2015/22

The Φ-Martingale

Frédéric Vrins and Monique Jeanblanc

CORE

DISCUSSION PAPER

Center for Operations Research
and Econometrics

Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium
http://www.uclouvain.be/core

CORE
Voie du Roman Pays 34, L1.03.01
B-1348 Louvain-la-Neuve, Belgium.
Tel (32 10) 47 43 04
Fax (32 10) 47 43 01
E-mail: immaq-library@uclouvain.be
http://www.uclouvain.be/en-44508.html

CORE DISCUSSION PAPER
2015/22

The Φ-Martingale

Frédéric Vrins∗ and Monique Jeanblanc⋆

April 2015

Abstract
In this paper we focus on continuous martingales evolving in the unit interval [0,1]. We first review some results about the martingale property of solution to one-dimensional driftless stochastic differential equations. We then provide a simple way to construct and handle such processes. One of these martingales proves to be analytically tractable, and received the specific name of Φ-martingale. It is shown that up to shifting and rescaling constants, it is the only martingale (with the trivial constant, Brownian motion and Geometric Brownian motion) having a separable coefficient $\sigma(t, x) = g(t)h(x)$ that can be obtained via a time-homogeneous mapping of Gaussian processes. The approach is applied to the modeling of stochastic survival probabilities.

Keywords: continuous stochastic processes, Gaussian processes, bounded martingales, local mar- tingales, Az´ema supermartingale, credit risk modeling
JEL classification: G13, C63
.

∗Contact author. Université catholique de Louvain. Louvain School of Management, Chaussée de Binche 151, B-7000 Mons, Belgium; CORE, B-1348 Louvain-la-Neuve, Belgium; e-mail: frederic.vrins@uclouvain.be.
⋆ Université d'Evry Val d'Essonne, Département de mathématique LaMME, UMR CNRS 8071, Evry, France.

The Φ-Martingale

Frédéric Vrins* and Monique Jeanblanc⋆

April 2015

Abstract

In this paper we focus on continuous martingales evolving in the unit interval $[0,1]$. We first review some results about the martingale property of solution to one-dimensional driftless stochastic differential equations. We then provide a simple way to construct and handle such processes. One of these martingales proves to be analytically tractable, and received the specific name of Φ-martingale. It is shown that up to shifting and rescaling constants, it is the only martingale (with the trivial constant, Brownian motion and Geometric Brownian motion) having a separable coefficient $\sigma(t,x) = g(t)h(x)$ that can be obtained via a time-homogeneous mapping of Gaussian processes. The approach is applied to the modeling of stochastic survival probabilities.

Keywords: continuous stochastic processes, Gaussian processes, bounded martingales, local martingales, Azéma supermartingale, credit risk modeling

JEL classification: G13, C63

Acknowledgment

Frederic Vrins is grateful to Sylvain Benito, Olivier Lévêque, Damien Lamberton and Bernt Øksendal for useful discussions.

*Contact author. Université catholique de Louvain. Louvain School of Management, Chaussée de Binche 151, B-7000 Mons, Belgium; CORE, B-1348 Louvain-la-Neuve, Belgium; e-mail: frederic.vrins@uclouvain.be.

⋆ Université d'Evry Val d'Essonne, Département de mathématique LaMME, UMR CNRS 8071, Evry, France.

1 Introduction and motivation

Mathematical finance extensively relies on martingales. For instance, they are used to represent the dynamics of (non-dividend paying) asset prices, denominated in units of numéraire under the associated measure. In particular, asset price processes are proven to be governed by conditional risk-neutral expected values of future (discounted) cashflows. These are trivially martingales in the risk-neutral measure. Martingales are also central in measure change techniques (via Radon-Nikodym derivative processes).

Depending on the situation, the martingale process may be subjected to some constraints. Discounted stock prices and Radon-Nikodym derivative processes are positive. Therefore, exponential martingales, which meet the non-negativity constraint are very popular tools. But financial processes can be subjected to other constraints, like being bounded below *and* above. For instance when interest rates (short rate r_s) are prevented to be negative, discounted zero-coupon bond prices defined by the risk-neutral expectation

$$P_{t,T} = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}\Big|\mathcal{F}_t\right] = \mathbb{E}\left[1\frac{D_T}{D_t}\Big|\mathcal{F}_t\right] \ , \quad D_t := \mathrm{e}^{-\int_0^t r_s ds} \tag{1.1}$$

belong to $[0,1]$ almost surely. Consequently, so is the martingale $P_{t,T}D_t$. Similarly, survival probabilities $S_{t,T}$ (probability that a default event τ occurs after a given time T as seen from time t), defined as the expected value of survival indicators

$$S_{t,T} = \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t\right] \tag{1.2}$$

are also martingales:

$$\mathbb{E}\left[S_{t,T}|\mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t\right]|\mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_s\right] = S_{s,T} \tag{1.3}$$

This property remains true independently of the definition of the filtration $\mathcal{F}_t$ (including actual default or not). This can of course be generalized to other type of probabilities by replacing the argument of the indicator function $\mathbb{1}_{\{A\}}$.

In the literature, these processes are handled in two ways. Either directly, or indirectly. In the case of survival probabilities modeling for instance, practitioners often disregard inconsistencies, working directly with Gaussian processes (which are not constrained to evolve in the unit interval, see e.g. (3)). This also applies to many (indirect) standard approaches were default intensity is modeled as a shifted Ornstein-Uhlenbeck (Hull-White) or shifted square-root diffusion (SSRD, also known as CIR^{++}), as both can potentially lead to probabilities exceeding 1.[1]

[1] The shift in the square-root process is required in order to fit market (CDS or Bonds) quotes, and may indeed affect the positivity of the resulting stochastic intensity.

In spite of the above drawbacks, these methods remain popular, and bounded martingales themselves, received little attention. We believe this results from the lack of adequate and tractable alternatives to handle such processes. Our purposes here is precisely to propose a contribution to fill this gap and focus on martingales evolving between bounds.

The paper is an extension of (12), and is organized as follows. In Section 2. We recall some results related to existence, uniqueness and martingale property of driftless stochastic differential equations (SDE). We then discuss how bounded martingales can be constructed in Section 3. Section 4 is devoted to a more detailed study of one particular process which remains analytically tractable. Finally, we apply this process to the modeling of univariate survival probability modeling in Section 5 before concluding.

2 Setup

We consider the usual probability space equipped with a filtration $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{Q})$. We study the martingale property of the solution to the "driftless" SDE of the form

$$dY_t = \sigma(t, Y_t)dW_t \ , \ 0 \leqslant t \leqslant T \tag{2.4}$$

where W is a $\mathbb{Q}$-Brownian motion adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ and Y_t is restricted to be in some closed interval $[a, b]$. In the sequel, we shall omit the specification "$0 \leqslant t \leqslant T$", and the diffusion coefficient function $\sigma(t, x) : [0, T] \times \mathbb{R} \to A \subseteq \mathbb{R}^+$ is always assumed to be *Borel-measurable*. The initial condition is non-random and satsifies $Y_0 \in [a, b]$.

The above differential equation can be written in its (Ito) integral form:

$$Y_t = Y_0 + \int_0^t \sigma(s, Y_s)dW_s \tag{2.5}$$

In the remaining part of this section, we review the conditions for Y_t to be a martingale.

2.1 Martingale property of Y_t

The process Y_t in eq. (2.5) is a *local martingale*, but may fail to be a (global) martingale. Additional technical conditions are required, preventing the use of "too fancy" diffusion coefficient functions. However, $\sigma(.,.)$ does not need to be "that fancy" for the (global) martingale property to be lost. For instance, the solution to (2.4) is a global martingale if $\sigma(t, x) = x$ (Geometric Brownian motion) but not if $\sigma(t, x) = x^2$. In financial applications however, the distinction between local and global martingales is crucial in order to prevent arbitrage opportunities (4). To determine whether the solution to the above SDE is indeed a martingale, the following square integrability condition needs to be checked (see e.g. (5) and Section 4.9 of (11)) :

Theorem 1. *The stochastic (Ito) integral in the RHS of eq (2.5) is a martingale if*

$$\mathbb{E}\left[\int_0^T \sigma^2(s, Y_s)ds\right] < \infty \tag{2.6}$$

Condition (2.6) may be tedious to check, since it requires to have some detailed information about the solution to the SDE (which may even not exist). In this context, not a lot can be said at this stage about the martingale property of Y_t. Fortunately, tractable conditions ensuring existence and uniqueness of solutions to generic SDE of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad |X_0| < \infty \tag{2.7}$$

can be found based on the expression of the deterministic function $\sigma(.,.)$ only (see e.g. (6),(8)), that will be helpful in that respect. The next theorem is essentially Theorem 5.2.1 of (8).

Theorem 2. *Let $\sigma(t, x)$ be Lipschitz in x for all $t \geqslant 0$. In addition, suppose σ satisfies the sub-linearity condition*

$$|\sigma(t, x)| \leqslant C(1 + |x|) \tag{2.8}$$

for some constant $C < \infty$, then $\mathbb{E}\left[\int_0^t Y_s^2 ds\right] < \infty$ $\forall t \geqslant 0$.

Corollary 1. *Under the assumption of Theorem 2, Y_t is a martingale.*

Proof. Our purpose is to show that sub-linearity condition combined with $\mathbb{E}\left[\int_0^t Y_s^2 ds\right] < \infty$ implies $\mathbb{E}\left[\int_0^t \sigma^2(s, Y_s)ds\right] < \infty$. The sub-linearity condition yields

$$\sigma^2(t, x) \leqslant 2C^2(1 + x^2) \tag{2.9}$$

Setting $K := 2C^2$

$$\int_0^t \sigma^2(s, Y_s)ds \leqslant Kt + K \int_0^t Y_s^2 ds \tag{2.10}$$

implying

$$\mathbb{E}\left[\int_0^t \sigma^2(s, Y_s)ds\right] \leqslant Kt + K\,\mathbb{E}\left[\int_0^t Y_s^2 ds\right] \tag{2.11}$$

where the RHS is bounded for all $t \in [0, T]$ by virtue of Theorem 2. Finally, Y_t is proven to be a martingale by applying Theorem 1. □

We then present a simple version of *existence and uniqueness* theorem (which is essentially Theorem 4.5.3 of (6), p. 131, up to mild changes).

