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Efficient Hedging of Life Insurance Portfolio for Loss-Averse Insurers

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Abstract

This paper investigates the hedging of equity-linked life insurance portfolio for loss-averse insurers. We consider a general arbitrage-free financial market and an actuarial market composed of $n-$independent policyholders. As the combined market is incomplete, perfect hedging of any actuarial-financial payoff is not possible. Instead, we study the efficient hedging of $n-$size equity-linked life insurance portfolio for insurers who are only concerned with their losses. To this end, we consider stochastic control problems (under the real-world measure) in order to determine the optimal hedging strategies that either maximize the probability of successful hedge (called quantile hedging) or minimize the expectation for a class of shortfall loss functions (called shortfall hedging). Based on the super-replication theory and a duality approach, we show that the optimal strategies depend both on actuarial and financial risks. Moreover, these strategies adapt not only to the size of the insurance portfolio but also to the risk-aversion of the insurer. The numerical results show that, for loss-averse insurers, the strategies outperform the mean-variance hedging strategy, demonstrating the relevance of adopting the right strategy according to the insurers' risk aversion and portfolio size.

KEYWORDS: *Efficient hedging, Super hedging, Incomplete market, Insurance portfolio, Loss-averse.*

1 Introduction

Equity-linked life insurance contracts are characterized by their dependence on both hedgeable and non-hedgeable contingent claims. While the pricing and the fair-valuation of these contracts have been widely discussed in the literature, see among others [1, 9, 11, 26], the question of hedging, especially for loss-averse insurers, receives less attention. Nevertheless, this question is an essential one and is very important, if not more important than the question of pricing, as the insurance market is by definition incomplete. A perfect hedging strategy for any financial-insurance payoff does not exist, and therefore insurers are exposed to losses. The life insurers' profit and loss (P&L) depends both on the observed mortality of their portfolio and on how they invest the premiums received, which is the only thing they can control. The insurers must therefore think about how they will invest all the premiums received so as to honour their obligations to their policyholders and to minimize their losses, whatever adverse situations arise.

Different hedging approaches for equity-linked life insurance portfolios have been proposed in the literature. [20, 21] first introduced risk-minimizing hedging strategies based on quadratic hedging criteria. These risk-minimizing hedging results were then extended to other models and frameworks: [27] considered an incomplete financial market, [8] assumed systematic mortality risk, and [5] considered partially observable market models. Other approaches for hedging equity-linked life portfolios were also examined: [22] studied hedging strategies associated with financial variance and standard deviation principles; [3, 11] introduced convex hedging strategies; [2] studied robust hedging strategies for an insurer seeking to maximize its profit; and more recently, [4, 6] investigated loss-averse convex hedging approaches for insurers giving more importance to losses than to gains. Note that, as hedging of insurance portfolios is closely related to asset and liability management (ALM), we can also refer to [15, 18] that study optimal ALM strategies in expected utility frameworks. Nevertheless, the question of the optimal hedging of life insurance portfolios, considered as a whole, for loss-averse insurers that only focus on losses has not yet been addressed. This paper, therefore, aims to fill this gap. For this purpose, we will consider insurers who are not interested in maximizing their profits and whose sole aim is to minimize their losses. This framework can be seen as an extension of the frameworks considered in [4, 6], where the coefficient of loss aversion tends towards infinity. For this type of insurer, on the one hand, it is important to consider the global hedging of the portfolio and not individual hedging for each contract composing the portfolio; on the other hand, it is important to determine the optimal strategy by formulating the problem in

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terms of the real measure P, so as to include all possible situations, even the most severe, that insurers may have to face.

To reflect the profile of insurers who are concerned with the downside only, we consider the partial hedging of equity-linked life insurance portfolios using shortfall loss functions. The partial hedging strategies studied in this paper therefore aim to either maximize the probability of successful hedge (called quantile hedging) or minimize a function of the shortfall loss (called shortfall hedging) under the real measure P. These hedging techniques were primarily studied by [13, 14] and then extended to several fields and settings. In incomplete financial markets, [17] extended the strategy in a stochastic volatility environment, and [23] extended it to rough volatility models. In defaultable markets, such as the insurance market, [24, 28] considered the partial hedging of defaultable claims, while in actuarial sciences, [19] studied the efficient hedging of life insurance contracts. However, these different approaches, in defaultable markets, essentially consider only one default risk and not multiple default risks. In practice, insurers mutualize their risks by considering sufficiently large portfolios. The study of optimal hedging for portfolios subject to default therefore seems very relevant.

To the best of our knowledge, this paper is the first to investigate the optimal hedging of n-size life portfolio for loss-averse insurers that only focus on their losses. To this end, we consider quantile and shortfall hedging approaches applied in a general financial-actuarial mathematical framework. Although [19] discussed the risk-management of a life insurance portfolio in the context of quantile hedging, the optimal strategy is never deduced. Moreover, as the quantile hedging approach does not control the severity of the losses, it is quite interesting to also consider shortfall hedging strategies for insurers seeking to control the size of the loss. The hedging problem considered is thus formulated as a stochastic control problem with a stochastic target that depends both on financial risk (the value of the financial payoff) and actuarial risk (the number of alive policyholders). Based on the theory of super-replication, initially introduced by [7, 12], we show that optimal strategies are equivalent to (super) hedging strategies of modified payoffs that depend on both financial and actuarial risks. Moreover, using a duality approach, we derive (semi)-explicit forms for the optimal strategies, and prove that these strategies adapt to the size of the portfolio as well as to the insurers' aversion. However, apart from restricting ourselves to purely financial hedging strategies, the forms of optimal strategies depend on a dual process that is not explicitly known in general, making it difficult to implement the optimal strategies in practice. In order to consider more explicit strategies admissible in the combined market, we also construct fully explicit hedging strategies adapted to the combined market, that depend on financial risk factors as well as the evolution of the number of deaths through time. Finally, we demonstrate through numerical results that these strategies outperform the mean-variance hedging strategy, showing the relevance of using the right strategy depending on the size of the portfolio and the risk aversion of the insurers, especially if the size of the portfolio is large.

The paper is outlined as follows. In Section 2, the mathematical framework for financial and actuarial markets is presented, and the hedging problem is posed. Then, Section 3 considers the quantile hedging problem and is divided into two sub-sections: Section 3.1 considers the hedging problem in a very general case where the hedging strategies are admissible in the combined financial-actuarial market, while in Section 3.2, we restrict ourselves to the case where the hedging strategies are only admissible in the financial market. For both sub-sections, we characterize the optimal strategies, and derive the (semi)-explicit form of these strategies. Then, Section 4 considers the class of shortfall hedging problems in order to determine hedging strategies that account for the size of the losses. As for the quantile hedging case, we divide this section into two sub-sections depending on whether or not hedging strategies are restricted to purely financial strategies. We characterize the optimal strategies and derive the (semi)-explicit forms of strategies for shortfall and quadratic shortfall losses. In Section 5, we show how to construct explicit hedging strategies adapted to the combined market filtration, that can be readily implemented and interpreted in practice. Finally, in Section 6, we provide numerical applications in a Black and Scholes framework, compare the hedging strategies with the mean-variance hedging strategy and also analyze the information gain by incorporating the actuarial information into hedging strategies.

2 Mathematical framework

In this paper, we work with a financial market composed of two traded assets: a risky asset $S := (S_t)_{t\geq 0}$ and a risk-free asset $B := (B_t)_{t\geq 0}$. These assets are described by semi-martingale processes defined on the financial market probability space $(\Omega^f, \mathcal{F}^f, P^f)$ equipped with a filtration $\left(\mathcal{F}^f_t\right)_{t\geq 0}$ satisfying the usual conditions. We make the assumption that the financial market is arbitrage-free. Under this assumption, we know that there exists a set of equivalent measures under which the discounted value of the risky asset is a martingale. This set is denoted by $\mathcal{Q}^f$ such that

$$\mathcal{Q}^f := \{Q^f : \textit{equivalent to } P^f, \textit{ and } \frac{S}{B} \textit{ is a martingale under } Q^f\}.$$

Our aim in the following is to consider the optimal hedging of insurance-financial payoffs by trading in the financial market. In this sense, we consider a self-financing hedging portfolio denoted by $V := (V_t)_{t\geq 0}$ and defined, for $t \geq 0$, by

$$\begin{aligned} dV_t &= (V_t - \xi_t S_t)\frac{dB_t}{B_t} + \xi_t dS_t, \\ V_0 &= v_0 \geq 0, \end{aligned} \tag{2.1}$$

on $(\Omega^f, \mathcal{F}^f, P^f)$, where $\xi := (\xi_t)_{t\geq 0}$ is a dynamic hedging strategy. We say that ξ is an admissible financial hedging strategy if ξ is a $\mathcal{F}^f_t-$progressively measurable process such that, for all $t \geq 0$, $E(V_t^2) < +\infty$ and $V_t \geq 0$. At this early stage, we can first consider the hedging of purely positive financial payoffs of the form $H_T \in \mathcal{F}^f_T$ with $T > 0$ a fixed maturity. As the financial market can be incomplete, not every positive contingent claim $H_T \in \mathcal{F}^f_T$ can be perfectly replicated by a hedging portfolio, but super-replicating hedging strategies exist. As shown, for example, in [7, 12], for a $T-$maturity positive given claim $H_T \in \mathcal{F}^f_T$, if

$$v_0 = \sup_{Q^f \in \mathcal{Q}^f} E^{Q_f}\left(\frac{H_T}{B_T}\right),$$

then there exists a super-replicating hedging strategy i.e.

$$\exists \xi \in \mathcal{A}^f \ s.t. \ V_T \geq H_T \ P^f - a.s.$$

where $\mathcal{A}^f$ is the set of all admissible financial hedging strategies valued in $\mathbb{R}$. In particular, if in addition the financial market is assumed to be complete, such that the set of equivalent measures reduces to a singleton, i.e. $\mathcal{Q}^f = \left\{\tilde{Q}^f\right\}$, then for

$$v_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T}\right),$$

a perfect hedging strategy exists i.e.

$$\exists \xi \in \mathcal{A}^f \ s.t. \ V_T = H_T \ P^f - a.s.$$

and the value at time $t \in [0, T]$ of the perfect hedging portfolio is given by

$$V_t = E^{\tilde{Q}^f}\left(\frac{B_t}{B_T} H_T | \mathcal{F}^f_t\right).$$

We now introduce the life insurance market, described by the probability space $(\Omega^a, \mathcal{F}^a, P^a)$ equipped with a filtration $(\mathcal{F}^a_t)_{t\geq 0}$. We assume that the remaining lifetime of an $x-$year-old individual is a random variable denoted by ω_x and defined on $(\Omega^a, \mathcal{F}^a, P^a)$. The mortality rate is noted $(\kappa_{x+t})_{t\geq 0}$ and the probability of survival from age x till age $x + t$, for $t \geq 0$, is noted ${}_tp_x := E^{P^a}\left(\exp\left(-\int_0^t \kappa_{x+s} ds\right)\right)$. We assume that the insurer has a $n-$size portfolio with policyholders from the same cohort, such that for $i = 1, ..., n$, $x_i = x$ and $\kappa_{x_i+t} = \kappa_{x+t}$. In this case, the remaining lifetime of the i^{th} policyholder is denoted by ω_i and $\omega_1, ..., \omega_n$ are assumed to be independent. If we define, for $i = 1, ..., n$,

$$N^i_t := 1_{\{\omega_i > t\}},$$

then the total number of policyholders remaining at time t is defined by

$$N_t := \sum_{i=1}^{n} N^i_t.$$

In the following, we will focus on the combined model denoted $(\Omega, \mathcal{F}, P)$, which is defined as the product space of the two individual spaces, financial and actuarial. We let $\Omega := \Omega^f \times \Omega^a$ and $P := P^f \otimes P^a$. The $\sigma-$algebra $\mathcal{F}$ is generated by all subsets of null-sets from $\mathcal{F}^a \otimes \mathcal{F}^f$ and the associated filtration is denoted by $(\mathcal{F}_t)_{t\geq 0}$. We make the standard assumption that actuarial and financial risks are independent. In this setting, $\mathcal{F}^f$ and $\mathcal{F}^a$ martingales are martingales in $(\Omega, \mathcal{F}, P)$. Even if the financial market is assumed to be complete, the combined market is incomplete since the mortality risk cannot be perfectly hedged. Therefore, we know that the pricing measure is not unique and that there exists a class of equivalent martingale measures denoted by $\mathcal{Q}$ such

$$\mathcal{Q} := \{Q : equivalent\ to\ P,\ and\ \frac{S}{B}\ is\ a\ martingale\ under\ Q\}.$$

For any contingent claim of the form $g(S_T, N_T) \in \mathcal{F}_T$, with $T > 0$ a fixed maturity and $g(x, y)$ a positive function increasing in y, we can characterize the super-replication price as the super-replication price of a $\mathcal{F}^f_T-$measurable payoff. This is the purpose of the next proposition.

Proposition 2.1. *Let $g(x,y)$ a positive function increasing in y, then*

$$\sup_{Q\in\mathcal{Q}} E^{Q}\left(\frac{g(S_T,N_T)}{B_T}\right) = \sup_{Q^f\in\mathcal{Q}^f} E^{Q_f}\left(\frac{g(S_T,n)}{B_T}\right).$$

Proof. The proof relies on Propositions 9.1 and 9.2 in Appendix 9.1. First, by independence between financial and actuarial probability spaces, we observe that

$$\sup_{Q\in\mathcal{Q}} E^{Q}\left(\frac{g(S_T,N_T)}{B_T}\right) = \sup_{Q^f\in\mathcal{Q}^f}\ \sup_{Q^a\in\mathcal{Q}^a} E^{Q^f\otimes Q^a}\left(\frac{g(S_T,N_T)}{B_T}\right),$$

where $\mathcal{Q}^a$ is the set of measures equivalent to P^a. Then, on the one hand, since $g(x,y)$ is growing in y and $N_T \leq n$, we observe that, for all $Q^f \in \mathcal{Q}^f$,

$$\sup_{Q^a\in\mathcal{Q}^a} E^{Q^f\otimes Q^a}\left(\frac{g(S_T,N_T)}{B_T}\right) \leq E^{Q_f}\left(\frac{g(S_T,n)}{B_T}\right).$$

On the other hand, using Proposition 9.2, for $\varepsilon > -1$, for all $\mathcal{F}^a_t-$adapted process $(h_t)_{t\geq 0}$ such that for $\forall t \geq 0$, $h_t > -1$ almost surely and for all $Q^f \in \mathcal{Q}^f$, there exists an equivalent martingale measure $Q_{h^\varepsilon} \in \mathcal{Q}$ such that

$$\lim_{\varepsilon\to -1} E^{Q_{h^\varepsilon}}\left(\frac{g(S_T,N_T)}{B_T}\right) = E^{Q_f}\left(\frac{g(S_T,n)}{B_T}\right).$$

Therefore, we have that, $\forall Q^f \in \mathcal{Q}^f$,

$$\sup_{Q^a\in\mathcal{Q}^a} E^{Q^f\otimes Q^a}\left(\frac{g(S_T,N_T)}{B_T}\right) = E^{Q_f}\left(\frac{g(S_T,n)}{B_T}\right)$$

and we easily conclude that

$$\sup_{Q\in\mathcal{Q}} E^{Q}\left(\frac{g(S_T,N_T)}{B_T}\right) = \sup_{Q^f\in\mathcal{Q}^f} E^{Q_f}\left(\frac{g(S_T,n)}{B_T}\right).$$

□

Suppose now that the insurer guarantees to its policyholders a stochastic positive amount $H_T \in \mathcal{F}^f_T$ if they are alive at a predefined maturity $T > 0$. In this case, the total liability of the insurer at maturity is given by

$$H^{Total}_T := N_T \times H_T \in \mathcal{F}_T.$$

To meet its liabilities, the insurer constructs a hedging portfolio denoted by V that evolves as (2.1) on $(\Omega, \mathcal{F}, P)$. The hedging strategy $\xi = (\xi_t)_{0\leq t\leq T}$ adopts by the insurer is said to be admissible on the combined market if ξ is a $\mathcal{F}_t-$progressively measurable process such that, for all $t \geq 0$, $E(V_t^2) < +\infty$ and $V_t \geq 0$. In the best of all possible worlds, the insurer would like to find a strategy that perfectly replicates its liabilities. Due to the incompleteness of the global market, perfect hedging strategy does not exist for any positive contingent claim $H^{Total}_T \in \mathcal{F}_T$. However, if

$$v_0 = \sup_{Q\in\mathcal{Q}} E^{Q}\left(\frac{H^{Total}_T}{B_T}\right) = \sup_{Q^f\in\mathcal{Q}^f} E^{Q^f}\left(n \times \frac{H_T}{B_T}\right), \tag{2.2}$$

then there exists a super-replicating hedging strategy i.e.

$$\exists \xi \in \mathcal{A} \ s.t. \ V_T \geq H^{Total}_T \ P-a.s,$$

where $\mathcal{A}$ is the set of all admissible hedging strategies on the combined market valued in $\mathbb{R}$. The super-replication price of the total liability (2.2) is actually too expensive, even if the financial market is complete, to be competitive in practice, which is why the insurer is not going to build the super hedging strategy. In fact, the premium that the insurer has to charge its policyholders for super hedging is the price of the same product without the mortality risk. Policyholders would therefore have no interest in choosing the insurance product over the financial product since the financial product is less risky for them and, consequently, the insurer would not have any business.

In what follows, we will assume that the insurer sells its life product at a total price $\bar{V}_0$ such that

$$\bar{V}_0 \leq \sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(n \times \frac{H_T}{B_T}\right),$$

and that this agent is loss-averse. So, instead of seeking to maximize the final value of its P&L and therefore its profit, its main aim is to limit the events leading to losses and also to control the size of these losses. Based on its own risk aversion, the insurer can define an efficient hedging strategy.

