

LONG-TIME ASYMPTOTICS FOR AN INTEGRABLE EVOLUTION EQUATION WITH A 3×3 LAX PAIR

C. CHARLIER AND J. LENELLS

*Department of Mathematics, KTH Royal Institute of Technology,
100 44 Stockholm, Sweden.*

ABSTRACT. We derive a Riemann–Hilbert representation for the solution of an integrable nonlinear evolution equation with a 3×3 Lax pair. We use the derived representation to obtain formulas for the long-time asymptotics.

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1. INTRODUCTION

We consider the nonlinear evolution equation

$$iq_t - \frac{1}{\sqrt{3}}q_{xx} + 2\sqrt{3}\bar{q}q_x = 0, \quad (1.1)$$

where $q(x, t)$ is a complex-valued function of a space variable $x \in \mathbb{R}$ and a time variable $t \geq 0$. Equation (1.1) is a type of quadratic derivative nonlinear Schrödinger (NLS) equation: instead of the cubic nonlinearity $|q|^2q$ of the standard NLS equation, it features the quadratic nonlinearity $\bar{q}q_x$. One of the interesting properties of equation (1.1) is that it is integrable. The integrability of (1.1) is a consequence of the fact that it is a reduction of a system that was studied and shown to be integrable in [12]. While the Lax pair for the NLS equation involves 2×2 matrices, the Lax pair for (1.1) involves 3×3 matrices; this corresponds to the fact that the isospectral problem for (1.1) is a third-order rather than a second-order differential equation.

The simple structure of equation (1.1) in combination with its integrability suggests that it could be relevant for various applications (the wide applicability of integrable equations has been given a heuristic explanation by Calogero [5]). However, we will not attempt to find such applications here; instead we adopt the point of view that (1.1) is sufficiently interesting as a prototypical example of a nonlinear PDE with a 3×3 matrix Lax pair to warrant further study. In fact, within the class of integrable equations with 3×3 matrix Lax pairs—which includes equations such as the Boussinesq [1], Sasa–Satsuma [15], Sawada–Kotera [16], Kaup–Kupershmidt [11], Degasperis–Procesi [8], and 3-wave resonant interaction [17] equations—equation (1.1) is one of the simplest and most natural generalizations of the NLS equation, at least from the point of view of the Lax pair. Indeed, while the x and t parts of the Lax pair for NLS involve the matrices $k\sigma_3$ and $k^2\sigma_3$, respectively, where σ_3 is the diagonal matrix whose entries are the second roots of unity, i.e., $\sigma_3 = \text{diag}(1, -1)$, the x and t parts of (1.1) involve the matrices kJ and k^2J^2 , where J is the diagonal matrix whose entries are the third roots of unity, i.e., $J = \text{diag}(\omega, \omega^2, 1)$ with $\omega = e^{2\pi i/3}$. The Lax pairs of NLS and (1.1) also exhibit other

E-mail address: cchar@kth.se, jlenells@kth.se.

similarities. This suggests that (1.1) constitutes a prototypical example within the above class which is valuable as a mathematical playground.

In this paper, we first develop an inverse scattering transform formalism for the solution of the initial-value problem for (1.1). We then use steepest descent techniques to describe the asymptotic behavior of the solution for large times. More precisely, assuming that $q(x, t)$ is a Schwartz class solution of (1.1) with prescribed initial data $q_0(x) = q(x, 0)$ at time $t = 0$, we show that $q(x, t)$ can be expressed in terms of the solution $m(x, t, k)$ of a 3×3 matrix Riemann–Hilbert (RH) problem. The jump contour Γ of this RH problem consists of the three lines $\mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}$ where $\omega = e^{2\pi i/3}$, see Figure 1.1. The jump matrices are expressed in terms of two reflection coefficients, $r_1(k)$ and $r_2(k)$, which are defined in terms of the initial data $q_0(x)$ via the solution of linear integral equations (the direct scattering problem). The RH problem provides the solution of the inverse scattering problem in the sense that the value of $q(x, t)$ at any later time $t > 0$ can be recovered from $m(x, t, k)$ via the relation

$$q(x, t) = \lim_{k \rightarrow \infty} (km(x, t, k))_{13}.$$

The large t behavior of $m(x, t, k)$ can be extracted by performing a Deift–Zhou [9] steepest descent analysis of the RH problem and this leads to an asymptotic formula for $q(x, t)$.

Our main results are stated in Theorem 1. The main difficulty in the proof of Theorem 1 lies in the fact that the Lax pair—and hence also the RH problem—involves 3×3 matrices. This means that the implementation of the direct and inverse scattering, as well as the steepest descent analysis of the RH problem, in general requires substantial effort. However, it turns out that the Lax pair derived in [12] for (1.1) is similar in structure to the Lax pair of the so-called “good” Boussinesq equation, and this observation will save us a lot of effort. Indeed, the direct and inverse scattering problems for the “good” Boussinesq equation were studied in detail in [6] and a steepest descent analysis of the corresponding RH problem was carried out in [7]. Consequently, by formulating our problems appropriately, we will be able to recycle many of the results and proofs from [6, 7]. In particular, the steepest descent analysis of [7] can be applied almost unchanged to obtain asymptotics also for the solution of (1.1). The upshot is that by exploiting the connection to the “good” Boussinesq equation and drawing on [6, 7], we can obtain results for (1.1) with relatively little effort.

Other works analyzing the long-time asymptotics for integrable evolution equations with 3×3 Lax pairs include [10, 13, 14] (for the Sasa–Satsuma equation), [7] (for the “good” Boussinesq equation), and [2, 3] (for the Degasperis–Procesi equation).

Our main theorem is stated in Section 2. The direct scattering problem is considered in Section 3. The proof of the main theorem is presented in Section 4.

2. MAIN RESULTS

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz class of rapidly decreasing functions on the real line, and let $q_0 \in \mathcal{S}(\mathbb{R})$ be some given initial data for (1.1). The RH problem for m is formulated in terms of two spectral functions $r_1(k)$ and $r_2(k)$, which depend only on q_0 . The functions $r_1(k)$ and $r_2(k)$ are referred to as reflection coefficients and are defined as follows. Let $\omega = e^{2\pi i/3}$ and define $\{l_j(k), z_j(k)\}_{j=1}^3$ by

$$l_j(k) = \omega^j k, \quad z_j(k) = \omega^{2j} k^2, \quad k \in \mathbb{C}. \quad (2.1)$$

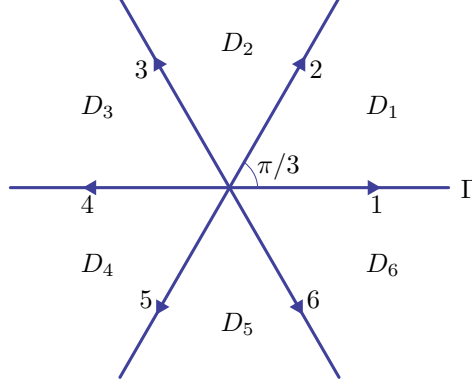


Figure 1.1. The contour Γ and the open subsets $\{D_n\}_1^6$ of the complex k -plane.

Let the matrix-valued function $\mathcal{U}(x)$ be given by

$$\mathcal{U}(x) = \begin{pmatrix} 0 & (1 - \omega^2)\overline{q_0(x)} & (1 - \omega)q_0(x) \\ (1 - \omega)q_0(x) & 0 & (1 - \omega^2)\overline{q_0(x)} \\ (1 - \omega^2)\overline{q_0(x)} & (1 - \omega)q_0(x) & 0 \end{pmatrix}. \quad (2.2)$$

Define the 3×3 -matrix valued eigenfunctions $X(x, k)$ and $X^A(x, k)$ as the unique solutions of the Volterra integral equations

$$X(x, k) = I - \int_x^\infty e^{(x-x')\widehat{\mathcal{L}(k)}} (\mathcal{U}(x')X(x', k)) dx', \quad (2.3a)$$

$$X^A(x, k) = I + \int_x^\infty e^{-(x-x')\widehat{\mathcal{L}(k)}} (\mathcal{U}(x')^T X^A(x', k)) dx', \quad (2.3b)$$

where $\mathcal{L} = \text{diag}(l_1, l_2, l_3)$, $\widehat{\mathcal{L}}$ denotes the operator which acts on a 3×3 matrix B by $\widehat{\mathcal{L}}B = [\mathcal{L}, B]$ (and thus $e^{\widehat{\mathcal{L}}}B = e^{\mathcal{L}}Be^{-\mathcal{L}}$), and $\mathcal{U}^T$ denotes the transpose of $\mathcal{U}$. Define $s(k)$ and $s^A(k)$ by

$$s(k) = I - \int_{\mathbb{R}} e^{-x\widehat{\mathcal{L}(k)}} (\mathcal{U}(x)X(x, k)) dx, \quad (2.4)$$

$$s^A(k) = I + \int_{\mathbb{R}} e^{x\widehat{\mathcal{L}(k)}} (\mathcal{U}(x)^T X^A(x, k)) dx. \quad (2.5)$$

The reflection coefficients $r_1(k)$ and $r_2(k)$ are defined by

$$\begin{cases} r_1(k) = \frac{(s(k))_{12}}{(s(k))_{11}}, & k \in (0, \infty), \\ r_2(k) = \frac{(s^A(k))_{12}}{(s^A(k))_{11}}, & k \in (-\infty, 0). \end{cases} \quad (2.6)$$

We note that the definition (2.6) of $r_1(k)$ and $r_2(k)$ coincides with the analogous definition in [6] except that the matrix U of [6] has been replaced by the matrix $\mathcal{U}$ defined in (2.2).

