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Sum-of-Squares methods for controlled invariant sets with applications to model-predictive control[☆,☆☆]

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ABSTRACT

We develop a method for computing controlled invariant sets of discrete-time affine systems using Sum-of-Squares programming. We apply our method to the controller design problem for switching affine systems with polytopic safe sets but our work also improves the state of the art for the particular case of LTI systems. The task is reduced to a semidefinite programming problem by enforcing an invariance relation in the dual space of the geometric problem. The paper ends with an application to safety critical model predictive control.

1. Introduction

The problem of computing a controlled invariant set is a paradigmatic challenge in the broad field of Hybrid Systems control. Indeed, it is for instance crucial in safety-critical applications, such as the control of a platoon of vehicles or air traffic management; see [1], where firm guarantees are needed on our ability to maintain the state in a safe region (e.g., with a certain minimal distance between vehicles). In other situations, the dynamical system might be too complicated to analyze exactly in every point of the state space, but yet it can be possible to confine the state within a guaranteed set. Such situations occur frequently in hybrid, embedded, event-triggered systems, because of the complexity of the dynamics.

1.1. Problem statement

A set is *controlled invariant* (sometimes also referred to as *viable*) if, any trajectory whose initial point is in the set can be kept inside it by means of a proper control action. Given a system with constraint specifications on the states and/or input, the controlled invariant set can be used to determine initial states such that trajectories with these initial conditions are guaranteed to meet the specifications. Moreover, in some situations, a state feedback control law can be derived from the knowledge of the controlled invariant set; see [2] for a survey.

The computation of invariant sets is usually achieved using either polyhedral computations or semidefinite programming. If the system contains a control input, the computational complexity of the problem becomes even more challenging.

☆ This paper extends our work on ellipsoidal controlled invariant sets presented at ADHS 2018 (Legat et al., 2018) to sets defined by polynomials of arbitrary degree.

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Indeed, this requires (see e.g., the procedure p. 201 in [3]) the computation of projections of polytopes when using polyhedral computations and as we show below semidefinite programming techniques are not directly applicable.

The semidefinite programming approach sacrifices exactness of the solution for the sake of algorithmic tractability. In the case of an uncontrolled system $x_{k+1} = Ax_k$, it consists in searching for an ellipsoidal set

$$\mathcal{E}_P = \{ x \in \mathbb{R}^n \mid x^\top Px \leq 1 \}$$

such that if $x^\top Px \leq 1$ then $x^\top A^\top PAx \leq 1$. Indeed, one can verify that it implies invariance of the set $\mathcal{E}_P$. The *S-procedure* (see Proposition 4) allows to formulate the search of P as a semidefine program; see [4] for a survey on the S-procedure. By the S-procedure, $x^\top Px \leq 1$ implies $x^\top A^\top PAx \leq 1$ if and only if

$$P - A^\top PA \text{ is positive semidefinite.} \tag{1}$$

With the presence of a disturbance $w \in W$ in the system $x_{k+1} = Ax_k + Bw_k$, the condition becomes:

$$\forall w, x^\top Px \leq 1 \Rightarrow (Ax + Bw)^\top P(Ax + Bw) \leq 1. \tag{2}$$

and the S-procedure can be applied again to provide a robust invariance LMI; see [5, (6.5)].

With the presence of the control u in the system $x_{k+1} = Ax_k + Bu_k$, the condition becomes:

$$x^\top Px \leq 1 \Rightarrow \exists u, (Ax + Bu)^\top P(Ax + Bu) \leq 1. \tag{3}$$

The control term u, or more precisely the existential quantifier $\exists$ prevents the S-procedure to be directly applied.

1.2. Related work

Methods based on polyhedral computations for hybrid control systems have been developed in [6–8]. Unfortunately, the problem of polyhedral projection is well known to severely suffer from the curse of dimensionality, see [9], and the additional complexity of the discrete dynamics in hybrid systems makes the problem even less scalable for these systems. Parametrizations of the set have been proposed to improve the scalability of the polyhedral approach; see [10]. Polyhedral computations are typically restricted to affine constraint specifications but it has been recently shown that it can also be applied to algebraic constraints; see [11].

In [12], the authors show how to compute an over- and under-approximation of the reachable sets of a hybrid control system. While they approximate *reachable sets* and do not compute *controlled invariant sets*, their approach bears similarities with the method presented in this paper. However, their technique does not rely on semidefinite programming as they propagate ellipsoidal sets and do not need to enforce any invariance property.

In [13], a semidefinite programming method is proposed for the computation of outer approximations of the *region of attraction*[1] (ROA) for polynomial control systems. Invariant set computation and ROA computation are different problems but the authors show how to adapt the method to the computation of outer approximations of the maximal controlled invariant set in [14]. While the set computed with this method can be a good approximation of the maximal controlled invariant set, it is an outer approximation and is not controlled invariant unless the approximation is exact. In [15], the authors show that in the uncontrolled case, an inner approximation of the ROA can be obtained as the complement of an outer approximation of the complement of the target set. The latter can be obtained using the technique developed in [13]. This may be extended to the computation of inner approximations of invariant sets in the uncontrolled case similarly to how [13] was adapted for invariance in [14]. However, this only applies to uncontrolled systems while our work tackles controlled systems.

Remark 1. The main challenge preventing the method developed in [13–15] to provide sufficient conditions for controlled invariance is the presence of the existential quantifier in (3). Indeed, adapting the method of [15] to invariance instead of ROA provides a generalization of (1) to polynomial systems and invariant polynomial sublevel sets. This technique can be generalized to systems with disturbance similarly to (2). In fact, given a control system for which we aim to compute a controlled invariant set inside a given target set, applying this generalization to the system with control input replaced by a disturbance and the target set replaced by its complement gives exactly the method developed in [14]. Indeed, the complement of any outer approximation of the maximal controlled invariant set is invariant for the system where the control input is replaced by a disturbance. Replacing the control input by a disturbance has the effect of replacing the existential quantifier of (3) with the universal quantifier of (2). This shows that the methods of [13–15] reformulate robust invariance conditions into LMIs. For this reason, their technique does not provide sufficient conditions for control invariance and hence does not seem to be readily applicable to the computation of controlled invariant sets.

On a similar note, the complement of any outer approximation of the minimal robust invariant set is invariant for the system where the disturbance is replaced by a control input. Therefore, the method presented in this paper can be used to obtain the complement of outer approximations of the minimal robust invariant set.

[1] Given a time T and a target set, the *region of attraction* (ROA) is the set of all initial conditions such that there exists an admissible trajectory whose state belongs to the target set at time T. Note that the ROA is not necessarily invariant if the target set is not invariant.

There is a well known technique to circumvent the presence of the existential quantifier ∃ in (3), which allows to formulate the search for an ellipsoidal controlled invariant set of controlled linear discrete systems as a semidefinite program. We describe this technique in the following paragraph for completeness.

Fixing the control to a linear state feedback $u(x) = Kx$ for some matrix K allows to fallback to the case of uncontrolled system $x_{k+1} = (A + BK)x_k$. Using the S-procedure and Schur lemma, the invariance condition can be formulated as a *Bilinear Matrix Inequality* (BMI) which is NP-hard to solve in general [16]. While the matrix inequality is bilinear in K and P, a clever algebraic manipulation allows to reformulate it as a Linear Matrix Inequality (LMI) in $Q := P^{-1}$ and $Y := KQ$, where the sought control-invariant ellipsoid is given by $\mathcal{E}_P$, see e.g. [17, Section 2.2.1]. The linear matrix inequality is

$$\begin{bmatrix} Q & QA^\top + Y^\top B^\top \\ AQ + BY & Q \end{bmatrix} \text{ is positive semidefinite.} \tag{4}$$

While the *uncontrolled* invariance LMI constraint (1) has size $n \times n$ where n is the state-space dimension, the *controlled* invariance LMI constraint (4) has size $(2n) \times (2n)$ which negatively affects the scalability of the computation. Moreover, as the algebraic manipulation which allows to reformulate the BMI into an LMI is done at the level of matrices, it is not clear how this approach can be generalized to sublevel sets of polynomials of higher degree. However, searching for ellipsoidal controlled invariant sets may be rather restrictive and the conservativeness is amplified for the class of hybrid system. For instance, in [18] the authors exhibit a simple example of hybrid systems for which no ellipsoidal set is invariant although it is shown in [19, Example 2.8] that there exists a quartic form with invariant sublevel set for this system.

The conservativeness may be reduced by considering intersection of ellipsoids with path-complete methods [20]. Given the LMI (4) for invariance of an ellipsoid or the LMI developed in this paper for invariant sets described by polynomials of higher degree, path-complete methods allow to generate LMIs for the invariance of intersection of such sets.

1.3. Contribution

An alternative approach to (4) for computing ellipsoidal controlled invariant sets was introduced in [21]. A key ingredient in this technique is that the problem is formulated in the dual space of the geometric problem. It leverages duality like [5,22] but contrary to them it combines it with a projection of the state space to its subspace that is not *directly* controllable. Compared to the method described above, with this approach the *controlled* invariance LMI constraint has size $n \times n$ and not $(2n) \times (2n)$. Both methods solve the problem in the dual space as the ellipsoid $\mathcal{E}_{P^{-1}}$ is the polar of the ellipsoid $\mathcal{E}_P$. However, the method of [21] is developed at the abstract level of sets instead of relying on algebraic manipulation at the level of matrices. As we show in this paper, this allows the method to be extended to compute controlled invariant sublevel sets of polynomials of higher degree.

