

ON A Γ -LIMIT OF WILLMORE FUNCTIONALS WITH ADDITIONAL CURVATURE PENALIZATION TERM

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ABSTRACT. We consider the Willmore functional on graphs, with an additional penalization of the area where the curvature is non-zero. Interpreting the penalization parameter as a Lagrange multiplier, this corresponds to the Willmore functional with a constraint on the area where the graph is flat. Sending the penalization parameter to ∞ and rescaling suitably, we derive the limit functional in the sense of Γ -convergence.

1. INTRODUCTION

1.1. Motivation: A constrained Willmore problem. The motivation for the present paper comes from a constrained Willmore problem. More precisely, let us consider smooth hypersurfaces $M \subset \mathbb{R}^3$ of a fixed topology, with constraints on the amount of surface area where the surface is flat and non-flat respectively. Here, by flat we mean that the second fundamental form vanishes. Within this class, we are interested in the variational problem

$$\inf_M \int_M (H^2 - 2K) \, d\mathcal{H}^2 = \inf_M \int_M (\kappa_1^2 + \kappa_2^2) d\mathcal{H}^2, \quad (1)$$

where κ_1, κ_2 denote the principal curvatures, $H = \kappa_1 + \kappa_2$ the mean curvature, $K = \kappa_1 \kappa_2$ the Gauss curvature, and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure.

Here we are going to simplify this problem in two ways: First, we are not going to consider arbitrary surfaces, but only graphs. Secondly, we are going to replace the constraint of having a fixed amount of non-flat surface area by a penalization of the non-flat part. This is the usual attempt of capturing constraints via the introduction of Lagrange multipliers. We will however not be able to prove a rigorous equivalence between the constrained variational problem and the problem involving Lagrange multipliers.

The latter consists in, for $\lambda > 0$ and a set of graphs M with fixed surface area, in the variational problems

$$\inf_M \int_M (H^2 - 2K) \, d\mathcal{H}^2 + \lambda \mathcal{H}^2(\{x \in M : S_M \neq 0\}), \quad (2)$$

where S_M denotes the second fundamental form of M . Additionally, boundary conditions or other constraints on M may be imposed.

Obviously, the shape of minimizers for such a problem depend on the penalization parameter λ . One expects that the concentration of curvature increases with λ , i.e., the area where the surface is flat becomes larger as λ increases (for configurations of low energy). The main purpose of the present paper is a rigorous investigation of the limit $\lambda \rightarrow \infty$ for the variational problem (2).

1.2. Statement of main result. For any Borel set $U \subset \mathbb{R}^n$, let $\mathcal{M}(U)$ denote the set of signed Radon measures on U . We denote by $\mathcal{M}(U; \mathbb{R}^p)$ the $\mathbb{R}^p$ valued Radon measures on U . Furthermore, let $\mathcal{M}(U; \mathbb{R}_{\text{sym}}^{n \times n})$ denote the space $\{\mu \in \mathcal{M}(U; \mathbb{R}^{n \times n}) : \mu_{ij} = \mu_{ji} \text{ for } i \neq j\}$. For $\mu \in \mathcal{M}(U; \mathbb{R}^p)$, let $|\mu|$ denote the total variation measure. For $\mu \in \mathcal{M}(U; \mathbb{R}^p)$, we have by the Radon-Nikodym differentiation Theorem (see Theorem 2 below) that for $|\mu|$ -almost every $x \in U$, the derivative $d\mu/d|\mu|$ exists. For any one-homogeneous function $h : \mathbb{R}^p \rightarrow \mathbb{R}$ and any $\mu \in \mathcal{M}(U; \mathbb{R}^p)$, we may hence define

$$h(\mu) = h\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

This is a well defined Borel measure.

For $\xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}$, let $\tau_1(\xi), \tau_2(\xi)$ denote the eigenvalues of ξ . We set

$$\rho^0(\xi) := \sum_{i=1}^2 |\tau_i(\xi)|.$$

We will repeatedly use the following estimates:

$$|\xi| \leq \rho^0(\xi) \leq 2|\xi|. \quad (3)$$

Note that $\rho^0 : \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}$ is sublinear and positively one-homogeneous.

For $u : \Omega \rightarrow \mathbb{R}$, let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth boundary, and let $u \in BH(\Omega)$; that is the space of $u \in L^1(\Omega)$ such that $\nabla u \in BV(\Omega; \mathbb{R}^2)$. We will use the usual notation for the BV function ∇u : $J_{\nabla u}$ denotes the jump set of ∇u . On $J_{\nabla u}$, there exists a measurable function $\nu_{\nabla u}$ with values in S^2 such that ∇u has well defined limits $\nabla u^\pm$ on both sides of the hyperplane defined by $\nu_{\nabla u}$. $S_{\nabla u}$ is the singular set of $D\nabla u$, i.e., the set where $D\nabla u$ is not absolutely continuous w.r.t. $\mathcal{L}^2$. Furthermore, $C_{\nabla u} := S_{\nabla u} \setminus J_{\nabla u}$. We have the decomposition

$$D\nabla u = \nabla^2 u \mathcal{L}^2 + (\nabla u_+ - \nabla u_-) \otimes \nu_{\nabla u} \llcorner J_{\nabla u} + D^s \nabla u \llcorner C_{\nabla u}.$$

For a sequence $u_j \in BV(\Omega)$, we say that $u_j \rightarrow u$ weakly $*$ in BV if $u_j \rightarrow u$ in L^1 and $Du_j \rightarrow Du$ weakly $*$ in the sense of measures, that is,

$$\lim_{j \rightarrow \infty} \int_{\Omega} u_j \operatorname{div} \varphi dx = \int_{\Omega} u \operatorname{div} \varphi dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

For $v \in \mathbb{R}^2$ and $\xi \in \mathbb{R}^{2 \times 2}$, we define

$$\begin{aligned} \mathbf{N}(v) &= (v, -1)/\sqrt{1 + |v|^2}, \\ \mathbf{g}(v) &= \operatorname{Id}_{2 \times 2} + v \otimes v, \\ \mathbf{H}(v, \xi) &= \frac{1}{\sqrt{1 + |v|^2}} \xi \\ S(v, \xi) &= \mathbf{g}(v)^{-1/2} \mathbf{H}(v, \xi) \mathbf{g}(v)^{-1/2}. \end{aligned}$$

By this definition, $S(\nabla u, \nabla^2 u) \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ is the second fundamental form (or shape operator) of the graph of u in matrix form; its eigenvalues are the principal curvatures of the graph. Let $F_\lambda : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ be defined by

$$F_\lambda(\xi) = \begin{cases} 0 & \text{if } \xi = 0 \\ |\xi|^2 + \lambda & \text{else.} \end{cases}$$

We define $\mathcal{F}_\lambda : W^{2,2}(\Omega) \rightarrow [0, +\infty)$ by

$$\mathcal{F}_\lambda(u) = \lambda^{-1/2} \int_{\Omega} F_\lambda(S(\nabla u, \nabla^2 u)) \sqrt{1 + |\nabla u|^2} dx. \quad (4)$$

Note that up to a normalizing factor, for smooth functions u , the right hand side is precisely the functional introduced in the previous subsection,

$$\lambda^{1/2} \mathcal{F}_\lambda(u) = \int_{\text{gr}(u)} (H^2 - 2K) d\mathcal{H}^2 + \lambda \mathcal{H}^2(\{x \in \text{gr}(u) : S_{\text{gr}(u)} \neq 0\}),$$

where $\text{gr}(u)$ denotes the graph of u . For $u \in W^{2,2}(\Omega)$, the right hand side in (4) is finite, since the Willmore integrand $\sqrt{1 + |\nabla u|^2} |S|^2$ is bounded from above by $|\nabla^2 u|^2$ (see Lemma 1 below). Let $\arccos : [-1, 1] \rightarrow [0, \pi]$ be the inverse function of $\cos : [0, \pi] \rightarrow [-1, 1]$, and for $v = (v_1, v_2) \in \mathbb{R}^2$, let $v^\perp = (-v_2, v_1)$. We define $\mathcal{F} : BH(\Omega) \rightarrow [0, +\infty)$ by

$$\begin{aligned} \mathcal{F}(u) &= 2 \int_{\Omega} \rho^0(S(\nabla u, \nabla^2 u)) \sqrt{1 + |\nabla u|^2} dx \\ &\quad + 2 \int_{C_{\nabla u}} \rho^0 \left(S \left(\nabla u, \frac{dD\nabla u}{d|D\nabla u|} \right) \right) \sqrt{1 + |\nabla u|^2} d|D\nabla u| \llcorner C_{\nabla u} \\ &\quad + 2 \int_{J_{\nabla u}} \arccos \mathbf{N}(\nabla u^+) \cdot \mathbf{N}(\nabla u^-) \sqrt{1 + |\nu_{\nabla u}^\perp \cdot \nabla u|^2} d\mathcal{H}^1. \end{aligned}$$

Again, the right hand side always exists and is finite, since $\rho^0(S(v, \xi)) \sqrt{1 + |v|^2} \leq 2|\xi|$, and hence the integrands can be estimated by the Lebesgue regular, jump and Cantor part of the measure $\rho(D\nabla u)$ respectively. Finally, let us write $\mathcal{A} = BH(\Omega) \cap W^{1,\infty}(\Omega)$.

Our main result is the following theorem, which establishes the Γ -convergence $\mathcal{F}_\lambda \rightarrow \mathcal{F}$ in the weak-* topology of $BH(\Omega)$.

Theorem 1. (i) Let u_λ be a sequence in $W^{2,2}(\Omega)$ with $\limsup_{\lambda \rightarrow \infty} \mathcal{F}_\lambda(u_\lambda) < \infty$, $\int_{\Omega} u_\lambda dx = 0$ and $\|\nabla u_\lambda\|_{L^\infty} < C$. Then there exists a subsequence (no relabeling) and $u \in \mathcal{A}$ such that

$$u_\lambda \rightarrow u \text{ in } W^{1,1}(\Omega), \quad \nabla u_\lambda \rightarrow \nabla u \text{ weakly }^* \text{ in } BV(\Omega; \mathbb{R}^2). \quad (5)$$

(ii) Let u_λ, u be as in (5). Then we have

$$\liminf_{\lambda \rightarrow \infty} \mathcal{F}_\lambda(u_\lambda) \geq \mathcal{F}(u).$$

(iii) Let $u \in \mathcal{A}$. Then there exists a sequence u_λ such that (5) is fulfilled and

$$\limsup_{\lambda \rightarrow \infty} \mathcal{F}_\lambda(u_\lambda) \leq \mathcal{F}(u).$$

Remark 1. (i) For $u \in C^2(\Omega)$, the limit functional $\mathcal{F}$ can be written as

$$\mathcal{F}(u) = \int_{\text{gr}(u)} 2\rho^0(S_{\text{gr}(u)}) d\mathcal{H}^2, \quad (6)$$

where $\text{gr}(u)$ denotes the graph of u . The formula for $\mathcal{F}$ from the statement of the theorem is a generalization for surfaces whose second fundamental form is a measure. We note that graphs of functions in $BH(\Omega)$ do not belong to the class of curvature varifolds as defined in [Hut86, Man96]. The latter do not allow for a Cantor part in the curvature measure.

- (ii) For the “geometrically linearized” functionals

$$\mathcal{G}_\lambda(u) = \lambda^{-1/2} \int_{\Omega} F_\lambda(\nabla^2 u) dx$$

we have shown in [Olb17] that the limit functional (again in the sense of Γ -convergence) is given by $\mathcal{G}(u) = 2 \int_{\Omega} d(\rho^0(D\nabla u))$. Here we merely replace the second derivative $\nabla^2 u$ by the second fundamental form $S(\nabla u, \nabla^2 u)$. However, the presence of lower order terms makes the analysis more difficult for several reasons. There exist a few different techniques for the proof of lower semicontinuity of integral functionals that depend on lower order terms starting from the results without those terms, see [Mar85, AF84, FMP98]. These techniques do not work here since we consider the convergence $\nabla u_\lambda \rightarrow \nabla u$ weakly $*$ in BV (and not in $W^{1,p}$ with $p > 1$ as in the quoted references). The lower semicontinuity in BV for integral functionals that depend on lower order terms has been treated in [FM93]. Their technique cannot be applied in a straightforward way here either, the reason being that for fixed λ the integrands of our functionals have 2-growth at infinity. Our technique will be a modification of the one from [FM93], choosing a cutoff that despite the 2-growth does not increase the energy by too much.

Carrying on with the comparison of our result with the one in [FM93], we would like to point out that we are able to determine the form of the Γ -limit on the jump part explicitly, which is not possible in the general situation treated in [FM93]. This requires the solution of a certain variational problem that we obtain through some geometric considerations (see Section 3.4).