Let $D_1, \dots, D_6$ be the open sectors shown in Figure 1.1. The functions $(s(k))_{11}$ and $(s^A(k))_{11}$, which appear in the denominators of (2.6), are analytic in D_1 and D_4 , respectively. The case when $(s(k))_{11}$ and $(s^A(k))_{11}$ have a finite number of simple zeros in D_1 and D_4 (related to the presence of solitons) can be handled as in [12], but to avoid unnecessary technicalities, we will assume that no such zeros are present.

Assumption 2.1 (Absence of solitons). Assume that $(s(k))_{11}$ and $(s^A(k))_{11}$ are nonzero for $k \in \bar{D}_1$ and $k \in \bar{D}_4$, respectively.

The next lemma, whose proof is given in Section 3.6, establishes several important properties of $r_1(k)$ and $r_2(k)$.

Lemma 2.2 (Properties of $r_1(k)$ and $r_2(k)$). *Suppose $q_0 \in \mathcal{S}(\mathbb{R})$ is such that Assumption 2.1 holds. Then $r_1 : (0, \infty) \rightarrow \mathbb{C}$ and $r_2 : (-\infty, 0) \rightarrow \mathbb{C}$ are well-defined by (2.6) and have the following properties:*

- (i) $r_1 \in C^\infty((0, \infty))$ and $r_2 \in C^\infty((-\infty, 0))$.
- (ii) The functions $r_1(k)$, $r_2(k)$, and their derivatives $\partial_k^j r_\ell(k)$ have continuous boundary values at $k = 0$ for $\ell = 1, 2$ and $j = 0, 1, 2, \dots$, and there exist expansions

$$r_1(k) = r_1(0) + r_1'(0)k + \frac{1}{2}r_1''(0)k^2 + \dots, \quad k \rightarrow 0, \quad k > 0, \quad (2.7a)$$

$$r_2(k) = r_2(0) + r_2'(0)k + \frac{1}{2}r_2''(0)k^2 + \dots, \quad k \rightarrow 0, \quad k < 0, \quad (2.7b)$$

which can be differentiated termwise any number of times.

- (iii) $r_1(k)$ and $r_2(k)$ are rapidly decreasing as $|k| \rightarrow \infty$, i.e.,

$$\max_{j=0,1,\dots,N} \sup_{k \in (0,\infty)} (1 + |k|)^N |\partial_k^j r_1(k)| < \infty, \quad (2.8a)$$

$$\max_{j=0,1,\dots,N} \sup_{k \in (-\infty,0)} (1 + |k|)^N |\partial_k^j r_2(k)| < \infty, \quad (2.8b)$$

for each integer $N \geq 0$.

- (iv) $|r_1(k)| < 1$ for all $k \geq 0$ and $|r_2(k)| < 1$ for all $k \leq 0$.

We will show that the solution $q(x, t)$ of (1.1) with initial data $q(x, 0) = q_0(x)$ can be recovered from the solution $m(x, t, k)$ of a 3×3 RH problem whose jumps are expressed in terms of r_1 and r_2 . To state the RH problem for m , let Γ be the contour consisting of the three lines $\mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}$ oriented away from the origin as in Figure 1.1. For $1 \leq i \neq j \leq 3$, define $\theta_{ij} = \theta_{ij}(x, t, k)$ by

$$\theta_{ij}(x, t, k) = (l_i - l_j)x + (z_i - z_j)t.$$

Define the jump matrix $v(x, t, k)$ for $k \in \Gamma$ by

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 & -r_1(k)e^{-\theta_{21}} & 0 \\ \frac{1}{r_1(k)}e^{\theta_{21}} & 1 - |r_1(k)|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & v_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - |r_2(\omega k)|^2 & -\overline{r_2(\omega k)}e^{-\theta_{32}} \\ 0 & r_2(\omega k)e^{\theta_{32}} & 1 \end{pmatrix}, \\ v_3 &= \begin{pmatrix} 1 - |r_1(\omega^2 k)|^2 & 0 & \overline{r_1(\omega^2 k)}e^{-\theta_{31}} \\ 0 & 1 & 0 \\ -r_1(\omega^2 k)e^{\theta_{31}} & 0 & 1 \end{pmatrix}, & v_4 &= \begin{pmatrix} 1 - |r_2(k)|^2 & -\overline{r_2(k)}e^{-\theta_{21}} & 0 \\ r_2(k)e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ v_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r_1(\omega k)e^{-\theta_{32}} \\ 0 & \frac{1}{r_1(\omega k)}e^{\theta_{32}} & 1 - |r_1(\omega k)|^2 \end{pmatrix}, & v_6 &= \begin{pmatrix} 1 & 0 & r_2(\omega^2 k)e^{-\theta_{31}} \\ 0 & 1 & 0 \\ -\overline{r_2(\omega^2 k)}e^{\theta_{31}} & 0 & 1 - |r_2(\omega^2 k)|^2 \end{pmatrix}, \end{aligned} \quad (2.9)$$

where v_j denotes the restriction of v to the part of Γ labeled by j in Figure 1.1.

RH problem 2.3 (RH problem for m). *Given $r_1 : (0, \infty) \rightarrow \mathbb{C}$ and $r_2 : (-\infty, 0) \rightarrow \mathbb{C}$, find a 3×3 -matrix valued function $m(x, t, k)$ with the following properties:*

- (a) $m(x, t, \cdot) : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{3 \times 3}$ is analytic.
- (b) The limits of $m(x, t, k)$ as k approaches $\Gamma \setminus \{0\}$ from the left (+) and right (−) exist, are continuous on $\Gamma \setminus \{0\}$, and are related by

$$m_+(x, t, k) = m_-(x, t, k)v(x, t, k), \quad k \in \Gamma \setminus \{0\}, \quad (2.10)$$

where v is defined in terms of r_1 and r_2 by (2.9).

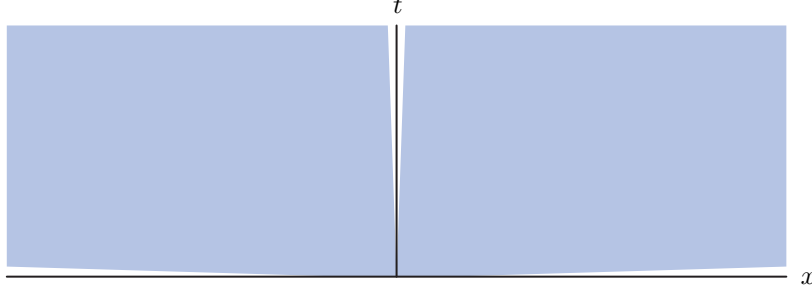


Figure 2.5. The shaded areas represent the sectors in the (x, t) -plane in which the asymptotic formulas of Theorem 1 apply.

(c) $m(x, t, k) = I + O(k^{-1})$ as $k \rightarrow \infty$, $k \notin \Gamma$.

(d) $m(x, t, k) = O(1)$ as $k \rightarrow 0$.

Before stating our main result, we introduce the notion of a Schwartz class solution.

Definition 2.4. We call $q(x, t)$ a *Schwartz class solution* of (1.1) with existence time $T \in (0, \infty]$ and initial data $q_0 \in \mathcal{S}(\mathbb{R})$ if

- (i) q is a smooth complex-valued function of $(x, t) \in \mathbb{R} \times [0, T)$.
- (ii) q satisfies (1.1) for $(x, t) \in \mathbb{R} \times [0, T)$ and

$$q(x, 0) = q_0(x), \quad x \in \mathbb{R}.$$

- (iii) q has rapid decay as $|x| \rightarrow \infty$ in the sense that, for each integer $N \geq 1$,

$$\sup_{\substack{x \in \mathbb{R} \\ t \in [0, T)}} \sum_{i=0}^N (1 + |x|)^N |\partial_x^i q(x, t)| < \infty.$$

We can now state our main theorem. The first part of the theorem establishes a representation for the solution $q(x, t)$ of (1.1) in terms of the solution of a RH problem; the second and third parts establish formulas for the long-time asymptotics of $q(x, t)$ for x/t in compact subsets of $(0, +\infty)$ and $(-\infty, 0)$, respectively, see Figure 2.5.

Theorem 1. Suppose $q(x, t)$ is a Schwartz class solution of (1.1) with existence time $T \in (0, \infty]$ and initial data $q_0 \in \mathcal{S}(\mathbb{R})$ such that Assumption 2.1 holds. Define the spectral functions $r_1(k)$ and $r_2(k)$ in terms of q_0 by (2.6).

- (i) The RH problem 2.3 has a unique solution $m(x, t, k)$ for each $(x, t) \in \mathbb{R} \times [0, T)$ and the formula

$$q(x, t) = \lim_{k \rightarrow \infty} (k m(x, t, k))_{13} \tag{2.11}$$

expressing $q(x, t)$ in terms of $m(x, t, k)$ is valid for all $(x, t) \in \mathbb{R} \times [0, T)$.

