Relations between steady-state timescales in the World Ocean

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Abstract. Properties of steady-state timescales in the World Ocean are established. It is demonstrated that they are non-negative. The average of the partial age over the related subdomain is seen to be equal to the average of the exposure time over the same subdomain. In the latter, the exposure time is greater than or equal to the residence time.

Motivation

At a steady state, partial ages in the World Ocean are related to the classical age, as well as the residence and exposure times. The objective of this working note is to establish relevant equalities and inequalities, which may be of use to interpret numerical simulation results — and, perhaps, some field data.

To obtain the aforementioned results, a positivity theorem is first established, which applies to the solution of a rather general steady-state transport problem.

Positivity theorem

The domain of interest, $\mathcal{V}$, is delimited by surface $\mathcal{S}$. The latter is split into several sub-regions, $\mathcal{S}^i$, $\mathcal{S}^s$ and $\mathcal{S}^o$, which do not overlap each other and obviously are such that $\mathcal{S} = \mathcal{S}^i \cup \mathcal{S}^s \cup \mathcal{S}^o$. On the domain boundary $\mathcal{S}$, $\mathbf{n}$ is the outward, unit, normal vector.

In the abovementioned domain, the position vector is denoted $\mathbf{x}$. Function $\varsigma(\mathbf{x})$ obeys the following partial differential equation

$$\nabla \bullet \left(\mathbf{D} \bullet \nabla \varsigma - \varsigma \mathbf{q} \right) + \mu = 0 \quad . \tag{1}$$

Tensor $\mathbf{D}(\mathbf{x})$ is symmetric,

$$\mathbf{D} = \mathbf{D}^T \quad , \tag{2}$$

and positive-definite,

$$\forall \, \mathbf{y} \neq 0 : \ \mathbf{y} \bullet \mathbf{D} \bullet \mathbf{y} > 0 \quad . \tag{3}$$

Vector $\mathbf{q}(\mathbf{x})$ is divergence-free,

$$\nabla \bullet \mathbf{q} = 0 \quad , \tag{4}$$

and satisfies the following boundary condition

$$\left[\mathbf{q} \bullet \mathbf{n} \right]_{\mathbf{x} \in \mathcal{S}^i \cup \mathcal{S}^s} = 0 \quad . \tag{5}$$

Source term $\mu(\mathbf{x})$ is non-negative,

$$\mu(\mathbf{x}) \geq 0 \quad . \tag{6}$$

Function $\varsigma(\mathbf{x})$ satisfies the following boundary conditions:

$$\left[(\mathbf{D} \bullet \nabla \varsigma) \bullet \mathbf{n} \right]_{\mathbf{x} \in \mathcal{S}^i} = 0 \tag{7}$$

and

$$\left[\varsigma(\mathbf{x})-\xi(\mathbf{x})\right]_{\mathbf{x}\in\mathcal{S}^s\cup\mathcal{S}^o} = 0 \quad \text{with} \quad \xi(\mathbf{x})\geq 0 \quad , \tag{8}$$

where $\xi(\mathbf{x})$ is a known function.

The relations above are meant to comprise all of the cases to be considered in the present working note. Accordingly, in some cases, not all of the three subregions of the domain boundary will be present. However, it is noteworthy that $\mathcal{S}^s\cup\mathcal{S}^o$ will never vanish. In other words, on a part of the boundary of the domain of interest, if not on the whole boundary, a Dirichlet condition will always be prescribed, causing function $\varsigma(\mathbf{x})$ to be non-negative on this subregion of the boundary.

It will be seen that function $\varsigma(\mathbf{x})$ is non-negative:

$$\varsigma(\mathbf{x}) \geq 0 \tag{9}$$

For the sake of the demonstration, it is first assumed that $\varsigma(\mathbf{x})$ does not satisfy inequality (10) at every point of the domain. Accordingly, its negative part is introduced, which reads

$$\varsigma^-(\mathbf{x}) = \frac{\varsigma^-(\mathbf{x})-\left|\varsigma^-(\mathbf{x})\right|}{2} \quad . \tag{10}$$

Clearly, this function is zero where $\varsigma(\mathbf{x})$ is positive and is equal to $\varsigma(\mathbf{x})$ where $\varsigma(\mathbf{x})$ is negative. As a consequence, proving that $\varsigma(\mathbf{x})$ is non-negative is tantamount to demonstrating that its negative part is zero at any location of the domain of interest.