Concerning the upper bound, this is more difficult here than in [Olb17] again because of the presence of lower order terms. In that reference, the upper bound follows directly from well known properties of approximations of BV functions by mollification. Here, we need to keep track of the behavior of the lower order terms in this approximation process, for which we need to use some results on the fine properties of BV functions.

- (iii) The requirement $\|\nabla u_\lambda\|_{L^\infty} < C$ in the compactness part of the theorem (statement (i)) may seem unnatural. Without such an assumption however, we are not able to obtain control of the BH -norm from the energy alone. This can be seen by considering graphs of functions with almost vertical parts. The energy of these almost vertical parts can be made arbitrarily small. In this way, we might obtain functions of arbitrarily large L^1 norm with bounded energy. This can be considered as an artefact of the restriction to graphs, and shows that a geometric description would be more appropriate. This will be the topic of future work.

The requirement $\int_{\Omega} u_\lambda dx = 0$ is included in the statement (i) to enforce the convergence $u_\lambda \rightarrow u$ in L^1 . Without such an assumption, we would still have the convergence $\nabla u_\lambda \rightarrow \nabla u$ weakly $*$ in BV (for a subsequence).

1.3. Scientific context. Vesicles of polyhedral shape play an important role in biology. Examples are virus capsids [CK62, LMN03], carboxysomes [YKH⁺08], cationic-ionic vesicles, and assembled supramolecular structures [MA99]. In [VSOdlC11], a model for the formation of polyhedral structures based on minimization of the free elastic energy of topologically spherical shells has been suggested. In the model, the free energy is a function of the deformation of the shell, and the material distribution of the two elastic components that the shell is made of.

Elastic inhomogeneities are known to exist in many virus capsids and for carboxysomes; in both of these cases, the vesicle shell is made up of different protein types. In [SOD1C12], it has been suggested that the inhomogeneities can act as the driving force for faceting. In this reference, it is assumed that the vesicle wall consists of two components, with different elastic properties (“soft” and “hard”), and the amount of soft and hard material available for the formation of the vesicle is fixed. The variational problems (1) and (2), interpreted as minimization problems for the free elastic energy, are models for such two-phase vesicles. Following this interpretation, we investigate here the limit in which the contrast between soft and hard phase is large (the hard phase does not bend at all), and there is a very small amount of soft material.

1.4. Comparison to the analogous one-dimensional problem. Consider the following variational problem, which is a lower dimensional analogue for problem (2), with the topology fixed to be that of a sphere instead of a graph:

$$\inf \left\{ \int_M \kappa^2 ds + \lambda \mathcal{H}^1(\{x \in M : \kappa \neq 0\}) : M \text{ homeomorphic to } S^1, \mathcal{H}^1(M) = 2\pi \right\}$$

Pulling the penalization term into the integral, we obtain

$$\inf \left\{ \int_M \tilde{f}_\lambda(\kappa) ds : M \text{ homeomorphic to } S^1, \mathcal{H}^1(M) = 2\pi \right\},$$

with

$$\tilde{f}_\lambda(\kappa) = \begin{cases} \lambda + \kappa^2 & \text{if } \kappa \neq 0 \\ 0 & \text{else.} \end{cases}$$

It is well known that such a problem requires relaxation to guarantee the existence of minimizers. The relaxed problem is obtained by replacing the integrand with its convex lower semicontinuous envelope,

$$\tilde{f}_\lambda^{**}(\kappa) = \begin{cases} 2\sqrt{\lambda}|\kappa| & \text{if } |\kappa| \leq \sqrt{\lambda} \\ \lambda + \kappa^2 & \text{else.} \end{cases}$$

We see immediately that the integrands $\lambda^{-1/2} \tilde{f}_\lambda^{**}$ are monotone decreasing, and converge to the function $\kappa \mapsto 2|\kappa|$. From this convergence, one deduces without difficulty the Γ -convergence of the respective integral functionals, with respect to weak * convergence of the curvatures. The limit functional $\mathcal{F}^{(1)} : M \mapsto 2 \int_M |\kappa| ds$ is also defined for one-spheres whose curvature is only a measure. Note that there is a large set of minimizers for $\mathcal{F}^{(1)}$: Any one-sphere with non-negative curvature will be a minimizer.

The situation in dimension two is completely different: From Theorem 1, it is natural to conjecture that one may define a limit functional in the sense of Γ convergence that for smooth surfaces is given by (6). For surfaces of convex bodies, this functional is the same as the total mean curvature. For sufficiently smooth surfaces, it is known that the only minimizer of this functional within the class of topological two-spheres is the round sphere, see [Min89, Bon26].

1.5. Some notation. The symbol “ C ” is used as follows: A statement such as “ $f \leq Cg$ ” is shorthand for “there exists a constant $C > 0$ such that $f \leq Cg$ ”. The value of C may change within the same line. For $f \leq Cg$, we also write $f \lesssim g$. Let $\mathbb{R}_{\text{sym}}^{n \times n}$ denote the symmetric $n \times n$ matrices. For $\xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}$, let $\tau_i(\xi)$, $i = 1, 2$ denote the eigenvalues of ξ . We denote the operator norm of ξ by

$$|\xi|_\infty = \max(|\tau_1(\xi)|, |\tau_2(\xi)|).$$

The two-dimensional Lebesgue measure is denoted by $\mathcal{L}^2$, the d -dimensional Hausdorff measure by $\mathcal{H}^d$. For $x = (x_1, x_2) \in \mathbb{R}^2$, we write $x^\perp = (-x_2, x_1)$. Let $Q = [-1/2, 1/2]^2$ and for $\nu \in S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$, let Q_ν be the unit cube in $\mathbb{R}^2$ with one of its sides parallel to ν , i.e., $Q_\nu = \{x \in \mathbb{R}^2 : \max(|x \cdot \nu|, |x \cdot \nu^\perp|) \leq 1/2\}$. For a set $K \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ and $\rho > 0$, we write $K(x_0, \rho) = x_0 + \rho K$. By $O(t)$, we denote terms $f(t)$ that satisfy $\limsup_{t \rightarrow \infty} t^{-1}|f(t)| < \infty$.

2. PRELIMINARIES

2.1. Properties of BV functions. Let $U \subset \mathbb{R}^n$ be open.

Theorem 2 (Proposition 2.2 in [ADM92]). *Let λ, μ be Radon measures in U with $\mu \geq 0$. Then there exists a Borel set $E \subset U$ with $\mu(E) = 0$ such that for any $x_0 \in \text{supp } \mu \setminus E$ we have*

$$\lim_{\rho \downarrow 0} \frac{\lambda(x_0 + \rho K)}{\mu(x_0 + \rho K)} = \frac{d\lambda}{d\mu}(x_0)$$

for any bounded convex set K containing the origin. Here, the set E is independent of K .

Theorem 3 (Theorem 2.3 in [ADM92]). *Let $u \in BV(U; \mathbb{R}^m)$ and for a bounded convex open set K containing the origin, and let ξ be the density of Du with respect to $|Du|$, $\xi = \frac{d(Du)}{d(|Du|)}$. For $x_0 \in \text{supp}(|Du|)$, assume that $\xi(x_0) = \eta \otimes \nu$ with $\eta \in \mathbb{R}^m$, $\nu \in \mathbb{R}^n$, $|\eta| = |\nu| = 1$, and for $\rho > 0$ let*

$$v^{(\rho)}(y) = \frac{\rho^{n-1}}{|Du|(x_0 + \rho K)} \left(u(x_0 + \rho y) - \int_{x_0 + \rho K} u(x') dx' \right).$$

Then for every $\sigma \in (0, 1)$ there exists a sequence ρ_j converging to 0 such that $v^{(\rho_j)}$ converges in $L^1(K; \mathbb{R}^m)$ to a function $v \in BV(K; \mathbb{R}^m)$ which satisfies $|Dv|(\sigma \bar{K}) \geq \sigma^n$ and can be represented as

$$v(y) = \psi(y \cdot \nu) \eta$$

for a suitable non-decreasing function $\psi : (a, b) \rightarrow \mathbb{R}$, where $a = \inf\{y \cdot \nu : y \in K\}$ and $b = \sup\{y \cdot \nu : y \in K\}$.

We recall that $BH(U)$ denotes the set of functions $u \in L^1(U)$ such that $\nabla u \in BV(U; \mathbb{R}^n)$. The set $BH(U)$ can be made into a normed space by setting

$$\|u\|_{BH(U)} = \|u\|_{W^{1,1}(U)} + |D\nabla u|(U).$$

We say that a sequence $u_j \in BH(U)$ converges weakly $*$ to $u \in BH(U)$ if $u_j \rightarrow u$ in $W^{1,1}(U)$ and $D\nabla u_j \rightarrow D\nabla u$ weakly $*$ in $\mathcal{M}(U; \mathbb{R}^{n \times n})$.

Theorem 4 ([Dem89]). *Let u_j be a bounded sequence in $BH(U)$. Then there exists a subsequence (no relabeling) and $u \in BH(U)$ such that*

$$u_j \rightarrow u \quad \text{weakly } * \text{ in } BH(U).$$

2.2. Relaxation of integral functionals that depend on higher derivatives. A function $f : \mathbb{R}^{m \times n^k} \rightarrow \mathbb{R}$ is called k -quasiconvex if

$$f(\xi) = \inf \left\{ \int_{[-1/2, 1/2]^n} f(\xi + \nabla^k \varphi) dx : \varphi \in W_0^{k, \infty}([-1/2, 1/2]^n; \mathbb{R}^m) \right\}, \quad (7)$$

see [Mey65].

The so-called k -quasiconvexification of $f : \mathbb{R}^{m \times n^k} \rightarrow \mathbb{R}$ is given by the right hand side above,

$$Q_k f(\xi) = \inf \left\{ \int_{[-1/2, 1/2]^n} f(\xi + \nabla^k \varphi) dx : \varphi \in W_0^{k, \infty}([-1/2, 1/2]^n; \mathbb{R}^m) \right\}.$$

In the case $k = 1$, one obtains the relaxation of integral functionals $u \mapsto \int f(\nabla u) dx$ by replacing f by its quasiconvex envelope $Q_1 f$.

2.3. Blow-up method. The main tool in our proof will be the so-called blow-up method. In the context of lower semicontinuity of integral functionals in BV , this has been developed by Fonseca and Müller.

Theorem 5 (Theorem 2.19 in [FM93]). *Let $f : \mathbb{R}^m \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ be quasiconvex and positively one-homogeneous¹ in the second argument. Assume that $v_j \rightarrow v$ weakly $*$ in $BV(\Omega)$ and $f(v_j, \nabla v_j) \mathcal{L}^n \rightarrow \mu$ weakly $*$ in the sense of measures, and that ζ_2, ζ_3 are defined as the Radon-Nikodym derivatives*

$$\zeta_2 = \frac{d\mu}{d(|D^s v| \llcorner C_v)}, \quad \zeta_3 = \frac{d\mu}{d(\mathcal{H}^1 \llcorner J_v)}.$$

Then

$$\begin{aligned} \zeta_2(x_0) &\geq f\left(v(x_0), \frac{dDv}{d|Dv|}(x_0)\right) \quad \text{for } |D^s v| \llcorner C_v \text{ a.e. } x_0 \in \Omega \\ \zeta_3(x_0) &\geq K_f(v^+(x_0), v^-(x_0), \nu_v(x_0)) \quad \text{for } |D^s v| \llcorner J_v \text{ a.e. } x_0 \in \Omega, \end{aligned}$$

where

$$\begin{aligned} K_f(a, b, \nu) &= \inf \left\{ \int_{Q_\nu} f(w, \nabla w) dx : w \in W^{1,1}(Q_\nu), \right. \\ &\quad \left. w(x) = a \text{ for } x \cdot \nu = +1/2, w(x) = b \text{ for } x \cdot \nu = -1/2 \right\}. \end{aligned}$$

3. SOME AUXILIARY LEMMAS

3.1. Relaxation and quasiconvexification. We consider the following integrands, defined for $v \in \mathbb{R}^2$, $\xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}$:

$$f_\lambda(v, \xi) = \lambda^{-1/2} F_\lambda(S(v, \xi)) \sqrt{1 + |v|^2}.$$

This choice implies $\mathcal{F}_\lambda(u) = \int_\Omega f_\lambda(\nabla u, \nabla^2 u) dx$.

In order to find the lower semicontinuous envelope of $\mathcal{F}_\lambda$, we will need to determine the 2-quasiconvexification of f_λ . In principle this is contained in [KS86, AK93], and the

¹When comparing our statement of the theorem with the one in [FM93], note that the assumption that f is positively one-homogeneous implies that the recession function for f is identical to f .

appendix of [Olb17] contains a detailed proof of the case $v = 0$. Hence we only point out the modifications that are necessary with respect to the latter; these changes can be found in the appendix to the present paper.

