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B E L G I U M

OLIGOPOLY EQUILIBRIA IN EXCHANGE ECONOMIES*

by

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Abstract

This paper extends the notion of Cournot–Walras equilibrium introduced by Codognato and Gabszewicz (1991) in a particular context (homogeneous oligopoly) to a general class of pure exchange economies. Some examples are considered, corresponding to oligopolistic structures analyzed in partial equilibrium. Finally, we discuss a possible extension of the concept to the case of a productive economy, and illustrate this discussion by means of an example embodying linear technologies.

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1. Introduction

In two recent papers, Codognato and Gabszewicz (1991) have analyzed the oligopolistic behaviour of economic agents in the framework of a pure exchange economy, with a particular structure of initial endowments, corresponding to a situation of *homogeneous* oligopoly. More precisely, they consider a situation in which the strategic traders (the "oligopolists") have a "corner" on a particular commodity, while the remaining agents, who behave competitively, share the initial ownership of the other goods. The oligopolists use quantity strategies, manipulating the fraction of their holdings they send to the market for trade: each oligopolist exerts a partial control over the equilibrium prices, via its individual supply. A concept of "Cournot–Walras Equilibrium" is introduced, namely a non-cooperative equilibrium of the game whose players are the oligopolists, strategies are their individual supplies, and whose payoffs are the utility levels which can be reached through the exchange mechanism.

The preceding analysis of quantity oligopoly was restrictive in at least three different ways. First, it envisions only situations of homogeneous oligopoly: all oligopolists manipulate the market for the same good. Second, the oligopolists are restricted to use quantity strategies on a single market only: the structure of initial endowments prevents them from acting strategically on the others, since they do not share the ownership of these other goods. Finally, some traders are assumed to behave competitively on all markets, introducing an unsatisfactory asymmetry in the behaviour of economic agents. In the present paper, we define a general notion of non-cooperative equilibrium for a quantity setting oligopoly in pure exchange economies, which circumvents the above restrictions, and covers, as particular cases, a wide variety of oligopolistic contexts considered in partial equilibrium analysis, like homogeneous and differentiated oligopoly, multilateral monopoly and oligopoly, a.s.o.... This notion is an extension of the Cournot Walras equilibrium concept, introduced in Codognato and Gabszewicz (1991), and is based on the following idea, which distinguishes between a competitive and noncompetitive behaviour of an agent in a particular market. In a competitive economy, exchange can be viewed as taking place in the following way. Each competitive trader comes with his total endowment in each good to a central market place, where the sum of these endowments for each good is supplied for trade. A price system is announced, which determines the income of each trader as the scalar product of this price vector with the vector of his initial endowments. Then each utility maximizing competitive trader buys back a bundle of the commodities, the value of which does not exceed his income. It the price system clears each market a competitive equilibrium obtains. By analogy with the above story, we shall say that a trader has a *competitive behaviour* in a particular market if he supplies the market place with his *total* initial endowment of the corresponding commodity. But it may be that a trader would like to supply one or several markets with only a restricted share of his initial holdings of the corresponding commodity, keeping for later consumption the remaining share. Then this remaining share does not transit through the market and accordingly, does not influence the market clearing process. If a trader uses this opportunity for a particular good, we say that he has a *non competitive* behaviour on the corresponding market. The interest for restricting his supply of one or several goods comes from the fact that the resulting equilibrium prices can give him better overall market opportunities than the equilibrium prices which would obtain, should he supply the market with his total initial endowment. But the influence of each non competitive trader on equilibrium prices is only partial; indeed, he may find other non competitive traders competing with him in the same manner and on the same markets. Accordingly, the equilibrium notion must reflect this complex system of rivalry among the non competitive traders on the various markets. This is exactly what does the non cooperative equilibrium notion we define below in section 2. The following sections are devoted to the study of particular illustrations of our equilibrium concept, corresponding to specific contexts of oligopolistic rivalry analyzed in partial equilibrium; homogeneous oligopoly is analyzed in section 3 and differentiated oligopoly in section 5. Section 4 provides an example for the case of multilateral oligopoly. In the closing section, we initiate a discussion on how to extend our equilibrium concept to the case of a productive economy. This discussion is illustrated by means of an example embodying linear technologies.

2. The Definition of an Oligopoly Equilibrium

Consider an exchange economy with ℓ *goods* h, $h = 1, \ldots, \ell$, and n *agents* (traders) i, $i = 1, \ldots, n$. Each agent i is represented by (1) his *initial endowment*, denoted ω_i, a vector of $\mathbb{R}^{\ell}_{+}$; (2) his *utility function*, denoted $U_i(x)$ and assumed to be monotone, continuous and strictly quasi-concave: $U_i(x)$ represents his preferences among the commodity bundles $x \in \mathbb{R}^{\ell}_{+}$; and (3) a subset H_i of markets, $H_i \subseteq \{h;\ 1 \leq h \leq \ell \text{ and } \omega_i^h > 0\}$: we call H_i the *markets' portfolio* of agent i. We shall assume that, for all $h = 1, \ldots, \ell$, there exists a trader i for whom

$h \notin H_i$. The definition of markets' portfolio is the formal counterpart of the idea developed in our introduction. On those markets h which are not in his portfolio, agent i adopts a competitive behaviour: He brings to these markets a quantity y^h which is equal to his total initial holdings of good h, i.e. $y^h = \omega_i^h$, $h \notin H_i$. On those markets h which are in his portfolio, agent i may either adopt a competitive behaviour, i.e. choose y^h equal to ω_i^h; or he may have a non competitive behaviour, restricting his supply to a share y^h, $0 < y^h < \omega_i^h$, of his total initial holdings of good h. In the sequel, the choice of a particular vector $y = (y^1, \ldots, y^h, \ldots, y^\ell)$, i.e., a choice of a particular supply y^h on each market h, will be considered as a *strategy* for agent i. Accordingly the set of strategies open to agent i is the set Y_i defined by

$$Y_i = \{y \mid 0 \leq y \leq \omega_i;\quad y^h = \omega_i^h \text{ for all } h \notin H_i\}:$$

supplies should be feasible ($0 \leq y^h \leq \omega_i^h$ for all h), and should be equal to initial holdings on the markets which are not in the portfolio of agent i.

Let each agent i choose a strategy y_i in Y_i, and consider a *price vector* $p = (p^1 \ldots p^h \ldots p^\ell)$, $p \neq 0$. Given the choice y_i the income of agent i (i.e., the value of resources brought to the market), is equal to $p \cdot y_i$, where $x \cdot y$ denotes the scalar product of x and y. If trader i, endowed with income $p \cdot y_i$, chooses to buy a commodity bundle x, he reaches a utility level equal to $U_i(\omega_i - y_i + x)$: $\omega_i - y_i$ is the bundle of goods still remaining available for consumption given the choice of strategy y_i, and x is the vector of purchases performed at the price system p. Accordingly, given p and y_i, the choice of x is dictated by the solution to the program

$$\max_x U_i(\omega_i - y_i + x) \quad \text{s.t. } p \cdot x \leq p \cdot y_i, \text{ and } x \geq 0. \tag{2.1}$$

Denote by $x_i(p, y_i)$ the solution to (2.1) : by monotonicity, continuity and strict quasi concavity of U_i, this solution exists and is unique.
Denote by $p(y_1 \ldots y_n)$ a price system which clears all markets in the exchange economy where traders i, $i = 1, \ldots, n$ are endowed with initial holdings y_i, i.e., $p(y_1 \ldots y_n)$ solves

$$\sum_{i=1}^{n} x_i(p, y_i) = \sum_{i=1}^{n} y_i$$

(standard assumptions guarantee the existence of a price system $p(y_1 \ldots y_n)$ for any n-tuple of strategies $(y_1 \ldots y_n)$, $y_i \in Y_i$, $i = 1 \ldots n$, we shall *assume* that this price system is unique).