- (ii) Suppose $T = \infty$. As $t \rightarrow \infty$, we have

$$q(x, t) = \frac{\sqrt{\pi} e^{-\frac{\pi \nu}{2}}}{3^{1/4} \sqrt{t}} \frac{e^{i\phi}}{r_1(k_0) \Gamma(i\nu)} + O(t^{-1} \ln t),$$

uniformly for $\zeta = x/t$ in compact subsets of $(0, \infty)$, where Γ denotes the Gamma function,

$$k_0 = k_0(\zeta) = \frac{\zeta}{2}, \quad \nu = \nu(\zeta) = -\frac{1}{2\pi} \ln(1 - |r_1(k_0)|^2) \geq 0,$$

and

$$\phi = \phi(\zeta) = \frac{11\pi}{12} + \nu \ln(6\sqrt{3}tk_0^2) - \sqrt{3}k_0^2t + \frac{1}{\pi} \int_{k_0}^{+\infty} \ln \left| \frac{s - k_0}{s - \omega k_0} \right| d \ln(1 - |r_1(s)|^2).$$

(iii) Suppose $T = \infty$. As $t \rightarrow \infty$, we have

$$q(x, t) = -\frac{\sqrt{\pi}e^{-\frac{\pi\tilde{\nu}}{2}}}{3^{1/4}\sqrt{t}} \frac{e^{i\tilde{\phi}}}{\tilde{r}_2(k_0)\Gamma(i\tilde{\nu})} + O(t^{-1} \ln t),$$

uniformly for $\zeta = x/t$ in compact subsets of $(-\infty, 0)$, where

$$k_0 = k_0(\zeta) = \frac{\zeta}{2}, \quad \tilde{\nu} = \tilde{\nu}(\zeta) = -\frac{1}{2\pi} \ln(1 - |r_2(k_0)|^2) \geq 0, \quad \tilde{r}_2(k_0) = \frac{(s^A(k_0))_{21}}{(s^A(k_0))_{11}},$$

and

$$\tilde{\phi} = \tilde{\phi}(\zeta) = \frac{11\pi}{12} + \tilde{\nu} \ln(6\sqrt{3}tk_0^2) - \sqrt{3}k_0^2t + \frac{1}{\pi} \int_{-\infty}^{k_0} \ln \left| \frac{s - k_0}{s - \omega k_0} \right| d \ln(1 - |r_2(s)|^2).$$

Proof. See Section 4. □

Remark 2.6. Theorem 1 provides the asymptotics of the solution $q(x, t)$ of (1.1) everywhere in the half-plane $t > 0$ except in narrow wedges containing the x and t -axes, see Figure 2.5. The asymptotics in these wedges can also be analyzed via a steepest descent analysis of the RH problem for m , but this falls outside the scope of the present paper. As $\zeta = x/t \rightarrow 0$, the critical points in the steepest descent analysis merge at the origin, leading to qualitatively different asymptotics in the wedge containing the t -axis. Near the x -axis, the solution has decay consistent with the decay of the Schwartz class initial data.

Remark 2.7. For many integrable PDEs with 2×2 matrix Lax pairs, such as the KdV and mKdV equations, the reflection coefficient is bounded above by 1 for all $k \in \mathbb{R}$. For these equations, this property is a direct consequence from the fact that the scattering matrix, which is of size 2×2 , has unit determinant and satisfies a certain symmetry. For integrable PDEs with 3×3 matrix Lax pairs, the situation is considerably richer, because there is in general no simple argument that ensures an upper bound for the reflection coefficients. In Appendix A, we explore this issue in the context of equation (1.1). Under Assumption 2.1, we have shown in part (iv) of Lemma 2.2 that $|r_1(k)| < 1$ for $k \geq 0$ and $|r_2(k)| < 1$ for $k \leq 0$. We show in Appendix A that these bounds on r_1 and r_2 hold even without Assumption 2.1 in the case of compactly supported initial data. (For general initial data $q_0 \in \mathcal{S}(\mathbb{R})$, we establish the slightly weaker bounds $|r_1(k)| \leq 1$ for $k \geq 0$ and $|r_2(k)| \leq 1$ for $k \leq 0$.) This is in contrast to other equations with 3×3 Lax pairs such as the “good” Boussinesq equation, for which the absolute values of the reflection coefficients are, in general, not bounded above by 1 unless further assumptions are imposed, see [6].

Remark 2.8. Assumption 2.1 can be relaxed slightly in the case of compactly supported initial data. More precisely, it follows from the arguments in Appendix A that $(s(k))_{11}$ and $(s^A(k))_{11}$ are automatically nonzero for $k \in \omega^2(-\infty, 0] \subset \bar{D}_1$ and $k \in \omega^2[0, \infty) \subset \bar{D}_4$, respectively, if q_0 has compact support.

Remark 2.9. The proof of the asymptotic formula valid for $\zeta = x/t > 0$ given in part (ii) of Theorem 1 relies on a steepest descent analysis of the RH problem for m . The asymptotics for $\zeta < 0$ can be derived in a similar way. However, for simplicity we choose to instead derive the asymptotics for $\zeta < 0$ from the asymptotics for $\zeta > 0$ and the fact that equation (1.1) is invariant under the symmetry $q(x, t) \rightarrow -q(-x, t)$, see Section 4.3.

3. DIRECT SCATTERING

3.1. Lax pair. Let $\omega := e^{2\pi i/3}$, define $\{l_j(k), z_j(k)\}_{j=1}^3$ by (2.1), and define J , $\mathcal{L} = \mathcal{L}(k)$, and $\mathcal{Z} = \mathcal{Z}(k)$ by

$$J = \text{diag}(\omega, \omega^2, 1), \quad \mathcal{L} = kJ = \text{diag}(l_1, l_2, l_3), \quad \mathcal{Z} = k^2 J^2 = \text{diag}(z_1, z_2, z_3).$$

Equation (1.1) is the compatibility condition of the following Lax pair (see [12])

$$\begin{cases} X_x - [\mathcal{L}, X] = \mathcal{U}X, \\ X_t - [\mathcal{Z}, X] = \mathcal{V}X, \end{cases} \quad (3.1)$$

where $k \in \mathbb{C}$ is the spectral parameter, $X(x, t, k)$ is a 3×3 -matrix valued eigenfunction, and $\mathcal{U} = \mathcal{U}(x, t, k)$ and $\mathcal{V} = \mathcal{V}(x, t, k)$ are defined by

$$\mathcal{U} = \begin{pmatrix} 0 & (1 - \omega^2)\bar{q} & (1 - \omega)q \\ (1 - \omega)q & 0 & (1 - \omega^2)\bar{q} \\ (1 - \omega^2)\bar{q} & (1 - \omega)q & 0 \end{pmatrix}, \quad (3.2a)$$

$$\begin{aligned} \mathcal{V} = k & \begin{pmatrix} 0 & (\omega^2 - 1)\bar{q} & (1 - \omega^2)q \\ (\omega - 1)q & 0 & (1 - \omega)\bar{q} \\ (\omega - \omega^2)\bar{q} & (\omega^2 - \omega)q & 0 \end{pmatrix} \\ & + (\bar{q}_x - 3q^2) \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega \\ \omega & 0 & 0 \end{pmatrix} + (q_x - 3\bar{q}^2) \begin{pmatrix} 0 & 0 & \omega^2 \\ \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}. \end{aligned} \quad (3.2b)$$

Indeed, a straightforward calculation shows that the compatibility condition of (3.1), i.e. $X_{xt} - X_{tx} = 0$, is equivalent to (1.1). Assuming that $q(x, t)$ is a Schwartz class solution of (1.1), we have

$$\lim_{x \rightarrow \pm\infty} \mathcal{U} = 0, \quad \lim_{x \rightarrow \pm\infty} \mathcal{V} = 0. \quad (3.3)$$

The Lax pair (3.1) possesses two convenient symmetries: If F is any of the matrices $\mathcal{L}, \mathcal{Z}, \mathcal{U}, \mathcal{V}$, then

$$F(k) = \mathcal{A}F(\omega k)\mathcal{A}^{-1} = \overline{\mathcal{B}F(\bar{k})\mathcal{B}}, \quad k \in \mathbb{C}, \quad (3.4)$$

where

$$\mathcal{A} := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{B} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 3.1. The Lax pair (3.1) for equation (1.1) is similar to the Lax pair for the “good” Boussinesq equation analyzed in [6]. For example, the symmetries (3.4) are the same as in [6, Eqs. (3.18) and (3.19)]. The main difference between (3.1) and the Lax pair analyzed in [6] is that whereas the matrices $\mathcal{U}$ and $\mathcal{V}$ in (3.1) are bounded at the origin, the analogs of $\mathcal{U}$ and $\mathcal{V}$ in [6] have double poles at the origin. There are also other, less essential, differences that we will indicate when necessary. Because of the evident similarities with [6], we will omit some proofs in what follows.

Remark 3.2. We point out that the x -part of the Lax pair (3.1) is similar in structure to the x -part associated with the Ostrovsky–Vakhnenko equation, see [4, Eq. (2.3a)]. In [4], a RH approach to the Ostrovsky–Vakhnenko equation was developed and used to obtain the principal term in the long-time asymptotics.