Proposition 1. *Let $v \in \mathbb{R}^2$. The 2-quasiconvexification of $f_\lambda(v, \cdot) = \lambda^{-1/2} F_\lambda(S(v, \cdot)) \sqrt{1 + |v|^2}$ is given by*

$$Q_2 f_\lambda(v, \xi) = \sqrt{1 + |v|^2} \begin{cases} 2\rho^0(S(v, \xi)) - 2 \frac{|\det S(v, \xi)|}{\sqrt{\lambda}} & \text{if } \rho^0(S(v, \xi)) \leq \sqrt{\lambda} \\ \frac{|S(v, \xi)|^2}{\sqrt{\lambda}} + \sqrt{\lambda} & \text{else.} \end{cases} \quad (8)$$

In the sequel we use the notation

$$g_\lambda(\xi) = \begin{cases} 2\rho^0(\xi) - \frac{|\det \xi|}{\sqrt{\lambda}} & \text{if } \rho^0(\xi) \leq \sqrt{\lambda} \\ \frac{|\xi|^2}{\sqrt{\lambda}} + \sqrt{\lambda} & \text{else.} \end{cases}$$

$$h_\lambda(v, \xi) = Q_2 f_\lambda(v, \xi) = g_\lambda(S(v, \xi)) \sqrt{1 + |v|^2}.$$

3.2. Properties of h_λ . The following straightforward estimate will be used repeatedly:

Lemma 1. *Let $v \in \mathbb{R}^2, \xi \in \mathbb{R}^{2 \times 2}$. Then*

$$|S(v, \xi)|^2 \leq (1 + |v|^2)^{-1} |\xi|^2.$$

Proof. This follows easily from the observation that $g(v)^{-1}$ is a symmetric matrix with eigenvalues 1 and $(1 + |v|^2)^{-1}$. $\square$

In the following lemma, we collect some properties of g_λ .

Lemma 2. (i) *Let $M > 1$. There exists a constant $C = C(M)$ such that whenever $A, B \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ with $|A| \leq M|B|$, we have*

$$g_\lambda(A) \leq C g_\lambda(B).$$

(ii) *For $A, B \in \mathbb{R}_{\text{sym}}^{2 \times 2}$, we have*

$$|g_\lambda(A) - g_\lambda(B)| \leq C|A - B| \left(1 + \frac{|A| + |B|}{\sqrt{\lambda}} \right).$$

(iii) *For every $\lambda > 0$, we have*

$$g_\lambda(\xi) \geq 2|\xi|_\infty.$$

Proof. We prove (i) by case distinction: If $\sqrt{\lambda} \leq \rho^0(A)$, then we have

$$g_\lambda(A) \leq 2\rho^0(A) \leq 4|A| \leq 4M|B| \leq 4Mg_\lambda(B).$$

If $\rho^0(B) \leq \sqrt{\lambda} \leq \rho^0(A)$, then we have

$$g_\lambda(A) = \sqrt{\lambda} + \frac{|A|^2}{\sqrt{\lambda}} \leq 2|A| + \frac{|A|^2}{M^{-1}|A|} \leq 3M^2|B| \leq 3M^2g_\lambda(B).$$

If $\sqrt{\lambda} \leq \min(\rho^0(A), \rho^0(B))$, then

$$g_\lambda(A) = \sqrt{\lambda} + \frac{|A|^2}{\sqrt{\lambda}} \leq M^2g_\lambda(B).$$

This completes the proof of (i).

To prove (ii) it suffices to observe that g_λ is piecewise differentiable. A direct computation yields

$$|\nabla g_\lambda(A)| \leq C \left(1 + \frac{|A|}{\sqrt{\lambda}} \right)$$

almost everywhere, which immediately implies (ii).

Finally we prove (iii). For $\xi = 0$, the inequality is trivial. So let $\xi \neq 0$, and denote the eigenvalues of ξ by τ_1, τ_2 . For $\rho^0(\xi) \leq \sqrt{\lambda}$, we have

$$\begin{aligned} g_\lambda(\xi) &= 2 \left(\rho^0(\xi) - \frac{|\det \xi|}{\sqrt{\lambda}} \right) \\ &\geq 2 \left(\rho^0(\xi) - \frac{|\det \xi|}{|\xi|_\infty} \right) \\ &\geq 2 \left(\rho^0(\xi) - \min(|\tau_1|, |\tau_2|) \right) \\ &= 2|\xi|_\infty. \end{aligned}$$

For $\rho^0(\xi) \geq \sqrt{\lambda}$, we have by the Cauchy-Schwarz inequality,

$$g_\lambda(\xi) = \sqrt{\lambda} + \frac{|\xi|^2}{\sqrt{\lambda}} \geq 2 \frac{|\xi|\sqrt{\lambda}}{\sqrt{\lambda}} \geq 2|\xi|_\infty.$$

This proves the lemma. $\square$

Lemma 3. *We have that*

$$\begin{aligned} |h_\lambda(v, \xi) - h_\lambda(\tilde{v}, \xi)| &\leq C|v - \tilde{v}| \max(h_\lambda(v, \xi), h_\lambda(\tilde{v}, \xi)) \\ |f_\lambda(v, \xi) - f_\lambda(\tilde{v}, \xi)| &\leq C|v - \tilde{v}| \max(f_\lambda(v, \xi), f_\lambda(\tilde{v}, \xi)) \end{aligned}$$

for all $v, \tilde{v} \in \mathbb{R}^2$, $\xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}$, where the constants C do not depend on λ .

Proof. We recall that $S(v, \xi)$ is given explicitly by

$$S(v, \xi) = (1 + |v|^2)^{-1/2} (\text{Id} + v \otimes v)^{-1/2} \xi (\text{Id} + v \otimes v)^{-1/2}.$$

We claim that

$$|\nabla_v S(v, \xi)| \lesssim \frac{S(v, \xi)}{\sqrt{1 + |v|^2}}. \quad (9)$$

Indeed, noting that

$$(\text{Id} + v \otimes v)^{-1/2} = \frac{1}{\sqrt{1 + |v|^2}} \frac{v \otimes v}{|v|^2} + \frac{v^\perp \otimes v^\perp}{|v|^2},$$

this follows from a direct calculation, which we omit here. Now we may estimate the partial derivative of $h_\lambda(v, \xi)$ using the chain rule and Lemma 2 (ii),

$$\begin{aligned} |\nabla_v h_\lambda(v, \xi)| &= \left| g_\lambda(S(v, \xi)) \nabla_v \sqrt{1 + |v|^2} + \sqrt{1 + |v|^2} \nabla g_\lambda(S(v, \xi)) \nabla_v S(v, \xi) \right| \\ &\lesssim g_\lambda(S(v, \xi)) + \sqrt{1 + |v|^2} \left(1 + \frac{S(v, \xi)}{\sqrt{\lambda}} \right) \frac{S(v, \xi)}{\sqrt{1 + |v|^2}} \\ &\lesssim g_\lambda(S(v, \xi)) \\ &\leq h_\lambda(S(v, \xi)). \end{aligned}$$

The analogous claim for f_λ is trivial for $\xi = 0$, and follows from (9) and the chain rule for $\xi \neq 0$. $\square$

The following lemma will provide the proof of the lower bound once the additional complication of the lower order terms has been treated.

Lemma 4. *Let $\Omega \subset \mathbb{R}^2$ be open and bounded, $v_0 \in \mathbb{R}^2$, $\xi_0 \in \mathbb{R}^{2 \times 2}$, $w_\lambda \rightarrow 0$ in $L^1(\Omega)$ as $\lambda \rightarrow \infty$, and $\|\nabla w_\lambda\|_{L^1} < C$. Then*

$$\liminf_{\lambda \rightarrow \infty} \int_{\Omega} h_\lambda(v_0, \xi_0 + \nabla w_\lambda) dx \geq 2\mathcal{L}^2(\Omega) \rho^0(S(v_0, \xi_0)) \sqrt{1 + |v_0|^2}.$$

Proof. Up to details, the proof is identical to the proof of Lemma 6.2 (i) in [Olb17]. There it is proved that

$$\liminf_{\lambda \rightarrow \infty} \int_{\Omega} g_\lambda(\xi_0 + \nabla w_\lambda) dx \geq 2\mathcal{L}^2(\Omega) \rho^0(\xi_0).$$

In that proof, one only needs to replace g_λ with $g_\lambda(S(v_0, \cdot))$. Apart from the additional dependence of some of the constants “ C ” on v_0 that appear in the proof, all arguments go through unchanged. $\square$

3.3. Blow-up of higher order gradients. Theorem 5 describes the behavior of integrands depending on gradients under the blow-up procedure. This will not be quite enough for our purposes: For the jump part, our proof will take advantage of the fact that we consider the second fundamental form of the graph, which in turn means that we need to consider integrands that depend on first and second derivatives.

Lemma 5. *Assume that $f : \mathbb{R}^m \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ fulfills the following properties:*

- (i) *f is quasiconvex and positively one-homogeneous in the second argument with $f(v, \xi) \leq C|\xi|$*
- (ii) *The functional $u \mapsto \int_{\Omega} f(\nabla u, \nabla^2 u) dx$ is continuous in $W^{2,2}(\Omega)$*

Furthermore assume that u_λ is a sequence in $W^{2,2}(\Omega)$, $u_\lambda \rightarrow u$ weakly $$ in $BH(\Omega)$, $f(\nabla u_j, \nabla^2 u_j) \mathcal{L}^n \rightarrow \mu$ weakly $*$ in the sense of measures, and that ζ_3 is defined as the Radon-Nikodym derivative*

$$\zeta_3 = \frac{d\mu}{d(\mathcal{H}^1 \llcorner J_{\nabla u})}.$$

Then

$$\zeta_3(x_0) \geq \tilde{K}_f(\nabla u^+(x_0), \nabla u^-(x_0), \nu_{\nabla u}(x_0)) \quad \text{for } |D^s \nabla u| \llcorner J_{\nabla u} \text{ a.e. } x_0 \in \Omega,$$

where

$$\tilde{K}_f(a, b, \nu) = \inf \left\{ \int_{Q_\nu} f(\nabla w, \nabla^2 w) dx : w \in \mathcal{A}_{a,b,\nu} \right\}$$

and

$$\begin{aligned} \mathcal{A}_{a,b,\nu} = & \left\{ w \in C^\infty(\overline{Q_\nu}) : \right. \\ & w(x) = a \cdot x \text{ in some neighborhood of } \left\{ x \in \partial Q_\nu : x \cdot \nu = \frac{1}{2} \right\}, \\ & w(x) = b \cdot x \text{ in some neighborhood of } \left\{ x \in \partial Q_\nu : x \cdot \nu = -\frac{1}{2} \right\}, \\ & \left. \nabla^k w(x + \nu^\perp) = \nabla^k w(x) \text{ for } x \cdot \nu^\perp = -\frac{1}{2} \text{ and } k = 1, 2, \dots \right\}. \end{aligned}$$

Proof. We write $\nu \equiv \nu_{\nabla u}(x_0)$. With

$$u_\lambda^{(\rho)}(x) = \rho^{-1} (u_\lambda(x_0 + \rho x) - u_\lambda(x_0)) , \quad U(x) = \begin{cases} \nabla u^+(x_0) \cdot x & \text{if } x \cdot \nu \geq 0 \\ \nabla u^-(x_0) \cdot x & \text{if } x \cdot \nu < 0 , \end{cases}$$

we have that for $|D\nabla u| \llcorner J_{\nabla u}$ almost every x_0 , $\lim_{\rho \rightarrow 0} \lim_{\lambda \rightarrow \infty} u_\lambda^{(\rho)} = U$ in $W^{1,1}(Q_\nu)$, see Theorem 3.77 in [AFP00]. Additionally,

$$\zeta_3(x_0) = \lim_{\rho \rightarrow 0} \lim_{\lambda \rightarrow \infty} \rho^{-1} \int_{Q_\nu(x_0, \rho)} f(\nabla u_\lambda, \nabla^2 u_\lambda) dx .$$

Choose $\rho_j \rightarrow 0, \lambda_j \rightarrow \infty$ such that $u_{\lambda_j}^{(\rho_j)} \rightarrow U$ in $W^{1,1}(Q_\nu)$ and

$$\zeta_3(x_0) = \lim_{j \rightarrow \infty} \rho_j^{-1} \int_{Q_\nu(x_0, \rho_j)} f(\nabla u_{\lambda_j}, \nabla^2 u_{\lambda_j}) dx .$$

We write $u_j := u_{\lambda_j}^{(\rho_j)}$. Let $\eta \in C_c^\infty(\mathbb{R}^2)$ be radially symmetric with $\int_{\mathbb{R}^2} \eta(x) dx = 1$ and $\eta_\varepsilon = \varepsilon^{-2} \eta(\cdot/\varepsilon)$. We set $U_j := \eta_{\rho_j} * U$. U_j is affine on the slices orthogonal to ν . With this notation, we have

$$\zeta_3(x_0) = \lim_{j \rightarrow \infty} \int_{Q_\nu} f(\nabla u_j, \nabla^2 u_j) dx .$$

Hence it remains to show

$$\tilde{K}_f(\nabla u^+(x_0), \nabla u^-(x_0), \nu) \leq \lim_{j \rightarrow \infty} \int_{Q_\nu} f(\nabla u_j, \nabla^2 u_j) dx . \quad (10)$$

By the continuity assumption (ii), we may assume that $u_j \in C^\infty(\overline{Q_\nu})$ in the proof of (10).