In this paper, we generalize the method introduced in [21] from ellipsoids to sublevel sets of polynomial of arbitrary degree. We detail the application of the method to two classes of hybrid systems: Discrete-Time Affine Hybrid Control System (HCS for short) and Discrete-Time Affine Hybrid Algebraic System (HAS for short). HAS are not control systems but the computation of invariant sets for such systems presents the same features than for HCS.

As a matter of fact, we show in Section 2 how to reduce the computation of controlled invariant sets for HCS to the computation of invariant sets for HAS. In Section 2.1, we define HCS and the notion of controlled invariance. In Section 2.2, we show how to reduce the computation of controlled invariant sets of a HCS with *constrained* input to controlled invariant sets of a HCS with *unconstrained* input. Then in Section 2.3, we give the reduction of the computation of controlled invariant sets of a HCS with unconstrained input to invariant sets of a HAS by projecting the state space to its subspace that is not directly controllable.

As the computation of controlled invariant sets of a HCS has been reduced to the computation of invariant sets of a HAS, we describe in Section 3 our method to compute invariant sets of HAS as ellipsoids or sublevel sets of polynomials. In Section 3.1, we detail the relation between the algebraic invariance condition of HAS on a convex set and its polar set. In Section 3.2, we show that using the results of Section 3.1, the invariance of ellipsoids for a *homogeneous* HAS, see Definition 4, can be formulated as a semidefinite program. In Section 3.3, we discuss how to lift the state space to handle non-homogeneity. In Section 3.4, we show how to use this homogenization technique to generalize the semidefinite program of Section 3.2 to non-homogeneous HAS. In Section 3.5, we generalize the semidefinite program of Section 3.2 to compute invariant sublevel sets of polynomials of arbitrary degree.

We end the paper with an application of the controlled invariant sets to safety critical model predictive control in Section 4. We show that precomputing such sets allows to guarantee safety of the model predictive controller thereby removing the need for long horizon.

Reproducibility. The code used to obtain the results is available at https://github.com/blegat/SwitchOnSafety.jl/tree/master/examples. The algorithms are part of the SwitchOnSafety Julia [23] package [24] which computes invariant sets for hybrid systems represented with the HybridSystems package [25]. The Sum-of-Squares programs are solved by Mosek v8 [26] through the SumOfSquares [27] and SetProg [28] extensions of JuMP [29]. The polyhedral computations are executed by CDD [30] through the Polyhedra package [31].

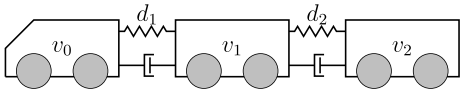

Fig. 1. Illustration for Example 1 with two trailers.

2. Controlled invariant set

In this section, we define HCS and HAS and give the invariance conditions for these two classes of hybrid systems. We detail the relation between controlled invariant sets of HCS and invariant sets of HAS.

2.1. Discrete-Time Affine Hybrid Control System

We will consider the following definition of Discrete-Time Affine Hybrid Control System.

Definition 1. A *Discrete-Time Affine Hybrid Control System (HCS)* is a system $S = (T, (A_\sigma, B_\sigma, c_\sigma)_{\sigma\in\Sigma}, (\mathcal{P}_q, U_q)_{q\in V})$ where $T = (V, \Sigma, \to)$, V is a finite set of nodes, Σ is a finite set of signals and $\to \subseteq V \times \Sigma \times V$ is a set of transitions. We denote $(q, \sigma, q') \in \to$ by $q \to_\sigma q'$.

Given a node $q \in V$, we denote the state dimension as $n_{q,x}$ and the input dimension as $n_{q,u}$. The set $\mathcal{P}_q \subseteq \mathbb{R}^{n_{q,x}}$ is the *safe set* corresponding to node q and the set $U_q \subseteq \mathbb{R}^{n_{q,u}}$ is the set of allowed inputs. For any transition $q \to_\sigma q'$, we have $A_\sigma \in \mathbb{R}^{n_{q',x}\times n_{q,x}}$, $B_\sigma \in \mathbb{R}^{n_{q',x}\times n_{q,u}}$ and $c_\sigma \in \mathbb{R}^{n_{q',x}}$.

A trajectory of S is a sequence $\{(x_k, u_k, \sigma_k)\}_{k\in\mathbb{N}}$ satisfying for all $k \in \mathbb{N}$:

$$x_{k+1} = A_{\sigma_k}x_k + B_{\sigma_k}u_k + c_{\sigma_k},$$
$$x_k \in \mathcal{P}_{q_k}, u_k \in \mathcal{U}_{q_k}, q_k \to_{\sigma_k} q_{k+1}.$$

The hybrid system defined in Definition 1 may be interpreted as a *hybrid automaton* [32] where the continuous dynamic at each node is $\dot{x} = 0$. We allow the state space of different nodes to differ as our method naturally extends to different state spaces but the reader may consider them to have identical dimension for simplicity.

We say that a HCS is *homogeneous* if $\mathcal{P}_q$ is symmetric for all $q \in V$ and c_σ is zero for every signal σ.

We illustrate this definition with the cruise control example of [6].

Example 1 (*See [6, Section 6.1]*)**.** We consider a truck with M trailers as represented by Fig. 1. There is a truck with mass m_0 and speed v_0 followed by multiple trailers, each with mass m. The speed of the ith trailer is denoted v_i. There is a spring with stiffness k_s and elongation d_1 (resp. d_i) and a damper with coefficient k_d between the truck and the first trailer (resp. the $(i-1)$th trailer and the ith trailer). The scalar input u controls the speed v_0 of the truck by creating a force m_0u. The dynamics of the system is given by the following equations:

$$\begin{aligned}
\dot{v}_0 &= \frac{k_d}{m_0}(v_1 - v_0) - \frac{k_s}{m_0}d_1 + u \\
\dot{v}_i &= \frac{k_d}{m}(v_{i-1} - 2v_i + v_{i+1}) + \frac{k_s}{m}(d_i - d_{i+1}) && 1 \le i < M \\
\dot{v}_M &= \frac{k_d}{m}(v_{M-1} - v_M) + \frac{k_s}{m}d_M \\
\dot{d}_i &= v_{i-1} - v_i && 1 \le i \le M.
\end{aligned} \tag{5}$$

The spring elongation should always remain between -0.5 m and 0.5 m and the speeds of the truck and trailers should remain between $5\,\mathrm{m\,s^{-1}}$ and $35\,\mathrm{m\,s^{-1}}$. Moreover, there are three speed limits $\bar{v}_a = 15.6\,\mathrm{m\,s^{-1}}$, $\bar{v}_b = 24.5\,\mathrm{m\,s^{-1}}$, $\bar{v}_c = 29.5\,\mathrm{m\,s^{-1}}$ and whenever the truck is informed of a new speed limit, it has 0.8 s to decrease v_i $(0 \le i \le M)$ below the speed limit.

We sample time with a period of 0.4 s and define an initial node q_{d0} and 6 nodes q_{ij} where $i \in \{a, b, c\}$ is the current speed limitation and $j \in \{0, 1\}$ is the number of sampling times left to satisfy the limit. The transitions are $q_{ij} \to_\sigma q_{\sigma 1}$ for each $i \in \{a, b, c, d\}$ and $\sigma \in \{a, b, c, d\} \setminus \{i\}$. The signal a (resp. b, c) represents that the truck sees a new speed limitation $\bar{v}_a$ (resp. $\bar{v}_b$, $\bar{v}_c$) and d represents that it does not see any new speed limitation. We suppose for simplicity that it is not possible to see a new speed limitation $\bar{v}_\sigma$ from a node $q_{\sigma j}$. The possible transitions are represented in Fig. 2.

The reset maps $(A_\sigma, B_\sigma, c_\sigma)$ are simply the integration of the dynamical system (5) over 0.4 s with a zero-order hold input extrapolation.

Let

$$P_0 = \{(d, v) \in \mathbb{R}^{2M+1} \mid -0.5 \le d \le 0.5, 5 \le v \le 35\},$$
$$P_i = \{(d, v) \in \mathbb{R}^{2M+1} \mid v \le \bar{v}_i\}, \quad i = a, b, c,$$

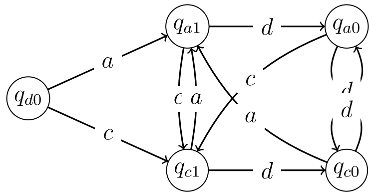

Fig. 2. Transitions and switchings between the nodes for Example 1. Nodes q_{b1} and q_{b0} are not shown for clarity.

where $d = (d_1, \dots, d_M)$, $v = (v_0, \dots, v_M)$ and inequalities in the two equations above are entrywise. The safe sets are $\mathcal{P}_{q_{d0}} = P_0$ and for $i = a, b, c$, $P_{q_{ij}} = P_0$ if $j > 0$ and $P_{q_{i0}} = P_0 \cap P_i$. The input set is $\mathcal{U}_{ij} = \{ u \in \mathbb{R} \mid -4 \le u \le 4 \}$ for each node q_{ij}.

Definition 2 (*Controlled Invariant Sets for a HCS*)**.** Consider a HCS S as in Definition 1. We say that sets $\mathcal{C} = (\mathcal{C}_q)_{q \in V}$ are *controlled invariant* for S if $\mathcal{C}_q \subseteq \mathcal{P}_q$ for each $q \in V$ and $\forall q \to_\sigma q', x \in \mathcal{C}_q$, $\exists u \in \mathcal{U}_q$ such that

$$A_\sigma x + B_\sigma u + c_\sigma \in \mathcal{C}_{q'}.$$

In view of Definition 2, a trajectory of a HCS should be interpreted as follows. Given initial conditions x_0, q_0, for each $k \ge 0$, a transition $q_k \to_{\sigma_k} q_{k+1}$ is first selected autonomously and then the input u_k can be controlled, knowing the selected transition.