An *oligopoly equilibrium* is a n-tuple of strategies $(y_1^* \dots y_n^*)$, $y_i^* \in Y_i$, such that $\forall\, i = 1 \dots n$, $\forall\, y_i \in Y_i : U_i(\omega_i - y_i^* + x_i(p(y_i^*, y_{-i}^*), y_i^*)) \geq U_i(\omega_i - y_i + x_i(p(y_i, y_{-i}^*), y_i))$ with y_{-i}^* denoting the $(n-1)$-tuple of strategies y_k^*, $k \neq i$.

At an oligopoly equilibrium each trader i chooses his supply on each market in his portfolio, in such a way that, given the supplies chosen by the other traders in their own markets' portfolios and the resulting equilibrium prices, no deviation from this choice can increase his utility.
Clearly the oligopoly equilibrium depends on the particular structure of markets' portfolios H_i available to the traders, since equilibrium prices can be manipulated only via markets belonging to the porfolio of at least one trader: otherwise all traders have a competitive behaviour on the corresponding market h, and total supply of this market must coïncide with the competitive supply $\sum_{i=1}^{n} \omega_i^h$. To illustrate the concept of oligopoly equilibrium, it is useful to examine particular cases corresponding to different structures of markets' porfolios. As we shall see, these particular cases are counterparts of oligopoly contexts which have been considered in partial or general equilibrium.

As a first case, let us consider the case where $H_i = \{\emptyset\}$ for all $i = 1, \dots, n$: no market is in the portfolio of any trader. Then the strategy set Y_i for each trader reduces to the singleton $\{\omega_i\}$: each trader has a competitive behaviour on every market. Then, given p, problem (2.1) reduces to

$$\max_x U_i(\omega_i - \omega_i + x) \text{ s.t. } p \cdot x \leq p \cdot \omega_i,$$

so that $x_i(p, y)$ coïncides with the walrasian demand at p. Accordingly the price system $p(\omega_1, \dots, \omega_i, \dots, \omega_n)$ which solves

$$\sum_{i=1}^{n} x_i(p, \omega_i) = \sum_{i=1}^{n} \omega_i,$$

is a competitive price system, and the resulting allocation $(x_i(p(\omega_1 \dots \omega_n); \omega_i)$ is a competitive allocation. Consequently an oligopoly equilibrium must coïncide with a competitive equilibrium whenever $H_i = \emptyset$ for all i: in that case, the strategy set of each trader coïncides with his initial endowment vector, and no strategic freedom is open to the traders in view of influencing equilibrium prices.

Our second case corresponds to the case of *homogeneous oligoply*, analyzed by Codognato and Gabszewicz (1991). In this case traders are divided into two

subsets: Traders i, with $i = 1, \dots, m < n$, have a market's porfolio $H_i = \{1\}$ (which implies $\omega_i^1 > 0$), while traders i, with $i = m+1, \dots, n$, have $H_i = \{\emptyset\}$. For $i = 1 \dots m$, the strategy sets Y_i is equal to $\{y; 0 < y^1 \leq \omega_i^1; y^h = \omega_i^h,\ h = 2, \dots, \ell\}$. If $\omega_i^1 = 0$, for $i = m+1, \dots, n$, we obtain a typical homogeneous oligopoly situation, with m oligopolists "having a corner" on the same commodity (here commodity 1), and $m - n$ competitive traders on the demand side of the market for commodity 1. Oligopolists behave strategically on this market. If $\omega_i^1 > 0$ for some traders i, with $i \in \{m+1, \dots, n\}$, we obtain a situation with m oligopolists acting strategically on the market for commodity 1, but now accompanied on the supply side of the market by a "competitive fringe" of traders.

The next case defines a situation of *differentiated oligopoly*. Each agent $i = 1, \dots, m < n$ has a market's portfolio $H_i = \{i\}$, while $H_i = \{\emptyset\}$ for $i = m+1, \dots, n$. For instance trader 1 has a monopoly position on commodity 1 ($\omega_1^1 > 0; \omega_i^1 = 0$ for $i \neq 1$) and trader 2 a monopoly position on commodity 2 ($\omega_2^2 > 0; \omega_i^2 = 0$ for $i \neq 2$). Whenever commodity 1 and 2 are substitutes we have a situation of differentiated duopoly. When $n = m = \ell$, each trader is a monopolist on a single market, and we obtain a structure of *multilateral monopoly*.

Finally, the most general case – a *complete multilateral oligopoly* – corresponds to the situation where $H_i = \{h;\ 1 \leq h \leq \ell \text{ and } \omega_i^h > 0\}$ for all $i = 1 \dots n$. Then each agent behaves non competitively on every market for which his initial holdings of the corresponding good is positive.

In the next sections we shall study a series of examples where the above structures will be successively examined. We start out in section 3 with the case of homogeneous oligopoly.

3. Homogeneous Oligopoly

Consider an exchange economy with 2 goods and $m + n$ agents defined by

$$U_i(x^1, x^2) = \log x^1 + \log x^2 = U(x^1, x^2)$$

$$\begin{aligned} \omega_i &= (\frac{1}{m}, 0), \quad i = 1 \dots m; \\ &= (\frac{\varepsilon}{n}, \frac{1}{n}), \quad i = m+1 \dots m+n,\ \varepsilon \geq 0; \\ H_i &= \{1\}, \quad i = 1 \dots m; \\ &= \{\emptyset\}, \quad i = m+1, \dots, m+n. \end{aligned}$$

The oligopolists have a non competitive behaviour on the market of good 1, while the remaining agents (the "competitive sector") behave competitively on both markets. When $\varepsilon > 0$, there is a "competitive fringe" of traders owning also commodity 1, but behaving competitively on both markets, and in particular on the market for good 1. We begin to study the case without a competitive fringe, i.e., $\varepsilon = 0$.

First the unique competitive equilibrium of this exchange economy is a price system $(\tilde{p}^1, \tilde{p}^2)$ with $\dfrac{\tilde{p}^1}{\tilde{p}^2} = 1$ and the allocation

$$\begin{aligned} \tilde{x}_i &= \left(\frac{1}{2m}, \frac{1}{2m}\right), \quad i = 1 \dots m; \\ &= \left(\frac{1}{2n}, \frac{1}{2n}\right), \quad i = m+1, \dots, m+n. \end{aligned}$$

Now we compute the oligopoly equilibrium. Let (p^1, p^2) be a price vector with $p^h > 0$, $h = 1, 2$, and consider the competitive sector $(i = m+1, \dots, m+n)$. For those traders, $Y_i = \{\omega_i\}$ so that $x_i((p^1, p^2), \omega_i)$ is the Walrasian demand at (p^1, p^2), i.e.

$$x_i(p, \omega_i) = \left(\frac{1}{2pn}, \frac{1}{2n}\right),$$

with $p = \dfrac{p^1}{p^2}$,

$$\text{and} \sum_{i=m+1}^{m+n} x_i^1(p, \omega_i) = \frac{1}{2p}. \tag{3.1}$$

Now, given p and a strategy $y_j = (y_j^1, y_2^j) \in Y_j = \left\{(y_j^1, y_j^2);\ 0 \leq y_j^1 \leq \dfrac{1}{m};\ y_j^2 = 0\right\}$, for $j = 1 \dots m$, each oligopolist j solves the problem

$$\max_{x^1, x^2} \log\left(\frac{1}{m} - y_j^1 + x^1\right) + \log x^2 \quad \text{s.t. } p^1 x^1 + p^2 x^2 \leq p^1 y_j^1;\ x^1 \geq 0,\ x^2 \geq 0. \tag{3.2}$$

Let us show that any utility level of oligopolist j reached at the solution $x_j(p, y_j)$ to problem (3.2) for $y_j \in Y_j$ can be reached if we restrict the strategy set Y_j to $\tilde{Y}_j = \left\{(y_j^1, y_j^2);\ 0 \leq y_j^1 \leq \frac{1}{2m};\ y_j^2 = 0\right\}$.