Multiplying governing equation (1) by $\varsigma^-(\mathbf{x})$, integrating over the domain of interest, using constraint (4) and the divergence theorem wherever appropriate, one obtains

$$\begin{aligned}&\overbrace{\int_{\mathcal{S}^i}\varsigma^-(\mathbf{D}\bullet\nabla\varsigma)\bullet\mathbf{n}\,d\mathcal{S}}^{=0,\text{ since }(\mathbf{D}\bullet\nabla\varsigma)\bullet\mathbf{n}=0\text{ on }\mathcal{S}^i;\text{ see (7)}} + \overbrace{\int_{\mathcal{S}^s\cup\mathcal{S}^0}\varsigma^-(\mathbf{D}\bullet\nabla\varsigma)\bullet\mathbf{n}\,d\mathcal{S}}^{=0,\text{ since }\varsigma\geq0\text{ on }\mathcal{S}^s\cup\mathcal{S}^o;\text{ see (8)}} + \overbrace{\int_{\mathcal{V}}(-\nabla\varsigma^-\bullet\mathbf{D}\bullet\nabla\varsigma^-)\,d\mathcal{V}}^{\leq0;\text{ see (3)}}\\ &-\underbrace{\frac{1}{2}\int_{\mathcal{S}^i\cup\mathcal{S}^s}(\varsigma^-)^2\,\mathbf{q}\bullet\mathbf{n}\,d\mathcal{S}}_{=0,\text{ since }\mathbf{q}\bullet\mathbf{n}=0\text{ on }\mathcal{S}^i\cup\mathcal{S}^s;\text{ see (5)}} - \underbrace{\frac{1}{2}\int_{\mathcal{S}^o}(\varsigma^-)^2\,\mathbf{q}\bullet\mathbf{n}\,d\mathcal{S}}_{=0,\text{ since }\varsigma\geq0\text{ on }\mathcal{S}^o;\text{ see (8)}} + \underbrace{\int_{\mathcal{V}}\varsigma^-\mu\,d\mathcal{V}}_{\leq0,\text{ since }\mu\geq0;\text{ see (6)}} = 0\end{aligned} \tag{11}$$

This expression simplifies to a sum of two non-positive terms,

$$\overbrace{\int_{\mathcal{V}}(-\nabla\varsigma^-\bullet\mathbf{D}\bullet\nabla\varsigma^-)\,d\mathcal{V}}^{\leq0;\text{ see (3)}} + \overbrace{\int_{\mathcal{V}}\varsigma^-\mu\,d\mathcal{V}}^{\leq0,\text{ since }\mu\geq0;\text{ see (6)}} = 0 \quad . \tag{12}$$

Clearly, both integrals must be zero, which is the case if and only if $\varsigma^-(\mathbf{x})$ is zero at every location of the domain of interest[1]. In other words, the negative part of function $\varsigma(\mathbf{x})$ is zero, implying that this function is non-negative. QED.

[1] Here one might object that if the source term μ is identically zero, which is not precluded by (6), then ς^- may be a non zero constant, causing the first integral of (12) to be zero. This is not acceptable, for, as explicitly mentioned above, it is assumed that a Dirichlet condition (with a non-negative value) is prescribed on at least a fraction of the domain boundary.

Age and partial age

Let three-dimensional domain Ω represent the World Ocean. The ocean-atmosphere interface and the rest of the boundary of Ω (which is impermeable) are denoted Γ^s and Γ^i, respectively. Then, to apply the above positivity theorem to the age of the water $a(\mathbf{x})$ (defined as the time elapsed since leaving the ocean surface), one sets $\varsigma(\mathbf{x}) = a(\mathbf{x})$ and

$$\begin{cases} \mathcal{V} = \Omega\ , \quad \mathcal{S}^i = \Gamma^i\ , \quad \mathcal{S}^s = \Gamma^s\ , \quad \mathcal{S}^o = \varnothing \\ \mathbf{D} = \mathbf{K}\ , \quad \mathbf{q} = \mathbf{v}\ , \quad \mu = 1 \end{cases} \tag{13}$$

where $\mathbf{K}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ denote the diffusivity tensor and the velocity, respectively. Therefore, by virtue of the above positivity theorem, water age $a(\mathbf{x})$ is non-negative.

To calculate partial (water) ages, the domain is split into K non-overlapping subdomains Ω^k, with $k = 1,2,\ldots K$. The boundary of each subdomain is denoted Γ^k. The related characteristic functions are defined as

$$\omega^k(\mathbf{x}) \;=\; \begin{cases} 1 \;\text{ if }\; \mathbf{x} \in \Omega^k \\ 0 \;\text{ if }\; \mathbf{x} \notin \Omega^k \end{cases} \tag{14}$$

The partial ages are denoted $a^k(\mathbf{x})$. Then, to apply the above positivity theorem to partial water age $a^k(\mathbf{x})$, one sets $\varsigma(\mathbf{x}) = a^k(\mathbf{x})$ and

$$\begin{cases} \mathcal{V} = \Omega\ , \quad \mathcal{S}^i = \Gamma^i\ , \quad \mathcal{S}^s = \Gamma^s\ , \quad \mathcal{S}^o = \varnothing \\ \mathbf{D} = \mathbf{K}\ , \quad \mathbf{q} = \mathbf{v}\ , \quad \mu = \omega^k \end{cases} \tag{15}$$

Therefore, partial water ages $a^k(\mathbf{x})$ are all non-negative.