On one hand, the insurer can seek to maximize the probability of successful hedge. Thus, in this case, the insurer constructs a strategy to maximize the probability (under P) that the terminal value of the hedging portfolio is above the total payoff. From a mathematical point of view, the problem reduces to the following stochastic control problem:

$$\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}} P(V_T \geq H_T^{Total}), \tag{2.3}$$

under the initial constraint

$$v_0 \leq \bar{V}_0.$$

On the other, the insurer can seek to minimize the loss resulting from the hedging operation. In this case, the insurer constructs a strategy to minimize the expectation of a class shortfall loss functions. This problem can be written as stochastic control problem of the form:

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}} E^P\left(L\Big((H_T^{Total} - V_T)_+\Big)\right), \tag{2.4}$$

with $L \in C^1$, under the initial constraint

$$v_0 \leq \bar{V}_0.$$

Such hedging strategies are called quantile hedging or shortfall hedging strategies and were initially studied by Föllmer and Leukert in [13, 14]. The difference with the Föllmer and Leukert setting is that here we have both financial and actuarial risks. Note that [19] has already studied efficient hedging of equity-linked insurance contracts and has shown that, when restricting ourselves to purely financial hedging strategies, the optimal quantile hedging strategy, when $n = 1$, is the quantile hedging strategy of the purely financial payoff $H_T \in \mathcal{F}_T^f$ (without actuarial risk). However, although the risk management of portfolios is discussed, the optimal hedging strategies are not deduced when $n > 1$.

In the following, we will study quantile and shortfall hedging strategies applied in the context of hedging a $n-$size life insurance portfolio for $n \geq 1$. Under suitable hypotheses, the stochastic control problems studied in this paper can be solved using dynamic programming techniques like solving the Hamilton-Jacobi-Bellman (HJB) equation as it was done in [17, 23]. Nevertheless, instead of applying those techniques, we will use the super hedging theory to characterize the optimal strategies. In this sense, we will transform stochastic control problems into static optimization problems. This allows us to derive that optimal hedging portfolios are actually the (super) replication hedging portfolios of $\mathcal{F}_T-$measurable modified payoffs. Moreover, using a duality approach, we can deduce semi-explicit forms for the optimal strategies. The advantage of this approach, compared with dynamic programming techniques, is that it allows to easily characterize the strategies and to derive semi-explicit forms of optimal hedging strategies even if the financial framework is not fully Markovian[1].

3 Optimal quantile hedging strategy

3.1 Characterization of the optimal strategy

In this section, we characterize the optimal hedging strategy associated with the introduced quantile hedging problem (2.3). To this end, for any $\xi \in \mathcal{A}$, we introduce the sets of successful financial hedge $A_1, ..., A_n \in \mathcal{F}_T$ defined as

$$A_i := \Big\{V_T \geq iH_T\Big\},\ i = 1, ..., n, \tag{3.1}$$

and we show that the dynamic optimization problem (2.3) is equivalent to a static optimization problem that allows to characterize the optimal strategy.

[1] There exist different non-Markovian financial markets, such as, for example, rough volatility or path-dependent volatility models.

Theorem 3.1. *Let $p_0 := P(N_T = 0)$ and $A_i^* \in \mathcal{F}_T$, $i = 1, ..., n$ the solution of the following problem*

$$\max_{A_1,...,A_n \in \mathcal{F}_T,\, A_{i+1} \subseteq A_i} E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i}\right) + p_0, \tag{3.2}$$

under the constraint

$$\sup_{Q \in \mathcal{Q}} E^Q\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i}\right) \leq \bar{V}_0. \tag{3.3}$$

Let ξ^{QH} denote the super-replicating hedging strategy for the modified payoff

$$H_T^{QH} := H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}. \tag{3.4}$$

Then $(\bar{V}_0, \xi^{QH})$ solves the stochastic control problem defined by (2.3) such that

$$\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}\ s.t.\ v_0 \leq \bar{V}_0} P(V_T \geq H_T^{Total}) = E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{N_T=i}\right) + p_0,$$

and $\xi^{QH} \in \mathcal{A}$.

Proof. The proof is given in Section 8. □

Remark 3.1. The existence of a solution to the problem (3.2) follows from the Proposition 9.3 in Appendix 9.2.
The previous theorem shows that the optimal quantile hedging of a $n-$size life insurance portfolio is linked to the super replication strategy of a modified payoff that takes into account both financial and actuarial risks. We can now go a step further and characterize the form of the optimal sets A_i^*, $i = 1, ..., n$. To this end, we apply a duality approach as proposed, for example, in [14], and consider in this sense the following sets:

$$\mathcal{Y} := \left\{(Y_t)_{t\in[0,T]} \geq 0\ P - a.s.:\ Y_0 = 1\ and \left(\frac{V_t}{B_t} Y_t\right)_{t\in[0,T]} is\ a\ P - supermartingale\right\},$$

and

$$\mathcal{D} := \left\{Z_T \in L^0(\Omega, \mathcal{F}, P):\ 0 \leq Z_T \leq Y_T\ for\ some\ (Y_t)_{t\in[0,T]} \in \mathcal{Y}\right\}.$$

Proposition 3.1. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ be the solution of*

$$\min_{\lambda \geq 0,\, Z_T \in \mathcal{D}} E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)}\left(\mathbb{1}_{N_T=i} - \lambda\, Z_T \frac{H_T}{B_T}\right) + \lambda\, \bar{V}_0\right), \tag{3.5}$$

with

$$A_i^*(\lambda, Z_T) = \left\{\alpha_i \geq \lambda \times Z_T \frac{H_T}{B_T}\right\},\ i = 1, ..., n,$$

such that:

$$\alpha_n = \min\left(\mathbb{1}_{N_T=n}, \frac{\mathbb{1}_{N_T=n} + \mathbb{1}_{N_T=n-1}}{2}, ..., \frac{\sum_{j=1}^n \mathbb{1}_{N_T=j}}{n}\right), \tag{3.6}$$

and for $i = 1, ..., n-1$,

$$\alpha_i = \max\left(\alpha_{i+1}, \min_{k=1,..,i}\left(\frac{\sum_{j=k}^i \mathbb{1}_{N_T=j}}{i-k+1}\right)\right). \tag{3.7}$$

Then, $A_i^* = A_i^*(\lambda^*, Z_T^*)$ is a solution of (3.2) such that

$$\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}\ s.t.\ v_0 \leq \bar{V}_0} P(V_T \geq H_T^{Total}) = E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{N_T=i}\right) + p_0.$$

Moreover,

$$\bar{V}_0 = E^P\left(Z_T^* \frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*}\right),$$

and ξ^{QH} is the admissible strategy associated to the super-replication hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$.

Proof. The proof is given in Section 8. $\square$

Remark 3.2. In Proposition 3.1, we consider the dual problem over the set $\mathcal{D}$ instead of the set of martingale density $\mathcal{L} := \left\{ \frac{dQ}{dP}|_T : Q \in \mathcal{Q} \right\} \subseteq \mathcal{D}$. We made this choice in order to guarantee the existence of a solution to the dual problem which is not the case if we consider the dual problem over the set $\mathcal{L}$. In addition, we show that there is no duality gap when considering the dual problem over the set $\mathcal{D}$.

From Proposition 3.1, we obtain semi-explicit expressions for the optimal sets A_i^*, $i = 1, ..., n$ that depend on $\lambda^* \in \mathbb{R}_+$ and $Z_T^* \in \mathcal{D}$ solving (3.5). We can observe that this quantile hedging strategy generalizes the "pure" quantile hedging strategy[2], taking into account the actuarial risk. However, in general, it is difficult to derive the optimal dual process[3] associated to the problem (3.5) and thus we cannot obtain more explicit expressions for the optimal sets. As shown in the following section, by restricting the set of admissible hedging strategies to the set of financial strategies, we are also able to characterize the solution of the problem and, by considering a complete financial market, the optimal dual process can be explicitly deduced. Moreover, we show in Section 5 how to construct an explicit $\mathcal{F}-$adapted quantile strategy that can be implemented in practice.

3.2 Restriction to the set of admissible financial hedging strategies $\mathcal{A}^f \subseteq \mathcal{A}$

In this section, we consider the quantile hedging problem by restricting the set of admissible hedging strategies to the set of financial hedging strategies. This particular setting corresponds, for example, to the case where the insurer has an incomplete information and has no information about the mortality evolution. Note that the developments made in this section will be very useful to construct a more optimal explicit $\mathcal{F}-$adapted quantile hedging strategy in Section 5. When $\xi \in \mathcal{A}^f$, the quantile hedging problem is reduced to

$$\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}^f \ s.t.\ v_0 \leq \bar{V}_0} P(V_T \geq H_T^{Total}), \tag{3.8}$$

such that

$$\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}^f \ s.t.\ v_0 \leq \bar{V}_0} P(V_T \geq H_T^{Total}) \leq \max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A} \ s.t.\ v_0 \leq \bar{V}_0} P(V_T \geq H_T^{Total}).$$

Let us now solve the quantile hedging problem in the set of financial hedging strategies defined by (3.8). To this end, for any $\xi \in \mathcal{A}^f$, we consider the sets of successful financial hedge $A_1, ..., A_n$ defined by (3.1) that are now $\mathcal{F}_T^f-$measurable, and we show that the dynamic optimization problem (3.8) is equivalent to a static optimization problem that allows us to characterize the optimal strategy.

Theorem 3.2. *Let $p_i := P(N_T = i)$ and $A_i^* \in \mathcal{F}_T^f$, $i = 1, ..., n$ the solution of the following problem*

$$\max_{A_1,...,A_n \in \mathcal{F}_T^f,\, A_{i+1} \subseteq A_i} E^P \left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times p_i \right) + p_0, \tag{3.9}$$

under the constraint

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f} \left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i} \right) \leq \bar{V}_0.$$

Let $\xi^{QH,\, f}$ denote the (super) replicating hedging strategy for the modified payoff

$$H_T^{QH} := H_T \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*}. \tag{3.10}$$

Then $(\bar{V}_0, \xi^{QH,\, f})$ solves the stochastic control problem defined by (3.8) such that

$$\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}^f \ s.t.\ v_0 \leq \bar{V}_0} P(V_T \geq H_T^{Total}) = E^P \left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times p_i \right) + p_0,$$

and $\xi^{QH,\, f} \in \mathcal{A}^f$.

Proof. We can easily deduced the announced result by using similar arguments than in the proof of Theorem 3.1 combined with the fact that $A_1, ..., A_n \in \mathcal{F}_T^f$ are independent of N_T. $\square$

[2] Strategy associated to the quantile hedging of $H_T \in \mathcal{F}_T^f$

[3] Except for $n = 1$ in a complete financial market setting, as shown in [24].

The previous theorem shows that the optimal financial quantile hedging of a $n-$size life insurance portfolio is linked to a (super) replication strategy of a modified financial payoff that takes into account the actuarial risk through the probabilities $p_i := P(N_T = i),\ i = 1,...,n$. As in the previous section, we can also go a step further and obtain semi-explicit expressions for the optimal sets $A_i^*,\ i = 1,...,n$ that depends on a dual process $(Z_t^*)_{0\leq t\leq T}$. Moreover, if the financial market is assumed to be complete i.e. $\mathcal{Q}^f = \{\tilde{Q}^f\}$, then we can deduce a fully explicit form of the optimal sets A_i^* since the optimal dual process can be derived. To this end, we apply once again duality approach and consider in this sense the following sets:

$$\mathcal{Y}^f := \left\{(Y_t)_{t\in[0,T]} \geq 0\, P^f - a.s.:\ Y_0 = 1\, and\, \left(\frac{V_t}{B_t}Y_t\right)_{t\in[0,T]} is\ a\ P^f - supermartingale\right\},$$

and

$$\mathcal{D}^f := \left\{Z_T \in L^0(\Omega^f, \mathcal{F}^f, P^f):\ 0 \leq Z_T \leq Y_T\ for\ some\ (Y_t)_{t\in[0,T]} \in \mathcal{Y}^f\right\}$$

Proposition 3.2. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}^f$ be the solution of*

$$\min_{\lambda\geq 0,\ Z_T\in\mathcal{D}^f} E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T, p)}\left(p_i - \lambda\, Z_T\frac{H_T}{B_T}\right) + \lambda\, \bar{V}_0\right), \tag{3.11}$$

with

$$A_i^*(\lambda, Z_T, p) = \left\{\alpha_i(p) \geq \lambda \times Z_T\frac{H_T}{B_T}\right\},\ i = 1,...,n, \tag{3.12}$$

such that:

- $\alpha_n(p) = \min\left(p_n, \frac{p_n+p_{n-1}}{2}, ..., \frac{\sum_{j=1}^n p_j}{n}\right)$,
- for $i = 1,...,n-1,\ \alpha_i(p) = \max\left(\alpha_{i+1}(p), \min_{k=1,..,i}\left(\frac{\sum_{j=k}^i p_j}{i-k+1}\right)\right)$.

Then, $A_i^* = A_i^*(\lambda^*, Z_T^*, p)$ is a solution of (3.9) such that

$$\max_{v_0\in\mathbb{R}_+,\ \xi\in\mathcal{A}^f\ s.t.\ v_0\leq\bar{V}_0} P(V_T \geq H_T^{Total}) = E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times p_i\right) + p_0.$$

Moreover,

$$\bar{V}_0 = E^P\left(Z_T^*\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*}\right),$$

and the optimal hedging strategy $\xi^{QH,\,f}$ is the admissible financial hedging strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. In particular, if $\mathcal{Q}^f = \{\tilde{Q}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$, then $Z_T^* = \tilde{Z}_T$ and $\xi^{QH,\,f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$.

Proof. The proof is very similar to the proof of Proposition 3.1 and it is sufficient to consider p_i instead of $\mathbb{1}_{N_T=i}$ to obtain the announced result. Moreover, if $\mathcal{Q}^f = \{\tilde{Q}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$, we observe that $\forall (Y_t)_{t\in[0,T]} \in \mathcal{Y}^f$, $\exists (A_t)_{t\in[0,T]} \in \mathcal{F}^f$, an increasing process with $A_0 = 0$ such that, almost surely, $Y_t = \tilde{Z}_t - A_t \leq \tilde{Z}_t$, $\forall t \in [0,T]$. In this case, we have that, for all $\lambda \geq 0$,

$$\min_{Z_T\in\mathcal{D}^f} E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T, p)}\left(p_i - \lambda\, Z_T\frac{H_T}{B_T}\right)\right) = E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, \tilde{Z}_T, p)}\left(p_i - \lambda\, \tilde{Z}_T\frac{H_T}{B_T}\right)\right),$$

and thus $Z_T^* = \tilde{Z}_T$. Finally,

$$\bar{V}_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*}\right),$$

and $\xi^{QH,\,f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. ∎

From Proposition 3.2, we deduce that, the optimal financial quantile hedging strategy of insurance-financial payoffs is equivalent to a super-hedging strategy of a sum of knock-out financial options that depend on the actuarial risk thought p_i, $i = 1, .., n$. In particular, for $n = 1$, we retrieve the results deduced in [19]. When $n > 1$, we can observe that the purely financial strategy associated with the quantile hedging of $n \times H_T$ is not optimal and it is preferable for the insurer to adopt the optimal quantile hedging strategy deduced for a $n-$size portfolio, especially if n is large. For a general incomplete financial market, it is difficult to derive the value at maturity of the optimal dual process Z_T^* solving (3.11). However, if the financial market is complete, then the optimal dual process is known and the optimal financial quantile hedging strategy is associated to a perfect hedging strategy of a modified financial payoff that can be readily implemented in practice.

4 Optimal shortfall hedging strategy

4.1 Characterization of the optimal strategy

In the previous section, we showed how to deduce the optimal quantile hedging strategy associated with the hedging of a life insurance portfolio. However, the main drawback of such hedging strategy is that it does not control the severity of the loss when the final value of the hedging portfolio is not sufficient to totally hedge the payoff H_T^{Total}. This quantile hedging approach therefore suffers from the same issue as optimal dynamic asset allocations under value-at-risk constraint (see [16]), and this can lead to very large infrequent losses. Therefore, in order to control the size of the shortfall loss, the insurer might be motivated to minimize the expectation of a class shortfall loss functions. To this end, we will solve in this section the introduced shortfall hedging problem (2.4). As in the previous sections, for any $\xi \in \mathcal{A}$, we consider the auxiliary variables $\phi_1, ..., \phi_n$ defined such that

$$\phi_i H_T := \begin{cases} 0, & if\ V_T \leq (i-1)H_T \\ H_T & if\ V_T \geq iH_T \\ V_T - (i-1)H_T & if\ (i-1)H_T < V_T < iH_T, \end{cases} \tag{4.1}$$

where ϕ_i can be interpreted as the success ratio for the hedging of the i^{th} financial contingent claim H_T. Using these introduced variables, we show that the dynamic optimization problem (2.4) is equivalent to a static optimization problem that enables us to characterize the class of optimal shortfall hedging strategies.

Theorem 4.1. *Let $\phi_i^* \in \mathcal{F}_T$, $i = 1, ..., n$ the solution of the following problem*

$$\min_{\phi_1,...,\phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i,\ \phi_0 = 1} E^P\bigg(\sum_{i=1}^{n} \mathbb{1}_{N_T = i} \times L\bigg(\sum_{j=1}^{i} H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\bigg)\bigg), \tag{4.2}$$

with

$$\mathcal{R} = \bigg\{\phi : \Omega \to [0,1], \mathcal{F}_T - measurable\bigg\}$$

under the constraint

$$\sup_{Q \in \mathcal{Q}} E^Q\bigg(\frac{H_T}{B_T}\sum_{i=1}^{n} \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\bigg) \leq \bar{V}_0.$$

Let ξ^{SH} denote the super replicating hedging strategy of the modified payoff

$$H_T^{SH} = H_T \times \sum_{i=1}^{n} \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}. \tag{4.3}$$

Then $(\bar{V}_0, \xi^{SH})$ solves the stochastic control problem defined by (2.4) such that

$$\min_{v_0 \in \mathbb{R}_+,\ \xi \in \mathcal{A}\ s.t.\ v_0 \leq \bar{V}_0} E^P\bigg(L\Big((H_T^{Total} - V_T)_+\Big)\bigg) = E^P\bigg(\sum_{i=1}^{n} \mathbb{1}_{N_T = i} \times L\bigg(\sum_{j=1}^{i} H_T(1 - \phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}})\bigg)\bigg)$$

and $\xi^{SH} \in \mathcal{A}$.