If $y_j^1 \leq \dfrac{1}{2m}$, the solution to problem (3.2) in Y_j coïncides with the solution to problem (3.2) in $\tilde{Y}_j$, and this solution is given by

$$x_j(p, y_j) = (0, p \cdot y_j^1). \tag{3.3}$$

If $y_j^1 > \dfrac{1}{2m}$, the solution to problem (3.2) in Y_j is given by

$$\left(y_j^1 - \frac{1}{2m}, \frac{p}{2m}\right),$$

and oligopolist j obtains a utility level equal to

$$U\left(\frac{1}{m} - y_j^1 + y_j^1 - \frac{1}{2m}, \frac{p}{2m}\right) = U\left(\frac{1}{2m}, \frac{p}{2m}\right).$$

Now assume that, if $y_j^1 > \dfrac{1}{2m}$, we replace the strategy $y_j = (y_j^1, 0)$ by the strategy $y_j' = \left(y_j'^1, 0\right) = \left(\dfrac{1}{2m}, 0\right)$. Then $y_j' \in \tilde{Y}_j$ and, according to (3.3), the solution to problem (3.2) in $\tilde{Y}_j$ is

$$x_j(p, y_j') = \left(0, \frac{p}{2m}\right),$$

so that the utility level of $x_j(p, y_j')$ is equal to $U\left(\dfrac{1}{m} - y_j'^1, \dfrac{p}{2m}\right) = U\left(\dfrac{1}{2m}, \dfrac{p}{2m}\right)$, which is the utility level obtained at the optimal solution in Y_j. Accordingly, to compute the oligopoly equilibrium, we may restrict ourselves to the set of strategies $\tilde{Y}_j$ and, in this set, the optimal solution to problem (3.2) is given by (3.3). On the other hand, given the strategies $(y_1, \dots, y_k, \dots, y_m)$, the price system (p^1, p^2) which clears both markets must satisfy

$$\frac{1}{2p} = \sum_{k=1}^{m} y_k^1$$

(see (3.1)), with $p = \dfrac{p^1}{p^2}$, i.e.

$$p = \frac{1}{2\sum_{k=1}^{m} y_k^1} = p(y_1^1, \dots, y_k^1, \dots, y_m^1).$$

Using (3.1) for p, and (3.3) for x^1 and x^2, the oligopolist j's problem (3.2) now rewrites as

$$\max_{y_j^1} \log\left(\frac{1}{m} - y_j^1\right) + \log\left(\frac{y_j^1}{2\sum_{k=1}^{m} y_k^1}\right)$$

By definition, an oligopoly equilibrium is a m-tuple of strategies $y_j^* = (y_j^{*1}, 0)$, $j = 1 \ldots m$, such that, for all j and $y_j = (y_j^1, 0) \in \tilde{Y}_j$,

$$U(x_j(p(y_1^{*1}, \ldots, y_m^{*1}), y_j^{*1}) = \log\left(\frac{1}{m} - y_j^{*1}\right) + \log\left(\frac{y_j^{*1}}{2\sum_{k=1}^{m} y_k^{*1}}\right) \geq$$

$$\log\left(\frac{1}{m} - y_j^1\right) + \log\left(\frac{y_j^1}{2(y_j^1 + \sum_{k \neq j} y_k^{*1})}\right) = U(x_j(p(y_j^1, y_{-j}^{*1}), y_j^1)).$$

Consider the derivative

$$\frac{\partial U}{\partial y_j^1} = \frac{1}{\frac{1}{m} - y_j^1} + \frac{1}{y_j^1} + \frac{1}{\sum_{k=1}^{m} y_k^1}.$$

Denoting by V_{-j} the value of $\sum_{k \neq j} y_k^1$, the expression $\frac{\partial U}{\partial y_j^1}$ has the same sign as the sign of the quadratic function $\mathcal{B}_j(y_j^1)$ defined by

$$\mathcal{B}_j(y_j^1) = V_{-j}\left(\frac{1}{m} - 2y_j^1\right) - (y_j^1)^2.$$

V_{-j} being fixed, $\mathcal{B}_j(y_j^1)$ is a decreasing function of y_j^1; which is positive at $y_j^1 = 0$ and negative at $y_j^1 = \frac{1}{2m}$; so that there exists a value $y_j^1(V_{-j})$ satisfying $\mathcal{B}_j(y_j^1(V_{-j})) = 0$. Since $\frac{\partial U}{\partial y_j^1}$ has the same sign as $\mathcal{B}_j(y_j^1)$, $U(x_j(p(y_j^1, y_{-j}^1), y_j^1)$ reaches it maximum at $y_j^1(V_{-j})$. Furthermore, at an oligopoly equilibrium, $\frac{\partial U}{\partial y_j^1} = 0$ for all $j = 1 \ldots m$, which implies that the m equilibrium quantities y_j^{*1} must be equal, i.e. $y_j^{*1} = y^{*1}$, for all $j = 1 \ldots m$ (all the values y_j^{*1} must satisfy the same expression with respect to the agregate supply $\sum_{k=1}^{m} y_k^{*1}$). Accordingly, *there exists a unique oligopoly equilibrium, which is symmetric*, with $y_j^* = (y^{*1}, 0)$, $j = 1 \ldots m$, and y^{*1} equal to the root of the equation:

$$\mathcal{B}_j(y^1) = (m-1)y^1\left(\frac{1}{m} - 2y^1\right) - (y^1)^2,$$

i.e.
$$y^{*1} = \frac{m-1}{m(2m-1)},$$

so that

$$p^* = \frac{2m-1}{2(m-1)}, \tag{3.6}$$

with p^* denoting the relative price corresponding to the oligopoly equilibrium quantities y^{*1}. First, when $m = 1$ (monopoly), the "monopoly" equilibrium is degenerate: supply is equal to zero and the price is infinite. This is reminiscent of the partial equilibrium monopoly solution with a unit elasticity market demand function. On the other hand we notice that $\lim_{m \to \infty} my^{*1} = \frac{1}{2}$, which is the agregate quantity of commodity 1 sold by the oligopolists at the competitive equilibrium. Consequently, although the oligopoly equilibrium differs from the competitive equilibrium, whatever the number of oligopolists, it tends to the competitive equilibrium when the number of the oligopolists increases.

The above analysis could easily be performed with the existence of a competitive fringe ($\varepsilon > 0$). In that case, we obtain

$$y^{*1} = a + \left(a^2 + \frac{\varepsilon}{2m(2m-1)}\right)^{1/2}$$

with a defined by

$$a = \frac{1}{4m-2}\left(\frac{m-1}{m} - \varepsilon\right).$$

Simple calculations show that

$$\frac{\partial my^{*1}}{\partial \varepsilon} > 0 \text{ and } \frac{\partial p^*}{\partial \varepsilon} < 0.$$

Furthermore $\lim_{\varepsilon \to \infty}(my^{*1}) = \frac{1}{2}$ and $\lim_{\varepsilon \to \infty} p^* = 1$: the existence of a competitive fringe increases competition and, if the size of the competitive fringe tends to infinity, the oligopoly equilibrium tends to the competitive equilibrium.