Remark 2. It is important to distinguish two types of switching: *autonomous switching* and *controlled switching*; see details in [33, Section 1.1.3]. Definition 2 is the definition of controlled invariance for autonomous systems and in this paper we only consider systems that switch autonomously. With controlled switching, "$\forall q \to_\sigma q'$" is replaced by "$\exists q \to_\sigma q'$" in Definition 2.

2.2. Handling controller constraints

We say that the input of a HCS is *unconstrained* if $\mathcal{U}_q = \mathbb{R}^{n_{q,u}}$ for all $q \in V$, otherwise we say that the input is *constrained*. The computation of controlled invariant sets for a HCS with constrained input can be reduced to the computation of invariant sets for a HCS with unconstrained input as shown by the following lemma.

Algorithm 1 Construct a HCS with unconstrained input given a HCS with constrained input

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Input: A HCS $S = (T, (A_\sigma, B_\sigma, c_\sigma)_{\sigma\in\Sigma}, (\mathcal{P}_q, \mathcal{U}_q)_{q\in V})$
Output: A HCS $S' = (T', (A_\sigma, B_\sigma, c_\sigma)_{\sigma\in\Sigma'}, (\mathcal{P}'_q, \mathcal{U}'_q)_{q\in V'})$ where $T' = (V', \Sigma', \to')$.
for all $q \in V$ **do**
  Add node q to V'
  Define $\mathcal{P}'_q := \mathcal{P}_q$
  Define $\mathcal{U}'_q := \mathbb{R}^{n_{q,u}}$
end for
for all $q \to_\sigma w$ **do**
  Add node q^σ to V'
  Define $\mathcal{P}'_{q^\sigma} := \mathcal{P}_q \times \mathcal{U}_q$
  Define $\mathcal{U}'_{q^\sigma} := \mathbb{R}^0$
  Add signal σ' to Σ'
  Define $A_{\sigma'} := \begin{bmatrix} A_\sigma & B_\sigma \end{bmatrix}$, $B_{\sigma'} \in \mathbb{R}^{n_{w,x}\times 0}$ and $c_{\sigma'} := c_\sigma$
  Add transition $q \to'_{q^0} q^\sigma$
  Add signal q^0 to Σ'
  Define $A_{q^0} := \begin{bmatrix} I & 0 \end{bmatrix}^\top$, $B_{q^0} := \begin{bmatrix} 0 & I \end{bmatrix}^\top$ and $c_{q^0} := 0$
  Add transition $q^\sigma \to'_{\sigma'} w$
end for

Proposition 1. *The sets $\mathcal{C} = (\mathcal{C}_q)_{q\in V}$ are* controlled invariant *for $S = (T, (A_\sigma, B_\sigma, c_\sigma)_{\sigma\in\Sigma}, (\mathcal{P}_q, U_q)_{q\in V})$ if and only if there exist controlled invariant sets $\mathcal{C}' = (\mathcal{C}'_q)_{q\in V'}$ such that $\mathcal{C}'_q = \mathcal{C}_q$ $\forall q \in V$ for the system returned by Algorithm 1 with input S.*

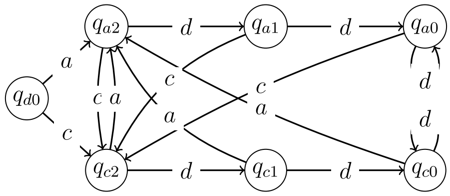


Fig. 3. Transitions and switchings between the nodes for Example 3. Nodes q_{b2}, q_{b1} and q_{b0} are not shown for clarity.

Proof. Consider controlled invariant sets $\mathcal{C}'$ for S' and let $\mathcal{C} = (\mathcal{C}'_q)_{q\in V}$. Given $x \in \mathcal{C}_q$ and $q \to_\sigma w$, the controlled invariance of $\mathcal{C}'$ ensures that there exists u such that $(x, u) \in \mathcal{C}'_{q^\sigma} \subseteq \mathcal{P}_q \times \mathcal{U}_q$ and $A_\sigma x + B_\sigma u + c_\sigma \in \mathcal{C}'_w = \mathcal{C}_w$. Hence $\mathcal{C}$ is controlled invariant for S.

Consider now controlled invariant sets $\mathcal{C}$ for S and let $\mathcal{C}' = (\mathcal{C}'_q)_{q\in V'}$ where $\mathcal{C}'_q = \mathcal{C}_q$ for each $q \in V$. Given $q \to_\sigma w$, for each $x \in \mathcal{C}'_q = \mathcal{C}_q$ the controlled invariance of $\mathcal{C}$ ensures that there exists $u \in \mathcal{U}_q$ such that $A_\sigma x + B_\sigma u + c_\sigma \in \mathcal{C}_w = \mathcal{C}'_w$, setting $\mathcal{C}'_{q^\sigma}$ to be the union of these pairs (x, u) makes $\mathcal{C}'$ controlled invariant for S'. □

Note that even if the increase of number of nodes in Proposition 1 induces more sets to compute, hence an increased computation time, the number of nodes added is at most the number of transitions, and thus this increase is limited.

Remark 3. For a given node q, let $\Sigma_q = \{\sigma \mid \exists q', q \to_q q'\}$. If Σ_q is a singleton $\{\sigma\}$, we can merge q and q^σ into one state hence have $\mathcal{P}'_q = \mathcal{P}_q \times \mathcal{U}_q$. In that case, $\mathcal{C}_q$ will be the projection of $\mathcal{C}'_q$ in its state space. Even if Σ_q is not a singleton, we can pick a single $\sigma \in \Sigma_q$ and merge q and q^σ into one state and use the reset map

$$A_{q^0} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad B_{q^0} = \begin{bmatrix} 0 \\ I \end{bmatrix} \qquad c_{q^0} = 0$$

so that switchings $\sigma' \in \Sigma_q \setminus \{\sigma\}$ ignore the part of the state of q that corresponds to the input to be used for σ.

We start by applying Proposition 1 on a simple example in Example 2 and then in Example 3 we detail its application to the system introduced in Example 1.

Example 2. Consider the discrete-time linear control system $x_{k+1} = x_k + u_k/2$ where the safe set for x_k is $[-1, 1]$ and the control u_k is constrained to be between -1 and 1. This system can be viewed as a HCS as defined in Definition 1 with only one node and one transition. As mentioned in Remark 3, as there is only one transition, we do not have to create an intermediate node hence the system constructed by Proposition 1 can also represented by a non-hybrid system. This system is planar and has the following dynamics

$$\begin{aligned} x_{k+1} &= x_k + \frac{u_k}{2} \\ u_{k+1} &= u'_k \end{aligned}$$

where the safe set for (x_k, u_k) is $[-1, 1]^2$ and the control u'_k is unconstrained. By Proposition 1 we know that a set $\mathcal{C}$ is controlled invariant for the scalar system if and only if it is the projection into the x-axis of a controlled invariant set $\mathcal{C}'$ of the planar system.

Example 3. We represent in Fig. 3 the application of the transformation described in Proposition 1 to the system of Example 1. We can use Remark 3 to avoid creating q^d for each q. Moreover, since $(A_\sigma, B_\sigma, c_\sigma)$ does not depend on σ, we can merge all the nodes q^a (resp. q^b, q^c) together into a common state that we name q_{a2} (resp. q_{b2}, q_{c2}).

2.3. Discrete-Time Affine Hybrid Algebraic System

Definition 3. A *Discrete-Time Affine Hybrid Algebraic System (HAS)* is a system $S = (T, (A_\sigma, E_\sigma, c_\sigma)_{\sigma\in\Sigma}, (\mathcal{P}_q)_{q\in V})$ where $T = (V, \Sigma, \to)$, V is a finite set of nodes, Σ is a finite set of signals and $\to \subseteq V \times \Sigma \times V$ is a set of transitions.

Given a node $q \in V$, we denote the state dimension as $n_{q,x}$. The set $\mathcal{P}_q \subseteq \mathbb{R}^{n_{q,x}}$ is the *safe set* corresponding to node q. For any transition $q \to_\sigma q'$, we have $A_\sigma \in \mathbb{R}^{n_{\sigma,p}\times n_{q,x}}$, $B_\sigma \in \mathbb{R}^{n_{\sigma,p}\times n_{q',x}}$ and $c_\sigma \in \mathbb{R}^{n_{\sigma,p}}$ for some natural number $n_{\sigma,p}$.

A trajectory of S is a sequence $\{(x_k, \sigma_k)\}_{k\in\mathbb{N}}$ satisfying for all $k \in \mathbb{N}$:

$$\begin{aligned} E_{\sigma_k} x_{k+1} &= A_{\sigma_k} x_k + c_{\sigma_k}, \\ x_k &\in \mathcal{P}_{q_k}, q_k \to_{\sigma_k} q_{k+1}. \end{aligned}$$

Definition 4. We say that a HAS is *homogeneous* if P_q is symmetric for all $q \in V$ and c_σ is zero for all signal σ.

Definition 5 (*Invariant Sets for a HAS*)**.** Consider a HAS S as in Definition 3. We say that sets $\mathcal{C} = (\mathcal{C}_q)_{q \in V}$ are *invariant* for S if $\mathcal{C}_q \subseteq \mathcal{P}_q$ for each $q \in V$ and for all $q \to_\sigma q'$,

$$A_\sigma \mathcal{C}_q + c_\sigma \subseteq E_\sigma \mathcal{C}_{q'}. \tag{6}$$

In view of Definition 5, a trajectory of a HAS should be interpreted as follows. Given initial conditions x_0, q_0, for each $k \geq 0$, a transition $q_k \to_{\sigma_k} q_{k+1}$ is first selected autonomously and then the state x_k such that $E_{\sigma_k} x_{k+1} = A_{\sigma_k} x_k + c_{\sigma_k}$ can be controlled, knowing the selected transition.