For $l \in \mathbb{N}$, let $K_l \in \mathbb{N}$ be the smallest integer that satisfies

$$K_l > l \sup_j \{ \|u_j\|_{W^{2,1}} + \|U_j\|_{W^{2,1}} \} ,$$

and

$$\alpha_l := \max \left(\frac{1}{l}, \sup \{ \|u_j - U_j\|_{W^{1,1}} : j > l \} \right) , \quad s_l := \frac{\alpha_l}{K_l} .$$

Note that $\alpha_l \rightarrow 0$ as $l \rightarrow \infty$. For $i = 0, \dots, K_l$, let

$$Q_{i,l} = (1 - \alpha_l + i s_l) Q_\nu .$$

Consider a family of cut-off functions $\{\varphi_{i,l} : i = 1, \dots, K_l\}$ with

$$\varphi_{i,l} \in C_c^\infty(Q_{i,l}), \quad 0 \leq \varphi_{i,l} \leq 1, \quad \varphi_{i,l} = 1 \text{ on } Q_{i-1,l}, \quad \|\nabla^k \varphi_{i,l}\|_{L^\infty} = O(s_l^{-k}) \text{ for } k = 1, 2 .$$

For $j > l$, we define

$$\tilde{u}_j^{i,l} := \varphi_{i,l} u_j + (1 - \varphi_{i,l}) U_j .$$

We have that $\tilde{u}_j^{i,l} \in \mathcal{A}_{\nabla u^+(x_0), \nabla u^-(x_0), \nu}$ (for j large enough). On $Q_{i,l} \setminus Q_{i-1,l}$, we have

$$\begin{aligned} \nabla^2 \tilde{u}_j^{i,l} &= (u_j - U_j) \nabla^2 \varphi_{i,l} + \nabla(u_j - U_j) \otimes \nabla \varphi_{i,l} \\ &\quad + \nabla \varphi_{i,l} \otimes \nabla(u_j - U_j) + \varphi_{i,l} \nabla^2(u_j - U_j) + \nabla^2 U_j . \end{aligned}$$

Now we may estimate, for every $i = 1, \dots, K_l$,

$$\begin{aligned}
\int_{Q_\nu} f(\nabla \tilde{u}_j^{i,l}, \nabla^2 \tilde{u}_j^{i,l}) dx &\leq \int_{Q_{i-1,l}} f(\nabla u_j, \nabla^2 u_j) dx + C \int_{Q_{i,l} \setminus Q_{i-1,l}} |\nabla^2 \tilde{u}_j^{i,l}| dx \\
&\quad + C \int_{Q_\nu \setminus Q_{i,l}} |\nabla^2 U_j| dx \\
&\leq \int_{Q_{i-1,l}} f(\nabla u_j, \nabla^2 u_j) dx \\
&\quad + C \int_{Q_{i,l} \setminus Q_{i-1,l}} s_l^{-2} |u_j - U_j| + s_l^{-1} |\nabla u_j - \nabla U_j| + |\nabla^2 u_j| + |\nabla^2 U_j| dx \\
&\quad + C \int_{Q_\nu \setminus Q_{i,l}} |\nabla^2 U_j| dx
\end{aligned} \tag{11}$$

We write $T_{i,l} = Q_{i,l} \setminus Q_{i-1,l}$, and choose an increasing sequence $j(l)$ with $j(l) > l$ such that for every $i = 1, \dots, K_l$,

$$\begin{aligned}
\left| \int_{T_{i,l}} |u_j - U_j| dx \right| &< s_l^2 \\
\left| \int_{T_{i,l}} |\nabla u_j - \nabla U_j| dx \right| &< s_l.
\end{aligned}$$

This is possible by $\|u_j - U_j\|_{W^{1,1}} \rightarrow 0$. With the help of these estimates, the second error term in (11) for $j = j(l)$ can be estimated as follows,

$$\begin{aligned}
&\int_{T_{i,l}} s_l^{-2} |u_j - U_j| + s_l^{-1} |\nabla u_j - \nabla U_j| + |\nabla^2 u_j| + |\nabla^2 U_j| dx \\
&\leq C \left(\|\nabla^2 u_j\|_{L^1(T_{i,l})} + \|\nabla^2 U_j\|_{L^1(T_{i,l})} + s_l \right).
\end{aligned}$$

Summing over all i and averaging, we obtain

$$\begin{aligned}
\frac{1}{K_l} \sum_{i=1}^{K_l} \int_{Q_\nu} f(\nabla \tilde{u}_j^{i,l}, \nabla^2 \tilde{u}_j^{i,l}) dx &\leq \int_{Q_\nu} f(\nabla u_j, \nabla^2 u_j) dx + \frac{C}{K_l} \int_{Q_\nu} |\nabla^2 U_j| dx \\
&\quad + \frac{C}{K_l} \int_{Q_\nu} (|\nabla^2 u_j| + |\nabla^2 U_j| + 1) dx + C s_l
\end{aligned}$$

Since the error terms vanish for $l \rightarrow \infty$, we can choose $i = i(l) \in \{1, \dots, K_l\}$ such that

$$\liminf_{l \rightarrow \infty} \int_{Q_\nu} f(\nabla \tilde{u}_j^{i,l}, \nabla^2 \tilde{u}_j^{i,l}) dx \leq \lim_{j \rightarrow \infty} \int_{Q_\nu} f(\nabla u_j, \nabla^2 u_j) dx.$$

Since $\tilde{u}_j^{i,l} \in \mathcal{A}_{\nabla u^+(x_0), \nabla u^-(x_0), \nu}$, the last equation proves (10). $\square$

3.4. Geometric considerations. We will need to apply Lemma 5 to the following particular choice of integrand:

$$G_\infty(v, \xi) = |S(v, \xi)|_\infty \sqrt{1 + |v|^2}.$$

By some geometric considerations, we are able to determine $\tilde{K}_{G_\infty}$ in Lemma 7 below. We start with a preparatory lemma. The assumptions are chosen such that we may apply the lemma to graphs of functions in $\mathcal{A}_{a,b,\nu}$ as defined in Lemma 5 with $\nu = e_2$, see Figure 1a.

Lemma 6. *Let M be an oriented C^2 submanifold of $\mathbb{R}^3$ with the following properties:*

- (i) *M is diffeomorphic to a square*
- (ii) *There exists $l > 0$ and for each $x_1 \in [0, l]$ there exists a C^2 curve γ_{x_1} contained in $\{x_1\} \times [0, 1] \times \mathbb{R}$ with its two endpoints in $\{x_1\} \times \{0\} \times \mathbb{R}$ and $\{x_1\} \times \{1\} \times \mathbb{R}$ respectively, such that*

$$M = \bigcup_{x_1 \in [0, l]} \gamma_{x_1}.$$

- (iii) *There exist $\mathbf{N}_0, \mathbf{N}_1 \in S^2$ such that for each $x_1 \in [0, l]$, the surface normals in the endpoints of γ_{x_1} are given by $\mathbf{N}_0, \mathbf{N}_1$ respectively.*

Then

$$\int_M |S_M|_\infty d\mathcal{H}^2 \geq l \arccos \mathbf{N}_0 \cdot \mathbf{N}_1,$$

and equality holds if any two curves $\gamma_{x_1}, \gamma_{x'_1}$ are parallel translations of each other in x_1 direction, and their curvature does not change sign.

Proof. Looking at the slices for $x_1 = \text{constant}$, we have that

$$\int_M |S_M|_\infty d\mathcal{H}^2 \geq \int_0^l \int_{\gamma_{x_1}} |S_M|_\infty d\mathcal{H}^1.$$

Denoting by N_{x_1} a differentiable choice of a normal to M along γ_{x_1} , we have that the derivative of the normal DN_{x_1} fulfills

$$|DN_{x_1}|_\infty \leq |S_M|_\infty.$$

Hence, by the fundamental theorem of calculus, and letting $\text{dist}_{S^2}(\cdot, \cdot)$ denote the geodesic distance on S^2 ,

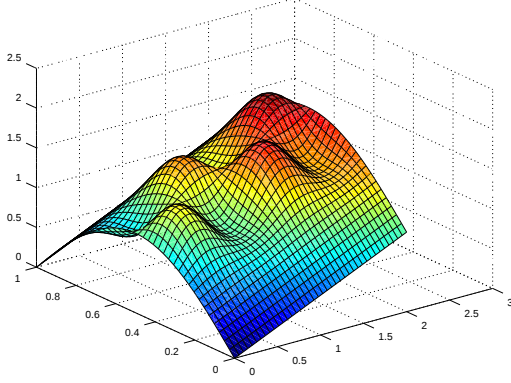
$$\int_{\gamma_{x_1}} |S_M|_\infty d\mathcal{H}^1 \geq \int_{\gamma_{x_1}} |DN_{x_1}| d\mathcal{H}^1 \geq \text{dist}_{S^2}(\mathbf{N}_0, \mathbf{N}_1) = \arccos \mathbf{N}_0 \cdot \mathbf{N}_1.$$

The claimed inequality follows. If the curves γ_{x_1} are parallel translations of each other in x_1 -direction and their curvature does not change sign, then the inequalities become sharp. $\square$

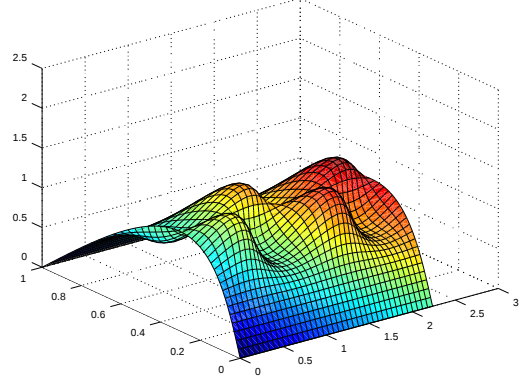
Lemma 7. *Let $a, b \in \mathbb{R}^2$, $\nu \in S^1$ with $a \cdot \nu^\perp = b \cdot \nu^\perp$, and $G_\infty(v, \xi) = 2|S(v, \xi)|_\infty \sqrt{1 + |v|^2}$. Then with $\tilde{K}$ defined as in the statement of Lemma 5, we have that*

$$\tilde{K}_{G_\infty}(a, b, \nu) = 2\sqrt{1 + |a \cdot \nu^\perp|^2} \arccos \mathbf{N}(a) \cdot \mathbf{N}(b).$$

Proof. Let $w \in \mathcal{A}_{a,b,\nu}$. After a rotation of the coordinate system, we may assume that $\nu = e_2$ and $a_1 = b_1$. Let M_1 denote the graph of w . By applying a suitable Euclidean motion (namely, a rotation with axis parallel to e_2 and a translation), we may map $\text{gr } w|_{[0,1] \times \{0\}}$ to $[0, \sqrt{1 + a_1^2}] \times \{(0, 0)\}$ and $\text{gr } w|_{[0,1] \times \{1\}}$ to $[0, \sqrt{1 + a_1^2}] \times \{(1, 0)\}$ respectively, see Figure 1b. Let us denote the resulting submanifold of $\mathbb{R}^3$ by M_2 . By the periodicity of $\nabla^k w$ for $k \in \{1, 2\}$ in x_1 -direction, we may translate $M_2 \cap [0, l] \times [0, 1] \times \mathbb{R}$ in x_1 -direction by $l = \sqrt{1 + a_1^2}$, and the resulting set will still be a C^2 submanifold, with $\int_{M_3} |S_{M_3}|_\infty d\mathcal{H}^2 =$



(A) The graph of some function w to which Lemma 6 may be applied.



(B) After applying a suitable Euclidean motion R , we have $R(\text{gr } w|_{[0,1] \times \{0\}}) \subset \mathbb{R} \times \{(0, 0)\}$ and $R(\text{gr } w|_{[0,1] \times \{1\}}) \subset \mathbb{R} \times \{(1, 0)\}$.