Theorem 3. *If the diffusion coefficient function satisfies the Lipschitz continuity and sub-linear properties, then the SDE (2.4) admits a pathwise unique (and thus strong) solution, which is a martingale.*

The above results guarantee that the solution to $dY_t = \sigma Y_t dW_t$ exists, is unique and is a martingale for $\sigma \in \mathbb{R}$. However, it is not enough to ensure that the solution to the square-root driftless SDE $dY_t = \sigma\sqrt{Y_t}dW_t$ exists and is unique ($\sqrt{x}$ is not Lipschitz at $x = 0$). Nonetheless, this is the case thanks to an alternative version of the theorem, in which Lipschitz continuity is replaced by the (weaker) Holder-1/2 continuity; see (16). It can be made even more general by relying on the *Yamada-Watanabe* condition only (see e.g. (6) Section 4.5 p.134-135 and (5)). These generalizations are however not necessary in the context of this paper.

2.2 Bounded martingales

In this section, we further specialize to (continuous) bounded martingales. By *bounded martingales*, we refer to processes that belong between two real constants. This is not to be confused with martingales bounded in L_2 for example (see e.g. (14)).

It is known that *bounded local martingales are martingales*. The following result focuses on the case where $Y_t \in [a, b]$.

Corollary 2. *Assume the diffusion coefficient $\sigma(t, x)$ is continuous in some interval $[a, b]$ and vanishes in $\{a, b\}$ for all $t \geqslant 0$. Then, the solution Y_t to SDE (2.5) is a martingale in $[a, b]$ provided that $Y_0 \in [a, b]$.*

Proof. This a straightforward corollary of Theorem 1. Clearly, the diffusion coefficient is bounded in $[a, b]$: for all $t \geqslant 0$, there exists $K(t) \in \mathbb{R} : \sigma^2(t, x) \leqslant K^2(t)$. On the other hand, $Y_t \in [a, b]$ results from a continuity argument: it cannot get out of $[a, b]$ as its activity (instantaneous volatility) vanishes on $\{a, b\}$. So that

$$\int_0^T \sigma^2(s, Y_s)ds \leqslant \int_0^T \max_{y\in[a,b]} \sigma^2(s, y)ds \leqslant \int_0^T K^2(s)ds < \infty \tag{2.12}$$

which proves inequality (2.6) by taking the expectations of the integrals. □

In the following, we shall focus on martingale processes Y_t evolving within the bounds $[0, 1]$, that we call $[0, 1]$-martingales. This is quite general since the later can be used to obtain a process Z_t evolving between any bounds $-\infty < a \leqslant b < \infty$ using $Z_t = a + (b - a)Y_t$ (so that the SDE oz Z_t is given by a change-of-variable technique). The key point here is that the range of Y_t is finite, and constant in time. In the sequel, we shall focus on separable diffusion coefficients, defined below.

Definition 1 (Separable diffusion coefficient). *A diffusion coefficient $\sigma(t,x)$ is said separable if it can be written as*

$$\sigma(t,x) = g(t)h(x) \tag{2.13}$$

It is obvious that when the diffusion coefficient is separable where the time component $g(t)$ is a bounded function for all $t \geqslant 0$, the solution Y_t to SDE (2.5) with initial condition $Y_0 \in [0,1]$ is a continuous martingale in $[0,1]$ provided that function $h(x)$ is continuous on $[0,1]$ and satisfies $|h(x)| > 0$ for $x \in (0,1)$ and $h(x) = 0$ for $x \in \{0,1\}$.

3 Construction of $[0,1]$-martingales

The above section explains how martingales evolving in the compact set $[a,b]$ can be obtained by adequately choosing the diffusion term $\sigma(t,x)$ in eq. (2.4). This framework is interesting theoretically but may be hard to deal with in practice. To illustrate this, suppose we want to construct a martingale process in $[0,1]$. To this end, we can choose $Y_0 \in (0,1)$, $\sigma(t,x) : 0 < \sigma(t,x) < \infty$ and $\sigma(t,0) = \sigma(t,1) = 0$ for all $t > 0$ (notice that $Y_0 \in \{0,1\}$ would lead to $Y_t = Y_0$ for all $t \geqslant 0$, so we can disregard this case). One simple smooth (Lipschitz continuous on $(0,1)$) function satisfying the required conditions is

$$\sigma(t,x) = \eta x(1-x) \tag{3.14}$$

where η is a constant, which is proven to satisfy the conditions of Theorem 2. This leads to the SDE:

$$dY_t = \eta Y_t(1-Y_t)dW_t \tag{3.15}$$

When $Y_0 \in (0,1)$, this SDE admits a unique strong solution. The diffusion coefficient is Lipschitz continuous on $(0,1)$ only, but this is precisely the range of the process Y (using a continuity argument). This will be shown more rigorously below. Solution Y_t is a (bounded) martingale as the range of this process is $[0,1]$; it is thus a $[0,1]$-martingale. However, even if numerical schemes can be worked out to estimate the distribution of Y_t, the analytical expression of such atypical SDE may not be trivial to find, if existing. Moreover, from a practical perspective, such schemes need to guarantee that all paths (for Monte-Carlo simulation, or the range of the distribution, for PDE solver) of Y_t remain in the range implied by the form of the diffusion coefficient function. Generally speaking, implicit schemes satisfying the boundedness conditions are required, but can be tedious to work out.

To address these two issues, we propose a specific construction scheme that yields the martingale Y_t as a transformation of a simpler (unconstrained, or "free") process X_t via some smooth functional $Y_t = F(t, X_t)$. The SDE and statistics of Y_t can thus directly be obtained through those of X_t.

3.1 Methodology

We now proceed with the next result, which is a central tool for constructing $[0, 1]$-martingales from latent processes.

Theorem 4 (Autonomous Mapped Martingales). *Let $F(x) : dom(F) \to [0, 1], x \mapsto F(x)$ be a strictly monotonic function of class $\mathcal{C}^2$ with $f(x) := F'(x)$. Define the Ito process X_t with range $R(X_t) \in dom(F)$ via its stochastic differential equation (SDE)*

$$dX_t = \frac{\eta^2(t, X_t)}{2}\psi(X_t)dt + \eta(t, X_t)dW_t \tag{3.16}$$

where $\psi(x) := -\frac{f'(x)}{f(x)}$ is the score function associated to F. Then, the process

$$Y_t = F(X_t) \tag{3.17}$$

is a $[0, 1]$-martingale.

Proof. Define the general Ito process $X_t = X_0 + \int_{s=0}^{t} \mu(s, X_s)ds + \int_{s=0}^{t} \eta(s, X_s)dW_s$ via its SDE:

$$dX_t = \mu(t, X_t)dt + \eta(t, X_t)dW_t \tag{3.18}$$

Because the domain of F includes the support of X_t, we can define the process $Y_t = F(X_t)$, which dynamics are given via Ito's lemma :

$$dY_t \quad = \quad \left(\mu(t, X_t) - \frac{\eta^2(t, X_t)}{2}\psi(X_t)\right) f(X_t)dt + f(X_t)\eta(t, X_t)dW_t \tag{3.19}$$

Because F maps X_t to $[0, 1]$, Y_t is restricted to the $[0, 1]$ interval. The drift term of the Y_t SDE vanishes when setting

$$\mu(t, X_t) = \frac{\eta^2(t, X_t)}{2}\psi(X_t) \tag{3.20}$$

The process $Y_t = F(X_t)$ is thus a local martingale, and because it is bounded, it is a martingale.

□

Corollary 3. *Let X_t be tha stochastic process defined by the SDE (3.18) with the initial condition X_0 and density f_{X_t}, and F be a bijection of class $\mathcal{C}^2$ with image $[0, 1]$. Then $Y_t = F(X_t)$ is a $[0, 1]$-martingale satisfying the SDE*

$$dY_t = \sigma(t, Y_t)dW_t \ , \quad \sigma(t, x) := f\left(F^{-1}(x)\right)\eta(t, F^{-1}(x)) \tag{3.21}$$

Moreover, the distribution of Y_t ($F_{Y_t}(y)$) and its density ($f_{Y_t}(y)$), can be obtained form that of the latent process X_t. In particular, if F is increasing:

$$
\begin{aligned}
F_{Y_t}(y) &= F_{X_t}(F^{-1}(y)) &(3.22)\\
f_{Y_t}(y) &= \frac{1}{f\left(F^{-1}(y)\right)} f_{X_t}\left(F^{-1}(y)\right) &(3.23)
\end{aligned}
$$

The existence of a solution Y_t to eq. (3.21) is conditioned upon the existence of a solution X_t to eq. (3.16). In the case where the SDE (3.16) admits a unique solution X_t, then SDE (3.21) admits a solution, given by $Y_t = F(X_t)$. Because Y_t is the solution of a driftless SDE, it is a local martingale, and because it is bounded, the solution Y_t to the above SDE is a martingale.

Remark 1. *Simple candidates for function F are cumulative distribution (or survival distribution) functions defined on the real line, admitting a continuously differentiable (non-vanishing) density.*

We now proceed with some examples in the next section.

3.2 Examples

1. Consider eq. (3.18) with $\eta(t,x) = \eta x$:

   $$dX_t = \mu(t, X_t)dt + \eta X_t dW_t \tag{3.24}$$

   Then, $Y_t = \exp(-\lambda X_t)$ is a martingale bounded in $[0,1]$ provided that $\mu(t,x) = \lambda(x\eta)^2/2$ and $\lambda > 0$. Indeed, $f(x) = -\lambda F(x)$ and $\psi(x) = \lambda$, so that the SDE of Y_t is given by

   $$dY_t = -\eta \ln(Y_t) Y_t dW_t \tag{3.25}$$

   The resulting SDE for X_t satisfies a variant of the *Verhulst* equation:

   $$dX_t = \mu X_t^2 dt + \eta X_t dW_t \tag{3.26}$$

   with $\mu = \lambda\eta^2/2 > 0$ which (non-negative) solution is given by

   $$X_t = \frac{\Theta_t}{1 - \frac{\lambda\eta^2}{2}\int_{s=0}^t \Theta_s ds} \text{ with } \Theta_t \doteq X_0 \exp\left\{-\eta^2/2t + \eta W_t\right\} \tag{3.27}$$

   This shows that $Y_t = \exp(-\lambda X_t)$ solves equation (3.25) if $Y_0 \in (0,1]$ and X_t is given above with $X_0 = -\ln(Y_0)/\lambda$. It can be shown that in spite of the quadratic form of the drift coefficient in eq. (3.26), this solution is unique at least up to a small time.

