

Very low energy matching of effective meson theories with QCD

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Abstract

A simple matching procedure is proposed to extract constraints on effective meson theories. In this way, a QCD prediction for the pion decay constant is found, $F_\pi = 2m_\pi/\pi \approx 90$ MeV. The same procedure also determines other mesonic observables, like the decay width of the sigma meson to two photons.

Finally, some information which can be gained about the hadronic light-by-light contributions to the muon anomalous moment are briefly commented.

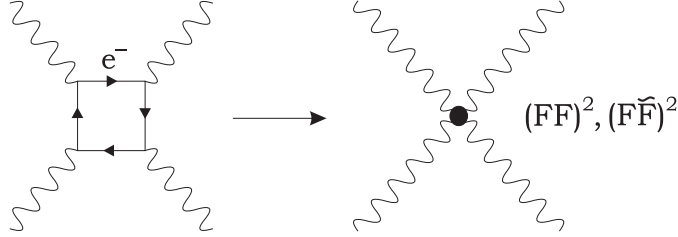
When the electron field is integrated out from the QED action, one gets an effective theory where only photons can propagate. In addition, virtual fermion loops generate an infinite tower of self-interactions among the photons. Schwinger calculated the one-loop effective action, for *constant* electromagnetic fields [1], to all orders

$$\mathcal{L}_{eff,fermion}^{(1)} = -\frac{1}{8\pi^2} \int_0^\infty \frac{d\tau}{\tau^3} e^{-m^2\tau} \left[(e\tau)^2 ab \frac{\cosh(ea\tau) \cos(eb\tau)}{\sinh(ea\tau) \sin(eb\tau)} - 1 \right] \quad (1)$$

with a, b solutions of $a^2 - b^2 = \mathbf{E}^2 - \mathbf{B}^2$ and $ab = \mathbf{E} \cdot \mathbf{B}$. When expanded with respect to the fermion mass (or α), one gets the four-photon interactions as described by the well-known Euler-Heisenberg effective Lagrangian [2]

$$\mathcal{L}_{EH,fermion}^{(1)} = \frac{\alpha^2}{90m^4} \left((F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right)$$

with the definition $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. Diagrammatically, the integration of the fermion fields can be represented as



The same treatment can be applied to get the photon effective couplings generated by virtual quark loops

$$\mathcal{L}_{EH,quarks}^{(1)} = e_Q^4 \frac{\alpha^2 N_c}{90m_Q^4} \left((F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right) \quad (2)$$

where N_c is the number of colors, e_Q is the quark charge (in unit of e) and m_Q its mass.

The starting point of the present letter is to assume that the relative strength of the $(F_{\mu\nu} F^{\mu\nu})^2$ and $(F_{\mu\nu} \tilde{F}^{\mu\nu})^2$ couplings is preserved through the complicate dressing by QCD of the quarks into hadrons. In other words, the hypothesis is that strong corrections will be absorbed into the quark mass, the only free parameter. The consequence is then that the same effective interaction among the photons should be obtainable starting from an effective meson theory. By matching the photon effective theories obtained by

integrating out meson fields to that obtained by integrating out the quark fields (2), we will get a set of constraints on the parameters of the effective meson theory. Ultimately, the validity of the basic hypothesis will be tested by comparison with experiment.

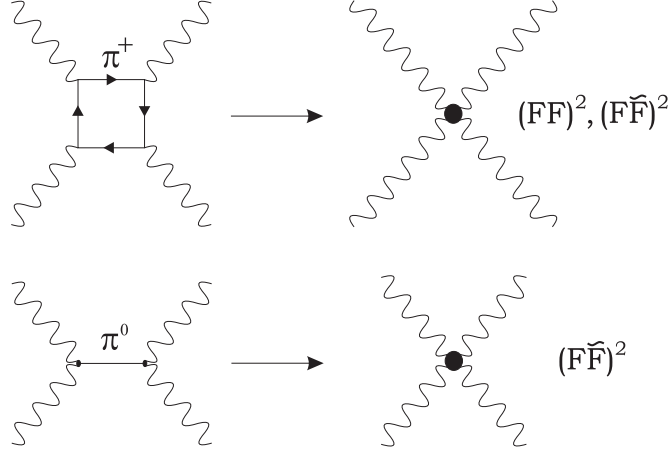
At very low energy, a few eV say, the contribution from pions will dominate. We will use the scalar QED Lagrangian for the charged pions