$\int_{M_1} |S_{M_1}|_\infty d\mathcal{H}^2$. To M_3 , we may apply Lemma 6 to obtain the claimed lower bound. If ∇w is constant in x_1 direction and $e_2 \cdot \nabla w$ is monotone in x_2 direction, the second part of that lemma yields that the bound is also attained. $\square$

4. PROOF OF THE MAIN THEOREM

4.1. Compactness.

Proof of Theorem 1 (i). Using $\|\nabla u_\lambda\|_{L^\infty} < C$, we have that

$$|\nabla^2 u_\lambda| \leq C |S(\nabla u_\lambda, \nabla^2 u_\lambda)| \quad (12)$$

By Lemma 2 (iii), we have that

$$|\xi| \leq g_\lambda(\xi) \quad \text{for all } \xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}. \quad (13)$$

From (12) and (13) it follows that

$$|\nabla^2 u_\lambda| \leq h_\lambda(\nabla u_\lambda, \nabla^2 u_\lambda),$$

and hence

$$\limsup \|\nabla^2 u_\lambda\|_{L^1(\Omega)} \leq C.$$

By Theorem 4, we obtain the weak $*$ convergence in BH for a subsequence. $\square$

4.2. Lower bound.

Proof of Theorem 1 (ii). The main tool of the proof is the blowup technique by Fonseca and Müller. We have that

$$D\nabla u = \nabla^2 u \mathcal{L}^2 + D^s \nabla u \llcorner C_{\nabla u} + (\nabla u_+ - \nabla u_-) \otimes \nu_{\nabla u} \mathcal{H}^1 \llcorner J_{\nabla u}.$$

In the sequel, we write $nu \equiv \nu_{\nabla u}$. After choosing a subsequence, we may assume that $\lim_{\lambda \rightarrow \infty} \mathcal{F}_\lambda(u_\lambda) = \liminf_\lambda \mathcal{F}_\lambda(u_\lambda)$, without increasing the $\liminf$. Since $h_\lambda = Q_2 f_\lambda \leq f_\lambda$, there exists a Radon measure μ such that (after passing to a further subsequence)

$$h_\lambda(\nabla u_\lambda, \nabla^2 u_\lambda) \mathcal{L}^2 \rightarrow \mu \quad \text{weakly } * \text{ in the sense of measures.}$$

Let $\zeta_1, \zeta_2, \zeta_3$ denote the Radon-Nikodym derivative of μ with respect to $\mathcal{L}^2$, $|D^s \nabla u| \llcorner C_{\nabla u}$ and $\mathcal{H}^1 \llcorner J_{\nabla u}$ respectively. By the non-negativity of μ , we have

$$\mu \geq \zeta_1 \mathcal{L}^2 + \zeta_2 |D^s \nabla u| \llcorner C_{\nabla u} + \zeta_3 \mathcal{H}^1 \llcorner J_{\nabla u}.$$

We will show that

$$\zeta_1(x) \geq 2\rho^0 \left(S(\nabla u(x), \nabla^2 u(x)) \right) \sqrt{1 + |\nabla u|^2} \quad \text{for } \mathcal{L}^2 - \text{a.e. } x \in \Omega \quad (14)$$

$$\zeta_2(x) \geq 2\rho^0 \left(S \left(\nabla u(x), \frac{d(D\nabla u)}{d|D\nabla u|}(x) \right) \right) \sqrt{1 + |\nabla u|^2} \quad \text{for } |D^s \nabla u| - \text{a.e. } x \in C_{\nabla u} \quad (15)$$

$$\zeta_3(x) \geq 2 \arccos(\mathbf{N}(\nabla u_+) \cdot \mathbf{N}(\nabla u_-)) \sqrt{1 + |\nu^\perp \cdot \nabla u|^2} \quad \text{for } \mathcal{H}^1 - \text{a.e. } x \in J_{\nabla u}. \quad (16)$$

This will prove the lower bound.

We will first prove (14).

We write $v_\lambda = \nabla u_\lambda$. For $\mathcal{L}^2$ -almost every x_0 , we may choose a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ converging to zero, such that $\mu(\partial Q(x_0, \varepsilon_j)) = 0$ for every $j \in \mathbb{N}$. When we write $\varepsilon \rightarrow 0$ in the sequel, we actually mean the limit $j \rightarrow \infty$ for such a sequence. For every j , we have

$$\lim_{\lambda \rightarrow \infty} \int_{Q(x_0, \varepsilon_j)} h_\lambda(v_\lambda, \nabla v_\lambda) dx = \mu(Q(x_0, \varepsilon_j)).$$

Moreover,

$$\lim_{j \rightarrow \infty} \lim_{\lambda \rightarrow \infty} \frac{Dv_\lambda(Q(x_0, \varepsilon_j))}{|Dv|(Q(x_0, \varepsilon_j))} = \frac{dDv}{d|Dv|}(x_0).$$

Note that by Theorem 2 we have

$$\begin{aligned} \zeta_1(x_0) &= \lim_{\varepsilon \rightarrow 0} \frac{\mu(Q(x_0, \varepsilon))}{\mathcal{L}^2(Q(x_0, \varepsilon))} \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} \int_{Q(x_0, \varepsilon)} h_\lambda(v_\lambda, \nabla v_\lambda) dx. \end{aligned} \quad (17)$$

We write $v_0 := v(x_0)$. For ε small enough, define $w_{\lambda, \varepsilon} : Q \rightarrow \mathbb{R}^2$ by

$$w_{\lambda, \varepsilon}(x) = \varepsilon^{-1} (v_\lambda(x_0 + \varepsilon x) - v_0).$$

Furthermore let $w_0(x) = \nabla v_0 \cdot x$. Using a change of variables, the Cauchy-Schwarz inequality and (36), we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} \|w_{\lambda, \varepsilon} - w_0\|_{L^1(Q)} &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_Q |v(x_0 + \varepsilon x) - v_0 - \nabla v_0 \cdot \varepsilon x| dx \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^3} \int_{Q(x_0, \varepsilon)} |v(x) - v_0 - \nabla v_0 \cdot (x - x_0)| dx \\ &\leq \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\int_{Q(x_0, \varepsilon)} |v(x) - v_0 - \nabla v_0 \cdot (x - x_0)|^2 dx \right)^{1/2} \\ &= 0. \end{aligned} \quad (18)$$

To obtain the last equality above, we have used that for $\mathcal{L}^2$ -almost every x_0 , $v = \nabla u$ is approximately differentiable at x_0 , see [AFP00] Theorem 3.83. Also note that we have

$$\int_{Q(x_0, \varepsilon)} h_\lambda(v_\lambda, \nabla v_\lambda) dx = \int_Q h_\lambda(v_0 + \varepsilon w_{\lambda, \varepsilon}(x), \nabla w_{\lambda, \varepsilon}(x)) dx.$$

By (17) and (18), it is possible to choose a sequence $\lambda_j \rightarrow \infty$ and a subsequence $\varepsilon_j \rightarrow 0$ (no relabeling) such that with $w_j := w_{\lambda_j, \varepsilon_j}$

$$\lim_{j \rightarrow \infty} \|w_j - w_0\| = 0$$

$$\lim_{j \rightarrow \infty} \int_Q h_{\lambda_j}(v_0 + \varepsilon_j w_j(x), \nabla w_j) dx = \zeta_1(x_0).$$

We need to modify w_j such that we get a suitable L^∞ -bound for fixing the lower order terms. Namely, we are going to construct $\tilde{w}_j$ such that $\|\tilde{w}_j\|_{L^\infty} \leq \varepsilon_j^{-1/2}$, and

$$\liminf_{j \rightarrow \infty} \int_Q h_{\lambda_j}(v_0 + \varepsilon_j \tilde{w}_j, \nabla \tilde{w}_j) dx \leq \liminf_{j \rightarrow \infty} \int_Q h_{\lambda_j}(v_0 + \varepsilon_j w_j, \nabla w_j) dx. \quad (19)$$

Let K_j be the largest integer smaller than $\log_2 \varepsilon_j^{-1/2}$. For $k = 1, \dots, K_j$, we define

$$E_k^j := \left\{ x \in Q : 2^{k-1} < |w_j - w_0| \leq 2^k \right\}.$$

Next we choose $k_j \in \{1, \dots, K_j\}$ such that

$$\int_{E_{k_j}^j} (1 + h_{\lambda_j}(v_0 + \varepsilon_j w_j, \nabla w_j)) dx \leq K_j^{-1} \int_Q (1 + h_{\lambda_j}(v_0 + \varepsilon_j w_j, \nabla w_j)) dx \quad (20)$$

and we define $\varphi_j : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \varphi_j(x) &= 1 \text{ for } x \leq 2^{k_j-1} \\ \varphi_j(x) &= 0 \text{ for } x \geq 2^{k_j} \\ |\varphi_j'| &\leq 2^{2-k_j}. \end{aligned}$$

Now we set

$$\tilde{w}_j = w_0 + \varphi_j(|w_j - w_0|)(w_j - w_0).$$

Note that $\|\tilde{w}_j - w_0\|_{L^\infty} \leq \varepsilon_j^{-1/2}$ by construction, and $\tilde{w}_j = w_0$ on $\{x \in \Omega : |w_j - w_0| \geq 2^{k_j}\}$. We have that

$$\begin{aligned} \liminf_{j \rightarrow \infty} \int_Q h_{\lambda_j}(v_0 + \varepsilon_j \tilde{w}_j, \nabla \tilde{w}_j) dx &\leq \liminf_{j \rightarrow \infty} \int_{\{|w_j - w_0| \leq 2^{k_j-1}\}} h_{\lambda_j}(v_0 + \varepsilon_j w_j, \nabla w_j) dx \\ &\quad + \int_{E_{k_j}^j} h_{\lambda_j}(v_0 + \varepsilon_j \tilde{w}_j, \nabla \tilde{w}_j) dx \\ &\quad + \int_{\{|w_j - w_0| \geq 2^{k_j}\}} h_{\lambda_j}(v_0 + \varepsilon_j w_0, \nabla w_0) dx. \end{aligned} \quad (21)$$

We claim that

$$\int_{E_{k_j}^j} h_{\lambda_j}(v_0 + \varepsilon_j \tilde{w}_j, \nabla \tilde{w}_j) dx \leq C(v_0, \nabla w_0) \int_{E_{k_j}^j} (1 + h_{\lambda_j}(v_0 + \varepsilon_j w_j, \nabla w_j)) dx. \quad (22)$$

Indeed, we have that on $E_{k_j}^j$, $|\varepsilon_j w_j - \varepsilon_j \tilde{w}_j| \lesssim \varepsilon_j^{1/2}$, and hence

$$\begin{aligned} |\mathbf{g}(v_0 + \varepsilon_j w_j) - \mathbf{g}(v_0 + \varepsilon_j \tilde{w}_j)| &\leq C(v_0) \varepsilon_j^{1/2}, \\ \left| \sqrt{1 + |v_0 + \varepsilon_j w_j|^2} - \sqrt{1 + |v_0 + \varepsilon_j \tilde{w}_j|^2} \right| &\leq C \varepsilon_j^{1/2}. \end{aligned} \quad (23)$$

Also,

$$\begin{aligned} |\nabla \tilde{w}_j| &= |\nabla w_0 + \varphi_j'(w_j - w_0) \otimes \nabla |w_j - w_0| + \varphi_j \nabla(w_j - w_0)| \\ &\leq C \left(|\nabla w_0| + (2^{-k_j} |w_j - w_0| + 1) |\nabla(w_j - w_0)| \right) \\ &\leq C(\nabla w_0) (|\nabla w_j| + 1). \end{aligned}$$

Hence, we have that

$$|S(v_0 + \varepsilon_j \tilde{w}_j, \nabla \tilde{w}_j)| \leq C(v_0, \nabla w_0) (|S(v_0 + \varepsilon_j w_j, \nabla w_j)| + 1)$$

and it follows from Lemma 2 (i) that

$$g_{\lambda_j}(S(v_0 + \varepsilon_j \tilde{w}_j, \nabla \tilde{w}_j)) \leq C(v_0, \nabla w_0) (g_{\lambda_j}(S(v_0 + \varepsilon_j w_j, \nabla w_j)) + 1). \quad (24)$$

Our claimed inequality (22) now follows from (23) and (24).

Using (20), (22) and the fact $K_j \rightarrow \infty$ as $j \rightarrow \infty$, as well as $w_j \rightarrow w_0$ in L^1 , the right hand side of (21) can be estimated from above by

$$\begin{aligned} \liminf_{j \rightarrow \infty} \left(\int_Q h_{\lambda_j}(v_0 + \varepsilon_j w_j, \nabla w_j) dx + C(v_0, \nabla w_0) \mathcal{L}^2 \left(\{|w_j - w_0| \geq 2^{k_j}\} \right) |\nabla w_0| \right) \\ = \liminf_{j \rightarrow \infty} \int_Q h_{\lambda_j}(v_0 + \varepsilon_j w_j, \nabla w_j) dx. \end{aligned}$$

This proves (19).

Now we have by Lemma 3 and Lemma 4,

$$\begin{aligned} \liminf_{j \rightarrow \infty} \int_Q h_{\lambda_j}(v_0 + \varepsilon_j \tilde{w}_j, \nabla \tilde{w}_j) dx &\geq \liminf_{j \rightarrow \infty} \frac{1 - C(v_0, \nabla w_0) \varepsilon_j^{1/2}}{1 + C(v_0, \nabla w_0) \varepsilon_j^{1/2}} \int_Q h_{\lambda_j}(v_0, \nabla \tilde{w}_j) dx \\ &\geq 2\sqrt{1 + |v_0|^2} \rho^0(S(v_0, \nabla w_0)). \end{aligned}$$

This proves equation (14).