3.2. The eigenfunctions X and Y . In the remainder of this section, we will set $t = 0$ and abuse notation by writing $\mathcal{U}(x)$ instead of $\mathcal{U}(x, 0)$. We start by considering the x -part of (3.1) evaluated at $t = 0$:

$$X_x - [\mathcal{L}, X] = \mathcal{U}X. \quad (3.5)$$

We define two 3×3 -matrix valued solutions $X(x, k)$ and $Y(x, k)$ of (3.5) as the solutions of the linear Volterra integral equations

$$X(x, k) = I - \int_x^\infty e^{(x-x')\widehat{\mathcal{L}(k)}} (\mathcal{U}(x')X(x', k)) dx', \quad (3.6a)$$

$$Y(x, k) = I + \int_{-\infty}^x e^{(x-x')\widehat{\mathcal{L}(k)}} (\mathcal{U}(x')Y(x', k)) dx'. \quad (3.6b)$$

The six open subsets $\{D_n\}_1^6$ of Figure 1.1 can be written as

$$\begin{aligned} D_1 &= \{k \in \mathbb{C} \mid \operatorname{Re} l_1 < \operatorname{Re} l_2 < \operatorname{Re} l_3\}, & D_2 &= \{k \in \mathbb{C} \mid \operatorname{Re} l_1 < \operatorname{Re} l_3 < \operatorname{Re} l_2\}, \\ D_3 &= \{k \in \mathbb{C} \mid \operatorname{Re} l_3 < \operatorname{Re} l_1 < \operatorname{Re} l_2\}, & D_4 &= \{k \in \mathbb{C} \mid \operatorname{Re} l_3 < \operatorname{Re} l_2 < \operatorname{Re} l_1\}, \\ D_5 &= \{k \in \mathbb{C} \mid \operatorname{Re} l_2 < \operatorname{Re} l_3 < \operatorname{Re} l_1\}, & D_6 &= \{k \in \mathbb{C} \mid \operatorname{Re} l_2 < \operatorname{Re} l_1 < \operatorname{Re} l_3\}. \end{aligned}$$

We let $\mathcal{S} := \{k \in \mathbb{C} \mid \arg k \in (\frac{2\pi}{3}, \frac{4\pi}{3})\}$ denote the interior of $\bar{D}_3 \cup \bar{D}_4$.

Proposition 3.3 below collects some basic properties of X and Y . The statement is almost identical to [6, Proposition 3.1] and we omit the proof; the only major difference is that in our case X and Y are well-defined at the origin.

Proposition 3.3. *Suppose $q_0 \in \mathcal{S}(\mathbb{R})$. Then the equations (3.6) uniquely define two 3×3 -matrix valued solutions X and Y of (3.5) with the following properties:*

- (a) *The function $X(x, k)$ is defined for $x \in \mathbb{R}$ and $k \in (\omega^2 \bar{\mathcal{S}}, \omega \bar{\mathcal{S}}, \bar{\mathcal{S}})$, that is, the first column of $X(x, k)$ is defined for $x \in \mathbb{R}$ and $k \in \omega^2 \bar{\mathcal{S}}$, the second column of $X(x, k)$ is defined for $x \in \mathbb{R}$ and $k \in \omega \bar{\mathcal{S}}$, etc. For each $k \in (\omega^2 \bar{\mathcal{S}}, \omega \bar{\mathcal{S}}, \bar{\mathcal{S}})$, $X(\cdot, k)$ is smooth and satisfies (3.5).*
- (b) *The function $Y(x, k)$ is defined for $x \in \mathbb{R}$ and $k \in (-\omega^2 \bar{\mathcal{S}}, -\omega \bar{\mathcal{S}}, -\bar{\mathcal{S}})$. For each $k \in (-\omega^2 \bar{\mathcal{S}}, -\omega \bar{\mathcal{S}}, -\bar{\mathcal{S}})$, $Y(\cdot, k)$ is smooth and satisfies (3.5).*
- (c) *For each $x \in \mathbb{R}$, the function $X(x, \cdot)$ is continuous for $k \in (\omega^2 \bar{\mathcal{S}}, \omega \bar{\mathcal{S}}, \bar{\mathcal{S}})$ and analytic for $k \in (\omega^2 \mathcal{S}, \omega \mathcal{S}, \mathcal{S})$.*
- (d) *For each $x \in \mathbb{R}$, the function $Y(x, \cdot)$ is continuous for $k \in (-\omega^2 \bar{\mathcal{S}}, -\omega \bar{\mathcal{S}}, -\bar{\mathcal{S}})$ and analytic for $k \in (-\omega^2 \mathcal{S}, -\omega \mathcal{S}, -\mathcal{S})$.*
- (e) *For each $x \in \mathbb{R}$ and each $j = 1, 2, \dots$, the partial derivative $\frac{\partial^j X}{\partial k^j}(x, \cdot)$ has a continuous extension to $(\omega^2 \bar{\mathcal{S}}, \omega \bar{\mathcal{S}}, \bar{\mathcal{S}})$.*
- (f) *For each $x \in \mathbb{R}$ and each $j = 1, 2, \dots$, the partial derivative $\frac{\partial^j Y}{\partial k^j}(x, \cdot)$ has a continuous extension to $(-\omega^2 \bar{\mathcal{S}}, -\omega \bar{\mathcal{S}}, -\bar{\mathcal{S}})$.*
- (g) *For each $n \geq 1$, there are bounded smooth positive functions $f_+(x)$ and $f_-(x)$ of $x \in \mathbb{R}$ with rapid decay as $x \rightarrow +\infty$ and $x \rightarrow -\infty$, respectively, such that the following estimates hold for $x \in \mathbb{R}$ and $j = 0, 1, \dots, n$:*

$$\left| \frac{\partial^j}{\partial k^j} (X(x, k) - I) \right| \leq f_+(x), \quad k \in (\omega^2 \bar{\mathcal{S}}, \omega \bar{\mathcal{S}}, \bar{\mathcal{S}}), \quad (3.7a)$$

$$\left| \frac{\partial^j}{\partial k^j} (Y(x, k) - I) \right| \leq f_-(x), \quad k \in (-\omega^2 \bar{\mathcal{S}}, -\omega \bar{\mathcal{S}}, -\bar{\mathcal{S}}). \quad (3.7b)$$

(h) X and Y obey the following symmetries for each $x \in \mathbb{R}$:

$$X(x, k) = \mathcal{A}X(x, \omega k)\mathcal{A}^{-1} = \overline{\mathcal{B}X(x, \bar{k})}\mathcal{B}, \quad k \in (\omega^2\bar{\mathcal{S}}, \omega\bar{\mathcal{S}}, \bar{\mathcal{S}}), \quad (3.8a)$$

$$Y(x, k) = \mathcal{A}Y(x, \omega k)\mathcal{A}^{-1} = \overline{\mathcal{B}Y(x, \bar{k})}\mathcal{B}, \quad k \in (-\omega^2\bar{\mathcal{S}}, -\omega\bar{\mathcal{S}}, -\bar{\mathcal{S}}). \quad (3.8b)$$

(i) If $q_0(x)$ has compact support, then, for each $x \in \mathbb{R}$, $X(x, k)$ and $Y(x, k)$ are defined and analytic for $k \in \mathbb{C}$ and $\det X = \det Y = 1$.

3.3. Asymptotics of X and Y as $k \rightarrow \infty$. Following [6, Section 3.3], we first consider the following formal series

$$\begin{aligned} X_{\text{formal}}(x, k) &= I + \frac{X_1(x)}{k} + \frac{X_2(x)}{k^2} + \cdots, \\ Y_{\text{formal}}(x, k) &= I + \frac{Y_1(x)}{k} + \frac{Y_2(x)}{k^2} + \cdots, \end{aligned}$$

normalized at $x = \infty$ and $x = -\infty$, respectively:

$$\lim_{x \rightarrow \infty} X_j(x) = \lim_{x \rightarrow -\infty} Y_j(x) = 0, \quad j \geq 1. \quad (3.9)$$

Substituting X_{formal} into (3.5), we obtain the following recurrence relations for the coefficients $\{X_j\}_{j=1}^{+\infty}$ by identifying terms of the same order:

$$\begin{cases} [J, X_{j+1}] = \partial_x X_j^{(o)} - (\mathcal{U}X_j)^{(o)}, \\ \partial_x X_j^{(d)} = (\mathcal{U}X_j)^{(d)}, \end{cases} \quad j \geq 0, \quad (3.10)$$

where $X_0 = I$, and $A^{(d)}$ and $A^{(o)}$ denote the diagonal and off-diagonal parts of a 3×3 matrix A , respectively. The first coefficients X_1 and Y_1 are given by

$$X_1(x) = \begin{pmatrix} 0 & \omega\bar{q}_0 & q_0 \\ \omega^2 q_0 & 0 & \bar{q}_0 \\ \omega^2 \bar{q}_0 & \omega q_0 & 0 \end{pmatrix} - 3 \int_{-\infty}^x |q_0|^2 dx' \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.11a)$$

$$Y_1(x) = \begin{pmatrix} 0 & \omega\bar{q}_0 & q_0 \\ \omega^2 q_0 & 0 & \bar{q}_0 \\ \omega^2 \bar{q}_0 & \omega q_0 & 0 \end{pmatrix} - 3 \int_{-\infty}^x |q_0|^2 dx' \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.11b)$$

Propositions 3.4 shows that X and Y coincide with X_{formal} and Y_{formal} to all orders as $k \rightarrow \infty$. We omit the proof which is very similar to the proof of [6, Proposition 3.2].