Mean partial age and residence time in a subdomain

To obtain the mean residence time in Ω^k, it is first necessary to deal with the following tracer transport problem in Ω^k

$$\begin{cases} \dfrac{\partial C^k}{\partial t} \;=\; -\nabla \bullet (C^k \mathbf{v} - \mathbf{K} \bullet \nabla C^k) \\ C^k(0,\mathbf{x}) = 1\ , \quad \left[C^k(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma^k} = 0 \end{cases} \tag{16}$$

Then, from "concentration" $C^k(t,\mathbf{x})$, the following intermediate variable is to be evaluated

$$\eta^k(\mathbf{x}) \;=\; \int_0^\infty C^k(t,\mathbf{x})\, dt \quad . \tag{17}$$

Combining (16) and (17) leads to the partial differential from which $\eta^k(\mathbf{x})$ may be derived:

$$\begin{cases} \nabla \bullet (\mathbf{K} \bullet \nabla \eta^k - \eta^k \mathbf{v}) \;+\; 1 \;=\; 0 \\ \left[\eta^k(\mathbf{x})\right]_{\mathbf{x}\in\Gamma^k} = 0 \end{cases} \tag{18}$$

To prove $a^k(\mathbf{x}) - \eta^k(\mathbf{x})$ is non-negative, one sets $\varsigma(\mathbf{x}) = a^k(\mathbf{x}) - \eta^k(\mathbf{x})$ and

$$\begin{cases} \mathcal{V} = \Omega^k\ , \quad \mathcal{S}^i = \varnothing\ , \quad \mathcal{S}^s = \varnothing\ , \quad \mathcal{S}^o = \Gamma^k \\ \mathbf{D} = \mathbf{K}\ , \quad \mathbf{q} = \mathbf{v}\ , \quad \mu = 0\ , \quad \xi = \left[a^k(\mathbf{x})\right]_{\mathbf{x}\in\Gamma^k} \end{cases} \tag{19}$$

Therefore, $a^k(\mathbf{x})-\eta^k(\mathbf{x})$ is non-negative.

Finally, the mean partial age and residence time in Ω^k are obtained from

$$\left(\bar{a}^k,\bar{\theta}^k\right) = \frac{1}{V^k}\int_{\Omega^k}\left[a^k(\mathbf{x}),\eta^k(\mathbf{x})\right]d\Omega^k \quad , \tag{20}$$

where V^k is the volume of k-th subdomain, i.e.

$$V^k = \int_{\Omega^k} d\Omega^k \quad . \tag{21}$$

Since $a^k(\mathbf{x})-\eta^k(\mathbf{x})$ is non-negative, the mean residence time cannot be larger than the mean partial age

$$\bar{a}^k \geq \bar{\theta}^k \tag{22}$$

QED.

It is noteworthy that inequality (22) was obtained without having to prove the positivity of variable $\eta^k(\mathbf{x})$. For the sake of completeness, it will be seen that, in accordance with elementary physical intuition, $\eta^k(\mathbf{x})$ is non-negative. To do so, one sets $\varsigma(\mathbf{x})=\eta^k(\mathbf{x})$ and

$$\begin{cases}\mathcal{V}=\Omega^k\ ,\quad \mathcal{S}^i=\varnothing\ ,\quad \mathcal{S}^s=\varnothing\ ,\quad \mathcal{S}^o=\Gamma^k \\ \mathbf{D}=\mathbf{K}\ ,\quad \mathbf{q}=\mathbf{v}\ ,\quad \mu=1\ ,\quad \xi=0\end{cases} \tag{23}$$

As a consequence, $\eta^k(\mathbf{x})$ satisfies inequality

$$\eta^k(\mathbf{x}) \geq 0 \quad . \tag{24}$$

Residence and exposure times

The residence time in subdomain Ω^k, $\theta^k(\mathbf{x})$, may be obtained by setting $\varsigma(\mathbf{x})=\theta^k(\mathbf{x})$ and

$$\begin{cases}\mathcal{V}=\Omega^k\ ,\quad \mathcal{S}^i=\varnothing\ ,\quad \mathcal{S}^s=\varnothing\ ,\quad \mathcal{S}^o=\Gamma^k \\ \mathbf{D}=\mathbf{K}\ ,\quad \mathbf{q}=-\mathbf{v}\ ,\quad \mu=1\ ,\quad \xi=0\end{cases} \tag{25}$$

As a consequence, residence time $\theta^k(\mathbf{x})$ is non-negative.