Let $G(v, p) = 2\rho^0(S(v, p))\sqrt{1 + |v|^2}$, and $G_\infty(v, p) = 2|S(v, p)|_\infty \sqrt{1 + |v|^2}$. Let $v_\lambda \rightarrow v$ weakly $*$ in BV. We have by Lemma 2 (iii) that

$$h_\lambda(v_\lambda, \nabla v_\lambda) \geq G_\infty(v_\lambda, \nabla v_\lambda)$$

for all λ . By Theorem 5, we have that for $|D^s v| \llcorner C_v$ almost every point $x_0 \in \Omega$,

$$\zeta_2(x_0) \geq G_\infty \left(v(x_0), \frac{dDv}{d|Dv|}(x_0) \right)$$

which proves (15), since $\frac{dDv}{d|Dv|}(x_0)$ is rank one, and hence

$$2\rho^0 \left(S \left(v(x_0), \frac{dDv}{d|Dv|}(x_0) \right) \right) = 2 \left| S \left(v(x_0), \frac{dDv}{d|Dv|}(x_0) \right) \right|_\infty.$$

By Lemma 5, we have in a similar fashion for $|D^s v| \llcorner J_v$ almost every $x_0 \in \Omega$,

$$\zeta_3(x_0) \geq \tilde{K}_{G_\infty}(v^+(x_0), v^-(x_0), \nu(x_0)).$$

By Lemma 7, it follows

$$\zeta_3(x_0) \geq 2\sqrt{1 + |\nu^\perp \cdot v(x_0)|^2} \arccos \mathbf{N}(v^+(x_0)) \cdot \mathbf{N}(v^-(x_0)).$$

This proves (16) and completes the proof of the lower bound. $\square$

4.3. Upper bound. For the proof of the upper bound, we will need a modification of the well known result in the calculus of variations that states that the relaxation of integral functionals with suitable integrands that depend on $x, u, \nabla u$ is obtained by the quasiconvexification of the integrand with respect to the gradient variable. Here, we will need the analogous result for integrands that depend on $\nabla u, \nabla^2 u$.

Proposition 2. *Let $1 \leq p < \infty$, and let $f : \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n^2} \rightarrow \mathbb{R}$ such that*

$$\left. \begin{aligned} 0 &\leq f(v, \xi) \leq C(1 + |\xi|^p) \\ |f(v, \xi) - f(\tilde{v}, \xi)| &\leq C|v - \tilde{v}| \max(f(v, \xi), f(\tilde{v}, \xi)) \\ |Q_2 f(v, \xi) - Q_2 f(\tilde{v}, \xi)| &\leq C|v - \tilde{v}| \max(Q_2 f(v, \xi), Q_2 f(\tilde{v}, \xi)) \end{aligned} \right\} \forall v, \tilde{v} \in \mathbb{R}^{m \times n}, \xi \in \mathbb{R}^{m \times n^2}. \quad (25)$$

Furthermore, let $\Omega \subset \mathbb{R}^n$ be open and bounded, $u \in W^{2,p}(\Omega)$ and $\delta > 0$. Then there exists $w \in W^{2,p}(\Omega; \mathbb{R}^m)$ with

$$\begin{aligned} \|u - w\|_{W^{1,p}(\Omega; \mathbb{R}^m)} &< \delta \\ \int_{\Omega} f(\nabla w, \nabla^2 w) dx &< \int_{\Omega} Q_2 f(\nabla u, \nabla^2 u) dx + \delta. \end{aligned}$$

For the proof of the proposition, we are going to use

Lemma 8 (Theorem 3.7 in [Olb17]). *Let $\Omega \subset \mathbb{R}^2$ be open and bounded, and let $p \in [1, \infty)$. Furthermore let $u \in C^3(\Omega)$ and $\delta > 0$. Then there exists $w \in W^{2,\infty}(\Omega)$ and $\Omega_w \subset \Omega$ such that Ω_w is the union of mutually disjoint closed cubes, w is piecewise a polynomial of degree 2 on Ω_w , and furthermore*

$$\begin{aligned} \|u - w\|_{W^{2,p}(\Omega)} &< \delta, \\ \|w\|_{W^{2,\infty}} &\lesssim \|u\|_{W^{2,\infty}} \\ \int_{\Omega \setminus \Omega_w} (1 + |\nabla^2 u|^p + |\nabla^2 w|^p) dx &< \delta. \end{aligned}$$

Proof of Proposition 2. First we recall the well known fact that rank-one convex functions are locally Lipschitz continuous (see e.g. [Dac08]). This holds true in particular for $Q_2 f(v, \cdot)$ for any v , and hence the assumption (25) implies that $Q_2 f$ is locally Lipschitz continuous in both arguments. More precisely, with the assumed growth properties for f , we have $Q_2 f(v, \xi) \leq C(1 + |\xi|^p)$ and hence

$$|Q_2(v, \xi) - Q_2(v, \tilde{\xi})| \leq C|\xi - \tilde{\xi}| \left(1 + |\xi|^{p-1} + |\tilde{\xi}|^{p-1}\right) \quad \forall \xi, \tilde{\xi} \in \mathbb{R}^{m \times n^2} \quad (26)$$

where C is some constant that is independent of $v, \xi, \tilde{\xi}$ (see Proposition 2.32 in [Dac08]).

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier and $\eta_\varepsilon := \varepsilon^{-n}\eta(\cdot/\varepsilon^n)$. We set $u_\varepsilon := \eta_\varepsilon * u$ and claim that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} Q_2 f(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) dx = \int_{\Omega} Q_2 f(\nabla u, \nabla^2 u) dx. \quad (27)$$

Indeed, we have that $u_\varepsilon \rightarrow u$ in $W^{2,p}$, and hence by (26) and the assumption (25), we have

$$\begin{aligned} & \int_{\Omega} |Q_2 f(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) - Q_2 f(\nabla u, \nabla^2 u)| dx \\ & \leq \int_{\Omega} |Q_2 f(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) - Q_2 f(\nabla u_\varepsilon, \nabla^2 u)| + |Q_2 f(\nabla u_\varepsilon, \nabla^2 u) - Q_2 f(\nabla u, \nabla^2 u)| dx \\ & \leq C \int_{\Omega} |\nabla^2 u_\varepsilon - \nabla^2 u| (1 + |\nabla^2 u|^{p-1} + |\nabla^2 u_\varepsilon|^{p-1}) + \max(|\nabla u_\varepsilon - \nabla u|, 1) (1 + |\nabla^2 u|^p) dx \end{aligned} \quad (28)$$

For $\varepsilon \rightarrow 0$, the integral on the right hand side converges to 0, proving the claim (27).

Let $\Delta > 0$. We choose u_ε such that $\|u - u_\varepsilon\|_{W^{2,p}} < \Delta$. By Lemma 8, there exists $w_\varepsilon \in W^{2,\infty}$ and a union of disjoint closed cubes $\Omega_w \subset \Omega$ such that w_ε is a polynomial of degree 2 on each of the cubes, and

$$\begin{aligned} & \|w_\varepsilon - u_\varepsilon\|_{W^{2,p}} < \Delta \\ & |\Omega \setminus \Omega_w| (1 + \|w_\varepsilon\|_{W^{2,\infty}}^p) < \Delta. \end{aligned}$$

By the same kind of estimate as in (28), we obtain that additionally, we may choose w_ε, Ω_w such that

$$\int_{\Omega} Q_2 f(\nabla w_\varepsilon, \nabla^2 w_\varepsilon) dx < \int_{\Omega} Q_2 f(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) dx + \Delta. \quad (29)$$

Moreover, we may choose the cubes to be so small that on each cube $\tilde{Q}$ with center x_0 in the collection,

$$\sup_{x \in \tilde{Q}} |\nabla w_\varepsilon(x) - \nabla w_\varepsilon(x_0)| < \Delta.$$

Let $\tilde{Q}$ be a cube where w_ε is a quadratic polynomial, with midpoint x_0 and sidelength r . Choose $\xi \in W_0^{2,\infty}(\tilde{Q})$ such that

$$\int_{\tilde{Q}} f(\nabla w_\varepsilon(x_0), \nabla^2 w(x_0) + \nabla^2 \xi) dx \leq \text{vol}(\tilde{Q}) Q_2 f(\nabla w_\varepsilon(x_0), \nabla^2 w_\varepsilon(x_0)) + \frac{\Delta}{N},$$

where N is the total number of cubes. Let us write $\tilde{\xi}(x) = \xi(x_0 + rx)$ for $x \in [-1/2, 1/2]^2$, and define $\tilde{\xi}$ on $\mathbb{R}^2$ by 1-periodic extension. For $M \in \mathbb{N}$, let

$$\xi_M(x) = M^{-2} \tilde{\xi}(M(x - x_0)).$$

We have that $\|\xi\|_{W^{1,\infty}} \rightarrow 0$ for $M \rightarrow \infty$, $\|\nabla^2 \xi\|_{L^\infty} = \|\nabla^2 \xi_M\|_{L^\infty}$, and

$$\int_{\tilde{Q}} f(\nabla w_\varepsilon(x_0), \nabla^2 w(x_0) + \nabla^2 \xi) dx = \int_{\tilde{Q}} f(\nabla w_\varepsilon(x_0), \nabla^2 w(x_0) + \nabla^2 \xi_M) dx.$$

We choose M so large that $\|\nabla \xi_M\|_{L^\infty} < \Delta$. This implies

$$\|\nabla w_\varepsilon + \nabla \xi_M - \nabla w_\varepsilon(x_0)\|_{L^\infty(\tilde{Q})} < 2\Delta. \quad (30)$$

Using the local Lipschitz continuity in the first argument of f as assumed in (25),

$$\int_{\tilde{Q}} f(\nabla w_\varepsilon + \nabla \xi_M, \nabla^2 w(x_0) + \nabla^2 \xi_M) dx < \frac{1 + C\Delta}{1 - C\Delta} \int_{\tilde{Q}} f(\nabla w_\varepsilon(x_0), \nabla^2 w(x_0) + \nabla^2 \xi_M) dx.$$

We repeat the same for all cubes $\tilde{Q}$ in Ω_w , obtaining a corrector function $\xi_{\tilde{Q}} \in W^{2,\infty}(\tilde{Q})$ in each of them. We set $\bar{w} = w_\varepsilon + \sum_{\tilde{Q}} \xi_{\tilde{Q}}$. Denoting by $x_{\tilde{Q}}$ the center of the cube $\tilde{Q}$, we have

$$\begin{aligned} \int_{\Omega} f(\nabla \bar{w}, \nabla^2 \bar{w}) dx &\leq \sum_{\tilde{Q}} \int_{\tilde{Q}} f(\nabla \bar{w}, \nabla^2 \bar{w}) dx + \int_{\Omega \setminus \Omega_w} f(\nabla \bar{w}, \nabla^2 \bar{w}) dx \\ &\leq \sum_{\tilde{Q}} \frac{1 + C\Delta}{1 - C\Delta} \int_{\tilde{Q}} f(\bar{w}(x_{\tilde{Q}}), \nabla^2 \bar{w}) dx + C \int_{\Omega \setminus \Omega_w} (1 + |\bar{w}|^p) dx \\ &\leq \sum_{\tilde{Q}} \frac{1 + C\Delta}{1 - C\Delta} \left(\int_{\tilde{Q}} Q_2 f(\bar{w}(x_{\tilde{Q}}), \nabla^2 w_\varepsilon) dx + \frac{\Delta}{N} \right) + \Delta \\ &\leq \left(\frac{1 + C\Delta}{1 - C\Delta} \right)^2 \int_{\Omega} Q_2 f(\nabla w_\varepsilon, \nabla^2 w_\varepsilon) dx + C\Delta. \end{aligned}$$

Here we used again (30) in combination with the assumption (25) to obtain the last inequality. By $\|u - w_\varepsilon\|_{W^{2,p}} < 2\Delta$ and (27), this last estimate proves the claim by choosing Δ small enough. $\square$

Proof of Theorem 1 (iii). Just as for the lower bound, we will use the blow-up method for the proof of the upper bound, in combination with a suitable mollification. We assume that we are given a sequence $\lambda_j \rightarrow \infty$, and we will prove that for any subsequence, there exists a further subsequence fulfilling the upper bound. We omit the index j from our notation and write $\lambda \rightarrow \infty$ for $j \rightarrow \infty$.