Finally, we notice that the oligopoly equilibrium does not depend on the number n of competitive agents. This competitive sector could have been as well modeled as an atomless continuum. In this case, each individual trader in the continuum has a null influence on the exchange process. Consequently, the oligopoly equilibrium would be invariant if alternative behaviour assumptions were made about the agents in the continuum: Even if they would behave non competitively on both markets, the same oligopoly equilibrium would emerge.

In the next section, we extend the previous example in view of covering the situation of *bilateral oligopoly*: the agents $i = m+1, \dots, n+m$, will now be assumed to behave non competitively on the market for good 2, while agents $i = 1 \dots m$, still behave non competitively on the market for good 1.

4. Bilateral and Multilateral Oligopoly

Consider the same exchange economy as in section 3 with $\varepsilon = 0$, except that we now define the traders' markets' portfolios and initial endowments in the following manner:

$$H_i = \{1\}, \quad i = 1, \dots, m; \quad \omega_i = (\frac{1}{m}, 0),$$

$$H_i = \{2\}, \quad i = m+1, \dots, m+n; \quad \omega_i = (0, \frac{1}{n}):$$

Each agent behaves non competitively on every market for which his initial holdings of the corresponding good is positive. We obtain a bilateral oligopoly structure, which is a particular case of a complete multilateral oligopoly structure, as defined at the end of section 2: *all* agents behave strategically.

The strategy sets of the agents are

$$Y_j = \left\{(y^1, y^2);\ 0 \le y_j^1 \le \frac{1}{m};\ y_j^2 = 0\right\} \quad \text{for } j = 1 \dots m$$

and

$$Y_i = \left\{(y^1, y^2);\ y_i^1 = 0;\ 0 \le y_i^2 \le \frac{1}{n}\right\} \quad \text{for } i = m+1, \dots, m+n,$$

respectively.

For simplicity and without loss of generality, we shall identify the strategy sets Y_j to the interval $[0, \frac{1}{m}]$, $j = 1, \dots, m$, and Y_i to the interval $[0, \frac{1}{n}]$, $i = m+1, \dots, m+n$: in any case, oligopolists j, $j = 1 \dots m$, (respectively, oligopolists i, $i = m+1, \dots, m+n$), cannot manipulate the market for good 2 (respectively, good 1). Moreover, an argument similar to the reasoning held in the previous example shows that, for computing the oligopoly equilibrium, we may restrict ourselves to the strategy sets $\tilde{Y}_j = [0, \frac{1}{2m}]$, $j = 1 \dots m$, and $\tilde{Y}_i = [0, \frac{1}{2n}]$, $i = m+1, \dots, m+n$. Given a price vector (p^1, p^2) and a strategy $y_j^1 \in \tilde{Y}_j$, oligopolist j, $j = 1 \dots m$, solves the problem

$$\max_{x^1, x^2} \left[\log(\frac{1}{m} - y_j^1 + x^1) + \log x^2\right] \quad \text{s.t. } x^1 \ge 0\ x^2 \ge 0 \text{ and } px^1 + x^2 \le py_j^1, \tag{4.1}$$

with $p = \frac{p^1}{p^2}$.

Similarly, oligopolist i, with a strategy $y_i^2 \in \tilde{Y}_i$, $i = m+1, \dots, m+n$, solves the problem

$$\max_{x^1, x^2} \left[\log x^1 + \log(\frac{1}{n} - y_i^2 + x^2)\right] \quad \text{s.t. } x^1 \ge 0\ x^2 \ge 0 \text{ and } px^1 + x^2 \le y_i^2. \tag{4.2}$$

The solution to problem (4.1) is

$$x_j(p, y_j^1) = (0, py_j^1), \qquad j = 1 \dots m \tag{4.3}$$

and to problem (4.2) is

$$x_i(p, y_i^2) = (\frac{y_i^2}{p}, 0), \quad i = m+1, \dots, m+n. \tag{4.4}$$

On the other hand, given the strategies $(y_1^1 \dots y_m^1; y_{m-1}^2 \dots y_{n+m}^2)$, the relative price p which clears both markets must satisfy

$$p(y_1^1 \dots y_m^1; y_{m+1}^2 \dots y_{m-n}^2) = \frac{\sum_{\ell=m+1}^{m+n} y_\ell^2}{\sum_{k=1}^{m} y_k^1}. \tag{4.5}$$

Using (4.5) for p and (4.3) for x^1 and x^2, the oligopolists' j's problem (4.1) now rewrites as

$$\max_{y_j^1} \left[\log(\frac{1}{m} - y_j^1) + \log\left(\frac{\sum_{\ell=m+1}^{m+n} y_\ell^2}{\sum_{k=1}^{m} y_k^1}\right) \cdot y_j^1\right].$$

Similarly, using (4.5) for p and (4.4) for x^1 and x^2, the oligopolist i's problem (4.2) now rewrites as

$$\max_{y_i^2} \left[\log\left(\frac{\sum_{k=1}^{m} y_j^1}{\sum_{\ell=m+1}^{m+n} y_\ell^2}\right) \cdot y_i^2 + \log(\frac{1}{n} - y_i^2)\right].$$

Simple calculations show that, at the oligopoly equilibrium, we have

$$y_j^{*1} = y^{*1} = \frac{m-1}{m(2m-1)}, \qquad j = 1 \dots m;$$

and

$$y_i^{*2} = y^{*2} = \frac{n-1}{n(2n-1)}, \qquad i = m+1, \dots, m+n.$$

Consequently the relative price $p^*(y^{*1} \dots y^{*1}; y^{*2} \dots y^{*2}) = p^*$ at the oligopoly equilibrium is equal to $\frac{(n-1)(2m-1)}{(2n-1)(m-1)}$.

Notice that, when $n \to \infty$, the relative price p^* tends to $\frac{2m-1}{2(m-1)}$, which is the relative price corresponding to the oligopoly equilibrium in the case of a homogeneous oligopoly for commodity 1 with $\varepsilon = 0$ (see (3.6)). Accordingly the homogeneous oligopoly structure is a good approximation to the case of a bilateral oligopoly with all agents behaving non competitively, but with a large number of agents on only one side of the bilateral market: then, each agent on this side exerts a negligible strategic effect on the outcome. Nevertheless, both market structures homogeneous oligopoly and bilateral oligopoly do really coïncide only when the influence of each agent on one side of the market is null, i.e. only when these agents belong to an atomless continuum. The above should be contrasted with the result observed by letting both m and n tend to infinity. Then the relative price p^* tends to 1, which is the relative price at the competitive equilibrium, while the oligopoly supplies tend to the competitive supplies: When the number of oligopolists on *both* sides of the market are simultaneously increased, the oligopoly equilibrium now converges to the competitive equilibrium.