Proposition 3.4. *Suppose $q_0 \in \mathcal{S}(\mathbb{R})$. As $k \rightarrow \infty$, X and Y coincide to all orders with X_{formal} and Y_{formal} , respectively. More precisely, let $p \geq 0$ be an integer. Then the functions*

$$X_{(p)}(x, k) := I + \frac{X_1(x)}{k} + \cdots + \frac{X_p(x)}{k^p}, \quad (3.12)$$

$$Y_{(p)}(x, k) := I + \frac{Y_1(x)}{k} + \cdots + \frac{Y_p(x)}{k^p},$$

are well-defined and, for each integer $j \geq 0$,

$$\left| \frac{\partial^j}{\partial k^j} (X - X_{(p)}) \right| \leq \frac{f_+(x)}{|k|^{p+1}}, \quad x \in \mathbb{R}, \quad k \in (\omega^2\bar{\mathcal{S}}, \omega\bar{\mathcal{S}}, \bar{\mathcal{S}}), \quad |k| \geq 2, \quad (3.13a)$$

$$\left| \frac{\partial^j}{\partial k^j} (Y - Y_{(p)}) \right| \leq \frac{f_-(x)}{|k|^{p+1}}, \quad x \in \mathbb{R}, \quad k \in (-\omega^2\bar{\mathcal{S}}, -\omega\bar{\mathcal{S}}, -\bar{\mathcal{S}}), \quad |k| \geq 2, \quad (3.13b)$$

where $f_+(x)$ and $f_-(x)$ are bounded smooth positive functions of $x \in \mathbb{R}$ with rapid decay as $x \rightarrow +\infty$ and $x \rightarrow -\infty$, respectively.

3.4. The scattering matrix. The scattering matrix $s(k)$ was defined in (2.4). Our next proposition, whose proof is omitted, establishes several properties of $s(k)$. Properties (a)-(e) and (g) are very similar (though not identical) to their counterparts in [6, Proposition 3.5]. The behavior of $s(k)$ as $k \rightarrow 0$ is stated in item (f); this item differs significantly from the corresponding item in [6, Proposition 3.5] but can be easily proved using Proposition 3.3.

Proposition 3.5. *Let $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0)$. Suppose $q_0 \in \mathcal{S}(\mathbb{R})$. Then the scattering matrix $s(k)$ defined in (2.4) has the following properties:*

(a) *The entries of $s(k)$ are defined and continuous for k in*

$$\begin{pmatrix} \omega^2 \bar{\mathcal{S}} & \mathbb{R}_+ & \omega \mathbb{R}_+ \\ \mathbb{R}_+ & \omega \mathcal{S} & \omega^2 \mathbb{R}_+ \\ \omega \mathbb{R}_+ & \omega^2 \mathbb{R}_+ & \bar{\mathcal{S}} \end{pmatrix}, \quad (3.14)$$

that is, the (11) entry of $s(k)$ is defined and continuous for $k \in \omega^2 \bar{\mathcal{S}}$, etc.

(b) *The diagonal entries of $s(k)$ are analytic in the interior of their domains of definition as given in (3.14).*

(c) *For $j = 1, 2, \dots$, the derivative $\partial_k^j s(k)$ is well-defined and continuous for k in (3.14).*

(d) *$s(k)$ obeys the symmetries*

$$s(k) = \mathcal{A}s(\omega k)\mathcal{A}^{-1} = \overline{\mathcal{B}s(\bar{k})\mathcal{B}}. \quad (3.15)$$

(e) *$s(k)$ approaches the identity matrix as $k \rightarrow \infty$. More precisely, there are diagonal matrices $\{s_j\}_1^\infty$ such that*

$$\left| \partial_k^j \left(s(k) - I - \sum_{j=1}^N \frac{s_j}{k^j} \right) \right| = O(k^{-N-1}), \quad k \rightarrow \infty, \quad k \in \begin{pmatrix} \omega^2 \bar{\mathcal{S}} & \mathbb{R}_+ & \omega \mathbb{R}_+ \\ \mathbb{R}_+ & \omega \mathcal{S} & \omega^2 \mathbb{R}_+ \\ \omega \mathbb{R}_+ & \omega^2 \mathbb{R}_+ & \bar{\mathcal{S}} \end{pmatrix},$$

for $j = 0, 1, \dots, N$ and each integer $N \geq 1$. In particular, the off-diagonal entries of $s(k)$ have rapid decay as $k \rightarrow \infty$.

(f) *We have*

$$s(k) = s^{(0)} + s^{(1)}k + \dots, \quad k \rightarrow 0, \quad k \in \begin{pmatrix} \omega^2 \bar{\mathcal{S}} & \mathbb{R}_+ & \omega \mathbb{R}_+ \\ \mathbb{R}_+ & \omega \mathcal{S} & \omega^2 \mathbb{R}_+ \\ \omega \mathbb{R}_+ & \omega^2 \mathbb{R}_+ & \bar{\mathcal{S}} \end{pmatrix}, \quad (3.16)$$

and the expansion can be differentiated termwise any number of times.

(g) *If $q_0(x)$ has compact support, then $s(k)$ is defined and analytic for $k \in \mathbb{C}$, $\det s = 1$ for $k \in \mathbb{C}$, and*

$$X(x, k) = Y(x, k)e^{x\widehat{\mathcal{L}(k)}}s(k), \quad k \in \mathbb{C}. \quad (3.17)$$

3.5. Analysis of X^A , Y^A and s^A . Since the Lax pair associated to (1.1) is third order, the columns of X and Y alone are not sufficient to define a RH problem. Following [12], we need to consider X^A and Y^A defined as the solutions of the following Volterra equations:

$$X^A(x, k) = I + \int_x^\infty e^{-(x-x')\widehat{\mathcal{L}(k)}}(\mathcal{U}(x')^T X^A(x', k))dx', \quad (3.18a)$$

$$Y^A(x, k) = I - \int_{-\infty}^x e^{-(x-x')\widehat{\mathcal{L}(k)}}(\mathcal{U}(x')^T Y^A(x', k))dx', \quad (3.18b)$$

and the associated spectral function $s^A(k)$ defined in (2.5). If q_0 is compactly supported, all entries of X, X^A, Y, Y^A, s, s^A are well defined for all $k \in \mathbb{C}$, and X^A, Y^A , and s^A are simply the inverse transpose of X, Y and s , respectively. For general $q_0 \in \mathcal{S}(\mathbb{R})$, the

properties of X^A , Y^A , and s^A can be established by adapting [6, Sections 3.6-3.8] in a similar way as was done above for X , Y , and s . We omit this analysis.

3.6. Proof of Lemma 2.2. Suppose $q_0 \in \mathcal{S}(\mathbb{R})$ is such that Assumption 2.1 holds. Recall from (2.6) that $r_1 = s_{12}/s_{11}$. Statements (a) and (c) of Proposition 3.5 imply that $s_{12}(k)$ and $s_{11}(k)$ are smooth for $k \in (0, \infty)$. Since s_{11} has no zeros by Assumption 2.1, it follows that $r_1 \in C^\infty((0, \infty))$. Moreover, statement (e) of Proposition 3.5 implies that r_1 satisfies (2.8a), and statement (f) implies that r_1 and its derivatives have continuous boundary values at $k = 0$, showing that r_1 also satisfies (2.7). This completes the proof of parts (i), (ii) and (iii) of Lemma 2.2 for r_1 ; the proof for r_2 follows similarly from the properties of X^A , Y^A , and s^A .

It remains to prove Lemma 2.2 (iv), i.e. that

$$|r_1(k)| < 1 \quad \text{for all } k \geq 0 \quad \text{and} \quad |r_2(k)| < 1 \quad \text{for all } k \leq 0.$$

For $k \leq 0$, all four entries $\{s_{ij}^A(k)\}_{i,j=1}^2$ are well-defined, so by uniqueness of s we conclude that $s_{11}^A(k)s_{22}^A(k) - s_{12}^A(k)s_{21}^A(k) = s_{33}(k)$. Hence, using also the symmetries $s^A(k) = \mathcal{B}s^A(\bar{k})\mathcal{B}$ and (3.15), we find

$$1 - |r_2(k)|^2 = 1 - \left| \frac{s_{12}^A(k)}{s_{11}^A(k)} \right|^2 = \frac{s_{11}^A(k)s_{22}^A(k) - s_{12}^A(k)s_{21}^A(k)}{|s_{11}^A(k)|^2} = \frac{s_{33}(k)}{|s_{11}^A(k)|^2} = \frac{s_{11}(\omega^2 k)}{|s_{11}^A(k)|^2}. \quad (3.19)$$

Since the right-hand side of (3.19) is non-zero for all $k \leq 0$ by Assumption 2.1, and since the left-hand side is real and tends to 1 as $k \rightarrow -\infty$ by Lemma 2.2 (iii), we conclude that $1 - |r_2(k)|^2 > 0$ for all $k \leq 0$. The proof of $|r_1(k)| < 1$ for $k \geq 0$ is similar and we omit it.

4. PROOF OF THEOREM 1

Let $q(x, t)$ be a Schwartz class solution of (1.1) with existence time $T \in (0, \infty]$ and initial data $q_0 \in \mathcal{S}(\mathbb{R})$. Suppose Assumption 2.1 holds and define $r_1(k)$ and $r_2(k)$ in terms of q_0 by (2.6).

4.1. Proof of (i). We will construct the solution m of RH problem 2.3 explicitly in terms of solutions of certain Fredholm equations. Let m_n denote the restriction of m to D_n . For each $n = 1, \dots, 6$, define $m_n(x, k)$ for $k \in D_n$ as the solution of the following system of Fredholm integral equations:

$$(m_n)_{ij}(x, k) = \delta_{ij} + \int_{\gamma_{ij}^n} \left(e^{(x-x')\widehat{\mathcal{L}(k)}} (\mathcal{U}(x')m_n(x', k)) \right)_{ij} dx', \quad i, j = 1, 2, 3, \quad (4.1)$$

where the contours γ_{ij}^n , $n = 1, \dots, 6$, $i, j = 1, 2, 3$, are defined by

$$\gamma_{ij}^n = \begin{cases} (-\infty, x), & \operatorname{Re} l_i(k) < \operatorname{Re} l_j(k), \\ (+\infty, x), & \operatorname{Re} l_i(k) \geq \operatorname{Re} l_j(k), \end{cases} \quad \text{for } k \in D_n. \quad (4.2)$$

Let Z denote the set of zeros of the Fredholm determinants associated with (4.1). The next proposition establishes the basic properties of m_n .