Step 1: Mollify $u \in BH(\Omega)$ to obtain $u_\varepsilon \in C^\infty$, where $\varepsilon = \varepsilon(\lambda, u)$ is chosen such that

$$\begin{aligned} \|S(\nabla u_\varepsilon, \nabla^2 u_\varepsilon)\|_{L^\infty} &< \sqrt{\lambda}/2 \\ \|\nabla^2 u_\varepsilon\|_{L^\infty} &< \sqrt{\lambda}/2, \end{aligned} \tag{31}$$

and $\varepsilon(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Writing $u_\lambda = u_{\varepsilon(\lambda, u)}$, we have

$$\begin{aligned} h_\lambda(\nabla u_\lambda, \nabla^2 u_\lambda) &\leq 2\sqrt{1 + |\nabla u_\lambda|^2} \rho^0(S(\nabla u_\lambda, \nabla^2 u_\lambda)) \\ &\leq 2|\nabla^2 u_\lambda|. \end{aligned} \tag{32}$$

Hence we have that $h_\lambda(\nabla u_\lambda, \nabla^2 u_\lambda) \mathcal{L}^2$, after passing to a subsequence, converges to some measure μ .

Step 2. For every continuous non-negative function $\varphi \in C_c^0(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} \varphi d\mu &= \lim_{\lambda \rightarrow \infty} \int_{\Omega} \varphi h_\lambda(\nabla u_\lambda, \nabla^2 u_\lambda) dx \\ &\leq 2 \lim_{\lambda \rightarrow \infty} \int_{\Omega} \varphi |\nabla^2 u_\lambda| dx \\ &= 2 \int_{\Omega} \varphi d|D\nabla u|. \end{aligned}$$

Hence, μ is absolutely continuous with respect to $|D\nabla u|$, and the measure μ can be decomposed into mutually singular measures,

$$\mu = \zeta_1 \mathcal{L}^2 + \zeta_2 |D^s \nabla u| \llcorner C_{\nabla u} + \zeta_3 \mathcal{H}^1 \llcorner J_{\nabla u}.$$

We will prove

$$\zeta_1(x) \leq 2\rho^0 \left(S(\nabla u(x), \nabla^2 u(x)) \right) \sqrt{1 + |\nabla u|^2} \quad \text{for } \mathcal{L}^2 - \text{a.e. } x \in \Omega \quad (33)$$

$$\zeta_2(x) \leq 2\rho^0 \left(S\left(\nabla u(x), \frac{dD\nabla u}{d|D\nabla u|}(x)\right) \right) \sqrt{1 + |\nabla u|^2} \quad \text{for } |D^s \nabla u| - \text{a.e. } x \in C_{\nabla u} \quad (34)$$

$$\zeta_3(x) \leq 2 \arccos(\mathbf{N}(\nabla u_+) \cdot \mathbf{N}(\nabla u_-)) \sqrt{1 + |\nu^\perp \cdot \nabla u|^2} \quad \text{for } \mathcal{H}^1 - \text{a.e. } x \in J_{\nabla u}. \quad (35)$$

Once we have proved these inequalities, we have proved

$$\limsup_{\lambda \rightarrow \infty} \int_{\Omega} h_\lambda(\nabla u_\lambda, \nabla^2 u_\lambda) dx \leq \mathcal{F}(u).$$

The upper bound then follows by $Q_2 f_\lambda = h_\lambda$ and Proposition 2. Indeed, the assumptions of the proposition (with $p = 2$) are fulfilled for f_λ by Lemma 3.

Step 3. To prove (32), we use again the fact that for $\mathcal{L}^2$ a.e. x_0 , $v = \nabla u$ is approximately differentiable, i.e.,

$$\lim_{r \rightarrow 0} \frac{1}{r^2} \int_{Q(x_0, r)} |v(x) - v(x_0) - \nabla v(x_0) \cdot (x - x_0)|^2 dx = 0. \quad (36)$$

Fix some $\delta > 0$, and choose r so small that

$$\int_{Q(x_0, r)} |v(x) - v(x_0) - \nabla v(x_0) \cdot (x - x_0)|^2 dx < r^2 \delta.$$

Note that the affine function $V : x \mapsto v(x_0) + \nabla v(x_0) \cdot (x - x_0)$ is invariant under mollification. We choose $\varepsilon < r/4$ and have that on $Q(x_0, r/2)$,

$$\begin{aligned} \sup_x |\nabla v_\varepsilon(x) - \nabla V(x)| + |v_\varepsilon(x) - V(x)| &= \sup_x |\nabla v_\varepsilon(x) - \nabla V_\varepsilon(x)| + |v_\varepsilon(x) - V_\varepsilon(x)| \\ &\leq \left(1 + \frac{C}{r}\right) \|v - V\|_{L^1(Q(x_0, r))} \\ &\leq C \left(1 + \frac{1}{r}\right) r \|v - V\|_{L^2(Q(x_0, r))} \\ &\leq C \delta^{1/2} r^3. \end{aligned}$$

Here we have used Hölder's inequality to obtain the next to last inequality. In particular, this shows that for any given δ , there exists an r_0 and a λ_0 such that whenever $r < r_0$ and $\lambda > \lambda_0$, then

$$|\nabla u_\lambda(x) - \nabla u(x_0)| + |\nabla^2 u_\lambda(x) - \nabla^2 u(x_0)| < \delta \quad \text{for all } x \in Q(x_0, r).$$

By the continuity of G and the inequality $h_\lambda \leq G$, it follows that for any $\delta > 0$, there exist r_0, λ_0 such that whenever $r < r_0$ and $\lambda > \lambda_0$, then

$$\int_{Q(x_0, r)} h_\lambda(\nabla u_\lambda, \nabla^2 u_\lambda) dx < G(\nabla u(x_0), \nabla^2 u(x_0)) + \delta \quad (37)$$

By the Radon-Nikodym Theorem,

$$\zeta_1(x_0) = \lim_{r \rightarrow 0} \lim_{\lambda \rightarrow 0} \int_{Q(x_0, r)} h_\lambda(\nabla u_\lambda, \nabla^2 u_\lambda) dx,$$

and hence (37) proves our claim (33), since δ in (37) was arbitrary.

Step 4. For $|D^s \nabla u| \perp C_{\nabla u}$ -almost every x_0 , we have that by Theorem 3 the following holds true: There exists a sequence $\rho_l \downarrow 0$ and a monotone function $\psi \in BV(-1/2, 1/2)$ such that the rescaled functions

$$v^{(\rho_l)}(x) := \frac{\rho_l}{|D \nabla u|(Q_\xi(x_0, \rho_l))} \left(\nabla u(x_0 + \rho_l x) - \int_{Q_\xi(x_0, \rho_l)} \nabla u(x') dx' \right)$$

converge for $l \rightarrow \infty$ in $L^1(Q_\xi; \mathbb{R}^2)$ to the function

$$x \mapsto \xi \psi(x \cdot \xi), \quad (38)$$

where $\xi \in S^1$ fulfills $\frac{d(D \nabla u)}{d|D \nabla u|}(x_0) = \xi \otimes \xi$. From now on, in order to alleviate the notation, we are going to omit the index l from ρ_l , and we write $\lim_{\rho \rightarrow 0}$ for $\lim_{l \rightarrow \infty}$.

The function (38) is of course the gradient of some function $U \in BH(Q_\xi)$, so we have

$$v^{(\rho)} \rightarrow \nabla U \text{ in } L^1 \quad \text{for } \rho \rightarrow 0.$$

By $|Dv^{(\rho)}|(Q_\xi) = 1$ and the lower semicontinuity of total variation, we have that $|D \nabla U|(Q_\xi) = |D \psi|(-1/2, 1/2) \leq 1$.

Now we choose² $\rho(\lambda) \rightarrow 0$ such that $\mu(\partial Q(x_0, \rho(\lambda))) = 0$, $\varepsilon/\rho \rightarrow 0$, and

$$\zeta_2(x_0) = \lim_{\lambda \rightarrow 0} \frac{1}{|D \nabla u|(Q_\xi(x_0, \rho(\lambda)))} \int_{Q_\xi(x_0, \rho(\lambda))} h_\lambda(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) dx. \quad (39)$$

Writing $\rho \equiv \rho(\lambda)$, we note that

$$\begin{aligned} (v^{(\rho)})_{\varepsilon/\rho}(x) &= \frac{\rho}{|D \nabla u|(Q_\xi(x_0, \rho))} (\nabla u_\varepsilon)(x_0 + \rho x) \\ \left(\nabla (v^{(\rho)})_{\varepsilon/\rho} \right)(x) &= \frac{\rho^2}{|D \nabla u|(Q_\xi(x_0, \rho))} (\nabla^2 u_\varepsilon)(x_0 + \rho x). \end{aligned} \quad (40)$$

Since $|D \nabla u|(Q_\xi(x_0, \rho))/\rho \rightarrow 0$ for $\rho \rightarrow 0$ by the fact that $x_0 \notin J_{\nabla u}$, we have that

$$\left(x \mapsto \nabla u(x_0 + \rho x) - \int_{Q_\xi(x_0, \rho)} \nabla u(x') dx' \right) \rightarrow 0 \text{ in } W^{1,1}(Q_\xi; \mathbb{R}^2) \text{ as } \rho \rightarrow 0. \quad (41)$$

It follows from (41) and $\varepsilon/\rho \rightarrow 0$ that we have

$$(x \mapsto \nabla u_\varepsilon(x_0 + \rho x)) \rightarrow \nabla u(x_0) \text{ in } C^1(Q_\xi; \mathbb{R}^2). \quad (42)$$

Using $h_\lambda \leq G$ (which we have achieved by our choice of mollification $\varepsilon \equiv \varepsilon(\lambda)$), we have that

²More precisely, using indices, we choose an increasing sequence $l(j)$ such that $\mu(\partial Q(x_0, \rho_{l(j)})) = 0$, $\varepsilon(\lambda_j, u)/\rho_{l(j)} \rightarrow 0$ and the $\limsup_{j \rightarrow \infty} \limsup_{l \rightarrow \infty}$ becomes a $\limsup_{j \rightarrow \infty}$.

$$\begin{aligned}
 & \limsup_{\lambda \rightarrow \infty} \frac{1}{|D^s \nabla u|(Q_\xi(x_0, \rho))} \int_{Q_\xi(x_0, \rho)} h_\lambda(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) \, dx \\
 & \leq \limsup_{\lambda \rightarrow \infty} \frac{1}{|D^s \nabla u|(Q_\xi(x_0, \rho))} \int_{Q_\xi(x_0, \rho)} G(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) \, dx.
 \end{aligned} \tag{43}$$

By a change of variables, we obtain

$$\int_{Q_\xi(x_0, \rho)} G(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) \, dx = \rho^2 \int_{Q_\xi} G(\nabla u_\varepsilon(x_0 + \rho y), \nabla^2 u_\varepsilon(x_0 + \rho y)) \, dy. \tag{44}$$

By the convergence (42), we have that

$$\begin{aligned}
 & \limsup_{\lambda \rightarrow \infty} \int_{Q_\xi} G(\nabla u_\varepsilon(x_0 + \rho y), \nabla^2 u_\varepsilon(x_0 + \rho y)) \, dy \\
 & = \limsup_{\lambda \rightarrow \infty} \int_{Q_\xi} G(\nabla u(x_0), \nabla^2 u_\varepsilon(x_0 + \rho y)) \, dy.
 \end{aligned} \tag{45}$$

Inserting (41), (44) and (45) in (43), we obtain

$$\begin{aligned}
 & \limsup_{\lambda \rightarrow \infty} \frac{1}{|D^s \nabla u|(Q_\xi(x_0, \rho))} \int_{Q_\xi(x_0, \rho)} h_\lambda(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) \, dx \\
 & \leq \limsup_{\lambda \rightarrow \infty} \int_{Q_\xi} G\left(\nabla u(x_0), \nabla\left(v^{(\rho)}\right)_{\varepsilon/\rho}\right) \, dx.
 \end{aligned} \tag{46}$$

By the fact $v^{(\rho)} \rightarrow \nabla U$ in $L^1(Q_\xi)$, we have that

$$\left(v^{(\rho)}\right)_{\varepsilon/\rho} - \nabla U_{\varepsilon/\rho} \rightarrow 0 \quad \text{in } C^\infty,$$

and hence

$$\limsup_{\lambda \rightarrow \infty} \int_{Q_\xi} G\left(\nabla u(x_0), \nabla\left(v^{(\rho)}\right)_{\varepsilon/\rho}\right) \, dx = \limsup_{\lambda \rightarrow \infty} \int_{Q_\xi} G(\nabla u(x_0), \nabla^2 U_{\varepsilon/\rho}) \, dx. \tag{47}$$

Since $G(\nabla u(x_0), \cdot)$ is positively one-homogeneous and $\nabla^2 U_{\varepsilon/\rho} = (\xi \otimes \xi : \nabla^2 U_{\varepsilon/\rho}) \xi \otimes \xi$, with $(\xi \otimes \xi : \nabla^2 U_{\varepsilon/\rho})$ either non-negative or non-positive for ε/ρ small enough, we obtain

$$\begin{aligned}
 \limsup_{\lambda \rightarrow \infty} \int_{Q_\xi} G(\nabla u(x_0), \nabla^2 U_{\varepsilon/\rho}) \, dx &= \int_{Q_\xi} G\left(\nabla u(x_0), \frac{D \nabla U}{|D \nabla U|}\right) \, d|D \nabla U| \\
 &= G(\nabla u(x_0), \xi \otimes \xi) |D \nabla U|(Q_\xi) \\
 &\leq G\left(\nabla u(x_0), \frac{dD \nabla u}{d|D \nabla u|}(x_0)\right).
 \end{aligned} \tag{48}$$

Recalling (39), we have that (46), (47) and (48) prove (34).

