Making three-dimensional vortex methods divergence-preserving and resilient to distortion through a novel Lagrangian lattice-based scheme

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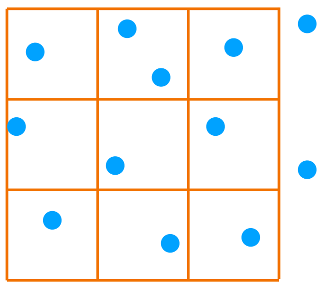
The VPM method



The Vortex Particle-mesh (VPM) a.k.a. Particle in Cell, …:

- Discretises vorticity using Lagrangian particles for advection and a fixed grid for differential operators.
- Low numerical dissipation, large allowable time steps, and straightforward treatment of unbounded domains.

The challenge: **Vorticity solenoidality**

Balty et al. [1] introduced connectivity between the particles, enabling a multiresolution framework. Drawing inspiration from this idea:

- We store vorticity as face-integrated fluxes on a deforming Lagrangian lattice
- Solenoidality becomes an exact algebraic identity that is preserved regardless of particle or lattice distortion.

Eulerian / Lagrangian viewpoint

The key idea is to work in the Lagrangian space where the deformation $\mathbb{J}$ links the two frames.

In physical space:

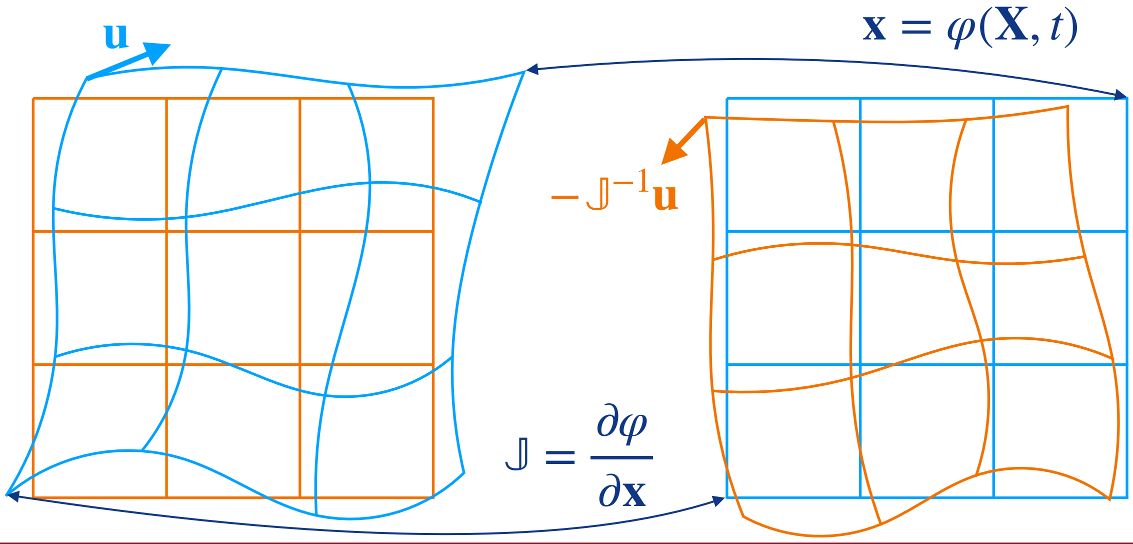
- Fixed Eulerian grid
- Lagrangian grid is advected by the velocity field $\mathbf{u}$

The physical fields are:

Velocity: $\mathbf{u}$

Vorticity: ω

$$\Rightarrow \omega = \nabla \times \mathbf{u}$$



In Lagrangian space:

- Fixed Lagrangian grid
- Eulerian grid is advected by the velocity field $-\mathbb{J}^{-1}\mathbf{u}$

Fields in the Lagrangian space ($\hat{\cdot}$) are linked to those in physical space by the covariant and contravariant Piola transforms:

Velocity: $\hat{\mathbf{u}} = \mathbb{J}^T \mathbf{u}$

Vorticity: $\hat{\omega} = \det(\mathbb{J})\mathbb{J}^{-1}\omega$

$$\Rightarrow \hat{\omega} = \hat{\nabla} \times \hat{\mathbf{u}}$$

Discretisation

Vorticity integrals on faces:

$$\omega_F = \int_F \omega \cdot \mathbf{n}\, dS$$

Velocity integrals on edges:

$$\mathbf{u}_E = \int_E \mathbf{u} \cdot \mathbf{t}\, dl$$

Helmholtz' and Stokes' theorems enforced exactly:

$$\frac{D\omega_F}{Dt} = 0$$

$$\omega_F = \sum_{E \in \partial F} \mathbf{u}_E$$

Face-integrated vorticity is thus a material invariant, and $\nabla \cdot \omega = 0$ reduces to the algebraic identity:

$$\sum_{F \in \partial C} \omega_F = 0.$$

Algorithm steps

Poisson solves: compute $\hat{\mathbf{u}}$ from $\hat{\omega}$ via two Poisson equations, where the first recovers a vorticity-consistent velocity and the second corrects it to enforce $\nabla \cdot \mathbf{u} = 0$ in physical space:

$$\hat{\nabla}^2 \hat{\mathbf{u}}_0 = -\hat{\nabla} \times \hat{\omega}$$

$$\hat{\nabla} \cdot \left(\det(\mathbb{J})\ \left(\mathbb{J}^T\mathbb{J}\right)^{-1} \hat{\nabla}\hat{\phi}\right) = -\hat{\nabla} \cdot \left(\det(\mathbb{J})\ \left(\mathbb{J}^T\mathbb{J}\right)^{-1} \hat{\mathbf{u}}_0\right)$$

$$\Rightarrow \hat{\mathbf{u}} = \hat{\mathbf{u}}_0 + \nabla\hat{\phi}$$

Advection: update flow map φ and Eulerian grid $\mathbf{X}_E$:

$$\frac{\partial \varphi}{\partial t} = \mathbf{u} \qquad \frac{\partial \mathbf{X}_E}{\partial t} = -\mathbb{J}^{-1}\mathbf{u}$$

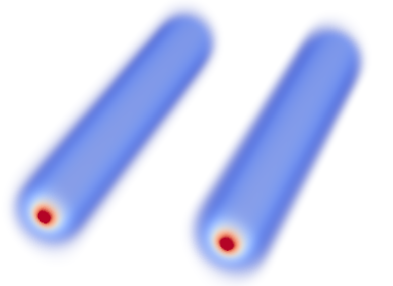
Remeshing: periodically project $\hat{\mathbf{u}}$ back to the Eulerian grid.

Results

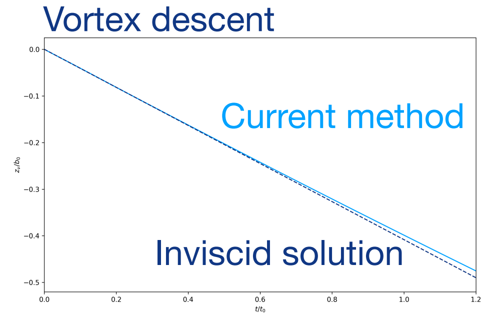
Vortex pair

Fully periodic domain with $b_0/L = 0.4$ and $b_0/r_c = 8$.

Exact circulation conservation

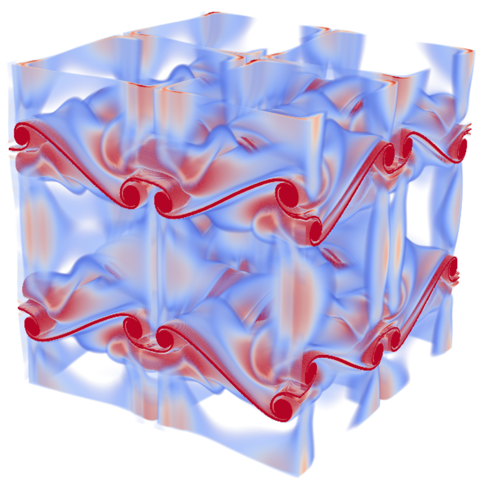


Vortex descent



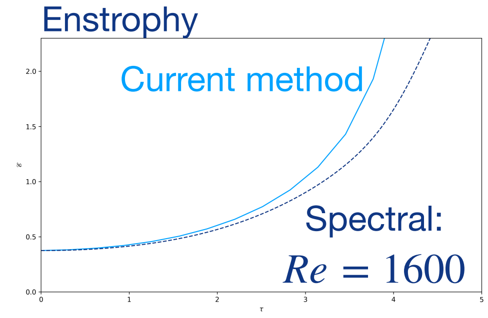
Taylor Green vortex

384 points per direction

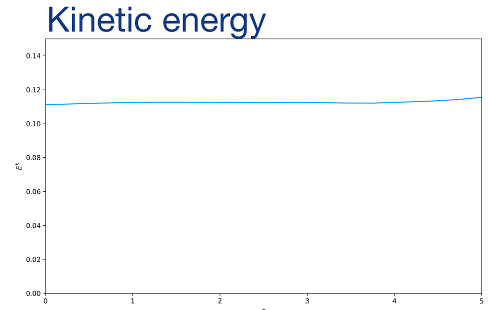


Volume rendering at the non-dimensional time $\tau = 5$

Enstrophy



Kinetic energy



Work in progress

- Assess the energy conservation properties of the solver and identify controls to enforce it
- Add the diffusion term for the full Navier-Stokes solver

[1] P. Balty, M. Duponcheel, and P. Chatelain, 'An adaptive multiresolution vortex particle-mesh method for the simulation of unbounded incompressible flows', Computer Methods in Applied Mechanics and Engineering, vol. 450, p. 118638, Mar. 2026.

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