2. Consider the unconstrained (latent) process X_t defined by the SDE (3.18). Define $Y_t = F(X_t)$ where

   $$F(x) = \frac{\mathrm{e}^x}{1+\mathrm{e}^x}\ , \quad \psi(x) = 2F(x) - 1 \tag{3.28}$$

Then, Y_t is a $[0,1]$-martingale provided that $\mu(t,x) = \frac{\eta^2}{2}\tanh\left(\frac{x}{2}\right)$ and $\lambda > 0$. The corresponding SDEs become

$$\begin{aligned} dX_t &= \frac{\eta^2}{2}\tanh\left(\frac{X_t}{2}\right)dt + \eta dW_t && (3.29)\\ dY_t &= \eta Y_t(1-Y_t)dW_t && (3.30)\end{aligned}$$

which agrees with eq. (3.15). Because the drift and diffusion coefficients of eq. (3.29) meet the required conditions, this SDE admits a unique strong solution, and so does eq. (3.30) since both are linked by the bijection $F(x)$ given in (3.28).

3. It is obvious that for a same latent process X_t, one can find several mappings F such that the resulting SDE of $Y_t = F(X_t)$ is driftless. For instance, consider the SDE (3.29) but choose $F(x) = \tanh(x/2)$. This leads to:

$$dY_t = \frac{\eta}{2}(1 - Y_t^2)dW_t \tag{3.31}$$

The reason why a same process X_t can lead to several martingale processes Y_t is that different mappings $F(X)$ can have the same score function. For instance, $F(x)$ and $1 - F(x)$ are functions with image being the unit interval, and have the same score function. The score function of $F(x) = \tanh(x/2)$ corresponds with its score function, which agrees with that of the logistic distribution $F(x) = \mathrm{e}^x/(1+\mathrm{e}^x) = (1+\tanh(x/2))/2$ that led to SDE (3.30).

3.3 Practical considerations

As explained above, martingales evolving in $[a,b]$ can be obtained by specifying the form of the diffusion coefficient $\sigma(t,x)$. However, the resulting SDE's are most often not analytically tractable, so that numerical schemes need to be used. The above examples show that one can construct bounded martingales $Y_t \in [a,b]$ by mapping some latent processes $X_t \in \mathbb{R}$ through a monotonic smooth function $F : \mathbb{R} \to [a,b]$. This change-of-variable approach is handy, because it provides a simulation scheme for generating paths for Y_t that will stay in the correct range. To that end, it is enough to draw sample paths $X_t(\omega)$ for the latent process, and we obtain $Y_t(\omega) = F(X_t(\omega)) \in image(F)$.

However, we have no clue how to choose the mapping function: we started from a latent process X_t with unspecified drift function $\mu(t,x)$, and choosing the mapping F specifies both $\mu(t,x)$ (the dynamics of the latent processes) and the diffusion coefficient $\sigma(t,x)$ of Y_t. The only thing we know is that the range of Y_t will be defined by the image of F, and that Y_t will be a martingale.

The above methodology is interesting for simulation purposes. Nevertheless, the drift of the latent process can be quite weird, rending the design of schemes (for simulation or numerical solution approaches) tedious. In that respect, it is interesting to have indications about how to choose F with a given image ($[0,1]$, say) such that the dynamics of X_t have a specific, simple and tractable form. The above considerations suggest that it may be better to first (analytically or numerically) try to solve the SDE of an underlying free process $X_t = G(Y_t)$, and then get the solution of Y_t via the mapping $F = G^{-1}$. If G is chosen in a clever way, it could be that the second SDE is more easy to deal with (more standard, evolving in $\mathbb{R}$ instead of e.g. $[0,1]$). At least we would get the correct range.

This is the purpose of the next theorem, which tells us how to choose $G = F^{-1}$ so that $X_t = G(Y_t)$ takes a specific, more appealing form. More specifically, the developed methodology allows us to write the solution (if it exists) of the SDE (2.4) when the instantaneous volatility is separable (in the sense of eq. (2.13)) as a mapping F of another process with specific drift but diffusion coefficient $g(t)$ depending on time only. The below theorem states sufficient conditions for the solution Y_t to the SDE (2.4) to be given by $F(X_t)$ where X_t is as desired.

Theorem 5. *Consider the process Y_t and satisfying the SDE (2.4) where the diffusion coefficient is separable in the sense of (2.13). Assume that the function $g(t)$ is continuous and satisfies $g^2(t) < \infty$ for all t and $h(x)$ is strictly positive of class $\mathcal{C}^2$ in the open interval (a,b) and vanishes at the boundaries $\{a,b\}$. Let $y = F(x) \subseteq [a,b]$ solves the first order autonomous non-linear ODE*[2]

$$\frac{dy}{dx} = h(y) \tag{3.32}$$

If $\psi(x) = -F''(x)/F'(x)$ is Lipschitz continuous, then (2.4) admits a strong pathwise unique solution $Y_t = F(X_t)$ which is a $[a,b]$-martingale, and where X_t is the unique strong solution to the SDE (3.18) with initial value $X_0 := F^{-1}(Y_0)$, diffusion coefficient $\eta(t,x) = g(t)$ and drift $\mu(t,x) = \frac{g^2(t)}{2}\psi(x)$.

Proof. It is easy to see by a continuity argument that $Y_t \in [a,b]$ a.s. if $Y_0 \in (a,b)$. Let us first prove that the solution $y = F(x)$ to the non-linear differential equation (3.32) is invertible on $[a,b]$, of class $\mathcal{C}^2$. Clearly, for $x \in (a,b)$ the above ODE is equivalent to

$$G'(x) = \frac{1}{h(x)} \tag{3.33}$$

For any $x \in (a,b)$, the change of variable $x = G(y)$ makes sense since $h(x) > 0$ implies that so is the derivative of G: F is indeed invertible in the neighborhood of x and G is nothing but the local inverse of F. In fact, h specifies the derivative of the local inverse of F and because h does not vanish on $[a,b]$, F admits an inverse on $[a,b]$: $G(x) = F^{-1}(x)$. Moreover the derivative of F^{-1} is of class $\mathcal{C}^2$, and thus

[2]The solution to this ODE is proven to have the general form $y(x) = H(x+k)$ where k is the integration constant and $H(x)$ is the inverse of $\int_{\inf dom(h)}^{x} h^{-1}(u)du$

so is that of F: the first three derivatives of F are continuous. Any solution F to the above ODE is a strictly increasing continuous function on $[a, b]$. Therefore, it has the functional form $h(x) = f(F^{-1}(x))$, where F has the required smoothness for Ito's lemma to be used, and is invertible. Ito's lemma yields the dynamics of $F^{-1}(Y_t)$, which corresponds to the SDE (3.18) where $\eta(t, x) = g(t)$ and $\mu(t, x) = -\frac{g^2(t)}{2}\psi(x)$. From Theorem 3 this SDE, with finite initial value $X_0 = F^{-1}(Y_0)$ has a strong pathwise unique solution since the coefficients meet the standard requirements (Lipschitz continuity of $\psi(x)$ together with the boundedness of $g^2(t)$ for $t < \infty$ implies the linear growth bound condition on $g^2(t)\psi(x)$, and hence so is the drift coefficient $\mu(t, x)$). Finally, the solution Y_t is given by the mapping F: $Y_t = F(X_t) \in [a, b]$. Because it is a local martingale in $[a, b]$ with continuous diffusion coefficient, it is a martingale. □

Example 1. *We used this trick to obtain the solution of the SDE (3.15) in the case $g(t) = \eta$ and $h(y) = y(1 - y)$ by mapping X_t via F. Eq. 3.32 yields the (logistic) first-order non-linear differential equation*

$$f(x) = \frac{dF(x)}{dx} = F(x)(1 - F(x)) \tag{3.34}$$

having (3.28) as particular solution.

Remark 2. *It is worth noting that although we obtain the SDE of $X_t = F^{-1}(Y_t)$ from that of Y_t, the expression of F^{-1} is not needed; it does not enter the SDE of X_t. The drift of X_t is determined by the score function of F, which solves the ODE. It only matters for the sake of computing X_0 corresponding to a given initial value Y_0 required for the $[a, b]$-martingale.*

Example 2. *Although Theorem 5 focuses on the bounded martingale case, it can be transposed to martingales defined on the half-real line (e.g. $a = 0$ and $b = \infty$). Consider for instance the case of the exponential martingale with time dependent volatility, with SDE $dY_t = \eta(t)Y_t dW_t$, $\eta^2(t) < \infty$ continuous. The above theorem tells us how we can obtain Y_t as a function of a latent process X_t. Setting $g(t) = \eta(t)$ and $h(x) = x$, Y_t can be obtained by $F(X_t)$ where $F(x) = e^x + k$ is a solution to (3.32) and X_t satisfies SDE (3.18) with $\mu(t, x) = -\eta(t)^2/2$ and $\eta(t, x) = \eta(t)$. Because the later SDE has a unique strong solution (as per Theorem 4.5.3 of (6)), so does that of Y_t. In the case where $\eta(t) = \eta$, one could equivalently choose $g(t) = 1$ and $h(x) = \sigma x$, in which case $F(x) = e^{\eta x} + k$ and $\mu(t, x) = -\eta/2$.*

4 The Φ-martingale

In this section, we use Theorem 5 to define the dynamics of a $[0, 1]$-martingale that remains analytically tractable, that is, with known distribution. The starting point is to observe that the most general form of a SDE associated to a Gaussian process takes the form $dX_t = (a(t)X_t + b(t))dt + c(t)dW_t$. In particular, the drift function needs to be affine in x, and the diffusion coefficient does not depend on x. That means that by choosing a latent process X_t with diffusion coefficient $\eta(t)$ and a mapping $F(x) : \mathbb{R} \to [0,1]$ with linear score function, the martingale Y_t would be a simple transform of a Gaussian process.