$$\mathcal{L}_{\pi^\pm} = \partial_\mu \pi^\pm \partial^\mu \pi^\mp - m_\pi^2 \pi^\pm \pi^\mp - ieA_\mu (\pi^\pm \partial_\mu \pi^\mp - \partial_\mu \pi^\pm \pi^\mp) + e^2 A_\mu A^\mu \pi^\pm \pi^\mp$$

and for the neutral pions, we introduce the coupling to two photons

$$\mathcal{L}_{\pi^0} = \frac{1}{2} [\partial_\mu \pi^0 \partial^\mu \pi^0 - m_\pi^2 \pi^0 \pi^0] + g_\pi F_{\mu\nu} \tilde{F}^{\mu\nu} \pi^0$$

We now integrate out the pion fields from $\mathcal{L}_{\pi^\pm} + \mathcal{L}_{\pi^0}$. This generates contact interactions among photons:



(it is understood that seagull interaction contributions are included).

Schwinger computed the effective action obtained by integrating out the charged pions, with the result [1]

$$\mathcal{L}_{eff, \pi^\pm}^{(1)} = \frac{1}{16\pi^2} \int_0^\infty \frac{d\tau}{\tau^3} e^{-m^2\tau} \left[(e\tau)^2 ab \frac{1}{\sinh(ea\tau) \sin(eb\tau)} - 1 \right] \quad (3)$$

When expanded, the four-photon effective couplings are generated

$$\mathcal{L}_{EH, \pi^\pm}^{(1)} = \frac{\alpha^2}{90m_{\pi^\pm}^4} \left(\frac{7}{16} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{1}{16} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right) \quad (4)$$

To this effective Lagrangian, we add the effective interaction generated by a neutral pion exchange

$$\mathcal{L}_{EH,\pi^0}^{(1)} = \frac{g_\pi^2}{2m_{\pi^0}^2} \left(F^{\mu\nu} \tilde{F}_{\mu\nu} \right)^2 \quad (5)$$

Combining (4) with (5), we get

$$\mathcal{L}_{EH,\pi}^{(1)} = \frac{7}{16} \frac{\alpha^2}{90m_{\pi^\pm}^4} (F_{\mu\nu} F^{\mu\nu})^2 + \left(\frac{g_\pi^2}{2m_{\pi^0}^2} + \frac{1}{16} \frac{\alpha^2}{90m_{\pi^\pm}^4} \right) \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right)^2 \quad (6)$$

The core of the method is to match this effective Lagrangian to the fermionic one (2), to extract a prediction for g_π

$$\frac{\frac{g_\pi^2}{2m_{\pi^0}^2} + \frac{1}{16} \frac{\alpha^2}{90m_{\pi^\pm}^4}}{\frac{7}{16} \frac{\alpha^2}{90m_{\pi^\pm}^4}} = \frac{7}{4} \rightarrow g_\pi = \frac{\alpha}{8} \frac{m_{\pi^0}}{m_{\pi^\pm}^2}$$

For the decay rate $\pi^0 \rightarrow \gamma\gamma$, this gives

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = \frac{m_{\pi^0}^3}{4\pi} g_\pi^2 = \frac{\alpha^2}{256\pi} \frac{m_{\pi^0}^5}{m_{\pi^\pm}^4} = 7.7 \text{ eV}$$

To be compared to the experimental value $\Gamma^{\text{exp}}(\pi^0 \rightarrow \gamma\gamma) = (7.7 \pm 0.6) \text{ eV}$ [3]. The agreement is very good.