Proof. The proof is given in Section 8. □

Remark 4.1. From Proposition 9.4 in Appendix 9.2, the existence of a solution to the problem (4.2) is guaranteed.

As for the quantile hedging problem, the previous theorem shows that the optimal shortfall hedging of a $n-$size life insurance portfolio is linked to a super-replication strategy of a modified financial payoff that takes into account both the financial and actuarial risks. Moreover, as for the quantile hedging problem, for some particular loss functions $L(.)$, we can apply a duality approach and characterize the form of the optimal random variables ϕ_i^*, $i=1,...,n$. This is the purpose of the next propositions.

Proposition 4.1. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ be the solution of*

$$\max_{\lambda\geq 0,\, Z_T\in\mathcal{D}} E^P\left(H_T \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda,Z_T)}\left(\lambda\frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j}\right)\right) - \lambda\bar{V}_0, \tag{4.4}$$

with

$$A_i^*(\lambda, Z_T) = \left\{\sum_{j=i}^n \mathbb{1}_{N_T=j} \geq \lambda\frac{Z_T}{B_T}\right\},\ i=1,...,n. \tag{4.5}$$

Then $\mathbb{1}_{A_i^*} = \mathbb{1}_{A_i^*(\lambda^*,Z_T^*)}$ is a solution to the problem (4.2) for $L(x)=x$, such that

$$\min_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}\ s.t.\ v_0\leq\bar{V}_0} E^P\left((H_T^{Total}-V_T)_+\right) = E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i}\left(\sum_{j=1}^i (1-\mathbb{1}_{A_j^*})H_T\right)\right).$$

Moreover,

$$\bar{V}_0 = E^P\left(Z_T^*\frac{H_T}{B_T}\times\sum_{i=1}^n \mathbb{1}_{A_i^*}\right),$$

and the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the super-hedging of the modified payoff $H_T\times\sum_{i=1}^n \mathbb{1}_{A_i^*}$.

Proof. The proof is given in Section 8. □

Proposition 4.2. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ be the solution of*

$$\max_{\lambda\geq 0,\, Z_T\in\mathcal{D}} E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i}\times\frac{1}{2}\left(\sum_{j=1}^i H_T(1-\phi_j^*(\lambda,Z_T)\mathbb{1}_{\{\phi_{j-1}^*(\lambda,Z_T)=1\}})\right)^2 + \lambda Z_T\frac{H_T}{B_T}\sum_{i=1}^n\phi_i^*(\lambda,Z_T)\right) - \lambda\bar{V}_0, \tag{4.6}$$

with

$$\phi_n^*(\lambda,Z_T) = \mathbb{1}_{\left\{\frac{\lambda Z_T}{B_T H_T}\leq \mathbb{1}_{\{N_T=n\}}\right\}}\left(1-\frac{\lambda}{\mathbb{1}_{\{N_T=n\}}}\frac{Z_T}{B_T H_T}\right)\ on\,\{H_T>0\},$$

and, for $i\in\{1,...,n-1\}$,

$$\phi_i^*(\lambda,Z_T) = \mathbb{1}_{A_i(\lambda,Z_T)} + \mathbb{1}_{A_i^c(\lambda,Z_T)}\left(1-\left(\frac{1}{\sum_{j=i}^n \mathbb{1}_{\{N_T=j\}}}\left(\frac{\lambda Z_T}{B_T H_T} - \sum_{j=i+1}^n \mathbb{1}_{\{N_T=j\}}\right)\wedge 1\right)\right)\ on\,\{H_T>0\}, \tag{4.7}$$

such that

$$A_i(\lambda,Z_T) = \left\{\frac{\lambda Z_T}{B_T H_T} \leq \sum_{j=i+1}^n\sum_{k=j}^n \mathbb{1}_{\{N_T=k\}}\right\}.$$

Then $\phi_i^* = \phi_i^*(\lambda^*, Z_T^*)$ is a solution to the problem (4.2) for a quadratic function $L(x)=\frac{1}{2}x^2$ such that

$$\min_{v_0\in\mathbb{R}_+,\, \xi\in A\ s.t.\ v_0\leq\bar{V}_0} E^P\left(\frac{1}{2}\left((H_T^{Total}-V_T)_+\right)^2\right) = E^P\left(\frac{1}{2}\sum_{i=1}^n \mathbb{1}_{\{N_T=i\}}\left(H_T\sum_{j=1}^i(1-\phi_j^*)\right)^2\right).$$

Moreover,

$$\bar{V}_0 = E^P\left(Z_T^*\frac{H_T}{B_T}\times\sum_{i=1}^n\phi_i^*\right), \tag{4.8}$$

and the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the super-hedging of the modified payoff $H_T\times\sum_{i=1}^n\phi_i^*$.

Proof. The proof is given in Section 8. $\square$

From Proposition 4.1 and 4.2, we obtain, when $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, an explicit characterization of the optimal variables ϕ_i^*, $i = 1, ..., n$ that depends on $\lambda^* \in \mathbb{R}_+$ and the $T-$value of a dual process $Z_T^* \in \mathcal{D}$. As for the quantile hedging problem, we observe that the optimal shortfall strategies of a $n-$size portfolio for $L(x) = x$ and $L(x) = \frac{1}{2}x^2$, generalizes the "pure" shortfall hedging strategy deduced in [14]. In particular, if $n = 1$, we recover the results deduced in [24]. We prove that the optimal shortfall hedging strategies depend on both financial and actuarial risks. Nevertheless, once again, it is difficult to derive the optimal dual process[4] associated to the problems (4.4) or (4.6) and thus, as for the quantile hedging problem, we cannot obtain, in general, more explicit expressions for the optimal variables ϕ_i^*. By restricting the set of admissible hedging strategies to the set of financial strategies, we are still able to characterize the solution of the problem when $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, and, for complete financial markets, the optimal dual process can be explicitly deduced. Furthermore, we show in Section 5 how to construct explicit $\mathcal{F}-$adapted shortfall strategies that can be readily implemented in practice.

4.2 Restriction to the set of admissible financial hedging strategy $\mathcal{A}^f \subseteq \mathcal{A}$

In this section, we consider the shortfall hedging problem by restricting the set of admissible hedging strategies to the set of financial hedging strategies. As for the quantile hedging problem, the results of this section will be used to derive more optimal explicit $\mathcal{F}-$adapted shortfall hedging strategies in Section 5. When $\xi \in \mathcal{A}^f$, the shortfall hedging problem is reduced to

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}^f \; s.t. \; v_0 \leq \bar{V}_0} E^P\Big(L\Big((H_T^{Total} - V_T)_+\Big)\Big), \tag{4.9}$$

such that

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}^f \; s.t. \; v_0 \leq \bar{V}_0} E^P\Big(L\Big((H_T^{Total} - V_T)_+\Big)\Big) \geq \min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A} \; s.t. \; v_0 \leq \bar{V}_0} E^P\Big(L\Big((H_T^{Total} - V_T)_+\Big)\Big).$$

Let us now solve the shortfall hedging problem in the set of financial hedging strategies defined by (4.9). Once again, we consider the auxiliary variables $\phi_1, ..., \phi_n$ defined by (4.1) that are now $\mathcal{F}_T^f-$measurable since $\xi \in \mathcal{A}^f$. Using these introduced variables, we show that the dynamic optimization problem (4.9) is equivalent to a static optimization problem that enables us to characterize the class of optimal shortfall hedging strategies.

Theorem 4.2. *Let $p_i := P(N_T = i)$ and $\phi_i^* \in \mathcal{F}_T^f$, $i = 1, ..., n$ the solution of the following problem*

$$\min_{\phi_1,...,\phi_n \in \mathcal{R}^f, \phi_{i+1} \leq \phi_i,\, \phi_0 = 1} E^P\Bigg(\sum_{i=1}^n p_i \times L\Bigg(\sum_{j=1}^i H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\Bigg)\Bigg), \tag{4.10}$$

with

$$\mathcal{R}^f = \Big\{\phi : \Omega^f \to [0,1], \mathcal{F}_T^f - measurable\Big\}$$

under the constraint

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T}\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\right) \leq \bar{V}_0.$$

Let $\xi^{SH,\,f}$ denote the (super) replicating hedging strategy of the modified payoff

$$H_T^{SH} = H_T \times \sum_{i=1}^n \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}. \tag{4.11}$$

Then $(\bar{V}_0, \xi^{SH,\,f})$ solves the stochastic control problem defined by (4.9) such that

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}^f \; s.t. \; v_0 \leq \bar{V}_0} E^P\Big(L\Big((H_T^{Total} - V_T)_+\Big)\Big) = E^P\Bigg(\sum_{i=1}^n p_i \times L\Bigg(\sum_{j=1}^i H_T(1 - \phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}})\Bigg)\Bigg)$$

and $\xi^{SH} \in \mathcal{A}^f$.

Proof. We can easily deduced the announced result by using similar arguments than in the proof of Theorem 4.1 combined with the fact that $\phi_1, ..., \phi_n \in \mathcal{F}_T^f$ are independent of N_T. $\square$

[4]Except for $n = 1$ in a complete financial market setting, as shown in [24].

The previous theorem shows that the optimal financial shortfall hedging of a $n-$size life insurance portfolio is linked to a (super) replication strategy of a modified financial payoff that takes into account the actuarial risk through the probabilities $p_i := P(N_T = i),\ i = 1, ..., n$. As in the previous section, we can also go a step further and, for $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, obtain explicit expressions for the optimal variables ϕ_i^*, $i = 1, ..., n$ depending on a dual process $(Z_t^*)_{0\le t\le T}$. Moreover, if the financial market is assumed to be complete i.e. $\mathcal{Q}^f = \{\tilde{Q}^f\}$, then the optimal dual process is known. This is the purpose of the next propositions.

Proposition 4.3. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}^f$ be the solution of*

$$\max_{\lambda\ge 0,\, Z_T\in\mathcal{D}^f} E^P\left(H_T \sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda,Z_T,p)} \left(\lambda\frac{Z_T}{B_T} - \sum_{j=i}^{n} p_j\right)\right) - \lambda\bar{V}_0,$$

with

$$A_i^*(\lambda, Z_T, p) = \left\{\sum_{j=i}^{n} p_j \ge \lambda\frac{Z_T}{B_T}\right\},\ i = 1, ..., n. \tag{4.12}$$

Then $\mathbb{1}_{A_i^*} = \mathbb{1}_{A_i^*(\lambda^*,Z_T^*,p)}$ is a solution to the problem (4.10) for $L(x) = x$, such that

$$\min_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}^f\ s.t.\ v_0\le\bar{V}_0} E^P\left((H_T^{Total} - V_T)_+\right) = E^P\left(\sum_{i=1}^{n} p_i\left(\sum_{j=1}^{i}(1-\mathbb{1}_{A_j^*})H_T\right)\right).$$

Moreover,

$$\bar{V}_0 = E^P\left(Z_T^*\frac{H_T}{B_T} \times \sum_{i=1}^{n}\mathbb{1}_{A_i^*}\right),$$

and the optimal hedging strategy $\xi^{SH,\,f}$ is the admissible financial strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^{n}\mathbb{1}_{A_i^*}$. In particular, if $\mathcal{Q}^f = \{\tilde{Q}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$, then $Z_T^* = \tilde{Z}_T$ and $\xi^{SH,\,f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^{n}\mathbb{1}_{A_i^*}$.

Proof. The proof is very similar to the proof of Proposition 4.1 and it is sufficient to consider p_i instead of $\mathbb{1}_{N_T=i}$ to obtain the announced result. Moreover, if $\mathcal{Q}^f = \{\tilde{Q}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$, we have that, for all $\lambda \ge 0$,

$$\max_{Z_T\in\mathcal{D}^f} E^P\left(H_T \sum_{i=1}^{n}\mathbb{1}_{A_i^*(\lambda,Z_T,p)}\left(\lambda\frac{Z_T}{B_T} - \sum_{j=i}^{n}p_j\right)\right) = E^P\left(H_T\sum_{i=1}^{n}\mathbb{1}_{A_i^*(\lambda,\tilde{Z}_T,p)}\left(\lambda\,\frac{\tilde{Z}_T}{B_T} - \sum_{j=i}^{n}p_j\right)\right),$$

and thus $Z_T^* = \tilde{Z}_T$. Finally,

$$\bar{V}_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n}\mathbb{1}_{A_i^*}\right),$$

$\xi^{SH,\,f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^{n}\mathbb{1}_{A_i^*}$. $\square$

Proposition 4.4. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}^f$ be the solution of*

$$\max_{\lambda\ge 0,\, Z_T\in\mathcal{D}^f} E^P\left(\sum_{i=1}^{n} p_i \times \frac{1}{2}\left(\sum_{j=1}^{i} H_T(1-\phi_j^*(\lambda,Z_T)\mathbb{1}_{\{\phi_{j-1}^*(\lambda,Z_T)=1\}})\right)^2 + \lambda Z_T\frac{H_T}{B_T}\sum_{i=1}^{n}\phi_i^*(\lambda,Z_T)\right) - \lambda\bar{V}_0,$$

with

$$\phi_n^*(\lambda, Z_T, p) = \left(1 - \frac{\lambda}{p_n}\frac{Z_T}{B_T H_T}\right)_+ \ on\, \{H_T > 0\},$$

and, for $i \in [1, ..., n-1]$,

$$\phi_i^*(\lambda, Z_T, p) = \mathbb{1}_{A_i(\lambda,Z_T,p)} + \mathbb{1}_{A_i^c(\lambda,Z_T,p)}\left(1 - \left(\frac{1}{\sum_{j=i}^{n}p_j}\left(\frac{\lambda Z_T}{B_T H_T} - \sum_{j=i+1}^{n}p_j\right)\wedge 1\right)\right)\ on\, \{H_T > 0\}, \tag{4.13}$$

such that

$$A_i(\lambda, Z_T, p) = \left\{\frac{\lambda Z_T}{B_T H_T} \le \sum_{j=i+1}^{n}\sum_{k=j}^{n}p_k\right\}.$$

Then $\phi_i^* = \phi_i^*(\lambda^*, Z_T^*)$ is a solution to the problem (4.10) for a quadratic function $L(x) = \frac{1}{2}x^2$ such that

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}^f \ s.t.\ v_0 \le \bar{V}_0} E^P\left(\frac{1}{2}\left((H_T^{Total} - V_T)_+\right)^2\right) = E^P\left(\frac{1}{2}\sum_{i=1}^n p_i \left(H_T \sum_{j=1}^i (1-\phi_j^*)\right)^2\right).$$

Moreover,

$$\bar{V}_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \phi_i^*\right),$$

and the optimal hedging strategy $\xi^{SH,f}$ is the admissible financial strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^n \phi_i^*$. In particular, if $\mathcal{Q}^f = \{\tilde{Q}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$, then $Z_T^* = \tilde{Z}_T$ and $\xi^{SH,\, f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^n \phi_i^*$.

Proof. The result is deduced using the same similar arguments as for the proof of Proposition 4.2 and 4.3. □

From Proposition 4.3 and 4.4, we deduce that, the optimal financial shortfall hedging strategies of insurance-financial payoffs, for $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, are equivalent to (super)-hedging strategies of a sum of knock-out financial options that depend on a dual process $(Z_t^*)_{0 \le t \le T}$. If the financial market is complete, the optimal dual process is known and then, the optimal financial shortfall hedging strategies, for $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, are associated to a perfect hedging strategies of a modified financial payoff that can be readily implemented in practice.

5 Enlarging the optimal $\mathcal{F}^f-$adapted hedging strategies with actuarial information

In the previous sections, we succeeded in characterizing optimal strategies by considering either $\xi \in \mathcal{A}$ or $\xi \in \mathcal{A}^f$. Moreover, when the financial market is assumed to be complete and $\xi \in \mathcal{A}^f$, fully explicit expressions for the optimal strategies are deduced. However, by considering only $\xi \in \mathcal{A}^f$, we have a non-realistic situation where the insurer cannot measure the evolution of the number of survivors through time. In practice, the insurer can measure the number of survivors through time and can adapt its hedging strategy taking into account this actuarial information. In this case, we observe that it is, in general, difficult to derive explicitly the optimal strategies. In order to obtain explicit $\mathcal{F}-$adapted hedging strategies, we propose to enlarge the optimal $\mathcal{F}^f-$adapted strategies with actuarial information. The idea is to adjust the optimal strategies for $\xi \in \mathcal{A}^f$ each time a death occurs. In this case, we obtain $\mathcal{F}-$adapted hedging strategies which are more optimal than the optimal $\mathcal{F}^f-$adapted hedging strategies. This is summarized by the following propositions, for which the proofs are easily deduced by recursive procedures. Note that for the sake of consistency, we have only included the proof for the quantile strategy, but the proof for the shortfall strategies can be directly deduced using the same arguments.