The only function F satisfying this condition is the normal distribution $F(x) = \Phi\left(\frac{x-\alpha}{\beta}\right)$ with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_0$. The particular case $(\alpha, \beta) = (0, 1)$ corresponds to a standard Normal distribution, which score function is the identity $\psi(x) = -\phi'(x)/\phi(x) = x$ where $\phi(x) = \Phi'(x)$ stands for the standard Normal probability density function. In this specific case, applying the above results, $Y_t = \Phi(X_t)$ is a $[0,1]$-martingale provided that

$$\mu(t,x) = \frac{\eta^2(t,x)}{2}x \tag{4.35}$$

and with diffusion coefficient

$$\sigma(t,y) = \phi\left(\Phi^{-1}(y)\right)\eta\left(t, \Phi^{-1}(y)\right) \tag{4.36}$$

The specific case $\eta(t,x) = \eta(t)$ (leading to the separable diffusion coefficient $\sigma(t,y) = \eta(t)\phi \circ \Phi^{-1}(y)$) is quite interesting in that in this case X_t is a Gaussian process ($\mu(t,x) = \frac{\eta^2(t)}{2}x$). The solution to the general extended Vasicek SDE with initial solution $X_s = x \in \mathbb{R}$ is, for $t \geqslant s$

$$dX_t = \kappa(t)(\theta(t) - X_t)dt + \eta(t)dW_t \tag{4.37}$$

is

$$\begin{aligned}
X_t &= G(s,t)(x + I(s,t) + J(s,t)) && (4.38)\\
G(s,t) &:= \mathrm{e}^{-\int_s^t \kappa(u)du} && (4.39)\\
I(s,t) &:= \int_s^t \kappa(u)\theta(u)G(u,s)du && (4.40)\\
J(s,t) &:= \int_s^t \eta(u)G(u,s)dW_u && (4.41)
\end{aligned}$$

which is Normally distributed with mean and variance

$$\begin{aligned}
m(t) &= G(s,t)(x + I(s,t)) && (4.42)\\
v(t) &= G^2(s,t)\int_s^t (\eta(u)G(u,s))^2\,du && (4.43)
\end{aligned}$$

Notice that in our case, we need to impose $\theta(t) \equiv 0$ which means that $I(s,t) = 0$ and $\kappa(t) = -\frac{\eta^2(t)}{2}$. Clearly, the solution to eq. (2.4) with diffusion coefficient (4.36) exists, is unique and is the martingale $Y_t = \Phi(X_t)$ where X_t is given in (4.38). This is, to our knowledge, the only $[0,1]$-martingale process being analytically tractable.

Remark 3. *It has been shown that the case (3.25) is also tractable since there is closed form expression for the latent process X_t (and thus for $Y_t = \exp(-\lambda X_t)$). Unfortunately, although the joint density of $\left(\Theta_t, \int_0^t \Theta_s ds\right)$ has been studied by Yor in (15), the law of X_t is tedious to deal with in practice.*

In the sequel, we focus on the constant volatility case $\eta(t) = \eta$ although the general case remain analytically tractable. The SDE of Y_t is given by

$$dY_t = \eta\phi\left(\Phi^{-1}(Y_t)\right) dW_t \tag{4.44}$$

and admits a strong unique solution $Y_t = \Phi(X_t)$ which is a $[0,1]$-martingale where X_t is a Vasicek process

$$X_t = X_0 \,\mathrm{e}^{\frac{\eta^2}{2}t} + \eta \,\mathrm{e}^{\frac{\eta^2}{2}t} \int_0^t \mathrm{e}^{-\frac{\eta^2}{2}s} \, dW_s \tag{4.45}$$

with constant diffusion coefficient η, zero long-term mean and negative speed of mean reversion $\eta^2/2$. Therefore, we can easily draw sample paths from this exact solution. Ten of them are shown in Fig. 1.

Since $X_t \sim \mathcal{N}(X_0 \,\mathrm{e}^{\eta^2 t/2}, \sqrt{\mathrm{e}^{\eta^2 t} - 1})$, the distribution of Y_t is known analytically:

$$f_{Y_t}(y) = \frac{1}{\phi(\Phi^{-1}(y))\sqrt{2\pi(\mathrm{e}^{\eta^2 t} - 1)}} \exp\left\{ -\frac{\left(\Phi^{-1}(y) - \Phi^{-1}(Y_0)\,\mathrm{e}^{\eta^2 t/2}\right)^2}{2(\mathrm{e}^{\eta^2 t} - 1)} \right\} \tag{4.46}$$

In particular, the distribution of Y_t is uniform provided that $X_0 = 0$ and $t = \ln(2)/\eta^2$.

4.1 Statistics and asymptotics

In the case where the SDE X_t with drift implied by F has an explicit solution, then the process Y_t can be studied in details. For instance, the asymptotic distribution of Y_t as $t \to \infty$ can be obtained. Moreover, one can also study the properties of disjoint increments of Y_t. They have zero-mean and are uncorrelated, as per the martingale property. Their variances and quantile functions can be computed as well. For the sake of comparison, we mention the corresponding results for the exponential martingale.

In the case of $Y_t = \Phi(X_t)$ where X_t is a Vasicek process with instantaneous variance η^2, zero long-term mean and positive mean reversion speed $\eta^2/2$, the variance of the martingale Y_t is given by $\mathbb{E}\left[Y_t^2\right] - Y_0^2$

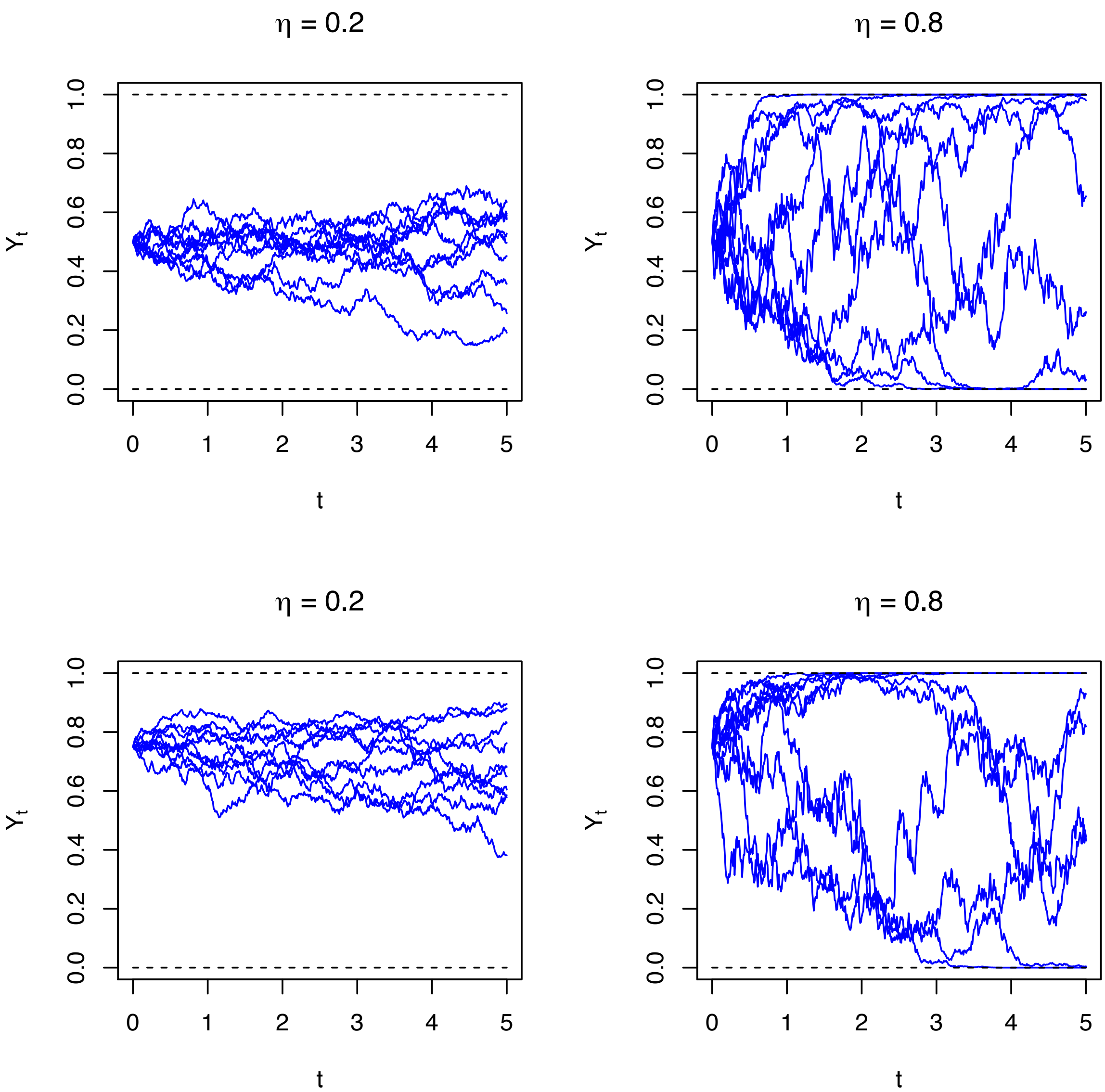


Figure 1: Ten sample paths of the Φ-martingale drawn from the exact solution $\Phi(X_t)$ for various diffusion scales η and initial conditions (top : $Y_0 = 1/2$, bottom: $Y_0 = 3/4$).

where

$$\begin{aligned}
\mathbb{E}\left[Y_t^2\right] &= \int_0^1 \frac{y^2}{f(F^{-1}(y))} f_{X_t}(F^{-1}(y))dy && (4.47)\\
&= \frac{1}{\sqrt{\mathrm{e}^{\eta^2 t}-1}} \int_0^1 \frac{y^2}{f(F^{-1}(y))} \phi\left(\frac{F^{-1}(y) - X_0\,\mathrm{e}^{\eta^2 t/2}}{\sqrt{\mathrm{e}^{\eta^2 t}-1}}\right) dy && (4.48)\\
&= \frac{1}{\sqrt{\mathrm{e}^{\eta^2 t}-1}} \int_{-\infty}^{\infty} \Phi^2(x) \phi\left(\frac{x - X_0\,\mathrm{e}^{\eta^2 t/2}}{\sqrt{\mathrm{e}^{\eta^2 t}-1}}\right) dx && (4.49)\\
&= \int_{v=-\infty}^{\infty} \Phi^2\left(X_0\,\mathrm{e}^{\eta^2 t/2} + v\sqrt{\mathrm{e}^{\eta^2 t}-1}\right)\phi(v)dv && (4.50)\\
&= \Phi_2\left(X_0, X_0; \frac{\mathrm{e}^{\eta^2 t}-1}{\mathrm{e}^{\eta^2 t}}\right) && (4.51)\\
&= \Phi_2\left(X_0, X_0; 1 - \mathrm{e}^{-\eta^2 t}\right) && (4.52)
\end{aligned}$$

where $\Phi_2(x, y; \rho)$ is the standard bivariate Normal cumulative distribution with correlation ρ. In particular, $\lim_{t\to\infty} \mathbb{E}\left[Y_t^2\right] - Y_0^2 = \Phi(X_0) - Y_0^2 = Y_0(1 - Y_0)$.