Finally, let us notice that the above example of bilateral oligopoly can easily be extended to cover a situation of *multilateral oligopoly* with an arbitrary number of goods. In this case we consider the following exchange economy: There are ℓ goods h, $h = 1, \dots, \ell$ and $n\ell$ traders, with n traders of type h, $h = 1, \dots, \ell$, the j-th trader of type h being characterized by a initial endowment

$$\omega_{jh} = (0, \dots, \frac{1}{n}, \dots, 0) \qquad j = 1, \dots, n \qquad h = 1, \dots, \ell$$

with $\frac{1}{n}$ as the h-th coordinate of the vector ω_{jh}; and a market's portfolio

$$H_{jh} = \{h\}, \quad j = 1 \dots n, \; h = 1 \dots \ell.$$

Furthermore $U_{jh}(x^1 \dots x^h \dots x^\ell) = \sum_{h=1}^{\ell} \log(x^h)$, for all $j = 1 \dots n$, $h = 1 \dots \ell$.

Accordingly, each agent of type h behaves non competitively on the market for commodity h, which is the sole market for which his initial holdings of the corresponding good is positive. Once again we obtain an example of a complete multilateral oligopoly structure. The above exchange economy has a unique competitive equilibrium, with all equilibrium prices being equal, and equal competitive supplies for all traders with $\tilde{y} = \frac{(\ell-1)}{n\ell}$. Simple calculations show that, at the oligopoly equilibrium, all agents have the same strategic supply y^* with $y^* = \frac{(\ell-1)(n-1)}{n[\ell + (\ell-1)(n-1)]}$. When the number n of traders in each type increases, the oligopoly equilibrium tends to the competitive equilibrium. Now we turn in section 5 to the case of differentiated oligopoly.

5. Differentiated Oligopoly

To capture in a general equilibrium setting the analog of a differentiated market in partial equilibrium analysis, we analyze in this section an exchange model combining our preceding multilateral oligopoly example with the existence of a competitive sector sharing the ownership of the remaining commodities. More precisely consider the following exchange economy, with $m+n$ agents and ℓ, $\ell > m$, goods. We shall index the first m agents by j, $j = 1 \dots m$, and the remaining agents by i, $i = m+1, \dots, m+n$. The initial endowment ω_j of trader j, $j = 1 \dots m$, is defined by

$$\omega_j^h \begin{cases} = 0, & \text{whenever } j \neq h; \\ = 1, & \text{if } j = h, \end{cases}$$

while the market's portfolio is $H_j = \{h\}$. Furthermore

$$U_j(x^1 \dots x^\ell) = \log x^j + \gamma \sum_{h=m+1}^{\ell} \log x^h.$$

As for a trader i, $i = m+1, \dots, m+n$, his initial endowment ω_i is defined by

$$\omega_i^h \begin{cases} = 0, & \text{if } h \le m \\ = \frac{1}{n}, & \text{if } m+1 \le h \le \ell; \end{cases}$$

while his utility function U_i is given by

$$U_i(x^1 \dots x^\ell) = \alpha \sum_{h=1}^{m} \log x^h + (1-\alpha) \log\left(\sum_{h=1}^{m} x^h\right) + \sum_{h=m+1}^{\ell} \log(x^h); \; \alpha \in [0, 1]$$

furthermore, his market's portfolio H_i is the empty set $\{\emptyset\}$. First, notice that if the number of goods (ℓ) would be equal to the number of oligopolists (m), the above exchange economy would be identical in terms of the initial endowments structure and markets' portfolios with the multilateral oligopoly situation considered at the end of section 4. The novelty here consists in adding a competitive sector sharing the ownership of commodities $m+1, \dots, \ell$. On the other hand, each oligopolist j has a utility function which depends only on his own good and the goods initially owned by the competitive sector, he is not interested in consuming the goods owned by the other oligopolists. As for the competitive traders, they are interested in consuming all goods. Notice that the parameter α, $\alpha \in [0,1]$, is a measure of the "degree of differentiation" among the goods owned by the oligopolists: this differentiation increases with α, while $\alpha = 0$ corresponds to the case of homogeneous oligopoly analyzed in section 3.

Now we derive the demand functions of the competitive sector faced with a price vector $(p^1, \dots, p^m; p^{m-1}, \dots, p^\ell)$. In fact we shall restrict ourselves to the case: $p^h = 1$ for all $h = m+1, \dots, \ell$: this is sufficient since we are interested only in equilibrium prices and, at equilibrium, all prices of these commodities owned by the competitive sector must be equal[(1)] with $p^h = 1$, for $h = m+1, \dots, \ell$. Each competitive agent i, $i = m+1, \dots, m+n$, solves the problem

$$\max_{x^1 \dots x^\ell} \alpha \sum_{j=1}^{m} \log x^j + (1-\alpha) \log\left(\sum_{j=1}^{m} x^j\right) + \sum_{h=m-1}^{\ell} \lg x^h \text{ s.t.}$$

$$\sum_{j=1}^{m} p^j x^j + \sum_{h=m-1}^{\ell} x^h = \frac{\ell - m}{n}. \tag{5.1}$$

First order conditions give

$$\frac{\alpha}{x^j} + \frac{(1-\alpha)}{\sum_{k=1}^{m} x^j} = \lambda p^j, \quad j = 1 \dots m; \tag{5.2}$$

$$\frac{1}{x^h} = \lambda, \quad h = m+1, \dots, \ell, \tag{5.3}$$

with λ the Lagrange parameter.

(1) On these markets, *all* agents have a competitive behaviour. On the other hand, for all agents, preferences are symmetric in the commodities corresponding to these markets, and the agregate endowments of each of these commodities are equal. Accordingly, at equilibrium, all prices on these markets must be equal.

From (5.2) and (5.3), and the budget constraint (5.1) we obtain

$$\frac{\ell - m}{n} = \sum_{j=1}^{m} \frac{1}{\lambda}\left(\alpha + \frac{(1-\alpha)x_j}{\sum_{k=1}^{m} x^k}\right) + \frac{\ell - m}{\lambda}$$
$$= \frac{1}{\lambda}(\ell - (m-1)(1-\alpha)),$$

so that, by (5.3)

$$x^h = \frac{\ell - m}{n(\ell - (m-1)(1-\alpha))} \quad \text{if } h = m+1, \dots, l; \tag{5.4}$$

and, by (5.2)

$$p^j = \frac{1}{\lambda}\left(\frac{\alpha}{x^j} + \frac{1-\alpha}{\sum_{k=1}^{m} x^k}\right), \quad \text{if } h = 1, \dots, m. \tag{5.5}$$

Since all agents i in the competitive sector have the same individual demand for each good owned by the oligopolists, it follows from (5.4) and (5.5) that their agregate demand $X^j = nx^j$ for commodity j, $j = 1 \dots m$, must satisfy the equality

$$p^j = \frac{\ell - m}{\ell - (m-1)(1-\alpha)}\left[\frac{\alpha}{X^j} + \frac{(1-\alpha)}{\sum_{k=1}^{m} X^k}\right] \tag{5.6}$$

Now let us consider the oligopolist j's problem, $j = 1 \dots m$. For simplicity and without loss of generality we identify his strategy set

$$Y_j = \left\{(y^1 \dots y^\ell);\ 0 \le y^j \le 1;\ y^h = 0,\ h \ne j\right\}$$

with the unit interval $[0,1]$: in any case he can only manipulate his supply on the market for commodity j. On the other hand, an argument similar to the reasoning held in section 3 shows that, for computing the oligopoly equilibrium, we may restrict ourselves to the strategy set $\tilde{Y}_j = \left[0, \frac{\gamma(\ell - m)}{1 + \gamma(\ell - m)}\right]$ where $\frac{\gamma(\ell - m)}{1 + \gamma(\ell - m)}$

corresponds to his net competitive supply of good j [2]. Faced with a price vector $(p^1, \dots p^j \dots p^m; p^{m-1} \dots p^\ell)$ and having chosen a strategy y^j, oligopolist j solves the problem

$$\max_{x^{m+1}\dots x^\ell} \mathrm{Lg}\,(1-y_j) + \gamma \sum_{h=m+1}^{\ell} \mathrm{Lg}\, x^h \quad \text{s.t.} \quad \sum_{h=m+1}^{\ell} p^h x^h \le p^j y^j;$$

$$\text{i.e.} \quad p^h x^h = \frac{1}{\ell - m} p^j y^j, \quad h = m+1, \dots, \ell,$$

so that the utility of oligopolist j is equal to

$$\mathrm{Lg}\,(1-y^j) + \gamma \sum_{h=m+1}^{\ell} \frac{1}{\ell - m} \frac{p^j}{p^h} y^j. \tag{5.7}$$