Definition 5.1. For $i = 0, ..., n$, let us define τ_i as the $\mathcal{F}^a-$stopping time associated to the i^{th} death with the convention $\tau_0 := 0$, and $p^{\tau_i} := (p_0^{\tau_i}, ..., p_n^{\tau_i})$ such that

$$p_j^{\tau_i} := P(N_T = j|\mathcal{F}_{\tau_i}),\ i, j = 0, ..., n.$$

Proposition 5.1. *Let us assume that $\mathcal{Q}^f = \{\tilde{Q}^f\}$such that $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$. Let $(\tilde{\xi}_t)_{t\in[0,T]}$ be a hedging strategy associated to a hedging portfolio $(\tilde{V}_t)_{t\in[0,T]}$ with $\tilde{V}_0 = \bar{V}_0$ and let us consider τ, the $\mathcal{F}-$stopping time defined by*

$$\tau := \inf\left\{\tau \in \{\tau_0 \wedge T, ..., \tau_n \wedge T\} : \tilde{V}_\tau 1_{\tau<T} \ge 1_{\tau<T} E^{\tilde{Q}^f}(H_T \times N_T|\mathcal{F}_\tau)\right\}.$$

Assume that for $i = 0, ..., n-1$, and $\tau_i < \tau$, $(\tilde{\xi}_t)_{t\in[\tau_i, \tau_{i+1}\wedge\tau)}$ is the perfect hedging strategy of

$$H_T \times \sum_{j=1}^{N_{\tau_i}} \mathbb{1}_{A_j^*(\lambda_{\tau_i}, \tilde{Z}_T, p^{\tau_i})},$$

where $A_i^*(.)$ is given by (3.12) and λ_{τ_i} satisfied

$$\tilde{V}_{\tau_i -} = E^{\tilde{Q}_f}\left(H_T \times \sum_{j=1}^{N_{\tau_i}} \mathbb{1}_{A_j^*(\lambda_{\tau_i}, \tilde{Z}_T, p^{\tau_i})}|\mathcal{F}_{\tau_i}\right),$$

and $(\tilde{\xi}_t)_{t\in[\tau,T)}$ *is the perfect hedging strategy of*

$$H_T \times N_\tau.$$

Then,

$$\tilde{V}_T = \begin{cases} H_T \times \sum_{j=1}^{N_T} \mathbb{1}_{A_j^*(\lambda_{\tau_{n-N_T}}, \tilde{Z}_T, p^{\tau_{n-N_T}})} & if\ \tau = T, \\ H_T \times N_\tau + \frac{B_T}{B_\tau}\Big(\tilde{V}_{\tau-} - E^{\tilde{Q}_f}(H_T \times N_\tau | \mathcal{F}_\tau)\Big) & if\ \tau < T, \end{cases}$$

and

$$\max_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}^f\ s.t.\ v_0\leq\bar{V}_0} P(V_T \geq H_T^{Total}) \leq P(\tilde{V}_T \geq H_T^{Total}) \leq \max_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}\ s.t.\ v_0\leq\bar{V}_0} P(V_T \geq H_T^{Total}). \tag{5.1}$$

Proof. The proof is given in Section 8. □

Proposition 5.2. *Let us assume that* $\mathcal{Q}^f = \{\tilde{Q}^f\}$*such that* $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$*. Suppose that* $L(x) = x$ *or* $L(x) = \frac{1}{2}x^2$*and let* $(\tilde{\xi}_t)_{t\in[0,T]}$ *be a hedging strategy associated to a hedging portfolio* $(\tilde{V}_t)_{t\in[0,T]}$ *with* $\tilde{V}_0 = \bar{V}_0$ *and let us consider* τ*, the* $\mathcal{F}-$*stopping time defined by*

$$\tau := \inf\Big\{\tau \in \{\tau_0 \wedge T, ..., \tau_n \wedge T\} : \tilde{V}_\tau 1_{\tau<T} \geq 1_{\tau<T} E^{\tilde{Q}^f}(H_T \times N_T|\mathcal{F}_\tau)\Big\}.$$

Assume that for $i = 0, ..., n-1$*, and* $\tau_i < \tau$*,* $(\tilde{\xi}_t)_{t\in[\tau_i,\tau_{i+1}\wedge\tau)}$ *is the perfect hedging strategy of*

$$H_T^{\tau_i} := \begin{cases} H_T \times \sum_{j=1}^{N_{\tau_i}} \mathbb{1}_{A_j^*(\lambda_{\tau_i}, \tilde{Z}_T, p^{\tau_i})} & if\ L(x) = x, \\ H_T \times \sum_{j=1}^{N_{\tau_i}} \mathbb{1}_{\phi_j^*(\lambda_{\tau_i}, \tilde{Z}_T, p^{\tau_i})} & if\ L(x) = \frac{1}{2}x^2, \end{cases}$$

where $A_i^*(.)$ *is given by (4.12),* $\phi_i^*(.)$ *is given by (4.13), and* λ_{τ_i} *satisfied*

$$\tilde{V}_{\tau_i-} = E^{\tilde{Q}_f}(H_T^{\tau_i}|\mathcal{F}_{\tau_i}), \tag{5.2}$$

and $(\tilde{\xi}_t)_{t\in[\tau,T)}$ *is the perfect hedging strategy of*

$$H_T \times N_\tau.$$

Then,

$$\tilde{V}_T = \begin{cases} H_T^{\tau_{n-N_T}} & if\ \tau = T, \\ H_T \times N_\tau + \frac{B_T}{B_\tau}\Big(\tilde{V}_{\tau-} - E^{\tilde{Q}_f}(H_T \times N_\tau | \mathcal{F}_\tau)\Big) & if\ \tau < T, \end{cases}$$

and

$$\min_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}\ s.t.\ v_0\leq\bar{V}_0} E^P\Big(L((H_T^{Total} - V_T)_+)\Big) \leq E^P\Big(L((H_T^{Total} - \tilde{V}_T)_+)\Big) \leq \min_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}^f\ s.t.\ v_0\leq\bar{V}_0} E^P\Big(L((H_T^{Total} - V_T)_+)\Big).$$

6 Application in Black and Scholes model

The various optimal strategies studied in this paper, admit explicit characterizations that depend on the $T-$value of a dual process Z_T^*. We observe that, when we restrict ourselves to the set of admissible financial hedging strategies $\mathcal{A}^f$ and consider complete financial markets, then the $T-$value of the optimal dual process Z_T^* is known and the hedging strategies can therefore be applied in practice. Moreover, we show in Section 5, how to enlarge these optimal $\mathcal{F}^f-$adapted strategies with actuarial information in order to obtain explicit $\mathcal{F}-$adapted hedging strategies. In this section, we illustrate these different strategies through numerical results by considering a complete financial market. We observe that the strategies are very relevant for hedging equity-linked life insurance portfolio when we consider loss-averse insurers. For the sake of simplicity, we consider a simple Black and Scholes model for the financial market, but the strategies can be applied in more sophisticated models, such as, for example, local volatility models, fully-endogenous (rough) stochastic volatility models, or path-dependent volatility models. Thus, in this section, we consider a financial market composed of the two traded assets that satisfy, for $t \in [0, T]$, the following dynamics:

$$dB_t = rB_t\, dt,$$

and

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t,$$

on $(\Omega^f, \mathcal{F}^f, P^f)$, with $r \in \mathbb{R}$ the risk-free interest rate, $\mu > r$, $\sigma \in \mathbb{R}_+$ respectively the return and the volatility of the risky asset, and $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion defined on P^f. This financial market is arbitrage-free and complete. Therefore, we know that there exists a unique equivalent martingale measure denoted by $\tilde{Q}^f$ and characterized by the following change of measure, for $t \in [0, T]$,

$$\begin{aligned}\tilde{Z}_t &:= \frac{d\tilde{Q}^f}{dP^f}|_t \\ &= \exp\left(-\frac{(\mu - r)}{\sigma} W_t - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} t \right). \\ &= K \times S_t^{-(\mu - r)/\sigma^2}.\end{aligned}$$

As the Black and Scholes market is complete, we also know that for any contingent claim of the form $H_T = f(S_T) \in \mathcal{F}_T^f$, we can construct a hedging portfolio $(V_t^{BS})_{0\leq t\leq T}$ driven by a hedging strategy $(\xi_t^{BS})_{0\leq t\leq T}$ such, P^f-almost surely,

$$V_T^{BS} = f(S_T)$$

with for $t \in [0, T]$,

$$V_t^{BS} = E^{\tilde{Q}^f}\left(e^{-r(T-t)} H_T | \mathcal{F}_t^f \right)$$

and

$$\xi_t^{BS} = \frac{\partial V_t^{BS}}{\partial S_t}.$$

For the insurance market, we consider a constant mortality rate $\kappa_{x+t} = \kappa \in \mathbb{R}_+$ such that ${}_{T-t}p_{x+t} = \exp\left(-\kappa(T - t)\right)$, $N_T|\mathcal{F}_t \sim Bin(N_t, {}_{T-t}p_{x+t})$ and for $i = 0, ..., N_t$, $p_i^t = P(N_T = i|\mathcal{F}_t) = \binom{N_t}{i} ({}_{T-t}p_{x+t})^i \times (1 - {}_{T-t}\, p_{x+t})^{N_t - i}$. In the numerical results, we will vary the initial survival probability ${}_Tp_x$[5] in order to analyze its effect on the hedging strategies.

We assume that the insurer has sold a life insurance product guaranteeing a stochastic positive amount at maturity given by a call payoff

$$H_T = (S_T - K_{call})_+,$$

and at an initial price[6]

$${}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+),$$

In this case, the initial constraint on the value of the hedging portfolio is given by

$$\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+).$$

We now construct the efficient hedging strategies for this type of $n-$size portfolio in a Black and Scholes setting. To this end, we consider three different loss-averse insurers:

1. A risk-averse insurer who seeks to avoid as much as possible the events of losses, but who does not particularly intend to control the size of those losses. For this agent, we consider the quantile hedging approach.
2. An insurer who controls the size of the losses and minimizes the expected shortfall loss. For this agent, we consider the shortfall hedging approach.
3. An insurer who minimizes the expected shortfall loss while also controlling the variance of this loss. For this agent, we consider the quadratic shortfall hedging approach.

Note that for the numerical results, without loss of generality, we consider the following parameters for the Black and Scholes model:

$$S_0 = 100,\ r = 0.02,\ \mu = 0.06,\ \sigma = 0.2.$$

[5] Which is equivalent to vary the mortality rate $\kappa \in \mathbb{R}_+$.

[6] Without loss of generality, we assume that the insurer does not consider safety-loading for the mortality risk.

6.1 Quantile hedging

Let us consider the quantile hedging of a $n-$size life insurance portfolio with call payoff as stochastic maturity value. First, we can observe that in the particular setting considered for numerical illustration, the modified payoff associated to the optimal quantile hedging strategy when $\xi \in \mathcal{A}^f$ can be more explicitly decomposed which allows us to more easily interpreted this hedging strategy. The same holds for the $\mathcal{F}-$adapted quantile hedging strategy constructed in Section 5. In fact, if we restrict ourselves to $\mathcal{A}^f$, from Proposition 3.2, we know that the success sets A_i^*, $i = 1, ..., n$, are given by

$$
\begin{aligned}
A_i^* &= \left\{ \alpha_i(p) \geq \lambda^* \times \tilde{Z}_T \frac{H_T}{B_T} \right\} \\
&= \left\{ e^{rT} \times \alpha_i(p) \times \frac{S_T^{(\mu-r)/\sigma^2}}{K} \geq \lambda^* (S_T - K_{call})_+ \right\},
\end{aligned}
$$

In general, we cannot obtain a more explicit expression for the modified payoff $H_T \sum_{i=1}^n \mathbb{1}_{A_i^*}$. Nevertheless, assuming that $\mu - r = \sigma^2$ holds, we can rewrite A_i^*, $i = 1, ..., n$ such that

$$
A_i^* = \left\{ \alpha_i(p) \times \frac{e^{rT}}{K\lambda^*} \times S_T \geq (S_T - K_{call})_+ \right\}
$$

with λ^* satisfying

$$
\bar{V}_0 = E^{\tilde{Q}^f} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right).
$$

In this particular case, the modified payoff becomes

$$
H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*} = \sum_{i=1}^n (S_T - K_{call})_+ \times \left(\mathbb{1}_{\left\{ S_T \leq \frac{K_{call}}{1 - \frac{e^{rT}}{K\lambda^*} \times \alpha_i(p)} \right\} \cap \left\{ 1 \geq \frac{e^{rT}}{K\lambda^*} \times \alpha_i(p) \right\}} + \mathbb{1}_{\left\{ 1 < \frac{e^{rT}}{K\lambda^*} \times \alpha_i(p) \right\}} \right).
$$

We therefore observe that the modified payoff associated with the quantile hedging of the $n-$size portfolio with call option is equal to a sum of up-and-out call options with out strikes larger than K_{call}. Moreover, as in Black and Scholes model, the price of up-and-out call options admits a closed form, the initial value of the modified payoff as well as the optimal quantile hedging loss are obtained by closed formula, and we can therefore easily deduce the optimal constant λ^*. Note that, when $\mu - r \neq \sigma^2$, we can use a stochastic algorithm (see Appendix 9.3) to deduce the optimal constant λ^*. When we enlarge the optimal $\mathcal{F}^f-$adapted quantile strategy with actuarial information and consider in this sense the $\mathcal{F}-$adapted quantile strategy proposed in Section 5, using similar reasoning and Proposition 5.1, we observe that, the modified payoff associated to this hedging strategy is also a combination of up-and-out call options where the out-strikes are updated at each jump in the process $(N_t)_{t\in[0,T]}$. Nevertheless, the quantile hedging loss associated to this strategy cannot be computed directly with closed formula. It can be computed through Monte Carlo simulations, where the point process is simulated and the strategy is updated at each jump as explained in Proposition 5.1. More precisely, we have to adjust the out-strikes i.e. the constant λ_{τ_i} at each stopping time $\tau_i < T$, $i = 1, ...n$.

To illustrate the efficiency of the quantile hedging strategies with numerical results, we will benchmark them against the mean-variance financial hedging strategy that is well known in actuarial sciences (see for example [3, 11]). This benchmark strategy consists of a perfect hedge of the financial payoff given by $n \times {}_Tp_x \times H_T$. Note that the probability of successful hedge for the benchmark strategy can be easily deduced and is given by

$$
P(V_T^{Bench} \geq H_T^{Total}) = P(N_T \leq n \times {}_Tp_x)\, P(H_T > 0) + P(H_T = 0).
$$

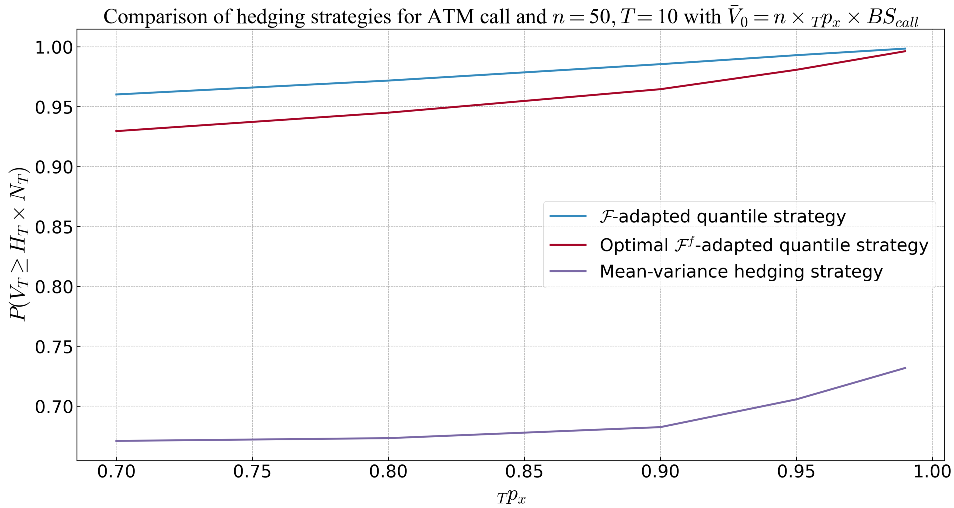

Figure 1: Success probability for the optimal quantile hedging strategy and the the benchmark strategy for ATM call payoff with respect to ${}_Tp_x$ with $n = 50$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

	$P(V_T \geq H_T^{Total})$		
${}_Tp_x$	$\mathcal{F}-$adapted quantile hedging	Optimal $\mathcal{F}^f-$adapted quantile hedging	Mean-variance hedging
0.99	0.9993	0.9964	0.7304
0.95	0.9909	0.9815	0.6789
0.9	0.9841	0.9712	0.6779

Table 1: Comparison of success probability for the quantile hedging strategies and the benchmark strategy for ATM call payoff with $n = 100$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

Figure 1 and Table 1 compare the probability of successful hedge with respect to the survival probability ${}_Tp_x$ for ATM call. We observe that, in terms of success probability, the quantile hedging strategies outperform the mean-variance hedging strategy. We observe that it is therefore in the insurer's interest to implement these strategies instead of the benchmark strategy if it wants to maximize its probability of successful hedge. The quantile strategies actually offer a better trade-off between actuarial and financial risks. Moreover, by comparing the two quantile hedging strategies proposed, we observe that there are a non-negligible gain from incorporating the actuarial information inside the trading strategy, especially when we consider low survival probabilities. The gain is less significant when higher survival probabilities are considered, but it is not zero and the insurer has an incentive to incorporate actuarial information into its strategy even for high survival probabilities.

We can also analyze the effect of the portfolio size n. Table 2 compares the effect of the size of the portfolio on the hedging strategy for a fixed survival probability ${}_Tp_x = 0.95$. Once again, we clearly observe that quantile hedging strategies always outperform the benchmark strategy in term of success probability. We also observe that by increasing the number of policyholders in the portfolio, the probability of successful hedge from the quantile strategies increases. We also notice that the gap between the $\mathcal{F}-$adapted strategy and the optimal strategy for $\xi \in \mathcal{A}^f$ slightly increases with the size of the portfolio. It is therefore in the insurer's interest to opt for the $\mathcal{F}-$adapted strategy when it has a large portfolio.

