does the choice of well-being measure matter empirically? An illustration with German data.
2014/51 François MANIQUET. Social ordering functions.
2014/52 Ivar EKELAND and Maurice QUEYRANNE. Optimal pits and optimal transportation.
2014/53 Luc BAUWENS, Manuela BRAIONE and Giuseppe STORTI. Forecasting comparison of long term component dynamic models for realized covariance matrices.
2014/54 François MANIQUET and Philippe MONGIN. Judgment aggregation theory can entail new social choice results.
2014/55 Pasquale AVELLA, Maurizio BOCCIA and Laurence A. WOLSEY. Single-period cutting planes for inventory routing problems.

Books

V. GINSBURGH and S. WEBER (2011), *How many languages make sense? The economics of linguistic diversity*. Princeton University Press.
I. THOMAS, D. VANNESTE and X. QUERRIAU (2011), *Atlas de Belgique – Tome 4 Habitat*. Academia Press.
W. GAERTNER and E. SCHOKKAERT (2012), *Empirical social choice*. Cambridge University Press.
L. BAUWENS, Ch. HAFNER and S. LAURENT (2012), *Handbook of volatility models and their applications*. Wiley.
J-C. PRAGER and J. THISSE (2012), *Economic geography and the unequal development of regions*. Routledge.
M. FLEURBAEY and F. MANIQUET (2012), *Equality of opportunity: the economics of responsibility*. World Scientific.
J. HINDRIKS (2012), *Gestion publique*. De Boeck.
M. FUJITA and J.F. THISSE (2013), *Economics of agglomeration: cities, industrial location, and globalization.* (2nd edition). Cambridge University Press.
J. HINDRIKS and G.D. MYLES (2013). *Intermediate public economics.* (2nd edition). MIT Press.
J. HINDRIKS, G.D. MYLES and N. HASHIMZADE (2013). *Solutions manual to accompany intermediate public economics.* (2nd edition). MIT Press.

CORE Lecture Series

R. AMIR (2002), Supermodularity and complementarity in economics.
R. WEISMANTEL (2006), Lectures on mixed nonlinear programming.
A. SHAPIRO (2010), Stochastic programming: modeling and theory.