Proposition 4.1. *Suppose $q_0 \in \mathcal{S}(\mathbb{R})$. Then the equations (4.1) uniquely define six 3×3 -matrix valued solutions $\{m_n\}_1^6$ of (3.5) with the following properties:*

- (a) *The function $m_n(x, k)$ is defined for $x \in \mathbb{R}$ and $k \in \bar{D}_n \setminus Z$. For each $k \in \bar{D}_n \setminus Z$, $m_n(\cdot, k)$ is smooth and satisfies (3.5).*
- (b) *For each $x \in \mathbb{R}$, the function $m_n(x, \cdot)$ is continuous for $k \in \bar{D}_n \setminus Z$ and analytic for $k \in D_n \setminus Z$.*

(c) For each $\epsilon > 0$, there exists a $C = C(\epsilon)$ such that

$$|m_n(x, k)| \leq C, \quad x \in \mathbb{R}, \quad k \in \bar{D}_n, \quad \text{dist}(k, Z) \geq \epsilon. \quad (4.3)$$

(d) For each $x \in \mathbb{R}$ and each $j = 1, 2, \dots$, the partial derivative $\frac{\partial^j m_n}{\partial k^j}(x, \cdot)$ has a continuous extension to $\bar{D}_n \setminus Z$.

(e) $\det m_n(x, k) = 1$ for $x \in \mathbb{R}$ and $k \in \bar{D}_n \setminus Z$.

(f) For each $x \in \mathbb{R}$, the sectionally analytic function $m(x, k)$ defined by $m(x, k) = m_n(x, k)$ for $k \in D_n$ satisfies the symmetries

$$m(x, k) = \mathcal{A}m(x, \omega k)\mathcal{A}^{-1} = \overline{\mathcal{B}m(x, \bar{k})\mathcal{B}}, \quad k \in \mathbb{C} \setminus Z. \quad (4.4)$$

Proof. We refer to [6, Proposition 4.1] for a detailed proof in a similar, though more complicated, situation. $\square$

Now we turn to the behavior of m as $k \rightarrow \infty$. Since $\gamma_{ii}^n = (\infty, x)$ for $i = 1, 2, 3$ and $n = 1, \dots, 6$, it follows that m has the same asymptotic behavior at ∞ as X . The proof of the next lemma consists of minor adaptations of [6, Lemma 4.3].

Lemma 4.2. *Suppose $q_0 \in \mathcal{S}(\mathbb{R})$ and $q_0 \not\equiv 0$. Given an integer $p \geq 1$, let $X_{(p)}(x, k)$ be the function defined in (3.12). Then there exists an $R > 0$ such that*

$$|m(x, k) - X_{(p)}(x, k)| \leq \frac{C}{|k|^{p+1}}, \quad x \in \mathbb{R}, \quad k \in \mathbb{C} \setminus \Gamma, \quad |k| \geq R. \quad (4.5)$$

Following [6, Lemmas 4.4, 4.5 and 4.6], we conclude that m satisfies the jumps of the RH problem 2.3, except at the points $k \in Z$.

Lemma 4.3 (Jump condition for m). *Let $q_0 \in \mathcal{S}(\mathbb{R})$. For each $x \in \mathbb{R}$, $m(x, k)$ satisfies the jump condition*

$$m_+(x, k) = m_-(x, k)v(x, 0, k), \quad k \in \Gamma \setminus Z,$$

where v is the jump matrix defined in (2.9).

We have not yet proved that m satisfies the RH problem 2.3, since m is only defined for $\mathbb{C} \setminus Z$. The next lemma shows that m can be explicitly written in terms of X , Y , s , X^A , Y^A , and s^A . Using the properties of these functions, Lemma 4.4 allows us to show that m can be analytically continued from $\mathbb{C} \setminus Z$ to $\mathbb{C}$. The proof is identical to the proof of [6, Lemma 4.7], so we omit it.

Lemma 4.4. *Let $q_0 \in \mathcal{S}(\mathbb{R})$. The function m_1 can be expressed in terms of the entries of X, Y, X^A, Y^A, s , and s^A as follows:*

$$m_1 = \begin{pmatrix} X_{11} & \frac{Y_{31}^A X_{23}^A - Y_{21}^A X_{33}^A}{s_{11}} & \frac{Y_{13}}{s_{33}^A} \\ X_{21} & \frac{Y_{11}^A X_{33}^A - Y_{31}^A X_{13}^A}{s_{11}} & \frac{Y_{23}}{s_{33}^A} \\ X_{31} & \frac{Y_{21}^A X_{13}^A - Y_{11}^A X_{23}^A}{s_{11}} & \frac{Y_{33}}{s_{33}^A} \end{pmatrix}, \quad (4.6)$$

for all $x \in \mathbb{R}$ and $k \in \bar{D}_1 \setminus Z$. There exist similar expressions for $m_2, \dots, m_6$.

Since q_0 satisfies Assumption 2.1, it follows from (4.6) that m is bounded for $k \in \bar{D}_1 \setminus Z$. Therefore, we can define the value of $m(x, t, k)$ at a point $k_j \in Z \cap \bar{D}_n$ by continuity:

$$m_n(x, k_j) = \lim_{\substack{k \rightarrow k_j \\ k \in \bar{D}_n \setminus Z}} m_n(x, k). \quad (4.7)$$

We conclude that Z can be replaced with the empty set in all the above results.

The next lemma establishes the asymptotics of m as $k \rightarrow 0$. The proof is similar to (but simpler than) the proof of [6, Lemma 4.9].

Lemma 4.5. *Suppose $q_0 \in \mathcal{S}(\mathbb{R})$ is such that Assumption 2.1 holds. Let $p \geq 1$ be an integer. Then there are 3×3 -matrix valued functions $\{\mathbf{m}_n^{(l)}(x)\}$, $n = 1, \dots, 6$, $l = 0, \dots, p$, with the following properties:*

(a) *The function m satisfies, for $x \in \mathbb{R}$,*

$$\left| m_n(x, k) - \sum_{l=0}^p \mathbf{m}_n^{(l)}(x) k^l \right| C \leq |k|^{p+1}, \quad |k| \leq \frac{1}{2}, \quad k \in \bar{D}_n.$$

(b) *For each n and each $l \geq 0$, $\mathbf{m}_n^{(l)}(x)$ is a smooth function of $x \in \mathbb{R}$.*

It follows from the above lemmas that $m(x, k)$ (or more precisely, $m(x, 0, k)$) defined by (4.1) satisfies the RH problem 2.3 with $t = 0$. Moreover, using Lemma 4.2 and recalling equations (3.11) and (3.12), we infer that (2.11) holds for $t = 0$. These arguments can be extended to any $t \in [0, T]$ as in [6, Section 5] by replacing $u_0(x)$ with $u(x, t)$ in the above steps. Since the jump matrix has unit determinant, uniqueness of the RH problem for m follows in the standard way. This completes the proof of part (i) of Theorem 1.

4.2. Proof of (ii). The RH problem 2.3 for m is a regularized version of the RH problem associated with the “good” Boussinesq equation obtained in [6], and the asymptotic analysis of this regularized RH problem was performed as a step in the derivation of long-time asymptotics for the “good” Boussinesq equation presented in [7]. More precisely, for $\zeta := x/t \in (\zeta_0, \infty)$, it follows from [7] that

$$\begin{aligned} \lim_{k \rightarrow \infty} (k m(x, t, k))_{13} &= \lim_{k \rightarrow \infty} (k \hat{m}(x, t, k) \Delta^{-1}(\zeta, k))_{13} \\ &= \lim_{k \rightarrow \infty} (k(\hat{m}(x, t, k) - I) \Delta^{-1}(\zeta, k))_{13} + \lim_{k \rightarrow \infty} (k \Delta^{-1}(\zeta, k))_{13}, \end{aligned}$$

where Δ^{-1} is given by

$$\Delta^{-1}(\zeta, k) = \begin{pmatrix} \frac{\delta_3(\zeta, k)}{\delta_1(\zeta, k)} & 0 & 0 \\ 0 & \frac{\delta_1(\zeta, k)}{\delta_5(\zeta, k)} & 0 \\ 0 & 0 & \frac{\delta_5(\zeta, k)}{\delta_3(\zeta, k)} \end{pmatrix}, \quad k \in \mathbb{C} \setminus ([k_0, +\infty) \cup \omega[k_0, +\infty) \cup \omega^2[k_0, +\infty))$$

with $k_0 = k_0(\zeta) = \zeta/2$ and

$$\delta_1(\zeta, k) = e^{-i\nu \ln_0(k-k_0)} e^{-\chi_1(\zeta, k)}, \quad k \in \mathbb{C} \setminus [k_0, +\infty), \quad (4.8)$$

$$\delta_3(\zeta, k) = \delta_1(\zeta, \omega^2 k), \quad k \in \mathbb{C} \setminus \omega[k_0, +\infty), \quad (4.9)$$

$$\delta_5(\zeta, k) = \delta_1(\zeta, \omega k), \quad k \in \mathbb{C} \setminus \omega^2[k_0, +\infty). \quad (4.10)$$