Remark 4. Definition 5 can be interpreted as stating that $\mathcal{C}$ is invariant if for each transition $q \to_\sigma q'$ and $x \in \mathcal{C}_q$,

there *exists* $y \in \mathcal{C}_{q'}$ such that $A_\sigma x + c_\sigma = E_\sigma y$.

A similar definition exists, see for instance [34], where this last part is replaced by

for *each* y such that $A_\sigma x + c_\sigma = E_\sigma y$, y must belong to $\mathcal{C}_{q'}$.

This is not equivalent to Definition 5 if A_σ and E_σ are not full rank. Moreover, computing ellipsoidal invariant sets according to this definition is much easier: it simply amounts to finding positive definite matrices Q_q such that $A_\sigma^\top Q_q A_\sigma \preceq E_\sigma^\top Q_{q'} E_\sigma$; see [35].

With this alternative definition of invariant sets, the invariance of sets of HAS would be equivalent to the robust invariance of sets for a system with disturbance. This differs from Proposition 2 for which the equivalence is with controlled invariance for a control system. This shows that, similarly to [13–15], the LMI found in [35] is related to robust invariance (2) and not control invariance (3).

We now show that the computation of controlled invariant sets of a HCS can be reduced to the computation of invariant sets of a HAS.

Proposition 2. *The sets $\mathcal{C} = (\mathcal{C}_q)_{q \in V}$ are* controlled invariant *for the HCS $S = (T, (A_\sigma, B_\sigma, c_\sigma)_{\sigma \in \Sigma}, (\mathcal{P}_q, \mathbb{R}^{n_{q,u}})_{q \in V})$ if and only if they are invariant sets for the HAS $S' = (T, (E_\sigma A_\sigma, E_\sigma, E_\sigma c_\sigma)_{\sigma \in \Sigma}, (\mathcal{P}_q)_{q \in V})$ where E_σ is a projection on* $\mathrm{Im}(B_\sigma)^\perp$.

Proof. As the input is unconstrained, for each $q \to_\sigma q'$ and $x \in \mathcal{P}_q$, there exists $u \in \mathbb{R}^{n_{q,u}}$ such that $A_\sigma x + B_\sigma u + c_\sigma \in \mathcal{C}_{q'}$ if and only if $E_\sigma A_\sigma x + E_\sigma c_\sigma \in E_\sigma \mathcal{C}_{q'}$. □

In the following example, we apply Proposition 2 to Example 2.

Example 4. Consider the planar system introduced in Example 2. In this system, $\mathrm{Im}(B)$ is the u-axis hence $\mathrm{Im}(B_\sigma)^\perp$ is the x-axis. Hence we have $A = \begin{bmatrix} 1 & 1/2 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

3. Computing controlled invariant sets

3.1. Duality correspondence for the invariance condition

Given a set $\mathcal{C}$ and a function f, we define the following notation:

$$\begin{aligned} f(\mathcal{C}) &= \{ f(x) \mid x \in \mathcal{C} \} \\ f^{-1}(\mathcal{C}) &= \{ x \mid f(x) \in \mathcal{C} \} \end{aligned} \tag{7}$$

Note that f does not need to be injective in these definitions. By slight abuse of notation, we also use the notation $A\mathcal{C}$, $A^{-1}\mathcal{C}$ and $A^{-\top}\mathcal{C} := [A^\top]^{-1}\mathcal{C}$ where A is a matrix.

Invariant sets can be computed numerically as *sublevel sets*[2] of polynomial functions using Sum-of-Squares programming [17,36]. One property of sublevel sets that is usually used can be formulated as follows:

Proposition 3. *If $\mathcal{C}$ is the ℓ-sublevel set of a function f then for any function g, $g^{-1}(\mathcal{C})$ is the ℓ-sublevel set of the function $f \circ g$.*

Proof. We have $\mathcal{C} = \{ x \mid f(x) \leq \ell \}$ and by (7), we have

$$g^{-1}(\mathcal{C}) = \{ x \mid g(x) \in \mathcal{C} \} = \{ x \mid f(g(x)) \leq \ell \}. \quad \square$$

[2] The ℓ-sublevel set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set $\{x \in \mathbb{R}^n \mid f(x) \leq \ell\}$.

Thanks to this property, computing a set $\mathcal{C}$ satisfying $A\mathcal{C} \subseteq \mathcal{C}$ for some linear map A can for example be achieved by searching for a set $\mathcal{C}$ being the 1-sublevel set of a polynomial $p(x)$. Indeed, the invariance constraint is equivalent to

$$\mathcal{C} \subseteq A^{-1}\mathcal{C} \tag{8}$$

which is equivalent to the following implication : for all x, $p(x) \leq 1 \Rightarrow p(Ax) \leq 1$. The latter proposition can be translated to a constraint of nonnegativity of a polynomial using the Sum-of-Squares formulation and the S-procedure. In the case of quadratic forms, this nonnegativity constraint is equivalent to the LMI (1).

We denote the subset of symmetric matrices of $\mathbb{R}^{n\times n}$ as $\mathcal{S}^n$.

Proposition 4 (*S-procedure [4]*)**.** *Given two symmetric matrices $Q_1, Q_2 \in \mathcal{S}^n$, the existence of a $\lambda \geq 0$ such that the matrix $\lambda Q_1 - Q_2$ is positive semidefinite is sufficient for the following proposition to hold:*

$$\text{for all } x \in \mathbb{R}^n, x^\top Q_1 x \leq 0 \Rightarrow x^\top Q_2 x \leq 0.$$

Moreover, if there exists $x \in \mathbb{R}^n$ such that $x^\top Q_1 x > 0$ then this condition is also necessary.

A well known corollary of the S-procedure states that λ may be fixed to 1 for ellipsoids centered at the origin.

Corollary 1. *Given two positive definite matrices $Q_1, Q_2 \succ 0$, the matrix $Q_1 - Q_2$ is positive semidefinite if and only if the following proposition hold:*

$$\text{for all } x \in \mathbb{R}^n, x^\top Q_1 x \leq 1 \Rightarrow x^\top Q_2 x \leq 1.$$

For HAS, we have in (6) an invariance constraint of the form $A\mathcal{C} \subseteq E\mathcal{C}$. In order to use Proposition 3, we aim to find an equivalent form with a pre-image as in (8). This can be achieved using the polar of the set $\mathcal{C}$ thanks to the following lemma.

Proposition 5 (*[37, Corollary 16.3.2]*)**.** *For any convex set $\mathcal{C}$ (resp. convex cone $\mathcal{K}$) and linear map A,*

$$(A\mathcal{C})^\circ = A^{-\top}\mathcal{C}^\circ$$
$$(A\mathcal{K})^* = A^{-\top}\mathcal{K}^*$$

where $\mathcal{C}^\circ$ denotes the polar of the set $\mathcal{C}$ and $\mathcal{K}^*$ denotes the dual of the cone $\mathcal{K}$.

Proposition 5 implies the following theorem that plays a key role in the reformulation of the computation of controlled invariant sets into semidefinite programs that is carried out in the following sections.

Theorem 1. *Consider a homogeneous HAS S as in Definition 3. The closed symmetric convex bodies $\mathcal{C} = (\mathcal{C}_q)_{q\in V}$ are* invariant *for S, as defined in Definition 5, if and only if $\mathcal{C}_q \subseteq \mathcal{P}_q$ for each $q \in V$ and for all $q \to_\sigma q'$,*

$$A_\sigma^{-\top}\mathcal{C}_q^\circ \supseteq E_\sigma^{-\top}\mathcal{C}_{q'}^\circ. \tag{9}$$

Proof. Since the HAS is homogeneous, c_σ is zero for each signal σ. Hence the invariance constraint (6) of Definition 5 is

$$A_\sigma\mathcal{C}_q \subseteq E_\sigma\mathcal{C}_{q'}.$$

By Proposition 5, this is equivalent to (9). □

3.2. Computation using ellipsoids for homogeneous systems

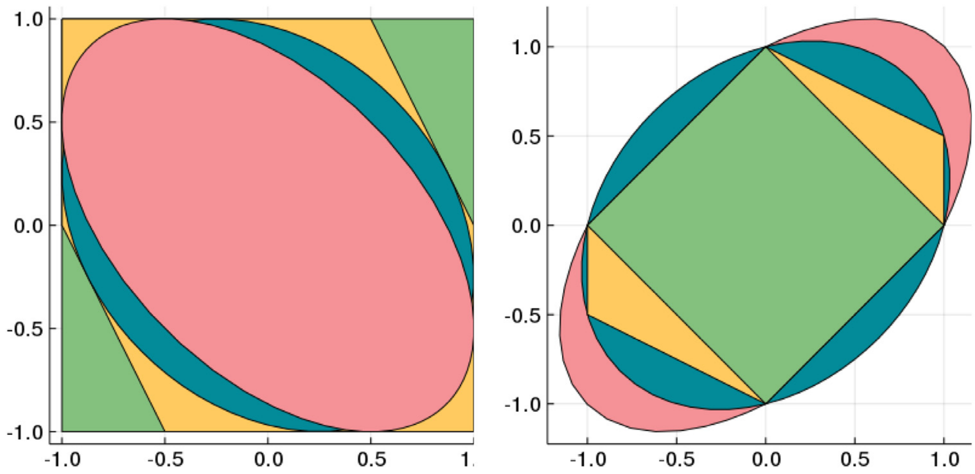
In this section, we show how to compute ellipsoidal controlled invariant sets using Theorem 1 in the particular case of homogeneous HCS and HAS. We show how to handle non-homogeneity in Section 3.4 and how to use more general sets in Section 3.5; we start by describing the ellipsoidal homogeneous case for clarity. This section details the semidefinite program needed to find these ellipsoidal invariant sets and shows its exactness in Theorem 2.