The exposure time in subdomain Ω^k, $\Theta^k(\mathbf{x})$, may be obtained by setting $\varsigma(\mathbf{x})=\Theta^k(\mathbf{x})$ and solving the following partial differential problem

$$\begin{cases}\mathcal{V}=\Omega\ ,\quad \mathcal{S}^i=\Gamma^i\ ,\quad \mathcal{S}^s=\Gamma^s\ ,\quad \mathcal{S}^o=\varnothing \\ \mathbf{D}=\mathbf{K}\ ,\quad \mathbf{q}=-\mathbf{v}\ ,\quad \mu=\omega^k\end{cases} \tag{26}$$

As a consequence, residence time $\Theta^k(\mathbf{x})$ is non-negative.

In subdomain Ω^k, the difference between the exposure time and the residence time is non-negative. To prove that this statement holds valid, one must set $\varsigma(\mathbf{x})=\Theta^k(\mathbf{x})-\theta^k(\mathbf{x})$ and

$$\begin{cases}\mathcal{V}=\Omega^k\ ,\quad \mathcal{S}^i=\varnothing\ ,\quad \mathcal{S}^s=\varnothing\ ,\quad \mathcal{S}^o=\Gamma^k \\ \mathbf{D}=\mathbf{K}\ ,\quad \mathbf{q}=-\mathbf{v}\ ,\quad \mu=0\ ,\quad \xi=\left[\Theta^k(\mathbf{x})\right]_{\mathbf{x}\in\Gamma^k}\end{cases} \tag{27}$$

Thus, in subdomain Ω^k, the inequalities

$$\Theta^k(\mathbf{x}) \geq \theta^k(\mathbf{x}) \geq 0 \tag{28}$$

hold valid. This result is far from unexpected.

Partial age and exposure time

To obtain the mean exposure time in Ω^k, it is first necessary to deal with the following tracer transport problem in Ω:

$$\begin{cases} \dfrac{\partial C^k}{\partial t} = -\nabla \bullet (C^k \mathbf{v} - \mathbf{K} \bullet \nabla C^k) \\ C^k(0,\mathbf{x}) = \omega^k(\mathbf{x}) \,,\ \left[(\mathbf{K} \bullet \nabla C^k) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^i} = 0 \,,\ \left[C^k(t,\mathbf{x})\right]_{\mathbf{x} \in \Gamma^s} = 0 \end{cases} \tag{29}$$

Then, from "concentration" $C^k(t,\mathbf{x})$, the following intermediate variable is to be evaluated

$$\eta^k(\mathbf{x}) = \int_0^\infty C^k(t,\mathbf{x})\, dt \quad . \tag{30}$$

Combining (29) and (30) leads to the partial differential from which $\eta^k(\mathbf{x})$ may be derived:

$$\begin{cases} \nabla \bullet (\mathbf{K} \bullet \nabla \eta^k - \eta^k \mathbf{v}) + \omega^k = 0 \\ \left[(\mathbf{K} \bullet \nabla \eta^k) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^i} = 0 \,,\ \left[\eta^k(\mathbf{x})\right]_{\mathbf{x} \in \Gamma^s} = 0 \end{cases} \tag{31}$$

Finally, the average over Ω^k of the exposure time is

$$\bar{\Theta}^k = \frac{1}{V^k} \int_{\Omega^k} \eta^k(\mathbf{x})\, d\Omega^k \quad . \tag{32}$$

Intermediate variable $\eta^k(\mathbf{x})$ is equal to partial age $a^k(\mathbf{x})$. This is easily seen by setting $\varsigma(\mathbf{x}) = \eta^k(\mathbf{x})$ and

$$\begin{cases} \mathcal{V} = \Omega \,,\ \mathcal{S}^i = \Gamma^i \,,\ \mathcal{S}^s = \Gamma^s \,,\ \mathcal{S}^o = \varnothing \\ \mathbf{D} = \mathbf{K} \,,\ \mathbf{q} = \mathbf{v} \,,\ \mu = \omega^k \end{cases} \tag{33}$$

Clearly, (33) is equivalent to (15). Therefore, by virtue of (20) and (32), the following relation holds valid

$$\bar{a}^k = \bar{\Theta}^k \tag{34}$$

Concluding remark

The above results are in line with elementary physical intuition. However, it must realised that they critically depend on the boundary conditions. If other boundary conditions were used, it is far from guaranteed that they would be left unchanged. In other words, should one wish to rely on other boundary conditions, all the developments should be remade — without prejudging their outcome.