Now consider a vector of supply strategies $(y^1 \dots y^j \dots y^m)$ for the m oligopolists and let $p = (p^1 \dots p^j \dots p^m; p^{m-1} \dots p^\ell)$ be a price system which clears all markets in the exchange economy. Then, $p^h = 1$ for $h = m+1 \dots \ell$, and for all $j = 1 \dots m$, we must have

$$X^j = y^j. \tag{5.8}$$

Using (5.6) and (5.8), we obtain at equilibrium

$$p^j = \frac{\ell - m}{\ell - (m-1)(1-\alpha)} \left[\frac{\alpha}{y^j} + \frac{(1-\alpha)}{\sum_{k=1}^{m} y^k} \right], \quad \text{if } j = 1, \dots, m \tag{5.9}$$

$$p^h = 1, \quad \text{if } h = m+1, \dots, \ell.$$

[2] Given the utility function of oligopolist j, the net competitive supply of commodity j is equal to $1 - x_j$ with x_j solution to the problem

$$\max_{x^j, x^h} \mathrm{Lg}\, x^j + \gamma \sum_{h=m+1}^{\ell} \mathrm{Lg}\, x^h$$

$$\text{s.t.} \ p^j x^j + \sum_{h=m+1}^{\ell} x^h \le p^j,$$

$$\text{i.e.} \ x^j = \frac{1}{1 + \gamma(\ell - m)}.$$

Consequently, considering the equilibrium prices given by (5.9), and a vector $(y^1 \dots y^j \dots y^m)$ of supply strategies chosen by the oligopolists, the utility of oligopolist j obtains as (see (5.7))

$$V_j(y^1 \dots y^j \dots y^m) \underset{def}{=} \mathrm{Lg}\,(1-y^j) + \gamma(\ell - m)\mathrm{Lg} \left(\frac{1}{\ell - m} p^j y^j \right)$$

with $p^j = p^j(y^1 \dots y^j \dots y^m)$ as given by (5.9).

Let us now study the oligopoly equilibrium. At an oligopoly equilibrium, we must have

$$\frac{\partial V_j}{\partial y^j} = 0, \qquad \text{for } j = 1 \dots m.$$

The derivative $\dfrac{\partial V_j}{\partial y^j}$ has the same sign as

$$S_j(y^j) = -\left(y^j + \alpha \sum_{k \ne j} y^k \right) \left(y^j + \sum_{k \ne j} y^k \right) + \gamma(\ell - m)(1-\alpha)(1-y^j) \sum_{k \ne j} y^k. \tag{5.10}$$

When $\alpha = 1$, $V_j(y^j)$ is negative for $y^j > 0$ and the maximal value of V_j is reached for $y^j = 0$. Accordingly the oligopoly equilibrium is "degenerate": all supplies are zero and prices are equal to infinity. When $0 < \alpha < 1$, the oligopoly equilibrium is characterized by the conditions $V^j(y^j) = 0$ and $y^j > 0$, $j = 1 \dots m$. This is equivalent to

$$y^j = y^* = \frac{\gamma(\ell - m)(1-\alpha)(m-1)}{\gamma(\ell - m)(1-\alpha)(m-1) + m(\alpha m + 1 - \alpha)}, \tag{5.11}$$

so that there exists a non degenerate oligopoly equilibrium, given by (5.11), whenever $0 \le \alpha < 1$ and $m \ge 2$.

In the particular case $\alpha = 0$, $\gamma = 1$, $\ell - m = 1$, we are led back to the case of a homogeneous oligopoly (without a competitive fringe) and the oligopoly equilibrium given by (5.11) is the same as the oligopoly equilibrium given by (3.6).

Substituting $y^j = y^*$, $j = 1 \dots m$, in the expressions (5.9), we obtain the prices $p^j = p^*$ corresponding to the non degenerate oligopoly equilibrium, i.e.

$$p^{*j} = p^* = \frac{(\alpha m + 1 - \alpha)[\gamma(\ell - m)(1-\alpha)(m-1) + m(\alpha m + 1 - \alpha)]}{\alpha \gamma m(\alpha m + 1 - \alpha + \ell - m)(1-\alpha)(m-1)}. \tag{5.12}$$

Finally we compute the competitive price $\tilde{p}^j = \tilde{p}$ for our exchange economy. The price is obtained from equation (5.9) by letting y^j be equal to the net competitive supply of commodity j by oligopolist j, i.e. $y^j = \dfrac{\gamma(\ell - m)}{1+\gamma(\ell-m)}$, so that

$$\tilde{p} = \frac{(m\alpha + 1 - \alpha)(1+\gamma(\ell - m))}{m\gamma(m\alpha + 1 - \alpha + \ell - m)}. \tag{5.13}$$

We are now in a position for discussing the dependence of the non degenerate oligopoly equilibrium with respect to the parameters of the model, i.e. α, ℓ, m and n. Let us start out with α. Define $A = \dfrac{m(\alpha m + 1 - \alpha)}{\gamma(\ell - m)(1-\alpha)(m-1)}$ so that $y^* = \frac{1}{1+A}$.
We have: $\dfrac{1}{A}\dfrac{\partial A}{\partial \alpha} = \dfrac{m-1}{\alpha m + 1 - \alpha} + \dfrac{1}{1-\alpha} > 0$, and $\dfrac{1}{y^*}\dfrac{\partial y^*}{\partial \alpha} = -\dfrac{1}{1+A}\dfrac{\partial A}{\partial \alpha} < 0$.
Consequently, y^* is a decreasing function of α. Using (5.11), we may rewrite p^*, given by (5.12), as

$$p^* = \frac{(\ell - m)(m\alpha + 1 - \alpha)}{(m\alpha + 1 - \alpha + \ell - m)my^*} \tag{5.14}$$

and we get

$$\frac{1}{p^*}\frac{\partial p^*}{\partial \alpha} = \frac{m-1}{m\alpha + 1 - \alpha} - \frac{m-1}{m\alpha + 1 - \alpha + \ell - m} - \frac{1}{y^*}\frac{\partial y^*}{\partial \alpha} > -\frac{1}{y^*}\frac{\partial y^*}{\partial \alpha} > 0 :$$