The quantities $\nu = \nu(\zeta) \geq 0$ and $\chi_1(\zeta, k)$ that appear in the definition of δ_1 are given by

$$\nu = -\frac{1}{2\pi} \ln(1 - |r_1(k_0)|^2), \quad \chi_1(\zeta, k) = \frac{1}{2\pi i} \int_{k_0}^{\infty} \ln_0(k-s) d \ln(1 - |r_1(s)|^2),$$

and $\ln_0(k) = \ln |k| + i \arg_0 k$ with $\arg_0 k \in [0, 2\pi)$. Since Δ^{-1} is a diagonal matrix, clearly

$$\lim_{k \rightarrow \infty} (k \Delta^{-1}(\zeta, k))_{13} = 0.$$

As $t \rightarrow \infty$, we have, by [7, Section 6],

$$\lim_{k \rightarrow \infty} k(\hat{m}(x, t, k) - I) = \frac{\sum_{j=0}^2 \omega^j \mathcal{A}^{-j} Y(\zeta, t) m_1^X(q(\zeta)) Y(\zeta, t)^{-1} \mathcal{A}^j}{3^{1/4} \sqrt{2} \sqrt{t}} + O(t^{-1} \ln t) \quad (4.11)$$

uniformly for $\zeta := x/t \in \mathcal{I}$ where $\mathcal{I}$ is any compact subset of (ζ_0, ∞) . The explicit expressions for $Y(\zeta, t)$, $q(\zeta)$ and m_1^X are recalled below. Note that $\Delta^{-1}(\zeta, k) = I + O(k^{-1})$ as $k \rightarrow \infty$. Hence, as $t \rightarrow \infty$

$$\begin{aligned} q(x, t) &= \lim_{k \rightarrow \infty} (k m(x, t, k))_{13} = \lim_{k \rightarrow \infty} (k(\hat{m}(x, t, k) - I)\Delta^{-1}(\zeta, k))_{13} = \lim_{k \rightarrow \infty} (k\hat{m}(x, t, k))_{13} \\ &= \frac{\sum_{j=0}^2 \omega^j (\mathcal{A}^{-j} Y(\zeta, t) m_1^X(q(\zeta)) Y(\zeta, t)^{-1} \mathcal{A}^j)_{13}}{3^{1/4} \sqrt{2} \sqrt{t}} + O(t^{-1} \ln t) \end{aligned} \quad (4.12)$$

uniformly for $\zeta \in \mathcal{I}$. Using that $q = q(\zeta) = r_1(k_0) \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and that

$$Y(\zeta, t) = \begin{pmatrix} d_0^{1/2}(\zeta, t) e^{-\frac{t}{2} \Phi_{21}(\zeta, k_0)} & 0 & 0 \\ 0 & d_0^{-1/2}(\zeta, t) e^{\frac{t}{2} \Phi_{21}(\zeta, k_0)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.13)$$

$$\Phi_{21}(\zeta, k_0) = \omega(\omega - 1)k_0^2 = -i\sqrt{3}k_0^2, \quad (4.13)$$

$$d_0(\zeta, t) = (2\sqrt{3}t)^{-i\nu} e^{2\chi_1(\zeta, k_0)} \delta_3(\zeta, k_0) \delta_5(\zeta, k_0), \quad (4.14)$$

$$m_1^X(q) = \begin{pmatrix} 0 & \beta_{12} & 0 \\ \beta_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where β_{12} and β_{21} are defined by

$$\beta_{12} = \frac{\sqrt{2\pi} e^{-\frac{\pi i}{4}} e^{-\frac{5\pi\nu}{2}}}{\bar{q}\Gamma(-i\nu)}, \quad \beta_{21} = \frac{\sqrt{2\pi} e^{\frac{\pi i}{4}} e^{\frac{3\pi\nu}{2}}}{q\Gamma(i\nu)},$$

we obtain after a straightforward calculation that

$$(\mathcal{A}^{-j} Y(\zeta, t) m_1^X(q(\zeta)) Y(\zeta, t)^{-1} \mathcal{A}^j)_{13} = \begin{cases} 0, & j = 0, 2, \\ \frac{\beta_{21} e^{t\Phi_{21}(\zeta, k_0)}}{d_0(\zeta, t)}, & j = 1. \end{cases}$$

Hence (4.12) yields

$$q(x, t) = \frac{\omega \beta_{21} e^{t\Phi_{21}(\zeta, k_0)}}{3^{1/4} d_0(\zeta, t) \sqrt{2t}} + O(t^{-1} \ln t) = \frac{\omega e^{t\Phi_{21}(\zeta, k_0)}}{3^{1/4} d_0(\zeta, t) \sqrt{2t}} \frac{\sqrt{2\pi} e^{\frac{\pi i}{4}} e^{\frac{3\pi\nu}{2}}}{q(\zeta) \Gamma(i\nu)} + O(t^{-1} \ln t) \quad (4.15)$$

as $t \rightarrow \infty$ uniformly for $\zeta \in \mathcal{I}$. Substituting (4.8)-(4.10) into (4.14), we get

$$\begin{aligned} d_0(\zeta, t) &= (2\sqrt{3}t)^{-i\nu} e^{2\chi_1(\zeta, k_0)} \delta_1(\zeta, \omega^2 k_0) \delta_1(\zeta, \omega k_0) \\ &= (2\sqrt{3}t)^{-i\nu} \exp \left(-i\nu \ln_0(\omega^2 k_0 - k_0) - i\nu \ln_0(\omega k_0 - k_0) \right. \\ &\quad \left. + 2\chi_1(\zeta, k_0) - \chi_1(\zeta, \omega^2 k_0) - \chi_1(\zeta, \omega k_0) \right). \end{aligned}$$

Hence, using that

$$\begin{aligned} \ln_0(\omega^2 k_0 - k_0) &= \ln |\omega^2 k_0 - k_0| + \frac{7\pi i}{6}, \quad \ln_0(\omega k_0 - k_0) = \ln |\omega k_0 - k_0| + \frac{5\pi i}{6}, \\ 2 \ln_0(k_0 - s) - \ln_0(\omega^2 k_0 - s) - \ln_0(\omega k_0 - s) &= 2 \ln \left| \frac{s - k_0}{s - \omega k_0} \right|, \end{aligned}$$

we obtain

$$d_0(\zeta, t) = \exp \left(-i\nu \ln(2\sqrt{3}t) - i\nu \ln(3k_0^2) + 2\pi\nu + \frac{1}{\pi i} \int_{k_0}^{+\infty} \ln \left| \frac{s - k_0}{s - \omega k_0} \right| d \ln(1 - |r_1(s)|^2) \right).$$

Substituting this expression into (4.15), and using (4.13) and $q = r_1(k_0)$, we arrive at

$$q(x, t) = \frac{\sqrt{\pi} e^{-\frac{\pi\nu}{2}}}{3^{1/4} \sqrt{t}} \frac{e^{i\phi}}{r_1(k_0) \Gamma(i\nu)} + O(t^{-1} \ln t), \quad t \rightarrow \infty, \quad (4.16)$$

uniformly for $\zeta \in \mathcal{I}$, where

$$\phi = \phi(\zeta) = \frac{11\pi}{12} + \nu \ln(6\sqrt{3}tk_0^2) - \sqrt{3}k_0^2 t + \frac{1}{\pi} \int_{k_0}^{+\infty} \ln \left| \frac{s - k_0}{s - \omega k_0} \right| d \ln(1 - |r_1(s)|^2).$$

This completes the proof of part (ii) of Theorem 1.

4.3. Proof of (iii). Let us indicate the dependence on the initial data $q_0 \in \mathcal{S}(\mathbb{R})$ explicitly by writing $X(x, k; q_0)$ for $X(x, k)$ etc. Let $q(x, t)$ be a Schwartz class solution of (1.1) with existence time $T \in (0, \infty]$ and initial data q_0 , and define $f(x, t) = -q(-x, t)$ and $f_0(x) = -q_0(-x)$. It is easy to check that $f(x, t)$ is a Schwartz class solution of (1.1) with existence time $T \in (0, \infty]$ and initial data $f_0 \in \mathcal{S}(\mathbb{R})$. Combining this symmetry with (4.16), we infer that

$$q(x, t) = - \frac{\sqrt{\pi} e^{-\frac{\pi\nu(-\zeta; f_0)}{2}}}{3^{1/4} \sqrt{t}} \frac{e^{i\phi(-\zeta; f_0)}}{r_1(-k_0; f_0) \Gamma(i\nu(-\zeta; f_0))} + O(t^{-1} \ln t), \quad t \rightarrow \infty, \quad (4.17)$$

uniformly for x/t in compact subsets of $(-\infty, 0)$. It follows from (2.2) that $\mathcal{U}(-x; f_0) = -\mathcal{U}(x; q_0)$. Since $\mathcal{L}(-k) = -\mathcal{L}(k)$, we deduce from (3.6) and (3.18) that

$$X(-x, -k; f_0) = Y(x, k; q_0), \quad X^A(-x, -k; f_0) = Y^A(x, k; q_0).$$

Hence, by (3.17), for all compactly supported $q_0 \in \mathcal{S}(\mathbb{R})$,

$$s(-k; f_0) = (s^A(k; q_0))^T, \quad s^A(-k; f_0) = (s(k; q_0))^T.$$

In particular, for $k \leq 0$, we have

$$r_1(-k; f_0) = \frac{s(-k; f_0)_{12}}{s(-k; f_0)_{11}} = \frac{s^A(k; q_0)_{21}}{s^A(k; q_0)_{11}} =: \tilde{r}_2(k; q_0).$$