	$P(V_T \geq H_T^{Total})$		
n	$\mathcal{F}-$adapted quantile hedging	Optimal $\mathcal{F}^f-$adapted quantile hedging	Mean-variance hedging
50	0.9892	0.9808	0.6019
100	0.9902	0.9814	0.6751
300	0.9958	0.9871	0.6701

Table 2: Success probability for ATM call payoff with respect to n with ${}_Tp_x = 0.95$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

6.2 Shortfall hedging

Let us now consider that the insurer minimizes the expected shortfall of its $n-$size portfolio with call payoff. As for the quantile strategy, we can observe that, in the particular setting considered, the modified payoff associated to the optimal shortfall hedging strategy when $\xi \in \mathcal{A}^f$ can be more explicitly decomposed which also allows us to more easily interpreted this hedging strategy. When we restrict $\mathcal{A}$ to $\mathcal{A}^f$, from Proposition 4.3, the optimal sets A_i^*, for $i = 1, ..., n$, are given by

$$\begin{aligned} A_i^* &= \left\{ \sum_{j=i}^{n} p_j \geq \lambda^* \frac{\tilde{Z}_T}{B_T} \right\} \\ &= \left\{ \sum_{j=i}^{n} p_j \geq \frac{\lambda^* K}{e^{rT}} S_T^{-(\mu-r)/\sigma^2} \right\} \\ &= \left\{ S_T \geq K_i \right\}, \end{aligned}$$

with $m := \frac{\mu-r}{\sigma^2}$, $K_i := \frac{\lambda^* K e^{-rT}}{\left(\sum_{j=i}^{n} p_j\right)^{1/m}}$ for $i = 1, ..., n$ and λ^* satisfying

$$\bar{V}_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \sum_{i=1}^{n} \mathbb{1}_{A_i^*} \right).$$

Thus, the modified payoff becomes

$$\sum_{i=1}^{n} H_T 1_{A_i^*} = \sum_{i=1}^{n} (S_T - K_{call})_+ \times \mathbb{1}_{\{S_T \geq K_i\}}.$$

We observe that the modified payoff associated with the optimal expected shortfall hedging is decomposed as a sum of up-and-in call options. If we compare the modified payoff with that for quantile hedging, we can see that they are very similar. Instead of sum of up-and-out call options for the quantile hedging strategy, we have a sum of up-and-in call options for the shortfall hedging strategy. This is simply explained by the fact that having up-and-in options instead of up-and-out makes it possible to limit losses in the event of default, which is the aim of the shortfall strategy. When we consider the $\mathcal{F}-$adapted shortfall hedging strategy constructed in Section 5, using similar reasoning, the modified payoff can also be decomposed as a sum of up-and-in call options, where the in-strikes are adjusted at each jump in the process $(N_t)_{t\in[0,T]}$. Thus, in this case, the shortfall loss associated to this strategy needs, once again, to be computed through Monte Carlo simulations, where the point process is simulated and the strategy is adapted at each jump as explained in Proposition 5.2 and as it was done for the quantile hedging strategy.

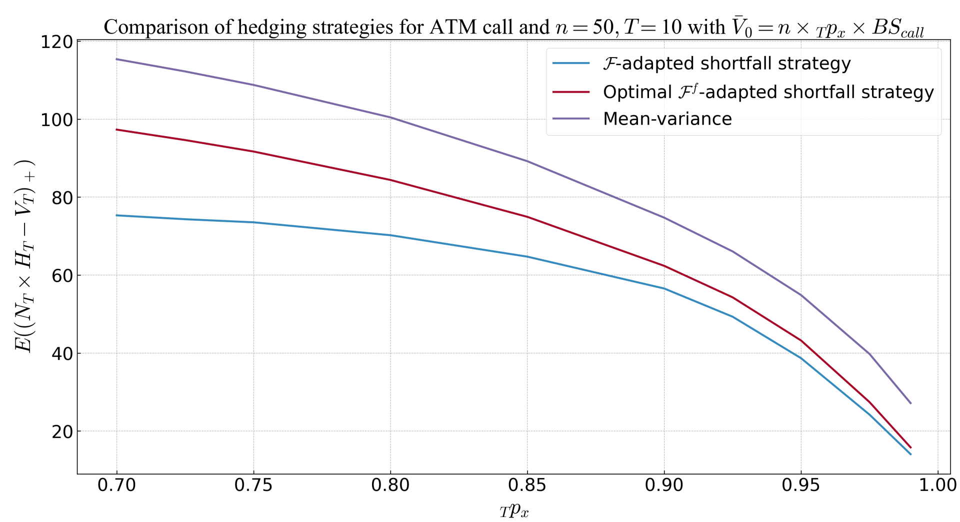

Figure 2: Expected shortfall for the optimal shortfall hedging strategy and the benchmark strategy for ATM call payoff with respect to ${}_Tp_x$ with $n = 50$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

	$E^P\left((H_T^{Total} - V_T)_+\right)$		
${}_Tp_x$	$\mathcal{F}-$adapted shortfall hedging	Optimal $\mathcal{F}^f-$adapted shortfall hedging	Mean-variance hedging
0.99	23.07	26.62	32.87
0.95	57.23	63.81	76.79
0.9	64.35	89.10	106.58

Table 3: Comparison of the expected shortfall for the optimal shortfall hedging strategy and the benchmark strategy for ATM call payoff with $n = 100$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

Let us now illustrate, with numerical results, the shortfall hedging strategies for a $n-$size life insurance portfolio with $H_T = (S_T - K_{call})_+$. We consider the same parameters as for the quantile hedging results. Figure 2 and Table 3 compare the expected shortfall with respect to the survival probability ${}_Tp_x$ for ATM call. We observe that, in terms of expected shortfall, the shortfall hedging strategies also outperform the benchmark strategy (mean-variance approach), and the gap is more pronounced for low values of ${}_Tp_{x.}$. We conclude that it is therefore in the insurer's interest to choose the shortfall hedging strategies instead of the benchmark strategy if it wants to minimize the expected shortfall loss. In addition, we once again notice an information gain by incorporating actuarial information in the shortfall strategy. This information gain is even more pronounced than in the case of quantile strategies.

We can also analyze the effect of the portfolio size n. Table 4 compares the effect of the size n of the portfolio on the hedging strategy for a fixed survival probability ${}_Tp_x = 0.95$ in terms of expected shortfall loss. We observe that by increasing the number of policyholders in the portfolio, the gap between the hedging strategies increases. An insurer with an aversion to shortfall losses should prefer the $\mathcal{F}-$adapted shortfall strategy, over the two other strategies, especially if it has a large portfolio.

	$E^P\left((H_T^{Total} - V_T)_+\right)$		
n	$\mathcal{F}-$adapted shortfall hedging	Optimal $\mathcal{F}^f-$adapted shortfall hedging	Mean-variance hedging
50	39.68	44.22	56.27
100	57.77	63.81	76.79
300	88.56	112.25	134.49

Table 4: Expected shortfall for ATM call payoff with respect to n with ${}_Tp_x = 0.95$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

6.3 Quadratic shortfall hedging

Finally, let us consider that the insurer minimizes the expected quadratic shortfall loss of its $n-$size portfolio. Unlike the two other strategies, it is not possible to obtain a more explicit form of modified payoffs when we consider quadratic shortfall hedging. To deduce the optimal constant λ^* that satisfies (4.8) (or λ_{τ_i} at each stopping time $\tau_i < T$, for the $\mathcal{F}-$adapted strategy that satisfies (5.2)), we use a stochastic algorithm (see Appendix 9.3), and the quadratic shortfall losses are computed via Monte Carlo method. This makes computation much more time consuming, especially when we simulate the P&L at maturity of the $\mathcal{F}-$adapted strategy, since at each jump time $\tau_i < T$, we have to perform a stochastic root finding to obtain the optimal value of λ_{τ_i}. Moreover, the smaller the ${}_Tp_x$ and the larger the portfolio size n, the more jumps there will be and, therefore, the higher the computation time. This is why, unlike the two other hedging strategies, we decide to only compare the quadratic shortfall strategies for high initial survival probabilities ${}_Tp_x > 0.9$ and relative low portfolio sizes $n < 50$. Let us now compare the quadratic shortfall hedging strategies with the mean-variance hedging strategy. To this end, we consider the same setting as before. Table 5 compares the expected quadratic shortfall with respect to the survival probability ${}_Tp_x$ for ATM call for fixed $n = 20$ and Table 6 shows the loss with respect to the portfolio size for fixed ${}_Tp_x = 0.95$. We observe that, in terms of quadratic shortfall loss, the quadratic shortfall strategies outperform the benchmark strategy. As for the two other hedging strategies, we conclude that it is therefore in the insurer's interest to implement the optimal strategies instead of the benchmark strategy if it wants to minimize the expected quadratic shortfall loss. Once gain, we also notice a gain in the quadratic loss when some actuarial information is incorporated into the hedging strategy, even if this gain decreases as ${}_Tp_x$ increases. Similarly, as the size of the portfolio increases, the gap between the optimal $\mathcal{F}^f-$adapted strategy and the $\mathcal{F}-$adapted strategy increases. Thus, the larger the size of the portfolio, the greater the insurer's interest in using the actuarial information to build its quadratic shortfall hedging strategy.

	$E^P\left(\frac{1}{2}(H_T^{Total} - V_T)_+^2\right)$		
${}_Tp_x$	$\mathcal{F}-$adapt. quad. shortfall hedging	Opt. $\mathcal{F}^f-$adapt. quad. shortfall hedging	Mean-variance hedging
0.99	49.12	55.12	371.22
0.95	851.62	1121.69	4067.61
0.9	2102.21	2643.55	8583.47

Table 5: Comparison of the expected quadratic shortfall for the optimal quadratic shortfall hedging strategy and the benchmark strategy for ATM call payoff with $n = 20$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

	$E^P\left(\frac{1}{2}(H_T^{Total} - V_T)_+^2\right)$		
n	$\mathcal{F}-$adapt. quad. shortfall hedging	Opt. $\mathcal{F}^f-$adapt. quad. shortfall hedging	Mean-variance hedging
50	3212.51	3619.01	11366.80
25	1284.86	1419.28	5182.42
10	327.53	372.24	1698.41

Table 6: Comparison of the expected quadratic shortfall for the optimal quadratic shortfall hedging strategy and the benchmark strategy for ATM call payoff with ${}_Tp_x = 0.95$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

6.4 Comparison of hedging strategies

Finally, we compare the different hedging strategies discussed in this paper using different metrics. To better analyze the quadratic shortfall hedging strategy, we can split the expected quadratic shortfall loss in terms of the quadratic mean and variance of the loss. Table 7 compares the different strategies using several metrics for fixed portfolio size $n = 25$ and survival probability ${}_Tp_x = 0.95$. Firstly, we clearly observe that even if the optimal quantile hedging strategy maximizes the probability of successful hedge, this strategy does not control the size of losses at all, and by following this strategy, the insurer is exposed to infrequent but very large losses, as revealed by comparing the expected shortfall and quadratic shortfall against other strategies. Next, the optimal shortfall strategy is effectively the one that minimizes the expected shortfall, but we can see that the variance of this shortfall can be significant. At last, we observe that, as expected, the quadratic hedging strategy appears to be the strategy leading to the best low mean-variance trade-off for the shortfall loss.

Hedging strategy	$P(\pi_T \geq 0)$	$E^P\left((-\pi_T)_+\right)$	$V^P\left((-\pi_T)_+\right)$	$E^P\left((-\pi_T)_+^2\right)$
$\mathcal{F}-$adapted quantile	**0.9886**	177.01	2 914 157.74	2 945 487.82
$\mathcal{F}-$adapted quad. shortfall	0.5643	28.88	**1735.32**	**2569.72**
$\mathcal{F}-$adapted shortfall	0.6465	**27.86**	3303.55	4079.72
Mean-variance	0.5286	39.25	8519.55	10 064.84

Table 7: Comparison of the hedging strategies for ATM call payoff with $\pi_T := V_T - H_T^{Total}$, ${}_Tp_x = 0.95$, $T = 10$, $n = 25$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

To get more insights about the distribution of the P&Ls generated by the hedging strategies, we can also analyze other metrics such as quantiles, kurtosis and skewness of the distributions. These metrics are provided at Table 8. First, we observe that quantile and shortfall hedging strategies lead to more asymmetric P&L distributions (left-skewed for the quantile strategy and right-skewed for the shortfall strategies) with fatter tails than the mean-variance hedging strategy. The P&L's distribution associated the quantile strategy has left fat tail indicating, once again, that although this strategy is the one that maximizes the probability of successful hedge, it leads to a risk of very large losses in the rare event of default. Regarding shortfall and quadratic shortfall hedging strategies, we observe that the P&L's distributions appear very similar which seems logical since they share the common goal of limiting the size of losses in the event of default and therefore explains the more pronounced right-skewed nature of their distribution.

Hedging strategy	$Q_{1\%}(\pi_T)$	$Q_{25\%}(\pi_T)$	$Q_{50\%}(\pi_T)$	$Q_{75\%}(\pi_T)$	$Q_{99\%}(\pi_T)$	$Kurtosis$[7]	$Skewness$[8]
$\mathcal{F}-$adapted quantile	-11799.19	0	0	52.31	485.92	118.43	-10.37
$\mathcal{F}-$adapted quad. shortfall	-133.04	-54.51	0	15.88	595.99	46.01	4.96
$\mathcal{F}-$adapted shortfall	-249.66	-32.11	0	12.53	580.02	30.83	3.97
Mean-variance	-464.35	-31.64	0	8.11	558.54	22.01	1.70

Table 8: Comparison of quantiles, kurtosis and skewness of P&Ls distributions for ATM call payoff with $\pi_T := V_T - H_T^{Total}$, ${}_Tp_x = 0.95$, $T = 10$, $n = 25$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$, where $Q_{\alpha\%}(.)$ is the quantile at level $\alpha\%$ under the probability measure P.

7 Conclusion

This paper investigates the efficient hedging of equity-linked life insurance portfolio. To this end, we consider loss-averse insurers and a general financial-actuarial framework. As the insurers are assumed to be loss-averse, their main goal is to minimize either the events leading to losses or the size of these losses. Therefore, we focus on optimal quantile and shortfall hedging strategies and posed the hedging problems as stochastic control problems. Based on the theory of super-replication and on a duality approach, we are able to characterize the optimal strategies with (semi)-explicit expressions that depend on a dual process. We show that these strategies depend not only on financial and actuarial risks but also on the size of the portfolio as well as the risk aversion of the insurers. When restricting ourselves to purely financial hedging strategies and complete financial markets, the optimal dual process is known and the forms of the strategies are fully explicit. Furthermore, in order to obtain hedging strategies in the combined market that could be easily implemented, we also construct $\mathcal{F}-$adapted strategies by enlarging the optimal $\mathcal{F}^f-$adapted strategies with actuarial information. Numerical results show satisfying results since, depending on the risk-aversion of the insurers, the strategies outperform the mean-variance hedging strategy. Hence, it demonstrates the relevance of using the right strategy for a given risk-aversion and portfolio size. We also observe that the $\mathcal{F}-$adapted strategies, by taking into account actuarial filtration, outperform the optimal $\mathcal{F}^f-$adapted strategies strategies, illustrating that there is a non-negligible gain to be obtained from incorporating actuarial information into hedging strategies.

8 Proofs

8.1 Theorem 3.1

Proof. 1) Let us denote a solution to the Problem (3.2) by $A_i^* \in \mathcal{F}_T$, $i = 1, ..., n$ such that $A_{i+1}^* \subseteq A_i^*$. For any admissible hedging strategy with $v_0 \leq \bar{V}_0$, we first observe that the associated success set is given by

$$\mathbb{1}_{\{V_T \geq H_T^{Total}\}} = 1_{N_T=0} + \sum_{i=1}^{n} \mathbb{1}_{\{V_T \geq i\, H_T\}} \mathbb{1}_{\{N_T = i\}}.$$

Moreover, we have that, $P-$almost surely,

$$V_T \geq H_T \times \sum_{i=1}^{n} \mathbb{1}_{\{V_T \geq i\, H_T\}}.$$

Therefore, for $A_i = \Big\{ V_T \geq iH_T \Big\} \in \mathcal{F}_T$ such that $A_{i+1} \subseteq A_i$, we observe that,

$$\bar{V}_0 \geq v_0 \geq \sup_{Q \in \mathcal{Q}} E^Q\left(\frac{V_T}{B_T}\right) \geq \sup_{Q \in \mathcal{Q}} E^Q\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right),$$

[7] $Kurtosis = E^P\left[\left(\frac{\pi_T - E^P(\pi_T)}{V^P(\pi_T)^{1/2}}\right)^4\right]$, measure of the tailedness of a distribution.