Step 5. For $\mathcal{H}^1 \llcorner J_{\nabla u}$ -almost every x_0 , we have that by Theorem 3 the following holds true: The rescaled functions

$$\nabla u^{(\rho)}(x) = \nabla u(x_0 + \rho x) - \int_{Q_\nu(x_0, \rho)} \nabla u(x') \, dx'$$

converge in $L^1(Q_\nu; \mathbb{R}^2)$ to the function

$$x \mapsto \begin{cases} \nabla u^+(x_0) & \text{if } x \cdot \nu > 0 \\ \nabla u^-(x_0) & \text{if } x \cdot \nu \leq 0 \end{cases} \quad (49)$$

The function (49) is of course the gradient of some function $U \in L^1(Q_\nu)$. It follows that

$$\nabla U_\varepsilon - \nabla u_\varepsilon^{(\rho)} = \eta_\varepsilon * (\nabla U - \nabla u^{(\rho)}) \rightarrow 0 \text{ in } C^\infty(Q_\nu; \mathbb{R}^2) \text{ for } \rho \rightarrow 0 \quad (50)$$

for every fixed ε . Now we choose $\rho(\lambda)$ such that $\rho \rightarrow 0$ and $\varepsilon/\rho \rightarrow 0$ as $\lambda \rightarrow \infty$, and we may write, again using $h_\lambda \leq G$,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{1}{\rho} \int_{Q_\nu(x_0, \rho)} h_\lambda(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) dx &\leq \liminf_{\lambda \rightarrow 0} \frac{1}{\rho} \int_{Q_\nu(x_0, \rho)} G(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) dx \\ &\leq \liminf_{\lambda \rightarrow 0} \int_{Q_\nu} \rho G(\nabla u_\varepsilon(x_0 + \rho x), \nabla^2 u_\varepsilon(x_0 + \rho x)) dx \\ &= \liminf_{\lambda \rightarrow 0} \int_{Q_\nu} G\left((\nabla u^{(\rho)})_{\varepsilon/\rho}, (\nabla(\nabla u^{(\rho)})_{\varepsilon/\rho})\right) dx. \end{aligned}$$

By (50), it follows from this last chain of inequalities that

$$\lim_{\lambda \rightarrow 0} \frac{1}{\rho} \int_{Q_\varepsilon(x_0, \rho)} h_\lambda(\nabla u_\varepsilon, \nabla^2 u_\varepsilon) dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{Q_\nu} G(\nabla U_\varepsilon, \nabla^2 U_\varepsilon) dx. \quad (51)$$

Next, we have that each component of $\nabla U_\varepsilon = \eta_\varepsilon * \nabla U$ is monotone in direction of ν , and constant in the direction orthogonal to ν . Additionally, we have that $\mathbf{N}(\nabla U_\varepsilon(x)) = \mathbf{N}(\nabla u^+(x_0))$ for $x \cdot \nu = +1/2$ and ε small enough, and analogous relation for $x \cdot \nu = -1/2$. Let us consider the extension of U_ε to the strip $\{x : |x \cdot \nu| \leq 1/2\}$, such that the gradient is constant in $\nu^\perp$ direction. The graph of this function is periodic in the sense that

$$y \in \text{gr}(U_\varepsilon) \Leftrightarrow y + \nu^\perp + e_3 \nu^\perp \cdot \nabla U_\varepsilon \in \text{gr}(U_\varepsilon)$$

Hence the graph $\text{gr}(U_\varepsilon)$ can be written as a union of curves, each of them contained in a hyperplane, that are parallel translates of each other. I.e., after a rotation of the coordinate axes, the conditions of the second part of Lemma 6 are fulfilled for $M = \text{gr}(U_\varepsilon)$, and we have (by the fact that $\nabla^2 U_\varepsilon$ is rank 1, and hence $G(\nabla U_\varepsilon, \nabla^2 U_\varepsilon) = G_\infty(\nabla U_\varepsilon, \nabla^2 U_\varepsilon)$)

$$\int_{Q_\nu} G(\nabla U_\varepsilon, \nabla^2 U_\varepsilon) dx = 2 \sqrt{1 + |\nu^\perp \cdot \nabla u|^2} \arccos(\mathbf{N}(\nabla u^+(x_0)) \cdot \mathbf{N}(\nabla u^-(x_0)))$$

for every $\varepsilon < 1/4$. Recalling that $\zeta_3(x)$ is given by the left hand side in (51), this proves (35) and hence the proof of the upper bound is complete. $\square$

APPENDIX A. PROOF OF PROPOSITION 1

Proof of Proposition 1. First we prove $\sqrt{1 + |v|^2} g_\lambda(S(v, \cdot)) \leq Q_2 f_\lambda(v, \cdot)$. Indeed, we prove the slightly stronger claim $\sqrt{1 + |v|^2} g_\lambda(S(v, \cdot)) \leq Q_1 f(v, \cdot)$, following the proof of Theorem 6.28 in [Dac08], where this is proved for $\lambda = 1$ and $v = 0$. The modifications that are necessary with respect to that proof are minor, so we will be brief.

First one shows that $g_1(S(v, \cdot))$ is polyconvex by defining

$$\theta(t) = \begin{cases} 2t & \text{if } t \leq 1 \\ 1 + t^2 & \text{else,} \end{cases}$$

and convex functions $H_{\pm} : \mathbb{R}^{2 \times 2} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$H_{\pm}(\xi, A) = \theta \left((|S(v, \xi)|^2 \pm 2 \det S(v, \xi))^{1/2} \right) \mp \frac{2A}{(1 + |v|^2)^2}.$$

Defining the convex function $H(\xi, A) = \max(H_+(\xi, A), H_-(\xi, A))$, one can then show that $g_1(S(v, \xi)) = H(\xi, \det \xi)$ and hence $g_1(S(v, \cdot))$ is polyconvex. Furthermore, we have that $g_{\lambda}(S(v, \cdot)) = \lambda g_1(S(v, \cdot/\sqrt{\lambda}))$ and hence $g_{\lambda}(S(v, \cdot))$ is polyconvex for every $\lambda > 0$. The inequality $\sqrt{1 + |v|^2} g_{\lambda}(S(v, \cdot)) \leq f_{\lambda}(v, \cdot)$ can be verified from the definitions. This proves $\sqrt{1 + |v|^2} g_{\lambda}(S(v, \cdot)) \leq Q_2 f_{\lambda}(v, \cdot)$. It remains to show the opposite inequality. To do so, we make the following definition:

Definition 1. Let $f : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$. We say that f is symmetric rank one convex if

$$f(t\xi_1 + (1-t)\xi_2) \leq tf(\xi_1) + (1-t)f(\xi_2)$$

for all $t \in [0, 1]$, and for all $\xi_1, \xi_2 \in \mathbb{R}_{\text{sym}}^{n \times n}$ such that $\xi_1 - \xi_2 = \alpha \eta \otimes \eta$ for some $\alpha \in \mathbb{R}$, $\eta \in \mathbb{R}^n$.

Furthermore, for $f : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$, we set

$$R^{\text{sym}} f(\xi) := \sup\{g(\xi) : g \leq f \text{ and } g \text{ is symmetric rank one convex}\}.$$

By Lemma B.5 of [Olb17], we have $Q_2 f_{\lambda}(v, \cdot) \leq R^{\text{sym}} f_{\lambda}(v, \cdot)$. In Lemma 9 below, the inequality $R^{\text{sym}} f_{\lambda}(v, \cdot) \leq h_{\lambda}(v, \cdot)$ is proved. This completes the proof of the proposition. $\square$

Lemma 9. We have

$$R^{\text{sym}} f_{\lambda}(v, \cdot) \leq h_{\lambda}(v, \cdot).$$

Proof. We write $\bar{f}_{\lambda} := (1 + |v|^2)^{-1/2} f_{\lambda}(v, \cdot)$, and need to show $R^{\text{sym}} \bar{f}_{\lambda} \leq g_{\lambda}(S(v, \cdot))$. Let $\xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}$. Then there exists an orthonormal basis $\tilde{e}_1, \tilde{e}_2$ and $x, y \in \mathbb{R}$ such that

$$S(v, \xi) = x \tilde{e}_1 \otimes \tilde{e}_1 + y \tilde{e}_2 \otimes \tilde{e}_2.$$

We may assume $|x| + |y| < \sqrt{\lambda}$, since otherwise we know $\bar{f}_{\lambda}(\xi) = g_{\lambda}(S(v, \xi)) = \sqrt{\lambda} + \frac{x^2 + y^2}{\sqrt{\lambda}}$. Similarly, we may assume $0 < |x| + |y|$, since otherwise $\bar{f}_{\lambda}(\xi) = g_{\lambda}(S(v, \xi)) = 0$. Let $\alpha, \beta \in (0, 1)$ to be chosen later, and set

$$S_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} x/\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} x & 0 \\ 0 & y/\beta \end{pmatrix},$$

where all matrices are in the $\tilde{e}_1, \tilde{e}_2$ basis. Observing that the map $\sigma \mapsto S(v, \sigma)$ is linear and invertible, we may choose ξ_i such that $S(v, \xi_i) = S_i$ for $i = 1, 2, 3$.

Note that $\beta S_3 + (1 - \beta)(\alpha S_2 + (1 - \alpha)S_1) = S(v, \xi)$, and $S_3 - (\alpha S_2 + (1 - \alpha)S_1), S_2 - S_1$ are both symmetric-rank-one. By linearity, we also have that $\beta \xi_3 + (1 - \beta)(\alpha \xi_2 + (1 - \alpha)\xi_1) = \xi$ and that $\xi_3 - (\alpha \xi_2 + (1 - \alpha)\xi_1), \xi_2 - \xi_1$ are symmetric-rank-one. (These relations provide a “laminate” of order two.) By Lemma B.7 in [Olb17] (which states that $R^{\text{sym}} \bar{f}_{\lambda}$ may be estimated from above by laminates of order k for any $k \in \mathbb{N}$), we have

$$\begin{aligned} R^{\text{sym}} \bar{f}_{\lambda}(\xi) &\leq \beta \bar{f}_{\lambda}(\xi_3) + (1 - \beta) (\alpha \bar{f}_{\lambda}(\xi_2) + (1 - \alpha) \bar{f}_{\lambda}(\xi_1)) \\ &= \beta \left(\sqrt{\lambda} + \frac{x^2 + y^2/\beta^2}{\sqrt{\lambda}} \right) + (1 - \beta) \alpha \left(\sqrt{\lambda} + \frac{x^2/\alpha^2}{\sqrt{\lambda}} \right). \end{aligned}$$

Now we assume $|x| > 0$. The right hand side in the last estimate is convex in α ; it attains its minimum at $\alpha = \frac{|x|}{\sqrt{\lambda}}$. Hence,

$$\begin{aligned} R^{\text{sym}} \bar{f}_\lambda(\xi) &\leq \beta \left(\sqrt{\lambda} + \frac{x^2 + y^2/\beta^2}{\sqrt{\lambda}} \right) + (1 - \beta)2|x| \\ &= 2|x| + \frac{\beta}{\sqrt{\lambda}} \left(\sqrt{\lambda} - \frac{|x|}{\sqrt{\lambda}} \right)^2 + \frac{y^2/\beta}{\sqrt{\lambda}} \end{aligned}$$

Choosing $\beta = |y|/(\sqrt{\lambda} - |x|)$, we obtain

$$R^{\text{sym}} \bar{f}_\lambda(\xi) \leq 2 \left(|x| + |y| - \frac{|xy|}{\sqrt{\lambda}} \right) = g_\lambda(S(v, \xi)).$$

It remains to prove the claim for the case $|x| = 0$. Then we have

$$\begin{aligned} R^{\text{sym}} \bar{f}_\lambda(\xi) &\leq \beta \bar{f}_\lambda(\xi_3) + (1 - \beta) \bar{f}_\lambda(\xi_1) \\ &= \beta \left(\sqrt{\lambda} + \frac{x^2 + y^2/\beta^2}{\sqrt{\lambda}} \right). \end{aligned}$$

Again setting $\beta = |y|/(\sqrt{\lambda} - |x|)$, we obtain the same conclusion as before. This proves the lemma. $\square$

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