As per properties of Ito integrals $\mathbb{E}\left[Y_s Y_t\right] = \mathbb{E}\left[Y_{s\wedge t}^2\right]$, the autocovariance of $\{Y_t, Y_{t+\delta}\}$ is equal to the variance of Y_t. The variance of the increments is then given by

$$\begin{aligned}
\operatorname{var}\left[Y_{t+\delta} - Y_t\right] &= \operatorname{var}\left[Y_{t+\delta}\right] - \operatorname{var}\left[Y_t\right] && (4.53)\\
&= \mathbb{E}\left[Y_{t+\delta}^2\right] - \mathbb{E}\left[Y_t^2\right] && (4.54)\\
&= \Phi_2\left(X_0, X_0; 1 - \mathrm{e}^{-\eta^2(t+\delta)}\right) - \Phi_2\left(X_0, X_0; 1 - \mathrm{e}^{-\eta^2 t}\right) && (4.55)
\end{aligned}$$

This expression converges to zero as $t \to \infty$. Intuitively, this means that the "activity" of the process (path by path) will decrease with time, and the process will converge to some constant level. By comparison, the variance of the increments of the exponential martingale Z defined as $Z_t = Z_0\,\mathrm{e}^{-\frac{\eta^2}{2}+\eta W_t}$ between t and $t+\delta$ is $Z_0\,\mathrm{e}^{\eta^2 t}\left(\mathrm{e}^{\eta^2 \delta}-1\right)$ and thus increases with t for a given δ:

$$\begin{aligned}
\operatorname{var}\left[Z_t - Z_s\right] &= \mathbb{E}\left[(Z_t - Z_s)^2\right] && (4.56)\\
&= \mathbb{E}\left[Z_s^2\left(\frac{Z_t}{Z_s} - 1\right)^2\right] && (4.57)\\
&= \mathbb{E}\left[Z_s^2\right]\mathbb{E}\left[\frac{Z_t^2}{Z_s^2} - 2\frac{Z_t}{Z_s} + 1\right] && (4.58)\\
&= \mathbb{E}\left[Z_s^2\right]\left(\mathbb{E}\left[\frac{Z_t^2}{Z_s^2}\right] - 2\,\mathbb{E}\left[\frac{Z_t}{Z_s}\right] + 1\right) && (4.59)\\
&= Z_0\,\mathrm{e}^{\eta^2 s}\,\mathbb{E}\left[\mathrm{e}^{-(2\eta)^2/2s+2\eta\sqrt{s}W_1}\right] && (4.60)\\
&\quad\times\left(\mathrm{e}^{\eta^2(t-s)}\,\mathbb{E}\left[\mathrm{e}^{-(2\eta)^2/2(t-s)+2\eta\sqrt{t-s}W_1}\right] - 2\,\mathbb{E}\left[\mathrm{e}^{-\eta^2/2(t-s)+\eta\sqrt{t-s}W_1}\right] + 1\right) && (4.61)\\
&= Z_0\,\mathrm{e}^{\eta^2 s}\left(\mathrm{e}^{\eta^2(t-s)}-1\right) && (4.62)
\end{aligned}$$

Because the paths of the Φ-martingale evolve between two bounds, a central question is to determine whether they collapse to the bounds, in which case the distribution of Y_t would have less and less mass in $(0, 1)$ in the sense that for any arbitrarily small threshold $\epsilon > 0$ and any probability level $0 < p < 1$, there exists a time t_ϵ such that for all $t > t_\epsilon$, $\mathbb{Q}\{Y_t \in [0, \epsilon) \cup (1 - \epsilon, 1]\} > p$; Y_t ends up in the neighborhood of the bounds with any desired confidence interval. This will be proven in the case when $F = \Phi$ and $\eta(t) = \eta$ (Section 4.1.2). An intuitive development is provided in Appendix (7.1) in the more general case of bounded martingales.

This might be an argument to show that this specific setup is not appropriate in many cases. However, this distribution behavior is shared by the quite popular geometric Brownian motion for example. The distribution of the geometric Brownian motion is collapsing to 0 as $t \to \infty$ (see Fig. 2 for an illustration of the quantiles $q(t, p)$ for the corresponding stochastic processes Z_t and Y_t). The fact that the variance of $Z_{t+\delta} - Z_t$ is not converging to zero as time passes in spite of this collapsing feature results from the fact that the right tail of the distribution of Z_t is unbounded.

4.1.1 Asymptotic distribution of the exponential martingale

The random variable Z_t defined above follows the lognormal distribution with $m = \ln(Z_0) - \eta^2/2t$ and variance $s^2 = \eta^2 t$.

The corresponding quantile function $q(t, p)$ defined according to $\Pr[Z_t \leqslant q(t, p)] = p$ is

$$q(t, p) = Z_0 \exp\left(\eta\sqrt{t}\Phi^{-1}(p) - \eta^2/2t\right) \tag{4.63}$$

and for $0 < p < 1$,

$$\lim_{t\to\infty} q(t, p) = Z_0 \lim_{t\to\infty} \exp\left(\sqrt{t}(a - b\sqrt{t})\right) \tag{4.64}$$

for some finite $a \doteq \eta\Phi^{-1}(p)$ and $b \doteq \eta^2/2 > 0$. For $t \geqslant t^\star \doteq \max(0, a/b)$, the expression in the RHS limit is strictly decreasing to 0 with respect to t. The $p = 50\%$ case (median) is precisely the largest p such that the curve is decreasing everywhere.[3]

[3]This contradicts the naive interpretation of martingales having "no tendency to raise or fall". This is clearly a wrong shortcoming. This is obviously true in expected value, but quite false otherwise since the exponential martingale *does* have a tendency to fall $\mathbb{Q}\{X_{t+\delta} < X_t\} > 50\%$. This remains true on a path by path basis since the median is (strictly) below the expected value.

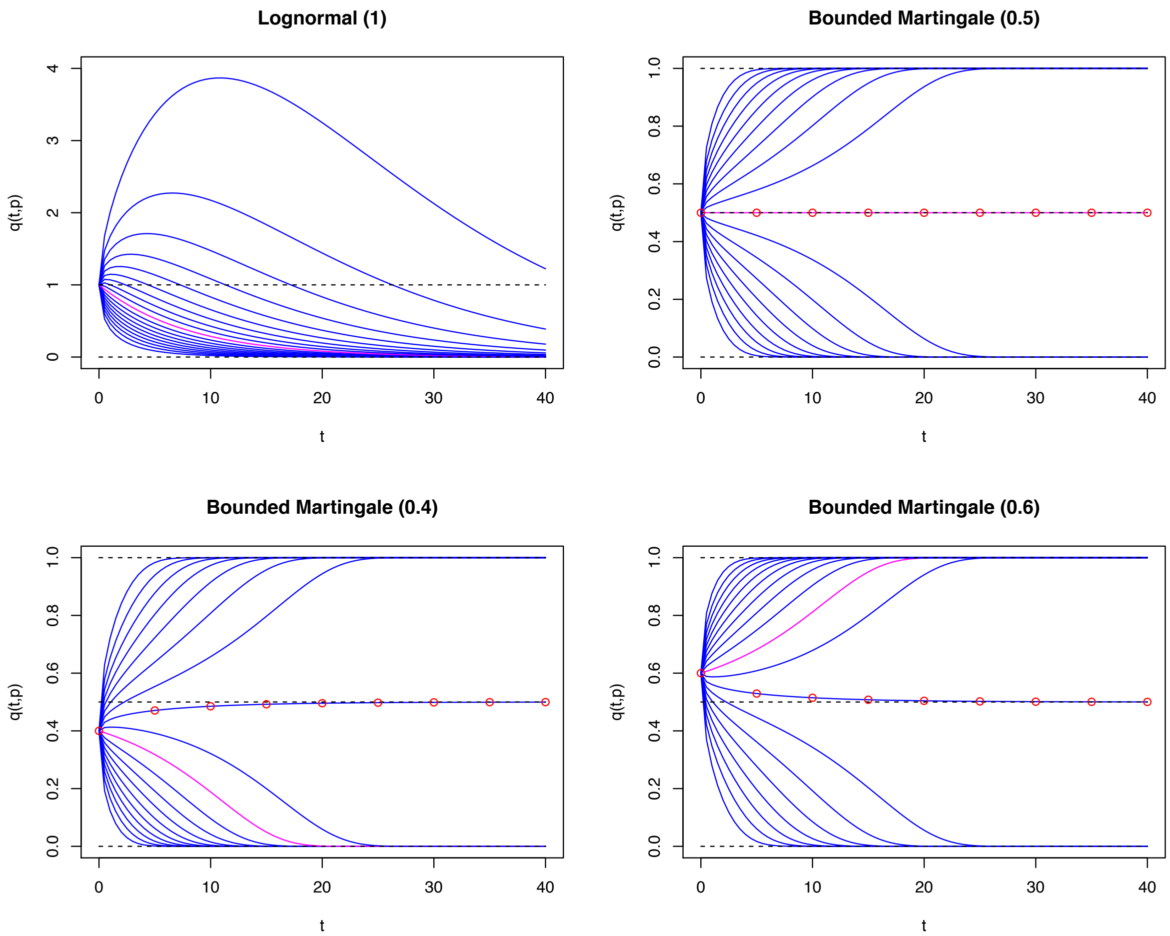


Figure 2: Quantile trajectories $q(t,p)$ of some distributions with $\eta = 50\%$: the exponential martingale Z_t for $Z_0 = 1$, $\eta = 0.5$ (top left) and bounded martingale for $F(.) = \Phi(.)$ and initial point value Y_0 of 0.5 (top right), 0.4 (bottom left) and 0.6 (bottom right). The curves are shown for $p \in \{5\%, 10\%, \ldots, 95\%\}$ probability levels (of course, the corresponding trajectories are ordered bottom up). For the exponential martingale case, the curve associated to the median is the largest non-increasing curve. For the bounded martingale case, the distribution collapses to 0 or 1 depending if $Y_0 < 0.5$ or $Y_0 > 0.5$, respectively. If $Y_0 = 0.5$, the distribution is equally splitted to the bounds. The median is shown in magenta, and the $(1 - Y_0)$-quantiles are emphasized with dots.

4.1.2 Asymptotic distribution of the Φ-martingale Y_t $(\eta(t) = \eta)$

We get

$$
\begin{aligned}
\lim_{t\to\infty} \mathbb{Q}\{Y_t \leqslant y\} &= \lim_{t\to\infty} \Phi\left(\Phi^{-1}(y), \Phi^{-1}(Y_0)\,\mathrm{e}^{\eta^2 t/2}, \sqrt{\mathrm{e}^{\eta^2 t} - 1}\right) && (4.65)\\
&= \lim_{t\to\infty} \Phi\left(\frac{\Phi^{-1}(y) - \Phi^{-1}(Y_0)\,\mathrm{e}^{\eta^2 t/2}}{\sqrt{\mathrm{e}^{\eta^2 t} - 1}}\right) && (4.66)\\
&= 0\,\mathbb{I}_{\{y=0\}} + \Phi\left(-\Phi^{-1}(Y_0)\right)\mathbb{I}_{\{0<y<1\}} + \mathbb{I}_{\{y=1\}} && (4.67)\\
&= \left(1 - \Phi\left(\Phi^{-1}(Y_0)\right)\right)\mathbb{I}_{\{0<y<1\}} + \mathbb{I}_{\{y=1\}} && (4.68)
\end{aligned}
$$

Therefore, Y_t converges in distribution to *Bernoulli* (Y_0) as $t \to \infty$. It is worth noting that *Bernoulli* (Y_0) corresponds to the distribution in $[0, 1]$ with maximum variance for a given mean $Y_0 \in [0, 1]$ (this is quite intuitive, but a formal proof is given in the Appendix (7.2)).