[8] $Skewness = E^P\left[\left(\frac{\pi_T - E^P(\pi_T)}{V^P(\pi_T)^{1/2}}\right)^3\right]$, measure of the asymmetry of a distribution.

for any $Q \in \mathcal{Q}$, since non-negative local $Q-$local martingale $\left(\frac{V_t}{B_t}\right)_{t\in[0,T]}$ is super martingale, this implies

$$E^P\left(\mathbb{1}_{\{V_T \geq H_T^{Total}\}}\right) = E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{\{N_T=i\}} + \mathbb{1}_{\{N_T=0\}}\right)$$
$$\leq E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{\{N_T=i\}}\right) + p_0 \tag{8.1}$$

and thus

$$\max_{v_0\in\mathbb{R}_+,\ \xi\in\mathcal{A}\ s.t.\ v_0\leq\bar{V}_0} P(V_T \geq H_T^{Total}) \leq E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{\{N_T=i\}}\right) + p_0. \tag{8.2}$$

2) Let now consider the hedging strategy $\xi^{QH} \in \mathcal{A}$ associated the (super) hedging for the modified payoff $H_T^{QH} = H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$ such that $\sup_{Q\in\mathcal{Q}} E^Q\left(\frac{H_T^{QH}}{B_T}\right) \leq \bar{V}_0$. In this case, if we denote $\bar{V}_T$ the portfolio with initial value $\bar{V}_0$ that evolves according to the hedging strategy ξ^{QH}, we have that

$$\bar{V}_T \geq H_T^{QH},$$

$P-$almost surely.
The corresponding maximal "success set" satisfies

$$E^P\left(\mathbb{1}_{\{\bar{V}_T\geq H_T^{Total}\}}\right) \geq E^P\left(\mathbb{1}_{\{H_T^{QH}\geq H_T^{Total}\}}\right)$$
$$= E^P\left(\mathbb{1}_{\{N_T=0\}} + \mathbb{1}_{\{N_T\geq 1\}}\mathbb{1}_{\{H_T\times\sum_{i=1}^n 1_{A_i^*}\geq N_T H_T\}}\right)$$
$$= p_0 + E^P\left(\sum_{i=1}^n \mathbb{1}_{\{N_T=i\}} \times \mathbb{1}_{\{H_T 1_{A_i^*}\geq H_T\}}\right)$$

For $i = 1, ..., n$, let define $\tilde{A}_i \in \mathcal{F}_T$ such that

$$\tilde{A}_i := \left\{H_T \mathbb{1}_{A_i^*} \geq H_T\right\} \supset A_i^*$$

and $\tilde{A}_{i+1} \subseteq \tilde{A}_i$.
Thus, we deduce that

$$E^P\left(\mathbb{1}_{\{\bar{V}_T\geq H_T^{Total}\}}\right) \geq E^P\left(\sum_{i=1}^n \mathbb{1}_{\tilde{A}_i} \times \mathbb{1}_{\{N_T=i\}}\right) + p_0 \tag{8.3}$$
$$\geq E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{\{N_T=i\}}\right) + p_0.$$

Combining (8.1), (8.2) and (8.3), we have that

$$\max_{v_0\in\mathbb{R}_+,\ \xi\in\mathcal{A}\ s.t.\ v_0\leq\bar{V}_0} P(V_T \geq H_T^{Total}) = E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{\{N_T=i\}}\right) + p_0$$
$$= E^P\left(\mathbb{1}_{\{\bar{V}_T\geq H_T^{Total}\}}\right),$$

and the optimality of the strategy $(\bar{V}_0, \xi^{QH})$ in Problem (2.3) follows. Finally, as ξ^{QH} is the hedging strategy associated to the super-replication of a positive payoff, we have that $\xi^{QH} \in \mathcal{A}$. □

8.2 Proposition 3.1

Proof. To prove this proposition, we use a duality approach as proposed, for example, in [14]. Let us define $\mathcal{C}$ by

$$\mathcal{C} := \left\{(A_1, ..., A_n) \in \mathcal{F}_T :\ A_{i+1} \subseteq A_i,\ \sup_{Q\in\mathcal{Q}} E^Q\left(\frac{H_T}{B_T}\sum_{i=1}^n \mathbb{1}_{A_i}\right) \leq \bar{V}_0\right\}.$$

We observe that, for any $(A_1, ..., A_n) \in \mathcal{C}$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, the following inequality holds

$$E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i}\right) + p_0 \le p_0 + E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T}\right)\right) + \lambda \bar{V}_0. \tag{8.4}$$

Moreover, for all $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, by taking the sets $(\tilde{A}_1(\lambda, Z_T), ..., \tilde{A}_n(\lambda, Z_T))$ such that

$$\tilde{A}_n(\lambda, Z_T) = \left\{\mathbb{1}_{N_T=n} \ge \lambda\, Z_T \frac{H_T}{B_T}\right\}, \tag{8.5}$$

and

$$\tilde{A}_i(\lambda, Z_T) = \left\{\mathbb{1}_{N_T=i} + \sum_{j=i+1}^n \mathbb{1}_{\tilde{A}_j(\lambda, Z_T)} \mathbb{1}_{N_T=j} \ge \lambda\, Z_T \frac{H_T}{B_T}\left(1 + \sum_{j=i+1}^n \mathbb{1}_{\tilde{A}_j(\lambda, Z_T)}\right)\right\}, \tag{8.6}$$

we observe that, for all $(A_1, ..., A_n) \in \mathcal{F}_T$,

$$\begin{aligned}
&E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T}\right)\right) \\
&\le E^P\left(\mathbb{1}_{\tilde{A}_1(\lambda, Z_T)}\left(\mathbb{1}_{N_T=1} - \lambda\, Z_T \frac{H_T}{B_T} + \mathbb{1}_{\tilde{A}_2(\lambda, Z_T)}\left(\mathbb{1}_{N_T=2} - \lambda\, Z_T \frac{H_T}{B_T} + ... + \mathbb{1}_{\tilde{A}_n(\lambda, Z_T)}\left(\mathbb{1}_{N_T=n} - \lambda\, Z_T \frac{H_T}{B_T}\right)...\right)\right)\right).
\end{aligned}$$

Thus by taking $A_1^*(\lambda, Z_T), ..., A_n^*(\lambda, Z_T) \in \mathcal{F}_T$ such that

$$A_i^*(\lambda, Z_T) := \bigcap_{j=1}^i \tilde{A}_j(\lambda, Z_T), \tag{8.7}$$

we obtain that $A_{i+1}^*(\lambda, Z_T) \subseteq A_i^*(\lambda, Z_T)$ and for any $(A_1, ..., A_n) \in \mathcal{F}_T$, $A_{i+1} \subseteq A_i$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$

$$E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T}\right)\right) \le E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)} \times \mathbb{1}_{N_T=i}\right) - \lambda\, E^P\left(Z_T \frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)}\right).$$

In this case, using inequality (8.4), we obtain that

$$\begin{aligned}
\max_{(A_1, ..., A_n) \in \mathcal{C}} E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i}\right) &\le \inf_{\lambda \in \mathbb{R}_+, Z_T \in \mathcal{D}} \max_{(A_1, ..., A_n) \in \mathcal{F}_T, A_{i+1} \subseteq A_i} E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T}\right)\right) + \lambda \bar{V}_0 \\
&= \inf_{\lambda \in \mathbb{R}_+, Z_T \in \mathcal{D}} E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T}\right)\right) + \lambda \bar{V}_0.
\end{aligned}$$

Furthermore, since $\mathcal{D}$ is convex and closed under almost sure convergence (see Theorem 7.2. in [14]), we know that there exists a solution $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ such that

$$\inf_{\lambda \in \mathbb{R}_+, Z_T \in \mathcal{D}} E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T}\right)\right) + \lambda \bar{V}_0 = E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*, Z_T^*)} \times \left(\mathbb{1}_{N_T=i} - \lambda^* Z_T^* \frac{H_T}{B_T}\right)\right) + \lambda^* \bar{V}_0,$$

and[9]

$$\bar{V}_0 = \sup_{Z_T \in \mathcal{D}} E^P\left(Z_T \frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*, Z_T)}\right) = E^P\left(Z_T^* \frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*, Z_T^*)}\right). \tag{8.8}$$

The previous equality (8.8) implies that

$$\bar{V}_0 \ge \sup_{Q \in \mathcal{Q}} E^Q\left(\frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*, Z_T^*)}\right),$$

[9] The constraint $\bar{V}_0 \le E^{\tilde{Q}^f}\left(n \times \frac{H_T}{B_T}\right)$ ensures that $\lambda^* \in \mathbb{R}_+$.

and as $\Big(A_i^*(\lambda^*, Z_T^*)\Big)_{i=1,..,n} \in \mathcal{F}_T$ such that $A_{i+1}^*(\lambda, Z_T) \subseteq A_i^*(\lambda, Z_T)$, we have that $\Big(A_i^*(\lambda^*, Z_T^*)\Big)_{i=1,..,n} \in \mathcal{C}$. We finally obtain that there is no duality gap since

$$\max_{(A_1,\dots,A_n)\in\mathcal{C}} E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i}\right) + p_0 = E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*,Z_T^*)} \times \mathbb{1}_{N_T=i}\right) + p_0.$$

Therefore, $(A_1^*, ..., A_n^*)$ defined as, for $i = 1, ..., n$,

$$A_i^* := A_i^*(\lambda^*, Z_T^*),$$

is solution of (3.2). Furthermore, using together (8.5), (8.6) and (8.7), we can easily deduce that, for $i = 1, ..., n$,

$$A_i^* = \left\{\alpha_i \geq \lambda^* \times Z_T^* \frac{H_T}{B_T}\right\},$$

where $(\alpha_1, .., \alpha_n)$ are defined by (3.6) and (3.7). In addition, the optimal hedging strategy ξ^{QH} is the admissible strategy associated to the super-replication hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. □

8.3 Theorem 4.1

Proof. 1) Let us denote a solution to the Problem (4.2) by $\phi_i^* \in \mathcal{R}$, $i = 1, ..., n$ such that $\phi_{i+1}^* \leq \phi_i^*$. Firstly, for any admissible hedging strategy with $v_0 \leq \bar{V}_0$, we can observe that

$$\begin{aligned}(H_T^{Total} - V_T)_+ &= H_T N_T - \min\Big\{V_T, N_T H_T\Big\} \\ &= \mathbb{1}_{\{N_T\geq 1\}} \sum_{i=1}^{N_T} H_T - \sum_{i=1}^n \min\Big\{V_T, iH_T\Big\}\mathbb{1}_{\{N_T=i\}}.\end{aligned}$$

For $i = 1, ..., n$, we postulate that

$$\Phi_i H_T := \min\Big\{V_T, iH_T\Big\},$$

with

$$\Phi_i \in \Big\{\phi : \Omega \to [0, i],\ \mathcal{F}_T - measurable\Big\}$$

such that $P-$almost surely

$$\Phi_{i+1} \geq \Phi_i,$$

and then

$$(H_T^{Total} - V_T)_+ = \mathbb{1}_{\{N_T\geq 1\}} \sum_{i=1}^{N_T} H_T - \sum_{i=1}^n \Phi_i H_T \mathbb{1}_{\{N_T=i\}}.$$

Moreover, for $i = 1, ..., n$, we observe that, for ϕ_i defined by (4.1),

$$(\Phi_i - \Phi_{i-1})H_T = \phi_i H_T$$

such that

$$\phi_i \in \Big\{\phi : \Omega \to [0, 1],\ \mathcal{F}_T - measurable\Big\}$$

and $P-$almost surely

$$\phi_{i+1} \leq \phi_i.$$

Thus, for $i = 1, ..., n$, we can rewrite Φ_i as

$$\Phi_i = \sum_{j=1}^i \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}},$$

with $\phi_0 := 1$. In this case, we deduce that

$$
\begin{aligned}
L\Big((H_T^{Total} - V_T)_+\Big) &= L\Big(\mathbb{1}_{\{N_T\geq 1\}}\sum_{i=1}^{N_T} H_T - \sum_{i=1}^{n}\Phi_i H_T \mathbb{1}_{\{N_T=i\}}\Big)\\
&= L\Big(\mathbb{1}_{\{N_T\geq 1\}}\sum_{i=1}^{N_T} H_T(1-\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}})\Big).
\end{aligned}
$$

Moreover, as we have that

$$
\min\Big\{V_T, H_T N_T\Big\} = \mathbb{1}_{\{N_T\geq 1\}}\sum_{i=1}^{N_T} H_T\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}},
$$

then

$$
V_T \geq \sum_{i=1}^{n} H_T\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}},
$$

and we observe that

$$
\bar{V}_0 \geq v_0 \geq \sup_{Q\in\mathcal{Q}} E^Q\Big(\frac{V_T}{B_T}\Big) \geq \sup_{Q\in\mathcal{Q}} E^Q\Big(\frac{H_T}{B_T}\sum_{i=1}^{n}\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}}\Big),
$$

for any $Q\in\mathcal{Q}$, since non-negative local $Q-$local martingale $\left(\frac{V_t}{B_t}\right)_{t\in[0,T]}$ is super martingale. This implies that

$$
\begin{aligned}
E^P\Big(L\Big((H_T^{Total} - V_T)_+\Big)\Big) &\geq E^P\Big(L\Big(\mathbb{1}_{\{N_T\geq 1\}}\sum_{i=1}^{N_T} H_T(1-\phi_i^*\mathbb{1}_{\{\phi_{i-1}^*=1\}})\Big)\Big)\\
&= E^P\Big(\sum_{i=1}^{n}\mathbb{1}_{\{N_T=i\}}\times L\Big(\sum_{j=1}^{i} H_T(1-\phi_j^*\mathbb{1}_{\{\phi_{j-1}^*=1\}})\Big)\Big),
\end{aligned}
$$

and thus

$$
\min_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}\ s.t.\ v_0\leq\bar{V}_0} E^P\Big(L\Big((H_T^{Total} - V_T)_+\Big)\Big) \geq E^P\Big(\sum_{i=1}^{n}\mathbb{1}_{\{N_T=i\}}\times L\Big(\sum_{j=1}^{i} H_T(1-\phi_j^*\mathbb{1}_{\{\phi_{j-1}^*=1\}})\Big)\Big). \tag{8.9}
$$

2) Let now consider the hedging strategy $\xi^{SH}\in\mathcal{A}$ associated to the (super) replicating hedging of the modified payoff $H_T^{SH} = H_T\times\sum_{i=1}^{n}\phi_i^*\mathbb{1}_{\{\phi_{i-1}^*=1\}}$ such that $\sup_{Q\in\mathcal{Q}} E^Q\left(\frac{H_T^{QH}}{B_T}\right)\leq\bar{V}_0$. In this case, if we denote $\bar{V}_T$ the portfolio with initial value $\bar{V}_0$ that evolves according to the hedging strategy ξ^{SH}, we have that

$$
\bar{V}_T \geq H_T^{SH},
$$

$P-$almost surely.
The expected shortfall loss of this hedging strategy satisfies

$$
\begin{aligned}
E^P\Big(L\Big((H_T^{Total} - \bar{V}_T)_+\Big)\Big) &\leq E^P\Big(L\Big((H_T^{Total} - H_T^{SH})_+\Big)\Big) \qquad (8.10)\\
&= E^P\Big(L\Big(\mathbb{1}_{\{N_T\geq 1\}}\sum_{i=1}^{N_T} H_T(1-\phi_i^*\mathbb{1}_{\{\phi_{i-1}^*=1\}})\Big)\Big)\\
&= E^P\Big(\sum_{i=1}^{n}\mathbb{1}_{\{N_T=i\}}\times L\Big(\sum_{j=1}^{i} H_T(1-\phi_j^*\mathbb{1}_{\{\phi_{i-1}^*=1\}})\Big)\Big)
\end{aligned}
$$

Combining (8.9) and (8.10), we have that

$$
\begin{aligned}
\min_{V_0,\, \xi\in\mathcal{A}\ s.t.\ V_0\leq\bar{V}_0} E^P\Big(L\Big((H_T^{Total} - V_T)_+\Big)\Big) =& E^P\Big(\sum_{i=1}^{n}\mathbb{1}_{\{N_T=i\}}\times L\Big(\sum_{j=1}^{i} H_T(1-\phi_j^*\mathbb{1}_{\{\phi_{i-1}^*=1\}})\Big)\Big)\\
=& E^P\Big(L\Big((H_T^{Total} - \bar{V}_T)_+\Big)\Big),
\end{aligned}
$$

and the optimality of the strategy $(\bar{V}_0, \xi^{SH})$ in Problem (2.4) follows. Finally, as ξ^{SH} is the hedging strategy associated to the super-replication of a positive payoff, we conclude that $\xi^{SH}\in\mathcal{A}$ $\square$

8.4 Proposition 4.1

Proof. To prove this proposition, we use once again a duality approach. Let us define $\mathcal{C}$ by

$$\mathcal{C} := \left\{ \phi_1, ..., \phi_n \in \mathcal{R} : \ \phi_{i+1} \leq \phi_i, \ \phi_0 = 1, \ \sup_{Q \in \mathcal{Q}} E^Q \left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \right) \leq \bar{V}_0 \right\}.$$

We observe that, for $L(x) = x$, for any $(\phi_1, ..., \phi_n) \in \mathcal{C}$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, the following inequality holds

$$\begin{aligned} & E^P \left(\sum_{i=1}^n \mathbb{1}_{N_T = i} \times \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right) \right) \\ & \geq E^P \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times i H_T \right) + E^P \left(H_T \left(\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) \right) - \lambda \bar{V}_0. \end{aligned} \tag{8.11}$$

Moreover, for $\phi_1, ..., \phi_n \in \mathcal{R}$ with $\phi_0 = 1$ and $\phi_{i+1} \leq \phi_i$, for any $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, we have that

$$\begin{aligned} & E^P \left(H_T \left(\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) \right) \\ \geq & E^P \left(H_T \left(\mathbb{1}_{A_1(\lambda, Z_T)} \left(\lambda \frac{Z_T}{B_T} - \sum_{i=1}^n \mathbb{1}_{N_T=i} + \mathbb{1}_{A_2(\lambda, Z_T)} \left(\lambda \frac{Z_T}{B_T} - \sum_{i=2}^n \mathbb{1}_{N_T=i} + ... + \mathbb{1}_{A_n(\lambda, Z_T)} (\lambda \frac{Z_T}{B_T} - \mathbb{1}_{N_T=n}) ... \right) \right) \right) \\ = & E^P \left(H_T \sum_{i=1}^n \phi_i^* (\lambda, Z_T) \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right), \end{aligned}$$

with

$$A_n(\lambda, Z_T) = \left\{ \mathbb{1}_{N_T=n} \geq \lambda \, \frac{Z_T}{B_T} \right\}, \tag{8.12}$$

for $i = 1, ..., n-1$,

$$A_i(\lambda, Z_T) = \left\{ \sum_{j=i}^n \mathbb{1}_{N_T=j} + \sum_{j=i+1}^n \mathbb{1}_{A_j(\lambda)} \left(\sum_{k=j}^n \mathbb{1}_{N_T=k} \right) \geq \lambda \left(1 + \sum_{j=i+1}^n \mathbb{1}_{A_j(\lambda)} \right) \frac{Z_T}{B_T} \right\}, \tag{8.13}$$

such that

$$A_i(\lambda, Z_T) \subseteq A_{i-1}(\lambda, Z_T),$$

and

$$\phi_i^*(\lambda, Z_T) = \prod_{j=1}^i \mathbb{1}_{A_j(\lambda, Z_T)} \tag{8.14}$$

Therefore, for $\lambda \in \mathbb{R}_+$, and $Z_T \in \mathcal{D}$ since $\phi_i^*(\lambda, Z_T) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda, Z_T) \leq \phi_i^*(\lambda, Z_T)$, we easily deduce that

$$\begin{aligned} & \min_{\phi_1, ..., \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i, \, \phi_0 = 1} E^P \left(H_T \left(\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) \right) \\ & \qquad = E^P \left(H_T \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right). \end{aligned}$$

Thus, we obtain the following inequality

$$\begin{aligned} & \min_{(\phi_1, ..., \phi_n) \in \mathcal{C}} E^P \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right) \right) \\ \geq & E^P \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times i H_T \right) + \sup_{\lambda \geq 0, \, Z_T \in \mathcal{D}} \left(E^P \left(H_T \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) - \lambda \bar{V}_0 \right). \end{aligned}$$