The optimization problem to solve is given in Program 1. We use the notation $p_q(x, z)$ to denote the evaluation of p_q at the vector $y = (x, z)$.

Program 1.

$$\max_{D_q \succ 0} \quad \sum_{q\in V} \log\det D_q$$
$$A_\sigma D_q A_\sigma^\top \preceq E_\sigma D_{q'} E_\sigma^\top, \qquad \forall q \to_\sigma q' \tag{10}$$
$$a^\top D_q a \leq \beta^2, \qquad \forall q \in V, (a, \beta) \in \mathcal{H}(\mathcal{P}_q) \tag{11}$$

where $\mathcal{H}(\mathcal{P}_Q)$ denotes the set of all (a, β) such that the half-space $a^\top x \leq \beta$ supports $\mathcal{P}_q$, which is commonly referred to as its H-representation [38].



(a) Primal space (b) Polar space

Fig. 4. Optimal solution of Program 1 for Example 5. The green set is the safe set $[-1, 1]^2$, the yellow set is the maximal controlled invariant set, the blue (resp. red) ellipsoid is the controlled invariant ellipsoid with maximal volume (resp. sum of squares of semi-axes). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The constraint (10) is the S-procedure applied to the condition (9). The constraint (11) ensures that $\mathcal{C}_q$ is contained in $\mathcal{P}_q$.

Remark 5. As we show in Theorem 2, (10) and (11) ensures that only invariant ellipsoids are feasible for Program 1. The most *relevant* solution among all feasible solutions depends on the application. Hence the objective function may involve some or all the ellipsoids and use one metric or another depending on the purpose of the optimization. One classical metric is the volume of the ellipsoid which is proportional to $\det D_q$ (optimized as $\log\det D_q$ or $\sqrt[n]{\det D_q}$ in order to have a convex program) and one well known alternative is the sum of the squares of the semi-axes of the ellipsoid given by the trace of D_q [39]. We consider the sum of the $\log\det D_q$ in Program 1 but it can be replaced by any of the variations listed above.

Theorem 2. *Consider a homogeneous HAS S as in Definition 3. The symmetric matrix D_q is feasible for Program 2 if and only if there exist invariant convex sets $\mathcal{C} = (\mathcal{C}_q)_{q\in V}$, as defined in Definition 5, such that $\mathcal{C}_q^\circ = \mathcal{E}_{D_q}$ for all $q \in V$. Moreover, the optimal solution of Program 2 is the solution that maximizes the sum of the logarithms of the volume of the ellipsoids.*

Proof. We first show that (10) is equivalent to the invariance of the sets $\mathcal{C}_q$. By Theorem 1, the invariance of the sets $\mathcal{C}_q$ is equivalent to (9). By Corollary 1, (9) is equivalent to (10).

We now show that (11) is equivalent to $\mathcal{C}_q \subseteq \mathcal{P}_q$. Since $\mathcal{P}_q$ is symmetric, it is the intersection of pairs of half-spaces $-\beta \le \langle a, x\rangle \le \beta$ and $\mathcal{C}_q \subseteq \mathcal{P}_q$ if and only if $a/\beta, -a/\beta \in \mathcal{C}_q{}^\circ$ for each pair of half-spaces. Since $\mathcal{C}_q{}^\circ = \mathcal{E}_{D_q}$, this is equivalent to (11). □

Example 5. Consider the algebraic system given in Example 4. In this system, there is only one node so only one ellipsoid $\mathcal{E}_{D^{-1}}$ (of polar $\mathcal{E}_D$). As the matrix $EDE^\top - ADA^\top$ has dimension 1×1, the LMI (10) is rewritten into the linear inequality $D_{1,2} + D_{2,2}/4 \le 0$. The optimal ellipsoid $\mathcal{E}_{D^{-1}}$ and its polar $\mathcal{E}_D$ are given respectively in Figs. 4(a) and 4(b).

3.3. Handling non-homogeneity

If the safe sets of the HAS do not all contain the interior, we cannot rely on (9) as the polar transformation is only defined for convex bodies with the origin in their interior. We handle this non-homogeneity by taking the conic hull of the lifted sets $\mathcal{C} \times \{1\}$. More precisely, we define

$$\tau(\mathcal{C}) = \{(\lambda x, \lambda) \mid \lambda \ge 0, x \in \mathcal{C}\} \tag{12}$$

$$r(A, c) = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix}.$$

It is easy to see that for any set $\mathcal{C}$, vector c and linear map A,

$$\tau(A\mathcal{C} + c) = r(A, c)\tau(\mathcal{C}). \tag{13}$$

Moreover, for any half-space $a^\top x \leq \beta$,

$$a^\top x \leq \beta, \forall x \in \mathcal{C} \Leftrightarrow (-a, \beta) \in \tau(\mathcal{C})^*. \tag{14}$$

Theorem 3. *Consider a HAS S as in Definition 3. The closed convex sets $\mathcal{C} = (\mathcal{C}_q)_{q\in V}$ are invariant for S, as defined in Definition 5, if and only if $\mathcal{C}_q \subseteq \mathcal{P}_q$ for each $q \in V$ and for all $q \to_\sigma q'$,*

$$r(A_\sigma, c_\sigma)^{-\top}\tau(\mathcal{C}_q)^* \supseteq r(E_\sigma, 0)^{-\top}\tau(\mathcal{C}_{q'})^*. \tag{15}$$

Proof. The invariance constraint (6) of Definition 5

$$A_\sigma \mathcal{C}_q + c_\sigma \subseteq E_\sigma \mathcal{C}_{q'}$$

can be rewritten, using (13), into

$$r(A_\sigma, c_\sigma)\tau(\mathcal{C}_q) \subseteq r(E_\sigma, 0)\tau(\mathcal{C}_{q'}). \tag{16}$$

As the sets $\mathcal{C}_q$ are closed and convex, so are the cones $\tau(\mathcal{C}_q)$ hence $\tau(\mathcal{C}_q)^{**} = \tau(\mathcal{C}_q)$. Therefore, by Proposition 5, (16) is equivalent to (15). □

3.4. Computation using ellipsoids for non-homogeneous systems

In this section, we show how to adapt Program 1 in the case of non-homogeneous systems using the homogenization technique detailed in Section 3.3.

We define the following notation for ellipsoids not necessarily centered at the origin.

$$\mathcal{E}_{Q,c} = \{\, x \mid (x-c)^\top Q(x-c) \leq 1 \,\}$$
$$\mathcal{E}_{D,d,\delta} = \{\, x \mid x^\top Dx + 2d^\top x + \delta \leq 0 \,\}.$$

The following lemma shows the relation between the two notations.

Proposition 6. *Let $Q, D \in \mathcal{S}^n$, $c, d \in \mathbb{R}^n$, $\delta \in \mathbb{R}$ with $Q \succ 0$. We have $\mathcal{E}_{Q,c} = \mathcal{E}_{D,d,\delta}$ if and only if $D \succ 0$ and there exists $\lambda > 0$ such that*

$$\lambda = d^\top D^{-1} d - \delta \tag{17}$$
$$c = -D^{-1} d \tag{18}$$
$$Q = D/\lambda. \tag{19}$$

Proof. Substituting Q and c using (18) and (19) in $(x-c)^\top Q(x-c) - 1$ gives $(x^\top Dx + 2d^\top x + d^\top D^{-1}d - \lambda)/\lambda$. We can conclude the "if" part of the proof with (17). We now show the "only if" part.

By Proposition 4, for $\mathcal{E}_{Q,c} = \mathcal{E}_{D,d,\delta}$ to hold, there must exist $\lambda > 0$ such that

$$x^\top Dx + 2d^\top x + \delta = \lambda((x-c)^\top Q(x-c) - 1).$$

This implies that

$$\delta = \lambda c^\top Qc - \lambda \tag{20}$$
$$d = -\lambda Qc \tag{21}$$
$$D = \lambda Q. \tag{22}$$

Eqs. (21) and (22) directly give (18) and (19). It remains to show (17). Eq. (21) is equivalent to $Q^{-1/2}d = -\lambda Q^{1/2}c$ which implies

$$d^\top Q^{-1} d = \lambda^2 c^\top Qc. \tag{23}$$

Combining (23) with (22), we get $\lambda c^\top Qc = d^\top D^{-1} d$ which, combined with (20), gives (17). □

We use the following corollary to represent the cones $\tau(\mathcal{C}_q)^*$ as the 0-sublevel set of quadratic forms $p(y) = p(x, z) = x^\top D_q x + 2d_q^\top xz + \delta_q z^2$.