If the description of $\pi^0 \rightarrow \gamma\gamma$ in terms of the axial anomaly is used ([4], [5]), we get a determination of F_π

$$g_\pi = \frac{\alpha N_c}{12\pi F_\pi} = \frac{\alpha}{8} \frac{m_{\pi^0}}{m_{\pi^\pm}^2} \rightarrow F_\pi = \frac{2N_c}{3\pi} \frac{m_{\pi^\pm}^2}{m_{\pi^0}} = 91.9 \text{ MeV} \quad (7)$$

again very close to the experimental value $F_\pi = 92.4 \pm 0.3 \text{ MeV}$ (obtained from $\Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu)$, see [3]).

A comment is in order. Usually, in chiral perturbation theory, the role played by m_π and F_π is radically different: F_π sets the scale of the Goldstone boson interactions, while m_π is only a small explicit breaking of the spontaneously broken symmetry; their physical content is therefore quite different and a priori unrelated. On the other hand, if the fermionic character of the underlying theory is assumed to be preserved through the passage from the quark picture to the hadron picture, the two turn out to be proportional.

As a by-product, we can also estimate the value of the constituent quark mass for which the two descriptions match, i.e. for which the four-photon couplings are the same in absolute magnitude. Setting $m_u = m_d$

$$e_u^4 \frac{\alpha^2 N_c}{90 m_u^4} + e_d^4 \frac{\alpha^2 N_c}{90 m_d^4} = \frac{7}{16} \frac{\alpha^2}{90 m_{\pi^\pm}^4} \rightarrow m_u = m_{\pi^\pm} \sqrt[4]{(e_u^4 + e_d^4) \frac{48}{7}} \approx 1.1 \times m_{\pi^\pm}$$

i.e. a constituent quark mass $m_u = m_d \approx 153$ MeV, quite close to m_π .

Introducing Higher Mass Particles

In principle, one can introduce the $\eta, \eta', K^\pm, \dots$, to get the effective Lagrangian

$$\begin{aligned} \mathcal{L}_{EH,PS}^{(1)} = & \left(\frac{g_\pi^2}{2m_{\pi^0}^2} + \frac{g_\eta^2}{2m_\eta^2} + \frac{g_{\eta'}^2}{2m_{\eta'}^2} + \frac{\alpha^2}{1440} \left(\frac{1}{m_{\pi^\pm}^4} + \frac{1}{m_{K^\pm}^4} \right) \right) (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \\ & + \frac{7\alpha^2}{1440} \left(\frac{1}{m_{\pi^\pm}^4} + \frac{1}{m_{K^\pm}^4} \right) (F_{\mu\nu} F^{\mu\nu})^2 \end{aligned} \quad (8)$$

with $g_{\eta^{(\prime)}}$ the coupling constant for $\eta^{(\prime)} \rightarrow \gamma\gamma$. We see that the pion contributions are by far the dominant ones: the corrections induced by heavier mesons are of a few percents, as can be seen by plugging in the experimental values $g_\eta \approx 6 \times 10^{-6} \text{ MeV}^{-1}$ and $g_{\eta'} \approx 8 \times 10^{-6} \text{ MeV}^{-1}$ [3]. Matching (8) to (2) and solving for F_π , we find $F_\pi \approx 96$ MeV. This shows that the matching should not be expected to work to a better accuracy than roughly 5%, despite the striking result (7).

We now turn to the introduction of resonances, and in particular of the sigma meson. Our point of view is to consider the sigma as a resonance occurring in the two-pion channel. We take as effective Lagrangian

$$\mathcal{L}_{EH,\pi^0,\sigma}^{(1)} = \frac{g_\sigma^2}{2m_\sigma^2} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{g_\pi^2}{2m_{\pi^0}^2} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \quad (9)$$

with g_σ the coupling constant for $\sigma \rightarrow \gamma\gamma$. To get this form, we assume that in a first approximation, the scalar channel (i.e. $(F_{\mu\nu} F^{\mu\nu})^2$) is saturated by the sigma, and we neglect the charged pion contribution to the pseudoscalar channel $(F_{\mu\nu} \tilde{F}^{\mu\nu})^2$ (which is roughly 10 times smaller than the neutral pion

contribution, see (6)). The effective photon Lagrangian (9) is now only generated by virtual π^0 and σ exchanges, which are particles respectively associated with the axial [4] and trace anomaly [6].