The oligopoly equilibrium price p^* is increasing with α. Accordingly, the larger the degree of differentiation between the commodities sold by the oligopolists (as measured by α), the weaker competition among them. With m fixed, the value of $l - m$ denotes the number of commodities initially owned by the competitive sector. An increase in ℓ means a widen variety of these commodities which are bought by the oligopolists at equilibrium. When $\ell \to \infty$, the oligopoly equilibrium strategy y^* tends to 1, which is the initial endowment of the oligopolists for the commodity they own. From (5.12) and (5.11) we get

$$\frac{p^*}{\tilde{p}} = \frac{\gamma(\ell - m)}{[1+\gamma(\ell - m)]\, y^*}$$

and, since $y^* \to 1$ when $\ell \to \infty$, we obtain

$$\lim_{\ell\to\infty} \frac{p^*}{\tilde{p}} = 1 :$$

When the number of commodities owned by the competitive sector tends to $+\infty$, the oligopoly equilibrium tends to the competitive equilibrium. This paradoxical

result follows from the fact that the desirability of the oligopolists for the goods owned by the competitive sector is increasing with ℓ. This desirability is measured by $\gamma(\ell - m)$, since their utility for equal quantities $\overline{x}$ of the competitive goods is given by $\gamma(\ell - m)\mathrm{Lg}\,\overline{x}$. If this "preference for variety" would not exist, so that the desirability would be constant with ℓ, say if $\gamma = \dfrac{1}{\ell - m}$, then the oligopoly equilibrium y^* would be also invariant with ℓ. (see (5.9)). In view of analyzing the dependence of the oligopoly equilibrium w.r.t. to the number m of oligopolists, assume that there is a fixed number, say $\overline{n}$, of goods in the competitive sector, so that

$$\ell = \overline{n} + m.$$

Substituting ℓ by its value in (5.13), the competitive price $\tilde{p}$ becomes:

$$\tilde{p} = \frac{(m\alpha + 1 - \alpha)(1+\gamma\overline{n})}{m\gamma(m\alpha + 1 - \alpha + \overline{n})}. \tag{5.15}$$

Similarly, we obtain from (5.12)

$$p^* = \frac{(\alpha m + 1 - \alpha)\,[\gamma\overline{n}(1-\alpha)(m-1) + m(\alpha m + 1 - \alpha)]}{\alpha\gamma m(\alpha m + 1 - \alpha + \overline{n})(1-\alpha)(m-1)}. \tag{5.16}$$

When the number m of oligopolists tends to $+\infty$, with $\overline{n}$ fixed, the competitive price $\tilde{p}$ tends to zero. On the contrary, we obtain from (5.16)

$$\lim_{m\to\infty} p^* = \frac{\alpha}{(1-\alpha)\gamma}.$$

It is only in the case of *homogeneous oligopoly* ($\alpha = 0$) that the oligopoly equilibrium price tends to the competitive price; otherwise ($\alpha > 0$), the convergence of the oligopoly equilibrium to the competitive equilibrium no longer holds. This problem arises from the fact that the dimension of the commodity space increases with m: should we have increased the number of oligopolists of each type, by letting fixed the number of goods, we would have obtained again the convergence property, as in the case of multilateral oligopoly.

Finally we notice that the size of the competitive sector – the number n of competitive agents – does not play any role at the oligopoly equilibrium. Once again, the competitive sector could have been modeled as well as an atomless continuum and, even with individual agents in the continuum behaving non competitively, the oligopoly equilibrium would have been identical.

6. Exchange with a productive sector

A natural question arises at the end of the above analysis: how to extend the oligopoly equilibrium concept when the economy includes a productive sector? We do not investigate this important problem in the framework of the present paper, devoted to the analysis of imperfectly competitive exchange. Neverthlesss we provide in this section a simple example, which can be viewed as a sensible starting point for such an investigation. The economy we consider includes two types of agents and two goods. The first good is initially owned by the first type of agents, who have a competitive behaviour on the exchange market. The second good does not exist a priori in the economy, but can be produced, out of the first good, by the second type of agents, who are each endowed with a linear technology transforming the first good into the second. These agents producing good 2 have a non competitive behaviour on the exchange market. More precisely, consider the following exchange situation. There are two goods and $n+m$ agents; all agents have the same utility function $U(x^1, x^2) = \text{Lg } x^1 + \text{Lg } x^2$. We shall index the first n agents by i, $i = 1, \dots, n$ and the remaining m agents by $j = 1, \dots, m$. The initial endowment of agent i, $i = 1, \dots, n$, is defined by

$$\omega_i = (\frac{1}{n}, 0)$$

and his market's portfolio by

$$H_i = \{\emptyset\}.$$

As for an agent j, $j = 1, \dots, m$, his initial endowment is $\omega_j = (0,0)$, so that he does not own initially any of the two goods. However he owns a linear technology, defined by

$$y_j = \frac{1}{\alpha_j} z_j, \; \alpha_j > 0 \tag{6.1}$$

where y_j denotes the amount of good 2 he can produce out of an amount z_j of good 1. An agent j, $j = 1, \dots, m$, has a market's portfolio H_j defined by $H_j = \{2\}$. The novelty here is that an agent j must now take *two* decisions: which amount y_j of good 2 to produce (which determines via (6.1) the amount z_j of good 1 to be bought), and which share q_j of the amount y_j produced of good 2 to supply in exchange of good 1. Accordingly, a *strategy* for an agent j, $j = 1, \dots, m$, is a pair (q_j, y_j) with $q_j \leq y_j$.

Given a price vector (p^1, p^2), each competitive agent i solves

$$\max_{x^1, x^2} \text{Lg } x^1 + \text{Lg } x^2 \quad \text{s.t.} \quad p^1 x^1 + p^2 x^2 \leq \frac{p^1}{n},$$

so that the individual demand of agent i is

$$x_i(p, \omega_i) = \left(\frac{1}{2n}, \frac{p^1}{2p^2 n}\right),$$

and agregate demand is

$$\left(\frac{1}{2}, \frac{p^1}{2p^2}\right). \tag{6.2}$$

Now consider the problem of oligopolist j when he faces a price system (p^1, p^2) and he has chosen a strategy (q_j, y_j). His profit $\Pi_j(q_j, y_j)$ is defined by

$$\begin{aligned} \Pi(q_j, y_j) &= p^2 q_j - p^1 z_j \\ &= p^2 q_j - p^1 \alpha_j y_j \end{aligned}$$

(see (6.1)). Accordingly he can buy any bundle (x^1, x^2), the value of which does not exceed $\Pi_j(q_j, y_j)$. Thus, having chosen (q_j, y_j) and faced with a price system (p^1, p^2), he solves the problem

$$\max_{x^1, x^2} \text{Lg } x^1 + \text{Lg } (y_j - q_j + x^2) \text{ s.t. } x^1 \geq 0,\; x^2 \geq 0 \text{ and } p^1 x^1 + p^2 x^2 \leq \Pi_j(q_j, y_j). \tag{6.3}$$

First, the positivity constraints imply that $\Pi_j(q_j, y_j) \geq 0$, or

$$\frac{p^1}{p^2} \alpha_j y_j \leq q_j. \tag{6.4}$$

On the other hand, using an argument similar to the argument used in section 3, it can be shown that, given y_j, any utility level which can be reached by choosing q_j smaller or equal to y_j, can also be reached by choosing q_j in such a way that the amount $y_j - q_j$ kept for later consumption is at least equal to the competitive demand of good 2, i.e. $\frac{1}{2} y_j \left(1 - \alpha_j \frac{p^1}{p^2}\right)$(3). Accordingly, we shall consider only strategies (q_j, y_j) satisfying the constraint

$$q_j \leq \frac{1}{2} y_j \left(1 + \alpha_j \frac{p^1}{p^2}\right). \tag{6.5}$$

(3) The competitive demand obtains from solving the problem

$$\max_{x^1, x^2} \text{Lg } x^1 + \text{Lg } x^2 \text{ s.t. } p^1 x^1 + p^2 x^2 \leq p^2 y_j - p^1 \alpha_j y_j,$$

which gives $x^2 = \frac{1}{2} y_j \left(1 - \alpha_j \frac{p^1}{p^2}\right)$.