Corollary 2. *Let $\mathcal{K} = \{\,(x, z) | x^\top Dx + 2d^\top xz + \delta z^2 \leq 0, z \geq 0\,\}$ be a cone that has a nonempty interior and no intersection with the hyperplane $\{(x, 0) | x \in \mathbb{R}^n\}$ except the origin. The cone $\mathcal{K}$ is convex if and only if $D \succ 0$.*

Proof. Let $\mathcal{C} = \mathcal{E}_{D,d,\delta}$. Since every point of the cone satisfies $z > 0$ except the origin, we have $\tau(\mathcal{C}) = \mathcal{K}$. Therefore, $\mathcal{K}$ is convex if and only if $\mathcal{C}$ is convex. Since $\mathcal{K}$ is nonempty,

$$\delta - d^\top Dd = \min_{x\in\mathbb{R}^n} x^\top Dx + 2d^\top x + \delta < 0.$$

We conclude with Proposition 6. □

In Corollary 2, we require the cone to have no intersection with a particular hyperplane (except the origin). However, the cone $\tau(\mathcal{C}_q)^*$ has no intersection with the hyperplane $\{(x, 0)|x \in \mathbb{R}^n\}$ if and only if the origin is contained in $\mathcal{C}_q$ which may not be the case. In order to alleviate this, the approach we suggest is to suppose that we know one point h_q in the interior of each $\mathcal{C}_q$ and we use Corollary 2 in a transformed space where h_q is mapped to the z-axis, i.e. the axis with direction vector $(0, 1)$. For this transformation we use the *Householder reflection* [40, Section 5.1.2]

$$H_h = I - \frac{2}{h^\top h} hh^\top.$$

Observe that the householder reflection is symmetric and orthogonal.

The optimization problem to solve is represented in Program 2. The transformation of this program to a semidefinite program can be done automatically using the standard Sum-of-Square procedure; see [17]. We use the notation $p_q(x, z)$ to denote the evaluation of p_q at the vector $y = (x, z)$.

Program 2.

$$\max_{\substack{D_q\in\mathcal{S}^n, d_q\in\mathbb{R}^n,\\ \delta_q\in\mathbb{R}, \lambda_{q\to_\sigma q'}\geq 0}} \sum_{q\in V} \log\det D_q$$

$$\begin{bmatrix} D_q & d_q \\ d_q^\top & \delta_q + 1 \end{bmatrix} \succ 0 \tag{24}$$

$$p_q(y) = y^\top H_{h_q} \begin{bmatrix} D_q & d_q \\ d_q^\top & \delta_q \end{bmatrix} H_{h_q} y \tag{25}$$

$$p_q(r(A_\sigma, c_\sigma)^\top y) \leq \lambda_{q\to_\sigma q'} p_{q'}(r(E_\sigma, 0)^\top y), \quad \forall q \to_\sigma q', y \in \mathbb{R}^{n_{q,x}+1} \tag{26}$$

$$p_q(-a, \beta) \leq 0, \quad \forall q \in V, (a, \beta) \in \mathcal{H}(\mathcal{P}_q) \tag{27}$$

$$p_q(0, 1) < 0, \quad \forall q \in V. \tag{28}$$

The constraint (24) ensures both convexity of $\tau(\mathcal{C}_q)^*$ and the fact that $\det D_q$ does not overestimate the volume of the ellipsoid transformed by the Householder reflection. The constraint (26) is the S-procedure applied to the condition (15). The constraint (27) uses (14) to ensure that $\mathcal{C}_q$ is contained in $\mathcal{P}_q$. The constraint (28) ensures that $\tau(\mathcal{C}_q)^*$ has non-empty interior. Note that if $\mathcal{P}_q$ has no unbounded subspace, (28) is not necessary since the non-empty interior condition will already be ensured by (27).

Theorem 4. *Consider a HAS S as in Definition 3 and points $(h_q \in \mathcal{P}_q)_{q\in V}$. The polynomial $p_q(x, z)$ is feasible for Program 2 if and only if there exist invariant convex sets $\mathcal{C} = (\mathcal{C}_q)_{q\in V}$, as defined in Definition 5, such that $h_q \in \mathcal{C}_q$ for each $q \in V$ and $\tau(\mathcal{C}_q)^*$ is the 0-sublevel set of $p_q(x, z)$. Moreover, the optimal solution of Program 2 is the solution that minimizes the sum of the logarithms of the volume of the intersection of the each cone $\tau(\mathcal{C}_q)^*$ with the hyperplane $\{\, x \mid \langle h_q, x\rangle = 1 \,\}$.*

Proof. Consider a solution $p = (p_q(x, z))_{q\in V}$ of Program 2. By Corollary 2, constraints (24) and (25) are satisfied if and only if there exist ellipsoids $\mathcal{C}_q$ such that $\tau(\mathcal{C}_q)^*$ is the 0-sublevel set of $p_q(x, z)$. By (14), constraint (27) is satisfied if and only if $\mathcal{C}_q \subseteq \mathcal{P}_q$. By Proposition 4, constraint (26) is satisfied if and only if (15) hold for all $q \to_\sigma q'$. Therefore, by Theorem 3, the solution p is a feasible solution of Program 2 if and only if the sets $\mathcal{C}_q$ are invariant for S.

We now prove the optimality of the solution. By Proposition 6, there exist Q_q, c_q such that $\mathcal{E}_{Q_q,c_q} = \mathcal{E}_{D_q,d_q,\delta_q}$ and $\lambda_q > 0$ such that $D_q = \lambda_q Q_q$. The volume of the intersection of $\tau(\mathcal{C}_q)^*$ with the hyperplane $\{\, x \mid \langle h_q, x\rangle = 1 \,\}$ is $-\det(Q_q)$. Therefore, it remains to show that $\lambda_q = 1$ for an optimal solution. We observe that without the constraint (24), for any feasible solution, D_q, d_q, δ_q can be scaled by any positive constant while remaining feasible but affecting the objective function. The *Schur complement* of the block D_q of the matrix in the left-hand side of constraint (24) is $\delta_q + 1 - d_q^\top D_q^{-1} d_q$ hence constraint (24) ensures that

$$d_q^\top D_q^{-1} d_q - \delta_q \leq 1.$$

Combining this inequality with Eq. (17) implies that $\lambda_q \leq 1$. Since the objective is to maximize $\det(D_q) = \lambda_q \det(Q_q)$, we know that if (D_q, d_q, δ_q) is optimal, then $\lambda_q = d_q^\top D_q^{-1} d_q - \delta_q = 1$. □

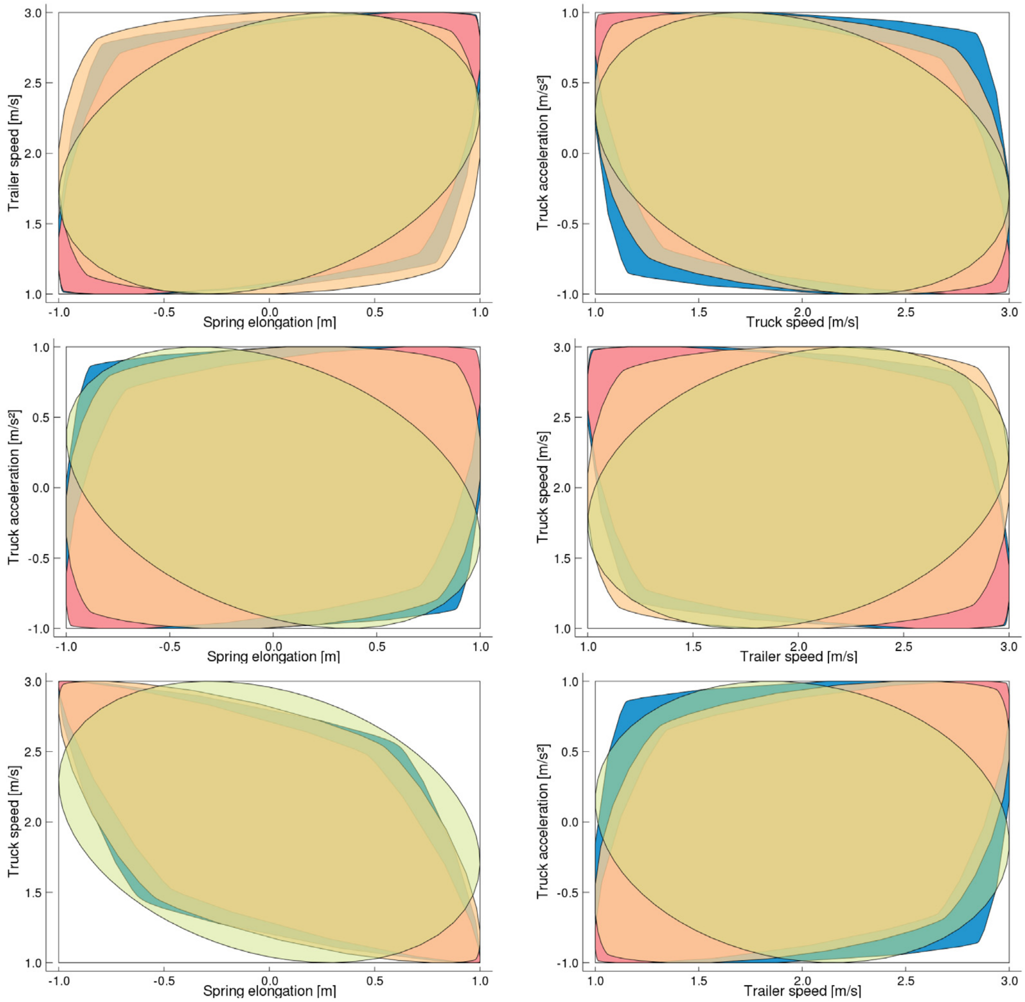


Fig. 5. Several projections of the optimal solution of Program 2 (in green) and Program 3 with the volume heuristic developed in [41] (in orange for quartic, red for sextic and blue for octic) for Example 6 at node q_{a0} for various numbers of trailers. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example 6. We apply Program 2 to Example 3 with the same values for the parameters as the ones used in [6], that is, $m_0 = 500\,\text{kg}$, $m = 1000\,\text{kg}$, $k_d = 4600\,\text{N s m}^{-1}$ and $k_s = 4500\,\text{N kg}^{-1}$. The values used for h_q are the same for each node $q \in V$: $u = d_i = 0$ and $v_0 = v_i = (5 + v_a)/2$ for $i = 1, \ldots, M$.