By matching (9) with (2), we can relate g_σ to g_π

$$\frac{g_\sigma^2/2m_\sigma^2}{g_\pi^2/2m_{\pi^0}^2} = \frac{4}{7} \rightarrow \frac{g_\sigma}{g_\pi} = \sqrt{\frac{4}{7}} \frac{m_\sigma}{m_{\pi^0}} \approx 2.8 \quad (10)$$

for $m_\sigma \approx 530$ MeV [7]. From (10), the sigma width to two photons is predicted to be

$$\Gamma(\sigma \rightarrow \gamma\gamma) = \frac{m_\sigma^3}{4\pi} g_\sigma^2 = \frac{1}{7\pi} \frac{m_\sigma^5}{m_{\pi^0}^2} g_\pi^2 = \frac{\alpha^2}{448\pi} \frac{m_\sigma^5}{m_{\pi^\pm}^4} \approx 4.1_{-2.1}^{+3.5} \text{ keV} \quad (11)$$

with $m_\sigma = (530 \pm 70)$ MeV [7]. This is compatible with

$$\Gamma(\sigma \rightarrow \gamma\gamma) = (3.8 \pm 1.5) \text{ keV} \quad (12)$$

found by a partial-wave analysis of $\gamma\gamma \rightarrow \pi\pi$ [8]. Note, by the way, that because m_σ appears to the fifth power in (11), this formula is in fact quite efficient in constraining m_σ once $\Gamma(\sigma \rightarrow \gamma\gamma)$ is known. For instance, from the experimental value (12), one finds $m_\sigma = 520_{-49}^{+36}$ MeV.

In conclusion, the assumption that the sigma resonance nearly saturates the two-pion scalar channel gives a reasonable prediction for $\Gamma(\sigma \rightarrow \gamma\gamma)$.

Conclusions and Perspectives

In this letter, we have shown how to get information on the coupling constant of effective meson theories from QCD. As said, the present framework relies entirely on the assumption that the strong interactions do not renormalize the relative strength of the $(F_{\mu\nu}F^{\mu\nu})^2$ and $(F_{\mu\nu}\tilde{F}^{\mu\nu})^2$ couplings (or at least that this renormalization is small). The fact that we found reasonable predictions for F_π and $\Gamma(\sigma \rightarrow \gamma\gamma)$ seems to validate this assumption. Further work should clarify the range of validity of the present method.

Encouraged by the success of the matching at $O(\alpha^2/m^4)$, one could now undertake an analysis at the next order. At $O(\alpha^3/m^8)$, the fermionic effective Lagrangian is [1], [9]

$$\mathcal{L}_{DKR,quarks}^{(1)} = e_Q^6 N_c \frac{\pi\alpha^3}{315m_Q^8} \left(-4(F_{\mu\nu}F^{\mu\nu})^3 - \frac{13}{2} (F_{\alpha\beta}F^{\alpha\beta}) (F_{\mu\nu}\tilde{F}^{\mu\nu})^2 \right)$$

At that order, the integration of the pseudoscalars is more involved, and it remains to be seen which kind of constraints can emerge. Also, the extension to non-constant electromagnetic fields may lead to interesting results. All these questions are left for future studies.

We close this letter with a comment on the computation of the hadronic light-by-light corrections to the muon anomalous moment. Even if it is true that the momentum configuration in that case and in our case is quite different, our approach can offer an interesting limiting case in which the various theoretical models can be tested. For instance, in the present approach, it appears that the pions and the constituent quarks do not contribute simultaneously. The same is true for the sigma meson and the charged pions. In the context of the muon anomalous moment, the constituent quark, charged pion and sigma meson contributions are all considered at the same time (see for example [10]). Whether this leads to double-counting or not is, in our opinion, not settled.

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