Furthermore, since $\mathcal{D}$ is convex and closed under almost sure convergence (see Theorem 7.2. in [14]), we know that there exists a solution $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ such that

$$\sup_{\lambda\geq 0,\, Z_T\in\mathcal{D}} E^P\left(H_T \sum_{i=1}^n \phi_i^*(\lambda, Z_T)\left(\lambda\frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j}\right)\right) - \lambda\bar{V}_0 = E^P\left(H_T \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*)\left(\lambda^*\frac{Z_T^*}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j}\right)\right) - \lambda^*\bar{V}_0,$$

and[10]

$$\bar{V}_0 = \sup_{Z_T\in\mathcal{D}} E^P\left(Z_T\frac{H_T}{B_T}\sum_{i=1}^n \phi_i^*(\lambda^*, Z_T)\right) = E^P\left(Z_T^*\frac{H_T}{B_T}\sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*)\right). \tag{8.15}$$

The previous equality (8.15) implies that

$$\bar{V}_0 \geq \sup_{Q\in\mathcal{Q}} E^Q\left(\frac{H_T}{B_T}\sum_{i=1}^n \mathbb{1}_{\phi_i^*(\lambda^*, Z_T^*)}\right),$$

and as $\phi_i^*(\lambda^*, Z_T^*) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda^*, Z_T^*) \leq \phi_i^*(\lambda^*, Z_T^*)$, we have that $\left(\phi_i^*(\lambda^*, Z_T^*)\right)_{i=1,..,n} \in \mathcal{C}$. We finally obtain that there is no duality gap since

$$\min_{(\phi_1,\ldots,\phi_n)\in\mathcal{C}} E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \left(\sum_{j=1}^i H_T(1-\phi_j\mathbb{1}_{\{\phi_{j-1}=1\}})\right)\right) = E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \left(\sum_{j=1}^i H_T(1-\phi_j^*(\lambda^*, Z_T^*))\right)\right).$$

Therefore, $(\phi_1^*, ..., \phi_n^*)$ defined as, for $i = 1, ..., n$,

$$\phi_i^* := \phi_i^*(\lambda^*, Z_T^*),$$

is solution of (4.2) for $L(x) = x$. Finally, using (8.12), (8.13) and (8.14), we can deduce that, for $i = 1, ..., n$, $\phi_i^* = \mathbb{1}_{A_i^*}$ with A_i^* defined by (4.5) for $\lambda = \lambda^*$ and $Z_T = Z_T^*$. Therefore, we have that

$$\min_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}\ s.t.\ v_0\leq\bar{V}_0} E^P\left((H_T^{Total} - V_T)_+\right) = E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i}\left(\sum_{j=1}^i (1-\mathbb{1}_{A_j^*})H_T\right)\right),$$

and the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. ☐

8.5 Proposition 4.2

Proof. Once again, let us take $\mathcal{C}$ defined by

$$\mathcal{C} := \left\{\phi_1, ..., \phi_n \in \mathcal{R} :\ \phi_{i+1} \leq \phi_i,\ \phi_0 = 1,\ \sup_{Q\in\mathcal{Q}} E^Q\left(\frac{H_T}{B_T}\sum_{i=1}^n \phi_i\right) \leq \bar{V}_0\right\}.$$

We observe that, for $L(x) = \frac{1}{2}x^2$, for any $(\phi_1, ..., \phi_n) \in \mathcal{C}$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, the following inequality holds

$$\begin{aligned} &E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1-\phi_j\mathbb{1}_{\{\phi_{j-1}=1\}})\right)^2\right) \\ &\geq E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1-\phi_j\mathbb{1}_{\{\phi_{j-1}=1\}})\right)^2 + \lambda Z_T\frac{H_T}{B_T}\sum_{i=1}^n \phi_i\right) - \lambda\bar{V}_0. \end{aligned} \tag{8.16}$$

Moreover, for $(\phi_1, ..., \phi_n) \in \mathcal{C}$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, we have that

$$\begin{aligned} &E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1-\phi_j\mathbb{1}_{\{\phi_{j-1}=1\}})\right)^2 + \lambda Z_T\frac{H_T}{B_T}\sum_{i=1}^n \phi_i\right) \\ &\geq \min_{\phi_1,\ldots,\phi_n\in\mathcal{R},\, \phi_{i+1}\leq\phi_i} E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1-\phi_j\mathbb{1}_{\{\phi_{j-1}=1\}})\right)^2 + \lambda Z_T\frac{H_T}{B_T}\sum_{i=1}^n \phi_i\right) \\ &= \min_{\phi_1,\ldots,\phi_n\in\mathcal{R},\, \phi_{i+1}\leq\phi_i} \mathcal{L}(\phi_1, ..., \phi_n; \lambda, Z_T). \end{aligned}$$

[10] The constraint $\bar{V}_0 \leq E^{\tilde{Q}^f}\left(n \times \frac{H_T}{B_T}\right)$ ensures that $\lambda^* \in \mathbb{R}_+$.

For any $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, since $\mathcal{L}$ is convex, using a first order condition, if there exists $\phi_1^*(\lambda, Z_T), ..., \phi_n^*(\lambda, Z_T) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda, Z_T) \leq \phi_i^*(\lambda, Z_T)$ that satisfy

$$\frac{\partial \mathcal{L}(\phi_1^*(\lambda, Z_T), ..., \phi_n^*(\lambda, Z_T); \lambda, Z_T)}{\partial \phi_i} = 0, \ i = 1, ..., n, \tag{8.17}$$

then

$$\min_{\phi_1, ..., \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i, \ \phi_0 = 1} \mathcal{L}(\phi_1, ..., \phi_n; \lambda, Z_T) = \mathcal{L}(\phi_1^*(\lambda, Z_T), ..., \phi_n^*(\lambda, Z_T); \lambda, Z_T).$$

We observe that, for $\phi_1^*(\lambda, Z_T), ..., \phi_n^*(\lambda, Z_T) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda, Z_T) \leq \phi_i^*(\lambda, Z_T)$, the condition (8.17) is equivalent, for $i = 1, ..., n$, to

$$E^P\left(H_T \mathbb{1}_{\{\phi_{i-1}^*(\lambda, Z_T)=1\}} \mathbb{1}_{\{H_T \phi_i^*(\lambda, Z_T)>0\}} \left(\sum_{j=i}^n \mathbb{1}_{N_T=j} \times \left(H_T \sum_{k=1}^j (1 - \phi_k^*(\lambda, Z_T) \mathbb{1}_{\{\phi_{k-1}^*(\lambda, Z_T)=1\}}\right) - \lambda \frac{Z_T}{B_T}\right)\right) = 0. \tag{8.18}$$

We can check that $\phi_1^*(\lambda, Z_T), ..., \phi_n^*(\lambda, Z_T)$ defined such as

$$\phi_n^*(\lambda, Z_T) := \mathbb{1}_{\left\{\frac{\lambda Z_T}{B_T H_T} \leq \mathbb{1}_{\{N_T=n\}}\right\}} \left(1 - \frac{\lambda}{\mathbb{1}_{\{N_T=n\}}} \frac{Z_T}{B_T H_T}\right) \text{ on } \{H_T > 0\}$$

and for $i \in [1, ..., n-1]$,

$$\phi_i^*(\lambda, Z_T) := 1 - \frac{1}{\sum_{j=i}^n \mathbb{1}_{N_T=j}} \left(\frac{\lambda Z_T}{B_T H_T} - \sum_{j=i+1}^n \left((1 - \phi_j^*(\lambda, Z_T)) \times \sum_{k=j}^n \mathbb{1}_{N_T=k}\right)\right) \wedge 1\right) \text{ on } \{H_T > 0\}, \tag{8.19}$$

solve the system of $n-$equations (8.18) with $\phi_0^* = 1$. From the inequality (8.16), we thus have that

$$\begin{aligned}
&\min_{(\phi_1, .., \phi_n) \in \mathcal{C}} E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\right)^2\right) \\
&\geq \sup_{\lambda \geq 0, \ Z_T \in \mathcal{D}} E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1 - \phi_j^*(\lambda, Z_T) \mathbb{1}_{\{\phi_{j-1}^*(\lambda, Z_T)=1\}})\right)^2 + \lambda Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda, Z_T)\right) - \lambda \bar{V}_0.
\end{aligned}$$

Furthermore, since $\mathcal{D}$ is convex and closed under almost sure convergence (see Theorem 7.2. in [14]), we know that there exists a solution $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ such that

$$\begin{aligned}
&\sup_{\lambda \geq 0, \ Z_T \in \mathcal{D}} E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1 - \phi_j^*(\lambda, Z_T) \mathbb{1}_{\{\phi_{j-1}^*(\lambda, Z_T)=1\}})\right)^2 + \lambda Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda, Z_T)\right) - \lambda \bar{V}_0 \\
&= E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1 - \phi_j^*(\lambda^*, Z_T^*) \mathbb{1}_{\{\phi_{j-1}^*(\lambda^*, Z_T^*)=1\}})\right)^2 + \lambda^* Z_T^* \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*)\right) - \lambda^* \bar{V}_0
\end{aligned}$$

and

$$\bar{V}_0 = \sup_{Z_T \in \mathcal{D}} E^P\left(Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T)\right) = E^P\left(Z_T^* \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*)\right). \tag{8.20}$$

The previous equality (8.20) implies that

$$\bar{V}_0 \geq \sup_{Q \in \mathcal{Q}} E^Q\left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*)\right),$$

and as $\phi_1^*(\lambda^*, Z_T^*), ..., \phi_n^*(\lambda^*, Z_T^*) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda^*, Z_T^*) \leq \phi_i^*(\lambda^*, Z_T^*)$, we have that $\left(\phi_1^*(\lambda^*, Z_T^*), ..., \phi_n^*(\lambda^*, Z_T^*)\right) \in \mathcal{C}$. We finally obtain that there is no duality gap and

$$\begin{aligned}
&\min_{(\phi_1, .., \phi_n) \in \mathcal{C}} E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\right)^2\right) \\
&= E^P\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2}\left(\sum_{j=1}^i H_T(1 - \phi_{j-1}^*(\lambda^*, Z_T^*) \mathbb{1}_{\{\phi_{j-1}^*(\lambda^*, Z_T^*)=1\}})\right)^2\right).
\end{aligned}$$

Therefore, $(\phi_1^*, ..., \phi_n^*)$ defined as, for $i = 1, ..., n$,

$$\phi_i^* := \phi_i^*(\lambda^*, Z_T^*),$$

is solution of (4.2) for $L(x) = \frac{1}{2}x^2$. Finally, by using the recursive relation (8.19) and the fact that, for $i = 1, ..., n-1$:

- if $\phi_i^* = 1$, then $\phi_{i-1}^* = 1$,
- if $0 < \phi_i^* < 1$ then $\phi_{i-1}^* = 1$ and $\phi_j^* = 0$ for $j = i+1, .., n$,
- if $\phi_i^* = 0$ then $\phi_{i-1}^* < 1$,

we can deduce that for $i = 1, ..., n-1$, ϕ_i^* satisfy (4.7) for $\lambda = \lambda^*$ and $Z_T = Z_T^*$. □

8.6 Proposition 5.1

Proof. First, it is easy to see that $(\tilde{\xi}_t)_{t\in[0,T]} \in \mathcal{A}$, and then we obviously observe that

$$P(\tilde{V}_T \geq H_T^{Total}) \leq \max_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A} \ s.t.\ v_0\leq\bar{V}_0} P(V_T \geq H_T^{Total}).$$

Let us now prove that

$$\max_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}^f \ s.t.\ v_0\leq\bar{V}_0} P(V_T \geq H_T^{Total}) \leq P(\tilde{V}_T \geq H_T^{Total}).$$

Using similar arguments that used in Section 3.2 and as done for example in [13, 14], we can deduce that, for any $t \in [0,T]$ and $n \in \mathbb{N}_0$ such that $N_t = n$,

$$\max_{V_t\in\mathbb{R}_+,\, \xi\in\mathcal{A}_t^f \ s.t.\ V_t\leq\bar{V}_t} P(V_T \geq H_T^{Total}|\mathcal{F}_t) = P(V_T^{f,t} \geq H_T^{Total}|\mathcal{F}_t), \tag{8.21}$$

where $\mathcal{A}_t^f$ is the set of all admissible financial hedging strategies valued from t to T in $\mathbb{R}$, and $(V_s^{f,t})_{s\in[t,T]}$, such that $V_t^{f,t} = \bar{V}_t$, is the hedging portfolio associated to the perfect hedging of the modified payoff

$$H_T \times \sum_{j=1}^n \mathbb{1}_{A_j^*(\lambda_t, \tilde{Z}_T, p^t)},$$

where $p^t = (p_1^t, ..., p_n^t)$ and λ_t satisfied

$$V_t^{f,t} \wedge E^{\tilde{Q}_f}(H_T \times n|\mathcal{F}_t) = E^{\tilde{Q}_f}\left(H_T \times \sum_{j=1}^n \mathbb{1}_{A_j^*(\lambda_t, \tilde{Z}_T, p^t)}|\mathcal{F}_t\right).$$

We can now prove the statement using a recursive approach. If $n = 1$, then we observe that

$$\max_{v_0\in\mathbb{R}_+,\, \xi\in\mathcal{A}^f \ s.t.\ v_0\leq\bar{V}_0} P(V_T \geq H_T^{Total}) = P(\tilde{V}_T \geq H_T^{Total}).$$

If $n = 2$, we have that

$$\begin{aligned} P(V_T^f \geq H_T^{Total}) &= E^P\left(\mathbb{1}_{\left\{N_T=2\right\}}\mathbb{1}_{\left\{V_T^f\geq 2H_T\right\}}\right) + E^P\left(\mathbb{1}_{\left\{N_T<2\right\}}\mathbb{1}_{\left\{V_T^f\geq N_T H_T\right\}}\right) \\ &= E^P\left(\mathbb{1}_{\left\{N_T=2\right\}}\mathbb{1}_{\left\{\tilde{V}_T\geq 2H_T\right\}}\right) + E^P\left(\mathbb{1}_{\left\{N_T<2\right\}}E^P\left(\mathbb{1}_{\left\{V_T^f\geq N_T H_T\right\}}|\mathcal{F}_{\tau_1}\right)\right). \end{aligned}$$

Using (8.21) and the definition of the hedging strategy $\tilde{\xi}$ in Proposition 5.1 with $n = 2$, we observe that

$$E^P\left(\mathbb{1}_{\left\{V_T^f\geq N_T H_T\right\}}|\mathcal{F}_{\tau_1}\right) \leq \max_{V_{\tau_1}\in\mathbb{R}_+,\, \xi\in\mathcal{A}_{\tau_1}^f \ s.t.\ V_{\tau_1}\leq\tilde{V}_{\tau_1}} P(V_T \geq H_T^{Total}|\mathcal{F}_{\tau_1}) = E^P\left(\mathbb{1}_{\left\{\tilde{V}_T\geq N_T H_T\right\}}|\mathcal{F}_{\tau_1}\right), \text{ on } \{N_T < 2\},$$

and thus

$$\max_{v_0\in\mathbb{R}_+,\,\xi\in\mathcal{A}^f\ s.t.\ v_0\le\bar{V}_0} P(V_T\ge H_T^{Total}) = P(V_T^f\ge H_T^{Total}) \le P(\tilde{V}_T\ge H_T^{Total}).$$

Let us assume now that the inequality (5.1) holds for $n=k\in\mathbb{N}_0$ and prove that it is still holds for $n=k+1$. For $n=k$, we have that

$$\max_{v_0\in\mathbb{R}_+,\,\xi\in\mathcal{A}^f\ s.t.\ v_0\le\bar{V}_0} P(V_T\ge H_T^{Total}) \le P(\tilde{V}_T\ge H_T^{Total}).$$

For any $t\in[0,T]$ such that $N_t=k$, let us consider, for $i=1,...,k$, $\tau_i^{k,t}$ as the $\mathcal{F}^a-$stopping time associated to the the i^{th} jump in the process $(N_s)_{s\in[t,T]}$ with the convention $\tau_0^{k,t}:=t$. Then, as done before, we can also prove that, for any $t\in[0,T]$ such that $N_t=k$,

$$\max_{V_t\in\mathbb{R}_+,\,\xi\in\mathcal{A}_t^f\ s.t.\ V_t\le\bar{V}_t} P(V_T\ge H_T^{Total}|\mathcal{F}_t) \le P(\tilde{V}_T^t\ge H_T^{Total}|\mathcal{F}_t), \tag{8.22}$$

where $(\tilde{V}_s^t)_{s\in[t,T]}$ such that $\tilde{V}_t^t=\bar{V}_t$ is the hedging portfolio associated to $(\tilde{\xi}_s^t)_{s\in[t,T]}$ defined, for $i=0,...,k-1$, with $\tau_i^{k,t}<\tau^{k,t}$, $(\tilde{\xi}_s^t)_{s\in[\tau_i^{k,t},\tau_{i+1}^{k,t}\wedge\tau^{k,t})}$ is the perfect hedging strategy of

$$H_T\times\sum_{j=1}^{N_{\tau_i^{k,t}}}\mathbb{1}_{A_j^*(\lambda_{\tau_i^{k,t}},\tilde{Z}_T,p^{\tau_i^{k,t}})},$$

where $\lambda_{\tau_i^{k,t}}$ satisfied

$$\tilde{V}^t_{\tau_i^{k,t}} = E^{\tilde{Q}_f}\Bigg(H_T\times\sum_{j=1}^{N_{\tau_i^{k,t}}}\mathbb{1}_{A_j^*(\lambda_{\tau_i^{k,t}},\tilde{Z}_T,p^{\tau_i^{k,t}})}|\mathcal{F}_{\tau_i^{k,t}}\Bigg).$$

and $(\tilde{\xi}_s^t)_{s\in[\tau^{k,t},T)}$ is the perfect hedging strategy of

$$H_T\times N_{\tau^{k,t}},$$

where

$$\tau^{k,t}:=\inf\Big\{\tau\in\{\tau_0^{k,t}\wedge T,...,\tau_n^{k,t}\wedge T\}:\tilde{V}_\tau^t 1_{\tau<T}\ge 1_{\tau<T}E^{\tilde{Q}^f}(H_T\times N_T|\mathcal{F}_\tau)\Big\}.$$