We vary the number of trailers M from 1 to 10. Fig. 5 represents the controlled invariant set at node q_{a0}. As we can see, the constraints on the trailers are propagated to the truck and, as the number M increases, the truck speed and acceleration become more constrained.

The time taken by Mosek 8.1.0.34 [26] to solve the problem is given by Fig. 6.[3]

3.5. Computation using higher degree polynomials

In this section, we generalize the results of Section 3.4 to sublevel sets of *homogeneous*[4] polynomials of degree $2d$. Note that the last variable of the polynomial is the *perspective* variable of the cone defined in (12). Therefore, it is not conservative to consider homogeneous polynomials as cones cannot be the sublevel set of non-homogeneous polynomials.

The main challenge of this generalization resides in constraint (24) ensuring both convexity of $\tau(\mathcal{C}_q)^*$ and the fact that $\det D_q$ does not overestimate the volume. Indeed, while checking the convexity and computing the volume of ellipsoids

[3] We set $\lambda_{q \to_\sigma q'}$ to 1 for each transition $q \to_\sigma q'$ to make the problem convex.

[4] A polynomial is homogeneous if all its monomials have the same total degree.

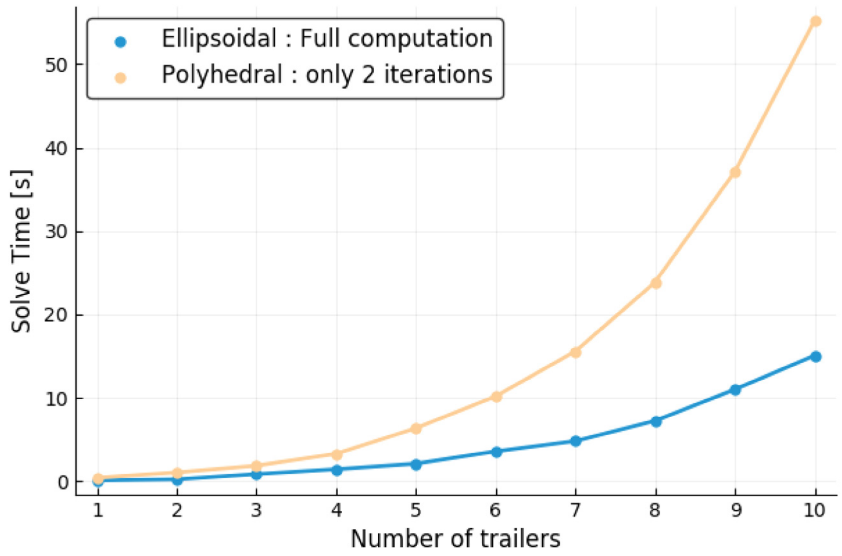


Fig. 6. Computation time with Mosek 8.1.0.34 for Program 2 with Example 6 with various numbers of trailers compared to two iterations of the polyhedral approach (see e.g., the procedure p. 201 in [3]) implemented with the CDD library [42]. Note that after two iterations, the polyhedral sets obtained are not controlled invariant. One needs to wait for the convergence of the algorithm to obtain a controlled invariant set. Moreover, iterations are usually increasingly slower as the number of facets of the polyhedral sets increases with the iterations.

can be done easily, it is more involved when using polynomials of higher degree. In fact, it has recently been shown that the convexity or quasi-convexity of a multivariate polynomial of degree at least four is NP-hard to decide [43]. However, the convexity constraint can be replaced by the tractable SOS-convexity constraint which is a sufficient condition for convexity [43] and heuristics can be used to approximately optimize the volume [41,44].

The following program searches homogeneous polynomials of degree $2dp_q(x)$ that corresponds to invariant sets. Combining Program 3 with volume heuristics as developed in [44] reduces to Program 2 for $2d = 2$. We use the notation $p_q(x, z)$ to denote the evaluation of p_q at the vector $y = (x, z)$.

Program 3.

$$s_q(x, z) + z^{2d} \text{ is SOS} \tag{29}$$

$$s_q(x, 1) \text{ is SOS-convex} \tag{30}$$

$$s_q(x, z) = p_q(H_{h_q}(x, z)), \qquad \forall x \in \mathbb{R}^{n_{q,x}}, z \in \mathbb{R} \tag{31}$$

$$p_q(r(A_\sigma, c_\sigma)^\top y) \le \lambda_{q \to_\sigma q'} p_{q'}(r(E_\sigma, 0)^\top y), \qquad \forall q \to_\sigma q', y \in \mathbb{R}^{n_{q,x}+1} \tag{32}$$

$$p_q(-a, \beta) \le 0, \qquad \forall q \in V, (a, \beta) \in \mathcal{H}(\mathcal{P}_q) \tag{33}$$

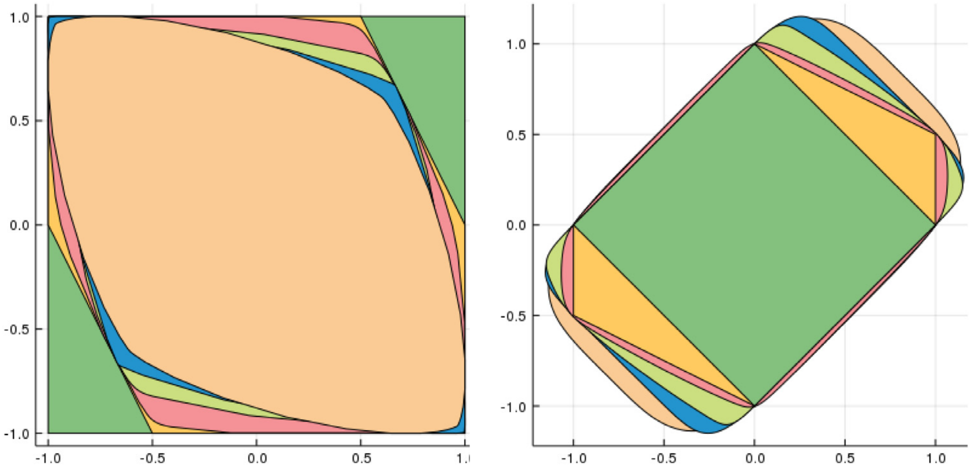
$$p_q(0, 1) < 0, \qquad \forall q \in V. \tag{34}$$

The constraint (30) ensure the convexity of $\tau(\mathcal{C}_q)^*$ and constraints (32)–(34) are identical to the corresponding constraints of Program 2. The constraint (29) is the generalization of (24) for polynomials of arbitrary degree. It certifies that $s_q(x, 1) + 1$ is nonnegative which is required for heuristics such as [41,44] to estimate the volume of its 1-sublevel set.

Let $k(n, d) = \binom{n+d-1}{n}$. The size of the semidefinite program generated by the Sum-of-Squares reformulation is as follows. The polynomial s_q is parametrized by a $k(n_{q,x}, d) \times k(n_{q,x}, d)$ positive semidefinite matrix to satisfy (29), the constraint (30) gives a $nk(n_{q,x}, d-2) \times nk(n_{q,x}, d-2)$ semidefinite constraint and the constraint (32) gives a $k(n_{\sigma,p}, d) \times k(n_{\sigma,p}, d)$ semidefinite constraint. The number of iterations required for an interior point algorithm to solve this semidefinite program has the complexity bound $\mathcal{O}(\sqrt{\nu}\ln(\nu/\epsilon))$ [45, Theorem 4.2.9] for an accuracy ϵ where

$$\nu = \sum_{q \in V} |\mathcal{H}(\mathcal{P}_q)| + k(n_{q,x}, d) + nk(n_{q,x}, d-2) + \sum_{q \to_\sigma q'} k(n_{\sigma,p}, d).$$

Theorem 5. *Consider a HAS S as in Definition 3 and points $(h_q \in \mathcal{P}_q)_{q \in V}$. The polynomial $p_q(x, z)$ is feasible for Program 3 if and only if there exist invariant convex sets $\mathcal{C} = (\mathcal{C}_q)_{q \in V}$, as defined in Definition 5, such that $h_q \in \mathcal{C}_q$ for each $q \in V$ and $\tau(\mathcal{C}_q)^*$ is the 0-sublevel set of $p_q(x, z)$.*



(a) Primal space (b) Polar space

Fig. 7. Optimal solution of Program 3 with the volume heuristic developed in [41]. The green set is the safe set $[-1, 1]^2$, the yellow set is the maximal controlled invariant set, the optimal controlled invariant sublevel set of a homogeneous polynomial of degree 4, (resp. 8, 18 and 22) is represented in orange (resp. blue, green and red). Note that, as discussed in [41, Remark 2], their heuristics coincides with the sum of squares of semi-axes for quadratic forms. For this reason, the optimal controlled invariant sublevel set of quadratic forms is the red ellipsoid in Fig. 4. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. Consider a solution $p = (p_q(x, z), s_q(x, z))_{q \in V}$ of Program 3. By (30), the 0-sublevel set of $s_q(x, z)$ is convex. As the 0-sublevel set of $p_q(x, z)$ is its image under the Householder transformation, it is also convex. Therefore there exist sets $\mathcal{C}_q$ such that $\tau(\mathcal{C}_q)^*$ is the 0-sublevel set of $p_q(x, z)$. By (14), constraint (27) is satisfied if and only if $\mathcal{C}_q \subseteq \mathcal{P}_q$. By Proposition 4, constraint (26) is satisfied if and only if (15) hold for all $q \to_\sigma q'$. Therefore, by Theorem 3, the solution p is a feasible solution of Program 2 if and only if the sets $\mathcal{C}_q$ are invariant for S. □

Example 7. The optimal solution of Program 3 using the volume heuristic developed in [41] for the algebraic system given in Example 4 is provided in Fig. 7.