4.2 Autonomous Gaussian martingales

We now address the following question: which types of continuous local martingales Y_t with separable diffusion coefficient can be obtained via a time-homogeneous (that is, autonomous) mapping F of a Gaussian process X_t ?

Definition 2 (Autonomous Gaussian Martingales)**.** *We say that the martingale Y_t is autonomous Gaussian if (i) it can be obtained by mapping a Gaussian process X_t, assumed to be the solution to a SDE, through an autonomous mapping $F(x)$ and (ii) the diffusion coefficient of Y_t is separable in the sense of (2.13).*

Let us assume $0 < g(t) < \infty$. We know that in order for X_t to be a Gaussian solution to an SDE, the drift $\mu(t, x)$ needs to be affine in x, that is $\mu(t, x) = a(t) + b(t)x$ and the diffusion coefficient must be a function of time only $\eta(t)$.[4]

Equating the (dt) and (dW_t) terms of the Y_t SDE 2.4 with that of the $F(t, X_t)$ SDE obtained using Ito, we get

$$
\begin{aligned}
(a(t) + b(t)x)F_x(t, x) + \frac{g^2(t)}{2}F_{xx}(t, x) &\overset{(dt)}{=} -F_t(t, x) && (4.69)\\
\eta(t)F_x(t, x) &\overset{(dW_t)}{=} g(t)h \circ F(t, x) && (4.70)
\end{aligned}
$$

Using a time-homogeneous mapping yields $F(t, x) = F(x)$, implying that $g(t) = \eta(t)$. The (dW_t) equation then corresponds to eq. (3.32) and its solution yields the space component $h(x)$ of the separable

[4] Note that there are Gaussian processes that cannot be expressed as the solution of a SDE. This is the case of the fractional Brownian motion, for example.

diffusion coefficient $\sigma(t,x) = \eta(t)h(x)$. In this exercise however, we are interested in the form of $F(x)$ that can be used so that X_t is a Gaussian process and $Y_t = F(X_t)$ a martingale. Setting $G(x) = F_x(x)$, the (dt) equation becomes

$$\frac{\eta^2(t)}{2} G_x(x) + (a(t) + b(t)x)G(x) = 0 \tag{4.71}$$

This is a first-order ODE which solution is given by

$$G(x,t) = k_1(t)\,\mathrm{e}^{-\frac{2a(t)x+b(t)x^2+c(t)}{\eta^2(t)}} \tag{4.72}$$

$$F(x,t) = k_1(t)\int_{-\infty}^{x} \mathrm{e}^{-\frac{2a(t)u+b(t)u^2+c(t)}{\eta^2(t)}}\,du + k_2(t) \tag{4.73}$$

As we considered time-independent mapping ($F(t,x) = F(x)$ for all x), the ratios $a(t)/\eta^2(t)$, $b(t)/\eta^2(t)$, $c(t)/\eta^2(t)$ and the integration constants $k_1(t), k_2(t)$ need all to be constant in order to get the required form for the $Y_t = F(X_t)$ SDE. We note them a, b, c, k_1, k_2. Clearly, the case $b < 0$ can be excluded as the integral in eq. (4.73) does not converge in this case. Notice that unless $k_1 = 0$, we can set $k_1 = \pm 1$ without loss of generality (by playing with c). We thus have the specific case $k_1 = 0$, and three cases for (a,b,c) (assuming $k_1 = \pm 1$ and $k_2 = k$):

- $k_1 = 0$: $F(x) = k$; we map X_t to a constant.
- $(0, 0, c \neq 0)$: $F(x) = \pm\,\mathrm{e}^c\,x + k$; $c(t) = c\eta^2(t)$ so that the mapping is an affine function of the form $F(x) = \alpha x + k$.
- $(a \neq 0, 0, c \neq 0)$: $F(x) = \pm\frac{\mathrm{e}^c}{2a}\,\mathrm{e}^{-2ax} + k$, the mapping is a shifted exponential $F(x) = \pm\frac{\mathrm{e}^c}{2a}\,\mathrm{e}^{-2ax} + k = \frac{\alpha}{\xi}\,\mathrm{e}^{\xi x} + k$.
- $(a \neq 0, b > 0, c \neq 0)$: $F(x) = \alpha\Phi(\frac{x-\beta}{\xi}) + k$ where $\alpha = \pm\sqrt{2\pi}\,\mathrm{e}^{a^2/b-c}$, $\beta = -a/b$ and $\xi = 1/\sqrt{2b}$. The mapping is a shifted and rescaled version of the Normal cumulative distribution function.

The above mappings are the only ones leading to null dt term and separable diffusion coefficient for $Y_t = F(X_t)$ when X_t is a Gaussian process. This also specifies the form of the space component $h(x)$ of the diffusion coefficient that can be obtained by mapping Gaussian processes through $F(x)$. From the (dW_t) equation, we get respectively

- Since $Y_t = F(X_t) = k$, $h(x) = 0$: this is the trivial martingale
- Since $a = b = 0$, X_t is a rescaled Brownian motion ($d\langle X, X\rangle_t = g^2(t)dt$) and $Y_t = \alpha X_t + k$ is a shifted and rescaled copy of X_t. In particular, $h(x) = h = \alpha$
- Y_t is the exponential of a Gaussian process with shift: $F_x(x) = \alpha\,\mathrm{e}^{\xi x}$ so that $h(x) = F_x(F^{-1}(x)) = \xi(y - k)$.

- Y_t is obtained by mapping the Gaussian process X_t through a Normal cumulative distribution, rescaling and shifting. The obtained process is a martingale bounded between k and $\alpha + k$. In this case, $h(x) = \frac{\alpha}{\xi}\phi\left(\Phi^{-1}\left(\frac{x-k}{\alpha}\right)\right)$.

We summarize these results in the theorem below.

Theorem 6 (Autonomous Gaussian Martingales)**.** *The only autonomous Gaussian martingales are (up to deterministic shift and scaling coefficients) i) the trivial martingale, ii) the Brownian motion, iii) the geometric Brownian motion and iv) the Φ-martingale. Interestingly, each resulting process has a specific range, namely: constant, unbounded, one-side bounded and two-sides bounded.*

This result says that if one wishes to construct a continuous local martingale Y_t with separable diffusion coefficient $\sigma(t,x) = \eta(t)h(x)$ and evolving in a given set by mapping a Gaussian process (i.e. with known analytical solution) via an invertible autonomous function (so that Y_t has also a known analytical expression), there are not many alternatives: only one family of mapping per type of range. In particular, the Φ-martingale $\Phi(X_t)$ is the only bounded continuous martingale with separable diffusion coefficient that can be obtained by mapping a Gaussian process X_t through a smooth function $F(x)$.

Note that if one relaxes the time homogeneity and invertibility constraints, other solutions are possible. For instance, in the case $a(t) = b(t) = 0$ and $\eta(t) = \eta$ ($X_t = X_0 + \eta W_t$) and setting $F(t,x) = x^2 - \eta^2 t$, we obtain $Y_t = X_t^2 - d\langle X, X\rangle_t$ which is a well known martingale (in $\mathbb{R}$). Similarly, other martingales can be obtained if the mapping is allowed to keep track of the whole history (for example, the Shiryaev-Yor process $\sup_{0<s\leqslant t} W_s - \sqrt{2t/\pi}$ is a martingale dominating the boundary $-\sqrt{2t/\pi}$, see (10)).

5 Application to Survival probabilities

We adopt the credit risk modeling setup and focus on the default time τ of some reference entity. In this framework, one usually defines the filtration $\mathcal{F}_t$, which represents the market information excluding default observation. The enlarged filtration is obtained by including the explicit information relevant to the default event: $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\mathbb{1}_{\{\tau>s\}}, 0 \leqslant s \leqslant t)$. In Cox models for example, the stochastic intensity process λ_t is $\mathcal{F}_t$-measurable, but conditional upon the path $(\lambda_t)_{t\geqslant 0}$, the occurrence of default $\mathbb{1}_{\{\tau\leqslant t\}}$ is an independent event. More generally, the latter is $\mathcal{G}_t$-measurable, but not $\mathcal{F}_t$-measurable. Literature on credit risk modeling emphasize that under some conditions, one can get rid of actual default modeling; default indicators can be replaced by stochastic default probabilities, working in the restricted filtration $\mathcal{F}_t$ instead of the complete filtration $\mathcal{G}_t$. We do not enter the details of this modeling approach, but refer the reader to (1) and (7) for more information.