Therefore, for $n=k+1$, we have that

$$\begin{aligned}
P(V_T^f\ge H_T^{Total}) &= E^P\Bigg(\mathbb{1}_{\Big\{N_T=k+1\Big\}}\mathbb{1}_{\Big\{V_T^f\ge(k+1)H_T\Big\}}\Bigg) + E^P\Bigg(\mathbb{1}_{\Big\{N_T\le k\Big\}}\mathbb{1}_{\Big\{V_T^f\ge N_TH_T\Big\}}\Bigg)\\
&= E^P\Bigg(\mathbb{1}_{\Big\{N_T=k+1\Big\}}\mathbb{1}_{\Big\{\tilde{V}_T\ge(k+1)H_T\Big\}}\Bigg) + E^P\Bigg(\mathbb{1}_{\Big\{N_T\le k\Big\}}E^P\Bigg(\mathbb{1}_{\Big\{V_T^f\ge N_TH_T\Big\}}|\mathcal{F}_{\tau_1}\Bigg)\Bigg),
\end{aligned}$$

and using (8.22) with $t=\tau_1=\tau_0^{k,\tau_1}$, as well as the definition of the strategy $\tilde{\xi}$ in Proposition 5.1 with $n=k+1$, we deduce that

$$E^P\Bigg(\mathbb{1}_{\Big\{V_T^f\ge N_TH_T\Big\}}|\mathcal{F}_{\tau_1}\Bigg) \le \max_{V_{\tau_1}\in\mathbb{R}_+,\,\xi\in\mathcal{A}^f_{\tau_1}\ s.t.\ V_{\tau_1}\le\tilde{V}_{\tau_1}} P(V_T\ge H_T^{Total}|\mathcal{F}_{\tau_1}) \le E^P\Bigg(\mathbb{1}_{\Big\{\tilde{V}_T\ge N_TH_T\Big\}}|\mathcal{F}_{\tau_1}\Bigg),\ \text{on }\{N_T\le k\},$$

and finally,

$$\max_{v_0\in\mathbb{R}_+,\,\xi\in\mathcal{A}^f\ s.t.\ v_0\le\bar{V}_0} P(V_T\ge H_T^{Total}) = P(V_T^f\ge H_T^{Total}) \le P(\tilde{V}_T\ge H_T^{Total}).$$

□

9 Appendix

9.1 Equivalent Martingale Measures in combined market

Proposition 9.1. *For $i = 1, ..., n$, suppose that*

$$M_t^i = (1 - N_t^i) - \int_0^t N_u^i \, \kappa_{x+u} \, du,$$

$$M_t := \sum_{i=1}^n M_t^i.$$

and $(L_t^h)_{t \geq 0}$ such that

$$\begin{aligned} dL_t^h &= L_{t^-}^h \, h_t \, dM_t, \\ L_0^h &= 1, \end{aligned} \tag{9.1}$$

for a given $\mathcal{F}_t^a-$adapted process $(h_t)_{t \geq 0}$ such that for all $t \geq 0$, $h_t > -1$ almost surely. Then, for all $\mathcal{F}_t^a-$adapted processes $(h_t)_{t \geq 0}$ and $Q^f \in \mathcal{Q}^f$ there exists a probability measure denoted by Q^h such that

$$\frac{dQ^h}{dP}|_t = L_t^h Z_t^f, \tag{9.2}$$

with $Z_t^f = \frac{dQ^f}{dP^f}|_t$ and

$$Q^h \in \mathcal{Q}.$$

Proof. Let $(h_t)_{t \geq 0}$ be a $\mathcal{F}_t^a-$adapted process such that for all $t \geq 0$, $h_t > -1$ almost surely and $Q^f \in \mathcal{Q}^f$ such that $Z_t^f = \frac{dQ^f}{dP^f}|_t$. For $i = 1, ..., n$, using the Doob-Meyer decomposition of $(N_t^i)_{t \geq 0}$, we observe that $(M_t^i)_{t \geq 0}$ is a $\mathcal{F}_t^a-$martingale and, as $M_t = \sum_{i=1}^n M_t^i$, M_t is also a $\mathcal{F}_t^a-$martingale. Since, $(L_t^h)_{t \geq 0}$ with $L_0^h = 1$ is a strictly positive $\mathcal{F}_t^a-$martingale and $(Z_t^f)_{t \geq 0}$ with $Z_0^f = 1$ is a strictly positive $\mathcal{F}^f-$martingale, we have that, for $0 \leq s \leq t$,

$$\begin{aligned} E^P(L_t^h Z_t | \mathcal{F}_s) &= E^{P_a}(L_t^h | \mathcal{F}_s^a) \, E^{P_f}(Z_t^f | \mathcal{F}_s^f) \\ &= L_s^h Z_s^f, \end{aligned}$$

and in particular,

$$E^P(L_t^h Z_t^f) = 1.$$

Hence the change of measure (9.2) is well defined and as $(L_t^h Z_t^f)_{t \geq 0}$ is a strictly positive process, the probability measure Q^h is equivalent to P. Moreover, we observe that, for $0 \leq s \leq t$,

$$\begin{aligned} E^{Q^h}\left(\frac{S_t}{B_t} | \mathcal{F}_s\right) &= \frac{E^P\left(L_t^h Z_t^f \frac{S_t}{B_t} | \mathcal{F}_s\right)}{L_s^h Z_s^f} \\ &= \frac{E^{P_a}(L_t^h | \mathcal{F}_s^a)}{L_s^h} \frac{E^{P_f}\left(Z_t^f \frac{S_t}{B_t} | \mathcal{F}_s^f\right)}{Z_s^f} \\ &= E^{Q_f}\left(\frac{S_t}{B_t} | \mathcal{F}_s^f\right) \\ &= \frac{S_s}{B_s}. \end{aligned}$$

We conclude that $\left(\frac{S_t}{B_t}\right)_{t \geq 0}$ is a martingale under Q^h and then, as $Q^h \sim P$, $Q^h \in \mathcal{Q}$. $\square$

Proposition 9.2. *For $\epsilon > -1$, for all $\mathcal{F}_t^a-$adapted processes $(h_t)_{t \geq 0}$ such that $\forall t \geq 0$, $h_t > -1$ almost surely and for all financial martingale measures $Q^f \in \mathcal{Q}^f$, consider the probability measure Q_{h^ϵ} such that*

$$\frac{dQ_{h^\varepsilon}}{dP}|_t := L_t^{h^\varepsilon} Z_t^f,$$

with $Z^f_t := \frac{dQ^f}{dP^f}|_t$, $(L^{h_\varepsilon}_t)_{t\geq 0}$ *defined by* (9.1) *and* $(h^\varepsilon_t)_{t\geq 0} := ((1+\varepsilon)h_t+\varepsilon)_{t\geq 0}$. *Then,*

$$Q_{h^\varepsilon} \in \mathcal{Q},$$

and

$$\lim_{\varepsilon\to -1} E^{Q_{h^\varepsilon}}\Big(f(S_T,N_T)\Big) = E^{Q_f}\Big(f(S_T,n)\Big).$$

Proof. Let $\varepsilon > -1$, consider a given $\mathcal{F}^a_t-$adapted process $(h_t)_{t\geq 0}$ such that for all $t\geq 0$, $h_t > -1$ almost surely and a financial martingale measure $Q^f \in \mathcal{Q}^f$. As $(h^\varepsilon_t)_{t\geq 0} = ((1+\varepsilon)h_t+\varepsilon)_{t\geq 0}$ is $\mathcal{F}^a_t-$adapted such that for all $t\geq 0$, $h^\varepsilon_t > -1$, using Proposition 9.1, we have that

$$Q_{h^\varepsilon} \in \mathcal{Q}.$$

We observe that, by independence between actuarial and financial risks,

$$\begin{aligned}
E^{Q_{h^\varepsilon}}\Big(f(S_T,N_T)\Big) &= \sum_{i=0}^n E^{Q_{h^\varepsilon}}\Big(f(S_T,i)\Big)\times Q_{h^\varepsilon}(N_T=i)\\
&= \sum_{i=0}^n E^{Q_f}\Big(f(S_T,i)\Big)\times Q_{h^\varepsilon}(N_T=i).
\end{aligned}$$

Using Ito's lemma, we obtain that

$$L^{h_\varepsilon}_t = \prod_{i=1}^n (1+h^\varepsilon_{\tau_i})^{1-N^i_t}\ \exp\Big(-\int_0^t h^\varepsilon_u\,\kappa_{x+u}N_u du\Big).$$

Therefore, under Q_{h^ε}, for $i=0,...,n$, we have

$$\begin{aligned}
Q_{h^\varepsilon}(N_T=i) =& E^{Q_{h^\varepsilon}}(\mathbb{1}_{\{N_T=i\}})\\
=& E^P(L^{h_\varepsilon}_T Z_T \mathbb{1}_{\{N_T=i\}})\\
=& E^P\Big(\mathbb{1}_{\{N_T=i\}}\prod_{j=1}^n (1+h^\varepsilon_{\tau_j})^{1-N^j_T}\ \exp\Big(-\int_0^T h^\varepsilon_u\,\kappa_{x+u}N_u du\Big)\Big).
\end{aligned}$$

In particular, for $i=n$, we obtain

$$\begin{aligned}
Q_{h^\varepsilon}(N_T=n) =& E^P\Big(\mathbb{1}_{\{N_T=n\}}\prod_{j=1}^n (1+h^\varepsilon_{\tau_j})^{1-N^j_T}\ \exp\Big(-\int_0^T h^\varepsilon_u\,\kappa_{x+u}N_u du\Big)\Big)\\
=& E^P\Big(\mathbb{1}_{\{N_T=n\}}\exp\Big(-n\int_0^T \kappa_{x+u}h^\varepsilon_u)du\Big)\Big)\\
=& E^P\Big(\mathbb{1}_{\{N_T=n\}}\exp\Big(-n\int_0^T \kappa_{x+u}((1+\varepsilon)h_u+\varepsilon)du\Big)\Big).
\end{aligned}$$

By monotone convergence, we have

$$\begin{aligned}
\lim_{\varepsilon\to -1} Q_{h^\varepsilon}(N_T=n) &= E^P\Big(\mathbb{1}_{\{N_T=n\}}\exp\Big(n\int_0^T \kappa_{x+u}du\Big)\Big)\\
&= 1,
\end{aligned}$$

and thus for $i=0,...,n-1$,

$$\lim_{\varepsilon\to -1} Q_{h^\varepsilon}(N_T=i) = 0,$$

We easily conclude that

$$\begin{aligned}
\lim_{\varepsilon\to -1} E^{Q_{h^\varepsilon}}\Big(f(S_T,N_T)\Big) &= \sum_{i=0}^n E^{Q_f}\Big(f(S_T,i)\Big)\times \lim_{\varepsilon\to -1} Q_{h^\varepsilon}(N_T=i)\\
&= E^{Q_f}\Big(f(S_T,n)\Big).
\end{aligned}$$

$\square$

9.2 Existence results for solutions to static optimization problems

Proposition 9.3. *There exists a solution $\tilde{A} = (\tilde{A}_1, ..., \tilde{A}_n) \in \mathcal{F}_T$ such that $\tilde{A}_{i+1} \subseteq \tilde{A}_i$, $i = 1, ..., n-1$ to the problem*

$$\max_{A_1,...,A_n \in \mathcal{F}_T,\ A_{i+1} \subseteq A_i} E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i}\right) + p_0, \tag{9.3}$$

under the constraint

$$\sup_{Q\in\mathcal{Q}} E^Q\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right) \leq \bar{V}_0. \tag{9.4}$$

Proof. The proof is just an adaptation of the proof of Proposition 3.1. in [14]. Let define $\mathcal{R}^n := \left\{(\mathbb{1}_{A_1}, ..., \mathbb{1}_{A_n}) |\ A_i \in \mathcal{F}_T,\ A_{i+1} \subseteq A_i,\ i = 1, ..., n-1\right\}$ and . In this case, we can rewrite the problem as

$$\max_{g\in\mathcal{R}^n} E^P\left(\sum_{i=1}^{n} g^i \times \mathbb{1}_{N_T=i}\right) + p_0,$$

under the constraint

$$\sup_{Q\in\mathcal{Q}} E^Q\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} g^i\right) \leq \bar{V}_0. \tag{9.5}$$

Let $\mathcal{R}^n_0$ the elements of $\mathcal{R}^n$ satisfying (9.5) and consider $(g_m)_{m\geq 1}$ a maximizing sequence for (9.3) in $\mathcal{R}^n_0$. Using Lemma A.1.1. in [10], we can choose $\tilde{g}_m \in conv(g_m, g_{m+1}, ...)$ such that $(\tilde{g}_m)_{m\geq1}$ converges $P-$almost surely to $\tilde{g} \in \mathcal{R}^n$. Therefore, we have that, by dominated convergence theorem[11],

$$E^P\left(\sum_{i=1}^{n} \tilde{g}^i_m \times \mathbb{1}_{N_T=i}\right) + p_0 \to \max_{g\in\mathcal{R}^n} E^P\left(\sum_{i=1}^{n} g^i \times \mathbb{1}_{N_T=i}\right) + p_0 = E^P\left(\sum_{i=1}^{n} \tilde{g}^i \times \mathbb{1}_{N_T=i}\right) + p_0$$

as m tends to infinity. Moreover, by Fatou's lemma, for all $Q \in \mathcal{Q}$, we observe that

$$E^Q\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \tilde{g}^i\right) \leq \liminf E^Q\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \tilde{g}^i_m\right) \leq \bar{V}_0,$$

then

$$\tilde{g} \in \mathcal{R}^n_0$$

and the existence of a solution to the problem (9.3) with constraint (9.4) follows. □

Proposition 9.4. *There exists a solution $\tilde{\phi} = (\tilde{\phi}_1, ..., \tilde{\phi}_n)$ such that $\tilde{\phi}_i \in \mathcal{R}$ and $\tilde{\phi}_i \geq \tilde{\phi}_{i-1}$, $i = 1, ..., n$, to the problem*

$$\min_{\phi_1,...,\phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i,\ \phi_0 = 1} E^P\left(\sum_{i=1}^{n} \mathbb{1}_{N_T=i} \times L\left(\sum_{j=1}^{i} H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\right)\right), \tag{9.6}$$

with

$$\mathcal{R} = \left\{\phi : \Omega \to [0,1], \mathcal{F}_T - measurable\right\}$$

under the constraint

$$\sup_{Q\in\mathcal{Q}} E^Q\left(\frac{H_T}{B_T} \sum_{i=1}^{n} \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\right) \leq \bar{V}_0. \tag{9.7}$$

Proof. The proof is just an adaptation of the proof of Proposition 3.1. in [14]. Let define $\mathcal{R}^n := \left\{\phi = (\phi_1, ..., \phi_n) |\ \phi_i \in \mathcal{R},\ \phi_i \leq \phi_{i-1},\ i = 1, ..., n\right\}$. In this case, the problem is equivalent to

$$\min_{\phi\in\mathcal{R}^n} E^P\left(\sum_{i=1}^{n} \mathbb{1}_{N_T=i} \times L\left(\sum_{j=1}^{i} H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\right)\right),$$

[11] Since $\forall m \geq 1, \forall i = 1, ..., n,\ 0 \leq \tilde{g}^i_m \leq 1$

under the constraint

$$\sup_{Q\in\mathcal{Q}} E^Q\left(\frac{H_T}{B_T}\sum_{i=1}^{n}\phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\right) \leq \bar{V}_0. \tag{9.8}$$

Let $\mathcal{R}_0^n$ the elements of $\mathcal{R}^n$ satisfying (9.8) and consider $(g_m)_{m\geq 1}$ a minimizing sequence for (9.6) in $\mathcal{R}_0^n$. By using similar arguments than in the proof of Proposition 9.3, we can easily deduce the existence of a solution to the problem (9.6) with constraint (9.7). □

9.3 Stochastic algorithm for root finding

Assume that we want to find the root for a function of the form $h(\lambda) = E^P(H(\lambda, Z))$ with $\lambda \in \mathbb{R}$ and Z a random variable defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. To this end, we can determine a stochastic algorithm that will converge (from any starting point) to λ^*. The stochastic algorithm can be seen as an extension of the Newton-Raphson algorithm. Let $(Z_n)_{n\geq 1}$ simulations of Z and $(\gamma_n)_{n\geq 1}$ a deterministic sequence such that $\sum_{n\geq 1}\gamma_n = +\infty$, $\sum_{n\geq 1}\gamma_n^2 < +\infty$. We consider a stochastic sequence $(\lambda_n)_{n\geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ and propose the following algorithm for root finding of $h(\lambda) = E^P(H(\lambda, Z))$:

$$\begin{aligned} \lambda_{n+1} &= \lambda_n - \gamma_{n+1}H(\lambda_n, Z_{n+1}), \\ \lambda_0 &\in L^2(P). \end{aligned} \tag{9.9}$$

Proposition 9.5. *(Robbins-Monro algorithm*, see Corollary 6.1 in [25])
Assume that the mean function $h(\lambda) = E^P(H(\lambda, Z))$ is continuous and satisfies

$$\forall \lambda \in \mathbb{R},\ \lambda \neq \lambda^*,\ \ (\lambda - \lambda^*)h(\lambda) > 0$$

Suppose furthermore that $\lambda_0 \in L^2(P)$ and that H satisfies

$$\forall \lambda \in \mathbb{R}, ||H(\lambda, Z)||_{2,P} \leq C(1 + |\lambda|).$$

Assume that $(\gamma_n)_{n\geq 1}$ satisfies

$$\sum \gamma_n = +\infty, \sum \gamma_n^2 < +\infty.$$

Then $(\lambda_n)_{n\geq 1}$ defined in (9.9) is such that

$$\lambda_n \xrightarrow{a.s.} \lambda^*,$$

and in every $L^p(P)$, $p \in (0, 2)$.

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