4. Application to model predictive control

As mentioned in the introduction, the controlled invariant sets can be used to derive a feedback control law. We illustrate this with a Model Predictive Control (MPC) numerical experiment. We consider a truck with one trailer ($M = 1$) as in Example 6. The truck starts with speeds $v_0 = v_1 = 2\,\mathrm{m\,s^{-1}}$ and spring elongation $d = 0\,\mathrm{m}$ and has as objective to maximize the distance covered in 20 s. The maximal speed is initially $4\,\mathrm{m\,s^{-1}}$ but after 10 s, it drops to $v_a = 3\,\mathrm{m\,s^{-1}}$.

In a classical MPC controller, the truck acceleration u is controlled by solving a constrained optimal control problem up to horizon H. We observe that if $H \le 2.5$ s, the controller is at some point unable to find values of u satisfying input constraints such that the state remains in the safe set.

For safety-critical applications, this lack of guarantee is not acceptable as it is necessary to be certain that the system can remain in the safe set. Moreover, in a real-time context, the need to pick a large horizon is problematic as it increases the cost of online computations. In our setting, we constrain the state to remain in the controlled invariant sets computed in Example 6[5] and thereby solve both issues; safety is guaranteed for arbitrarily long simulations and the length of the horizon does not influence safety so smaller length can be used. Note that the controlled invariant sets can be computed offline so if it allows to reduce the horizon length, it enables online computational cost to be moved offline. Besides, constraining the state variables to belong to the controlled invariant sets obtained as solution of Programs 2 or 3 reduces to a convex program as shown by Proposition 7 and Theorem 6. The results of the experiment can be found in Figs. 8 and 9.

Proposition 7 (*[46, Section 3.3.1]*)**.** *Given a positive semidefinite matrix $Q \in \mathcal{S}^n$ and a vector $c \in \mathbb{R}^n$, the constraint $x \in \mathcal{E}_{Q,c}$ is second order cone representable.*

5 Example 6 corresponds to an MPC controller of horizon 0.8 s. An MPC controller of different horizon computes different controlled invariant sets by updating the hybrid system accordingly.

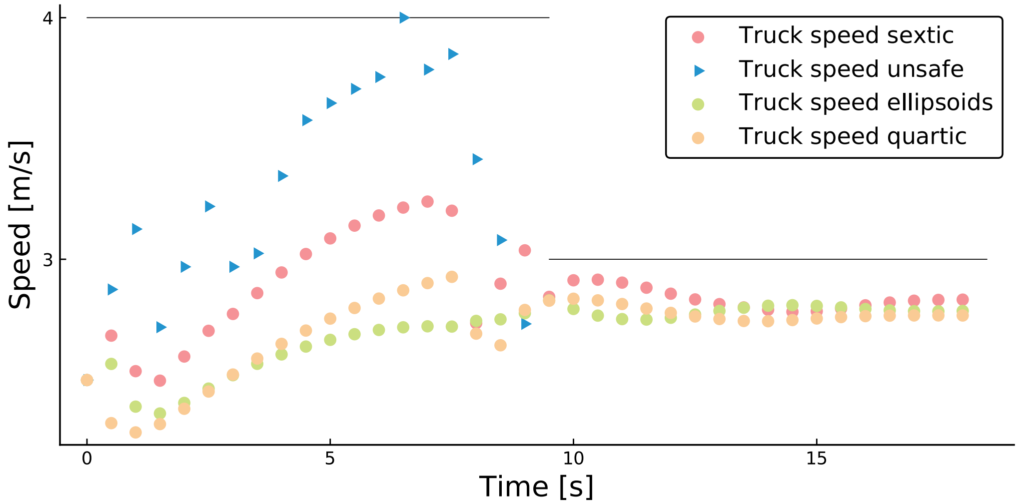


Fig. 8. Evolution with time of the speed of the truck for various MPC strategies. In the legend, *ellipsoids* (resp. *quartic*, *sextic*) designates our MPC strategy using our computed invariant sets using Program 2 (resp. Program 3 with degree 4 and 6 and the volume heuristic developed in [41]), while *unsafe* designates a classical MPC approach. The piecewise horizontal line represents the speed limitation at time t. One can see that the MPC approach with invariant sets allows to remain in the safe set with a conservativeness that decreases with increasing degree. Moreover, the unsafe controller can fail to find feasible values, as shown in Fig. 9.

Proof. Consider a Cholesky factorization $Q = L^\top L$, the inequality $(x-c)^\top Q(x-c) \le 1$ can be rewritten as $\|L(x-c)\|_2 \le 1$ where $\|\cdot\|_2$ is the Euclidean norm. □

Definition 6 (*[36, Section 3.2.1]*)**.** Given a positive integer d and a vector y of moments of all the monomials up to degree $2d$, the *moment matrix* is defined as the matrix $M_d(y)$, indexed by the exponents of monomials of degree d such that $[M_d(y)]_{\alpha,\beta} = y_{\alpha+\beta}$ where $y_{\alpha+\beta}$ is the moment of the monomial $x^\alpha x^\beta$.

Proposition 8 (*[47, Theorem 9]*)**.** *Given an SOS-convex polynomial $s(x) = \sum_\alpha s_\alpha x^\alpha$ of degree $2d$, the membership of the point x to the set $\{x \mid s(x) \le 0\}$ is equivalent to the existence of a vector y of moments of all the monomials up to degree $2d$ such that $y_0 = 1$, the moment matrix $M_d(y)$, defined in Definition 6 is positive semidefinite and $\sum_\alpha s_\alpha y_\alpha \le 0$.*

Theorem 6. *Given an SOS-convex polynomial $s(y)$, a point $c \in \mathbb{R}^{n+1}$ such that $s(c) < 0$ and a matrix $H \in \mathbb{R}^{(n+1)\times(n+1)}$, the membership of a vector $x \in \mathbb{R}^n$ to the set $\mathcal{C}$ satisfying*

$$\tau(\mathcal{C})^* = \{ Hy \mid s(y) \le 0 \} \tag{35}$$

is equivalent to the existence of $\lambda \ge 0$ such that

$$\lambda s(y) - \langle (x, 1), Hy \rangle \text{ is SOS.} \tag{36}$$

Proof. By (12), the constraint $x \in \mathcal{C}$ is equivalent to $(x, 1) \in \tau(\mathcal{C})$ which is equivalent to $\langle (x, 1), u \rangle$ $\forall u \in \tau(\mathcal{C})^*$ by definition of duality. Therefore, by (35), the constraint $x \in \mathcal{C}$ is equivalent to $0 \le \inf_{y \in \tau(\mathcal{C})^*} \langle (x, 1), Hy \rangle$. By Proposition 8, this minimization is semidefinite representable. Since c is strictly feasible for this program, strong duality holds and its optimal objective value is equal to the optimal objective value of its dual which gives (36). □

5. Conclusion

We have developed a methodology for computing controlled invariant sets of Discrete-Time Affine Hybrid Control System (HCS) and Discrete-Time Affine Hybrid Algebraic System (HAS) with *autonomous switching* (see Remark 2). This method can be combined with semidefinite programming in order to compute ellipsoidal controlled invariant sets. We have shown that our technique can be used as a building block in a model predictive control scheme. This allows, among other things, to reduce the online computational cost by precomputing controlled invariant sets.

We feel that we have only scratched the surface of the potential of the duality correspondence of Section 3.1. Many extensions of this work are possible such as hybrid systems with controlled switching, or the computation of controlled invariant intersection of sets to enrich the geometry of the possible invariant sets.

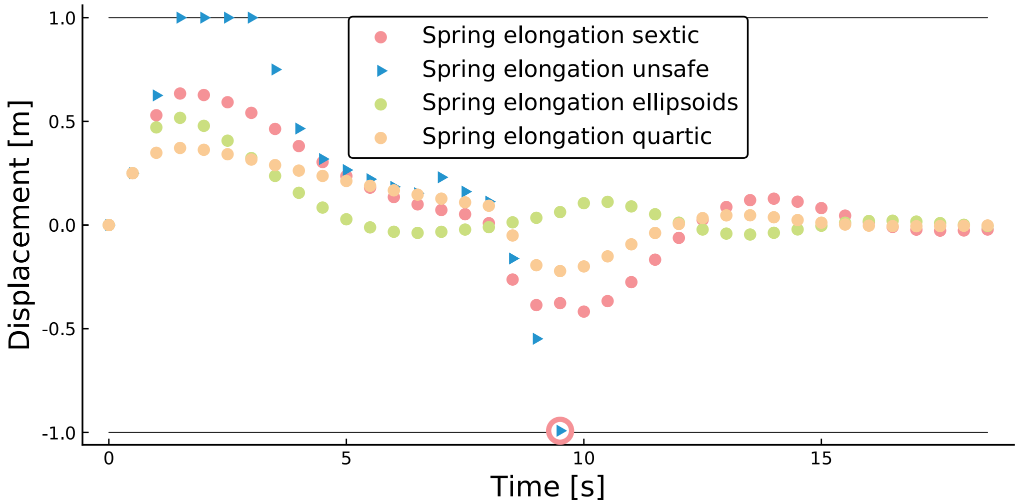


Fig. 9. Spring elongation in safe and unsafe modes. See Fig. 8 for the legend syntax. We see (just before $t = 10$ s) that the unsafe controller makes the trailer go too close to the truck.

The reformulation of the computation of controlled invariant sets of hybrid control system to the computation of invariant sets of hybrid algebraic system with Propositions 1 and 2 allows to have a more behavioral invariance relation. In the future, we would like to put our results in the framework of behavioral theory in order to investigate how to further generalize them; see [48].

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