We now move to the modeling of the martingale $S_{t,T}$ defined in eq. (1.2). This is useful in many circumstances, including the pricing of credit derivatives or to adjust the price of a derivatives portfolio to account for counterparty risk (credit value adjustment), see e.g (2), (3), (13). As an illustration of the above methodology, we set $S_{t,T} = \Phi(X_{t,T})$ where $X_{t,T}$ satisfies

$$dX_{t,T} = \mu X_{t,T} dt + \eta dW_t \tag{5.74}$$

Clearly, $S_{t,T}$ is a martingale with initial value $S_{0,T} = S_0(T)$ provided that

$$\mu = \eta^2/2 \tag{5.75}$$

$$X_{0,T} = \Phi^{-1}(S_0(T)) \tag{5.76}$$

The initial survival probability function $S_0(t)$ is assumed to be provided (in credit derivative applications, it is obtained by bootstrapping market quotes of financial instruments, like defaultable bonds or credit default swaps). It is decreasing and satisfies for all $t > 0$ $0 < S_0(t) < 1$, which means the the associated hazard rate is strictly positive and finite. This leads to

$$S_{t,T} = \Phi\left(m(t,T) + \eta Z_t\right) \tag{5.77}$$

$$Z_t = \int_0^t \mathrm{e}^{\mu(t-s)} dW_s \tag{5.78}$$

$$m(t,T) = X_{0,T}\,\mathrm{e}^{\mu t} \tag{5.79}$$

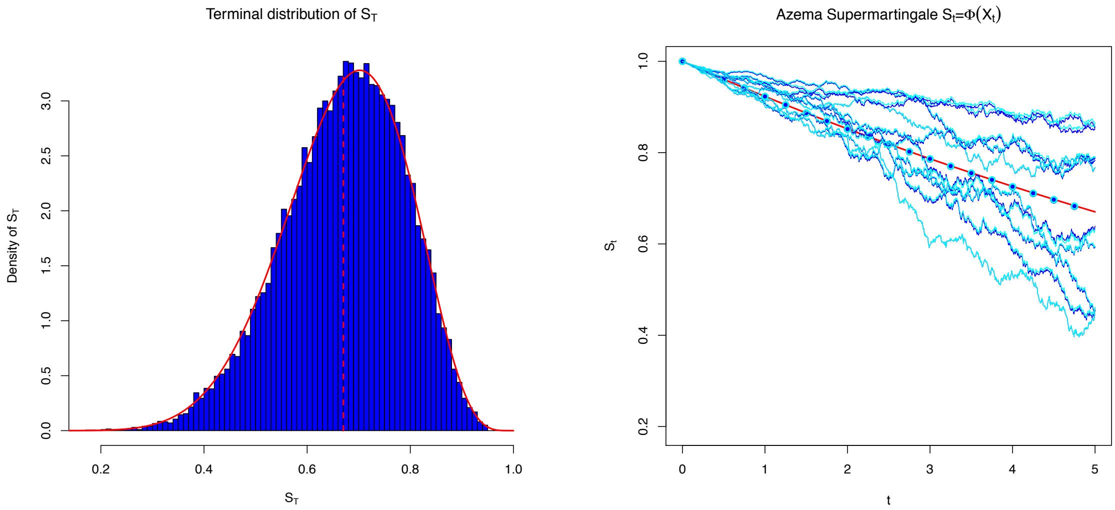
The Azéma supermartingale $S_t := S_{t,t}$ is often modeled either by using a simple Gaussian process (3) or by adopting the Cox process (2). However, the first approach clearly violates the $[0,1]$ condition, and the second corresponds to the specific case where S_t is a decreasing predictable process. However, the general Doob-Meyer decomposition (in its additive form) reveals that in all generality, we have (1), (9)

$$dS_t = dD_t + dM_t \tag{5.80}$$

where D_t is a decreasing $\mathcal{F}_t$-predictable process and M_t a martingale. The Cox setup is just one particular case. Moreover, as explained in the introduction, it is hard to find a positive stochastic intensity model which allows for analytical calibration to market quotes whilst preventing negative paths. This gives room to alternative modeling setups, and the Φ-martingale is one of them (12). The associated Azéma martingale is proven to be

$$S_t = 1 + \int_0^t \zeta_u dS_0(u) + \eta \int_0^t \phi(\Phi^{-1}(S_u)) dW_u \tag{5.81}$$

$$\zeta_t := \frac{\phi(\Phi^{-1}(S_t))\,\mathrm{e}^{\eta^2/2t}}{\phi(\Phi^{-1}(S_0(t)))} \tag{5.82}$$



(a) Density of S_T. Histogram (10k paths, Euler discretization) and theoretical density (red)

(b) Sample paths of S_t using the analytical solution $\Phi(X_t)$ (cyan) and Euler discretization of S_t SDE (5.81) (dark blue)

Figure 3: Distribution of S_T and sample paths of S_t. Parameters: $\eta = 0.15,\ \mu = \eta^2/2,\ S_0(t) = \mathrm{e}^{-ht},\ h = 8\%,\ T = 5$.

Sample paths of such a process and the empirical histogram and theoretical density of values snapped at $t = T$, $S_{T,T}$, is given in Fig. 3. It can be shown that ζ_t is a positive process satisfying $\mathbb{E}[\zeta_t] = 1$ for all $t > 0$:

$$\begin{aligned}
\mathbb{E}[S_t] &= \mathbb{E}\left[\int_0^t dS_u\right] + 1 && (5.83)\\
&= \int_0^t \mathbb{E}[\zeta_u] dS_0(u) + 1 && (5.84)\\
&= S_0(t) && (5.85)
\end{aligned}$$

where we have used the fact that Ito integrals have zero expectation and Fubini's theorem.

6 Conclusion

Bounded martingales are important in many applications. Surprisingly, however, they received little attention so far. This paper aimed at filling this gap, by providing ways to tackle those processes. In particular, we show that analytically tractable martingales bounded in $[0, 1]$ can be obtained by mapping a Gaussian process through a the standard Normal cumulative distribution. This specific process, which received the name of Φ-martingale, is the only $[0, 1]$-martingale that can be obtained by mapping a

Gaussian process in a time-homogeneous way. In the case where the mapped Gaussian process has a constant diffusion coefficient, the distribution of the $[0,1]$-martingale converges to that a Bernoulli distribution with parameter equal to the initial value of the process. We have shown how this setup can be used to simulate stochastic survival probabilities out of the Cox setup.

7 Appendix

7.1 Colapsing property of bounded martingales

It has been proven that when X_t is a Gaussian process with diffusion coefficient η and drift $\eta^2/2x$ then $Y_t = \Phi(X_t)$ is a martingale bounded in $[0,1]$ which converges in distribution to a $Bernoulli(Y_0)$. This proof was easy as the distribution of Y_t is known analytically. However, it is likely that autonomous martingales of the form $F(X_t)$ where X_t is a free process and the image of F is a compact interval $[a,b]$ will share the same "collapsing" feature. Although we do not give a formal proof, we provide an intuitive development below. We further discuss which forms of mappings F could potentially *not* have this feature.

Let F (assumed to be strictly increasing and $\mathcal{C}^2$) and f be unimodal (i.e. $f'(x)$ is first positive, then vanishes at some point $x^\star$ and then remains negative) when $\eta(t, X_t) = \eta$. Recall the SDE of X_t:

$$dX_t = -\frac{\eta^2}{2}\frac{f'(X_t)}{f(X_t)}dt + \eta dW(t) = \frac{\eta^2}{2}\psi(X_t)dt + \eta dW(t) \tag{7.86}$$

Because $f'(x) < 0$ for all $x > x^\star$ and $f'(x) > 0$ for all $x > x^\star$, we can see from the above SDE that X_t has a positive drift when being above $x^\star$ and a negative drift otherwise. As long as we choose F such that f is unimodal, then the process will tend to diverge, and same will hold true for $Y_t = F(X_t)$. In other words, bounded martingales Y_t obtained via an underlying process $Y_t = F(X_t)$ will be attracted towards one of the boundaries when the bounds (a,b) are constant and $f = F'$ is unimodal.

Therefore, Y_t collapses (asymptotically) to a or b unless F is chosen to have some specific properties, and the above development allows us to understand which properties may break this attraction.

A first possibility of course is to use a vanishing time-dependent diffusion coefficient $\eta(t)$. If $\eta(t)$ collapses to zero, the X_t process will be frozen, and so are the paths of the Y_t process. However, we argue that this is not the only way to prevent all paths to converge to one of the bounds. Although we do not give formal proof, we claim that this can be achieved by choosing F so that $F' = f$ is bimodal (i.e. f' changes sign between the two modes) and X_0 belongs to the interval defined by the two modes. Consider for example $F(x) = 1/2\left(\Phi\left(\frac{x-(X_0+\mu)}{s}\right) + \Phi\left(\frac{x-(X_0-\mu)}{s}\right)\right)$ for $s > 0$ and μ large enough to ensure that the sign of f' changes between the two modes. This function maps $\mathbb{R}$ to the unit interval $[0,1]$. Then, the fact that X_t falls in the valley $[X_0-\mu, X_0+\mu]$ will create a pulling effect such that X_t will tend to stay within the interval $[X_0-\mu, X_0+\mu]$.[5] The effect of the bimodal nature of f is to partly prevent all paths to collapse to one of the boundaries (the center of the distribution would not be empty anymore; the probability to be in arbitrarily small neighborhood of the bounds would be non-zero, but would not sum to 1 either). It remains to formally prove that $\mathbb{Q}\{X_t \in [X_0-\mu, X_0+\mu]\} > 0$ as $t \to \infty$ for any $\mu > 0$. This is cumbersome since we do not have an analytical expression for the distribution of X_t. However, Monte Carlo simulations (or PDE solver) confirm that this is effectively the case: only part of the paths collapse to the boundary. The "sharpness of the dip" (which can be tuned by playing with η) determines the probability that Y_t lies in $[F(X_0-\mu), F(X_0+\mu)]$ as $t \to \infty$. However, the process X_t built according to the above procedure has a stable stationary point (zero drift, or equivalently $f'(x) = 0$) at X_0 (zero drift, and in the neighborhood, the drift tends to pull X_t back to X_0), and unstable stationary points at $X_0 \pm \mu$ (zero drift, but when X_t moves around these points, the effect of the drift is to push X_t away from those). It is then likely that the paths, instead of collapsing to either 0 or 1 almost surely, now collapse to $\{0,1\}$ with some probability p, but X_t has a non-zero probability $1-p$ to be in the interval $[X_0-\mu, X_0+\mu]$ even in the limit $t \to \infty$. In particular, we expect to have $\lim_{t\to\infty} \mathbb{Q}\{Y_t \in (\epsilon, F(X_0-\mu)]\} = \lim_{t\to\infty} \mathbb{Q}\{[Y_t \in [F(X_0+\mu), 1-\epsilon)\} = 0$ for any $\epsilon > 0$, $\lim_{t\to\infty} \mathbb{Q}\{Y_t \in (1-\epsilon, 1]\} = \lim_{t\to\infty} \mathbb{Q}\{Y_t \in [0,\epsilon)\} > 0$ and $\lim_{t\to\infty} \mathbb{Q}\{Y_t \in [F(X_0-\mu), F(X_0+\mu)]\} > 0$.

7.2 Maximal variance of bounded distribution

We show that the variance of a random variable X defined on the compact set $[a,b]$ and with given mean m is smaller than the variance of $M = a + (b-a)B$ where B is a Bernoulli variable which parameter is chosen such that $\mathbb{E}[M] = \mathbb{E}[X]$, with equality only if X takes the same form as M.

Due to the scaling and shifting properties of mean and variance operators, it is equivalent to prove that the variance of B is larger than that of $Z = (X-a)/(b-a)$ (defined on the unit interval $[0,1]$), where the inequality is strict unless Z is itself a Bernoulli variable with same parameter than B.

Suppose Z has the probability density f defined on the compact set $[0,1]$. In order to keep the notations simple, we consider f as a "generalized" (mixed discrete-continuous) density, in the sense that it can contain probability mass concentrated at single points, represented with the help of dirac deltas.