In that case, the solution to problem (6.2) is given by $x^2 = 0$ and $x^1 = \frac{p^2}{p^1} q_j - \alpha_j y_j$. Consequently, for strategies (q_j, y_j) satisfying (6.4) and (6.5), the utility level reached by oligopolist j at the solution of (6.3) is

$$V(q_j, y_j) = \mathrm{Lg}\left(\frac{p^2}{p^1} q_j - \alpha_j y_j\right) + \mathrm{Lg}\,(y^j - q^j). \tag{6.6}$$

Let us now compute the oligopoly equilibrium. Given an m-tuple of strategies $((y^1, q^1), \ldots, (y^j, y^j), \ldots, (y^m, q^m))$, the equilibrium price vector must satisfy

$$\frac{p^1}{2p^2} = \sum_{k=1}^{m} q_k$$

(see (6.2)), so that

$$\frac{p^2}{p^1} = \frac{1}{2\sum_{k=1}^{m} q_k}. \tag{6.7}$$

Substituting (6.7) in (6.6), we obtain

$$V(q_j, y_j) = \mathrm{Lg}\left(\frac{q_j}{2\sum_{k=1}^{m} q_k} - \alpha_j y_j\right) + \mathrm{Lg}\,(y_j - q_j).$$

At an oligopoly equilibrium, V must be maximal with respect to q_j and y_j, given the strategies (q_k, y_k) chosen by the oligopolists k, $k \neq j$, and this must be satisfied for all j, $j = 1, \ldots, m$. Denoting by Q the value of $\sum_{k=1}^{m} q_k$, the optimality conditions with respect to q_j and y_j give

$$\frac{1}{y_j - q_j} = \frac{(Q - q_j)/2Q^2}{(q_j/2Q) - \alpha_j y_j} = \frac{\alpha_j}{(q_j/2Q) - \alpha_j y_j}.$$

From the second equality of the above equation, we obtain that, for all $j = 1, \ldots, m$, we have at equilibrium

$$1 - \frac{q_j}{Q} = 2\alpha_j Q, \quad j = 1, \ldots, m. \tag{6.8}$$

Summing up these equations, we get, at the oligopoly equilibrium, with $Q^* = \sum_{j=1}^{m} q_j^*$, and $\overline{\alpha} \underset{def}{=} \frac{1}{m}\sum_{j=1}^{m} \alpha_j$,

$$Q^* = \frac{m-1}{2m}\,\frac{1}{\overline{\alpha}} \tag{6.9}$$

and

$$p^* = \frac{m}{m-1}\,\overline{\alpha}, \tag{6.10}$$

with p^* denoting the relative price $\frac{p^2}{p^1}$ corresponding to the oligopoly equilibrium (see (6.7)). Furthermore we obtain

$$q_j^* = \left(1 - \frac{\alpha_j}{p^*}\right) Q^*$$

and

$$y_j^* = \frac{1}{2}\left(1 + \frac{p^*}{\alpha_j}\right) q_j^*.$$

If we identify the oligopolist j with a firm - say, firm j - the oligopoly equilibrium has m "active" firms if, and only if,

$$\alpha_j < p^* = \frac{m}{m-1}\overline{\alpha} = \frac{1}{m-1}\left(\sum_{k=1}^{m} \alpha_k\right). \tag{6.11}$$

First we notice that, if all firms are identical, i.e. $\alpha_j = \overline{\alpha}$, the m firms are active at the oligopoly equilibrium. Furthermore, when $m \to \infty$, $p^* = \frac{m}{m-1}\overline{\alpha} \to \overline{\alpha}$, which is the competitive price with zero profits. On the other hand, when the "average costs" α_j's are not equal, condition (6.1) does not necessarily hold for all j at the oligopoly equilibrium with m firms. If it is the case, all firms cannot be simultaneously active, and some oligopolists have to leave the market. In the case of two firms ($m = 2$), $p^* = \alpha_1 + \alpha_2$, and condition (6.11) is always satisfied. In the case of three firms, we have $p^* = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3)$, and $\alpha_j < p^* \Leftrightarrow \alpha_j < \alpha_k + \alpha_\ell$. Then an oligopolist whose "average cost" α_j exceeds the sum of the average costs of the two others would be excluded from the market. When m firms, with different α_j's, are competing at the oligopoly equilibrium, and one of these firms has an average cost which exceeds $p^*(m)$, it has to leave the market. Then, the $m - 1$ remaining firms are competing at the oligopoly equilibrium with $m - 1$ firms. If, for some firms j, we have $\alpha_j < p^*(m-1)$, these firms should in turn leave the market. Finally, proceeding by successive eliminations, one obtains an oligopoly equilibrium with an endogenous number of active firms. This description should be contrasted with the elimination process leading to the competitive solution, at which the only firm remaining active is firm j for which $\alpha_j = \min_{k=1,\ldots,m}\{\alpha_k\}$.

The above example reveals that, contrary to the decentralization property of the competitive process, the owner of a firm operating in an imperfectly competitive market, cannot separate his production from his consumption decisions: the level of production has to be determined by taking into account simultaneously both a consumption objective and a market objective, in view of exploiting at best the price response of the market to the oligopolists' supply, while in a competitive market, a firm realizes *de facto* the best market opportunities for its owners when it chooses its production level in accordance with the profit maximization rule. This lack of decentralization makes the treatment of production a very complex problem in the framework of general equilibrium, when markets operate under imperfectly competitive conditions. Strictly speaking, the enterprises should take explicitly into account, when making their production decisions, the preferences of their shareholders. If a firm is owned by a single individual, it can be done in the manner we did it in the previous example. When there is more than a single owner, and when the shareholders do not have the same preferences and/or the same shares, some alternative device must be studied, like an agregation rule of shareholders' preferences, or a voting procedure. In any case, the profit maximization criterion is no longer logically valid(4).

7. Conclusion

In this paper we have extended the notion of Cournot-Walras equilibrium introduced by Codognato and Gabszewicz (1991) in a particular context (homogeneous oligopoly) to a general class of pure exchange economies.(5) Some examples were considered, corresponding to oligopolistic structures analyzed in partial equilibrium. Finally, we have discussed a possible extension of the concept to the case of a productive economy, and illustrated this discussion by means of an example embodying linear technologies.

(4) Furthermore it is known that this criterion entails other difficulties in the context of imperfect competition. As shown in Gabszewicz and Vial (1972), the oligopoly equilibrium when defined with this criterion is not invariant with respect to the normalization rule used for the definition of the price system. On the other hand, the oligopoly equilibrium resulting from the profit maximization criterion can be Pareto-dominated from the viewpoint of the shareholders, by the choice of an alternative production plan.

(5) A general approach to the imperfect competition paradigm covering as a particular case the paper by Codognato and Gabszewicz (1991) has been recently proposed by d'Aspremont et al. (1992).

Even for the restrictive context of exchange, there are still important issues remaining unanswered. In particular, it is known that the existence of a non cooperative equilibrium in pure strategies for this class of models, poses extremely difficult problems, which are even met in the context of partial equilibrium analysis. One possible avenue for circumventing these difficulties consists in examining the asymptotic behaviour of oligopoly equilibria when the number of oligopolists increases indefinitily (see K. Roberts (1980)). When the sequence of oligopoly equilibria converges to a regular competitive equilibrium, local analysis applies and may provide the existence of an oligopoly equilibrium in the neighborhood of this competitive equilibrium. We plan to devote our attention to this problem in a near future.

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