[5] Observe that this pulling effect does not impact the martingale property of Y_t, since this is compensated by the function F, just like the fact that X_t is a diverging process when $F = \Phi$ is used did not impact the martingality of Y_t in the unimodal case.

If Z is not a Bernoulli random variable, then there exists a generalized density g defined on $[0,1]$ with zero mass on the bounds and a parameter $0 < p \leqslant 1$ satisfying:

$$f(x) = p_0\delta(x) + pg(x) + p_1\delta(x-1) \tag{7.87}$$

Define the zero, first and second moments of given density h as follows:

$$\mu_h(j) = \int_x x^j h(x)dx \tag{7.88}$$

Clearly, $0 < \mu_g(2) < \mu_g(1) < \mu_g(0)$ and since $\mu_f(0) = p_0 + p\mu_g(0) + p_1$ and $\mu_f(0) = \mu_g(0) = 1$, $p = 1 - (p_0 + p_1) > 0$.

Let us build $\bar{f}$ by distributing the probability mass $p > 0$ into $p\alpha$ at $x = 1$ and $p\bar{\alpha} = p(1-\alpha)$ at $x = 0$. So,

$$\mu_{\bar{f}}(0) = \underbrace{(p_0 + p\bar{\alpha})}_{\bar{p}_0} + \underbrace{(p_1 + p\alpha)}_{\bar{p}_1} = 1 \tag{7.89}$$

Setting $\alpha = \mu_g(1) \in (0,1)$, the mean of f is equal to that of $\bar{f}$:

$$\mu_{\bar{f}}(1) = \bar{p_1} = p_1 + p\alpha = p_1 + p\mu_g(1) = \mu_f(1) \tag{7.90}$$

Finally, the second moment of $\bar{f}$ is proven to be strictly larger than that of f since for $0 < p \leqslant 1$

$$\mu_f(2) = p\mu_g(2) + p_1 < p\mu_g(1) + p_1 = p\alpha + p_1 = \bar{p}_1 = \mu_{\bar{f}}(2) \tag{7.91}$$

Clearly, $\bar{f}$ is a Bernoulli distribution with parameter $\bar{p}_1$. In particular, the mean is equal to $\mu_{\bar{f}}(1) = \bar{p_1}$ and the variance is $\mu_{\bar{f}}(2) - \mu_{\bar{f}}^2(1) = \bar{p_1}(1-\bar{p_1})$.

8 References

[1] Bielecki, T., Jeanblanc, M., and Rutkowski, M. (2011). Credit risk modeling. Technical report, Osaka (Japan).

[2] Brigo, D. and Alfonsi, A. (2005). Credit default swaps calibration and option pricing with the SSRD stochastic intensity and interest rate model. *Finance and Stochastics*, 9:29–42.

[3] Cesari, G., Aquilina, J., Charpillon, N., Filipovic, Z., Lee, G., and Manda, I. (2009). *Modelling, pricing and hedging counterparty credit exposure - A Technical Guide*. Springer.

[4] Cox, A. and Hobson, D. (2005). Local martingales, bubbles and option prices. *Finance and Stochastics*, 9(4):477–492.

[5] Karatzas, I. and Shreve, S. (2005). *Brownian Motion and Stochastic Calculus.* Springer.

[6] Kloeden, P. E. and Platen, E. (1999). *Numerical solution of stochastic differential equation.* Applications of Mathematics (stochastic modelling and applied probability)). Springer.

[7] Lando, D. (2004). *Credit Risk Modeling: Theory and Applications.* Pinceton Series in Finance. Pinceton University Press.

[8] Øksendal, B. (2003). *Stochastic Differential Equations. An Introduction with Applications.* Universitext. Springer.

[9] Profeta, C., Roynette, B., and Yor, M. (2010). *Option Prices as Probabilities: A New look at Generalized Black-Scholes Formulae.* Springer Finance.

[10] Shiryaev, A. N. and Yor, M. (2004). On stochastic integral representations of functionals of brownian motion. *Theory of Probability and its Applications*, 48:304–313.

[11] Shreve, S. (2004). *Stochastic calculus for finance vol. II - Continuous-time models.* Springer.

[12] Vrins, F. (2014). Conic martingales. In *8th World Congress of the Bachelier Finance Society*, Brussels.

[13] Vrins, F. (2015). CVA and wrong-way risk: A model comparison from a measure-change perspective. *Submitted.*

[14] Williams, D. (1991). *Probability with Martingales.* Cambridge University Press.

[15] Yor, M. (1992). On some exponential functionals of brownian motion. *Advances in Applied Probability*, 24(3):509–531.

[16] Zvonkin, A. (1974). A transformation of the phase space of a process that removes the drift. *Math. USSR Sbornik*, 93(1):129–149.

Recent titles

CORE Discussion Papers

2014/59 Manuela BRAIONE and Nicolas K. SCHOLTES. Construction of Value-at-Risk forecasts under different distributional assumptions within a BEKK framework.
2014/60 Jörg BREITUNG and Christian M. HAFNER. A simple model for now-casting volatility series.
2014/61 Timo TERASVIRTA and Yukai YANG. Linearity and misspecification tests for vector smooth transition regression models.
2014/62 Timo TERASVIRTA and Yukai YANG. Specification, estimation and evaluation of vector smooth transition autoregressive models with applications.
2014/63 Axel GAUTIER and Nicolas PETIT. Optimal enforcement of competition policy: the commitments procedure under uncertainty.
2014/64 Sébastien BROOS and Axel GAUTIER. Competing one-way essential complements: the forgotten side of net neutrality.
2014/65 Jean HINDRIKS and Yukihiro NISHIMURA. On the timing of tax and investment in fiscal competition models.
2014/66 Jean HINDRIKS and Guillaume LAMY. Back to school, back to segregation?
2014/67 François MANIQUET et Dirk NEUMANN. Echelles d'équivalence du temps de travail: évaluation de l'impôt sur le revenu en Belgique à la lumière de l'éthique de la responsabilité.
2015/01 Yurii NESTEROV and Vladimir SHIKHMAN. Algorithm of Price Adjustment for Market Equilibrium.
2015/02 Claude d'ASPREMONT and Rodolphe DOS SANTOS FERREIRA. Oligopolistic vs. monopolistic competition: Do intersectoral effects matter?
2015/03 Yuuri NESTEROV. Complexity bounds for primal-dual methods minimizing the model of objective function.
2015/04 Hassène AISSI, A. Ridha MAHJOUB, S. Thomas MCCORMICK and Maurice QUEYRANNE. Strongly polynomial bounds for multiobjective and parametric global minimum cuts in graphs and hypergraphs.
2015/05 Marc FLEURBAEY and François MANIQUET. Optimal taxation theory and principles of fairness.
2015/06 Arnaud VANDAELE, Nicolas GILLIS, François GLINEUR and Daniel TUYTTENS. Heuristics for exact nonnegative matrix factorization.
2015/07 Luc BAUWENS, Jean-François CARPANTIER and Arnaud DUFAYS. Autoregressive moving average infinite hidden Markov-switching models.
2015/08 Koen DECANCQ, Marc FLEURBAEY and François MANIQUET. Multidimensional poverty measurement with individual preferences
2015/09 Eric BALANDRAUD, Maurice QUEYRANNE, and Fabio TARDELLA. Largest minimally inversion-complete and pair-complete sets of permutations.
2015/10 Maurice QUEYRANNE and Fabio TARDELLA. Carathéodory, helly and radon numbers for sublattice convexities.
2015/11 Takatoshi TABUSHI, Jacques-François THISSE and Xiwei ZHU. Does technological progress affect the location of economic activity.
2015/12 Mathieu PARENTI, Philip USHCHEV, Jacques-François THISSE. Toward a theory of monopolistic competition.
2015/13 A.B. TAYLOR, J.M. HENDRICKX and F. GLINEUR. Smooth strongly convex interpolation and exact worst-case performance of first-order methods.

Recent titles
CORE Discussion Papers – continued

2015/14 Christian HAFNER, Hans MANNER and Léopold SIMAR. The "wrong skewness" problem in stochastic frontier models: A new approach.
2015/15 Paul BELLEFLAMME, Nesssrine OMRANI Martin PEITZ. The Economics of Crowdfunding Platforms.
2015/16 Samuel FEREY and Pierre DEHEZ. Multiple Causation, Apportionment and the Shapley Value.
2015/17 Mehdi MADANI and Mathieu VAN VYVE. A MIP Framework for Non-convex Uniform Price Day-ahead Electricity Auctions.
2015/18 Yukihiro NISHIMURA and Pierre PESTIEAU. Efficient Taxation with Differential Risks of Dependence and Mortality.
2015/19 Louis N. CHRISTOFIDES, Michael HOY, Joniada MILLA and Thanasis STENGOS. Nature or Nurture in Higher Education? Inter-generational Implications of the Vietnam-Era Lottery.
2015/20 Philippe J. DESCHAMPS, Alternative Formulations of the Leverage Effect in a Stochatic Volatility Model with Asymmetric Heavy-Tailed Errors.
2015/21 Jean HINDRIKS and Yukihiro NISHIMURA. Equilibrium Leadership in Tax Competition Models with Capital Ownership: A Rejoinder.
2015/22 Frédéric VRINS and Monique JEANBLANC. The Φ-Martingale.

Books

W. GAERTNER and E. SCHOKKAERT (2012), *Empirical Social Choice*. Cambridge University Press.
L. BAUWENS, Ch. HAFNER and S. LAURENT (2012), *Handbook of Volatility Models and their Applications*. Wiley.
J-C. PRAGER and J. THISSE (2012), *Economic Geography and the Unequal Development of Regions*. Routledge.
M. FLEURBAEY and F. MANIQUET (2012), *Equality of Opportunity: The Economics of Responsibility*. World Scientific.
J. HINDRIKS (2012), *Gestion publique*. De Boeck.
M. FUJITA and J.F. THISSE (2013), *Economics of Agglomeration: Cities, Industrial Location, and Globalization.* (2nd edition). Cambridge University Press.
J. HINDRIKS and G.D. MYLES (2013). *Intermediate Public Economics.* (2nd edition). MIT Press.
J. HINDRIKS, G.D. MYLES and N. HASHIMZADE (2013). *Solutions Manual to Accompany Intermediate Public Economics.* (2nd edition). MIT Press.
J. HINDRIKS (2015). *Quel avenir pour nos pensions ? Les grands défis de la réforme des pensions.* De Boeck.

CORE Lecture Series

R. AMIR (2002), Supermodularity and Complementarity in Economics.
R. WEISMANTEL (2006), Lectures on Mixed Nonlinear Programming.
A. SHAPIRO (2010), Stochastic Programming: Modeling and Theory.