Furthermore, the symmetry $s^A(k) = \overline{\mathcal{B} s^A(\bar{k}) \mathcal{B}}$ implies that

$$|\tilde{r}_2(k; q_0)| = |r_2(k; q_0)|, \quad k \leq 0,$$

and therefore

$$\begin{aligned} \nu(-\zeta; f_0) &= -\frac{1}{2\pi} \ln(1 - |r_1(-k_0; f_0)|^2) = -\frac{1}{2\pi} \ln(1 - |r_2(k_0; q_0)|^2) =: \tilde{\nu}, \\ \phi(-\zeta; f_0) &= \frac{11\pi}{12} + \nu(-\zeta; f_0) \ln(6\sqrt{3}tk_0^2) - \sqrt{3}k_0^2 t + \frac{1}{\pi} \int_{-k_0}^{+\infty} \ln \left| \frac{s + k_0}{s + \omega k_0} \right| d \ln(1 - |r_1(s; f_0)|^2) \\ &= \frac{11\pi}{12} + \tilde{\nu} \ln(6\sqrt{3}tk_0^2) - \sqrt{3}k_0^2 t + \frac{1}{\pi} \int_{-\infty}^{k_0} \ln \left| \frac{s - k_0}{s - \omega k_0} \right| d \ln(1 - |r_2(s; q_0)|^2). \end{aligned}$$

This concludes the proof of part (iii) of Theorem 1.

APPENDIX A. UPPER BOUNDS ON $|r_1|$ AND $|r_2|$

We will prove that if $q_0 \in \mathcal{S}(\mathbb{R})$ has compact support, then

$$|r_1(k; q_0)| < 1 \quad \text{for all } k \geq 0 \quad \text{and} \quad |r_2(k; q_0)| < 1 \quad \text{for all } k \leq 0, \quad (\text{A.1})$$

where we have indicated the dependence of r_1 and r_2 on the initial data q_0 for later convenience. We will also prove that

$$|r_1(k; q_0)| \leq 1 \quad \text{for all } k \geq 0 \quad \text{and} \quad |r_2(k; q_0)| \leq 1 \quad \text{for all } k \leq 0 \quad (\text{A.2})$$

holds whenever $q_0 \in \mathcal{S}(\mathbb{R})$. Note from (2.1) and (2.2) that

$$\mathcal{L}(k) = -\mathcal{L}(-k), \quad \mathcal{U}(x; q_0) = -\mathcal{U}^T(x; \omega^2 \bar{q}_0).$$

This implies by (2.3), (2.4), and (2.5) that

$$X(x, k; q_0) = X^A(x, -k; \omega^2 \bar{q}_0) \quad \text{and} \quad s(k; q_0) = s^A(-k; \omega^2 \bar{q}_0),$$

which, by (2.6), in turn implies that $r_1(k; q_0) = r_2(-k; \omega^2 \bar{q}_0)$. Hence, to prove (A.1) and (A.2), it is sufficient to prove that

$$|r_2(k; q_0)| < 1 \quad \text{for all } k \leq 0 \text{ and for all compactly supported } q_0 \in \mathcal{S}(\mathbb{R}), \quad (\text{A.3})$$

$$|r_2(k; q_0)| \leq 1 \quad \text{for all } k \leq 0 \text{ and for all } q_0 \in \mathcal{S}(\mathbb{R}). \quad (\text{A.4})$$

Let us fix $q_0 \in \mathcal{S}(\mathbb{R})$ and $k \in (-\infty, 0]$, and assume that q_0 is compactly supported. We will prove that $|r_2(k; q_0)| < 1$ for $k \leq 0$. Let $x_l \in \mathbb{R}$ be a point to the left of the support of q_0 . By (3.6b), we have $Y(x_l, k) = I$. Hence, by taking the inverse transpose of (3.15) and (3.17), we see that

$$1 - |r_2(k)|^2 = 1 - \left| \frac{s_{12}^A(k)}{s_{11}^A(k)} \right|^2 = \frac{s_{11}^A(k)s_{22}^A(k) - s_{12}^A(k)s_{21}^A(k)}{|s_{11}^A(k)|^2} = \frac{s_{33}(k)}{|s_{11}^A(k)|^2} = \frac{X_{33}(x_l, k)}{|s_{11}^A(k)|^2}.$$

We will prove that $X_{33}(x, k) > 0$ for all $x \in \mathbb{R}$ and for all $k \leq 0$. Define $P = P(k)$ by

$$P = \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 k & \omega k & k \\ k^2 & k^2 & k^2 \end{pmatrix}$$

and define $\tilde{L} = \tilde{L}(x, k)$ by

$$\tilde{L} = P(\mathcal{L} + \mathcal{U})P^{-1} = \begin{pmatrix} (\omega - \omega^2)(q_0 - \bar{q}_0) & 1 & 0 \\ 0 & (\omega^2 - 1)(q_0 - \omega \bar{q}_0) & 1 \\ k^3 & 0 & (1 - \omega)(q_0 - \omega^2 \bar{q}_0) \end{pmatrix}.$$

Since X satisfies (3.1), the matrix $\tilde{X}(x, k) := P(k)X(x, k)e^{x\mathcal{L}(k)}$ satisfies

$$\tilde{X}_x = \tilde{L}\tilde{X}. \quad (\text{A.5})$$

Note that

$$X_{33} = \frac{1}{3}(f_1 + f_2 + f_3)e^{-xk}, \quad (\text{A.6})$$

where

$$f_1 = \tilde{X}_{13}, \quad f_2 = k^{-1}\tilde{X}_{23}, \quad f_3 = k^{-2}\tilde{X}_{33}.$$

Since the functions f_1 , f_2 , and f_3 can be rewritten as

$$f_1 = \sum_{j=1}^3 \omega^j X_{j3} e^{xk}, \quad f_2 = \sum_{j=1}^3 \omega^{2j} X_{j3} e^{xk}, \quad f_3 = \sum_{j=1}^3 X_{j3} e^{xk},$$

they are well-defined for $k = 0$. Clearly, $X_{33} > 0$ is equivalent to $f_1 + f_2 + f_3 > 0$. We will show that in fact

$$f_j(x, k) > 0 \quad \text{for all } j = 1, 2, 3, \quad x \in \mathbb{R}, \quad k \leq 0. \quad (\text{A.7})$$

From (A.5), we get

$$kf_2 = \alpha_1 f_1 + f_1', \quad \alpha_1(x) := (\omega^2 - \omega)q_0(x) + (\omega - \omega^2)\overline{q_0(x)}, \quad (\text{A.8a})$$

$$kf_3 = \alpha_2 f_2 + f_2', \quad \alpha_2(x) := (1 - \omega^2)q_0(x) + (1 - \omega)\overline{q_0(x)}, \quad (\text{A.8b})$$

$$kf_1 = \alpha_3 f_3 + f_3', \quad \alpha_3(x) := (\omega - 1)q_0(x) + (\omega^2 - 1)\overline{q_0(x)}, \quad (\text{A.8c})$$

together with the normalization conditions

$$\lim_{x \rightarrow +\infty} f_j(x, k) e^{-xk} = 1, \quad \lim_{x \rightarrow +\infty} f'_j(x, k) e^{-xk} = k, \quad \lim_{x \rightarrow +\infty} f''_j(x, k) e^{-xk} = k^2, \quad (\text{A.9})$$

where prime denotes differentiation with respect to x . If $k = 0$, this system decouples and can be solved explicitly:

$$f_1(x, 0) = e^{-\int_{+\infty}^x \alpha_1}, \quad f_2(x, 0) = e^{-\int_{+\infty}^x \alpha_2}, \quad f_3(x, 0) = e^{-\int_{+\infty}^x \alpha_3}.$$

Since the functions $\alpha_1, \alpha_2, \alpha_3$ are real-valued, (A.7) is proved for $k = 0$. Using (A.8) and the fact that

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 = -9|q_0|^2,$$

we infer that the functions f_1, f_2, f_3 satisfy the following third-order ordinary differential equations

$$\begin{aligned} f_1''' + (-9|q_0|^2 + 2\alpha_1' + \alpha_2')f_1' + (-k^3 + \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_1' + \alpha_3\alpha_1' + \alpha_1\alpha_2' + \alpha_1'')f_1 &= 0, \\ f_2''' + (-9|q_0|^2 + 2\alpha_2' + \alpha_3')f_2' + (-k^3 + \alpha_1\alpha_2\alpha_3 + \alpha_3\alpha_2' + \alpha_1\alpha_2' + \alpha_2\alpha_3' + \alpha_2'')f_2 &= 0, \\ f_3''' + (-9|q_0|^2 + 2\alpha_3' + \alpha_1')f_3' + (-k^3 + \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3' + \alpha_2\alpha_3' + \alpha_3\alpha_1' + \alpha_3'')f_3 &= 0. \end{aligned} \quad (\text{A.10})$$

If $\tilde{q}_0 = \omega q_0$, then $\overline{\tilde{q}_0} = \omega^2 \bar{q}_0$ and hence

$$\begin{aligned} \alpha_1(x; \omega q_0) &= (1 - \omega^2)q_0 + (1 - \omega)\bar{q}_0 = \alpha_2(x; q_0), \\ \alpha_2(x; \omega q_0) &= (\omega - 1)q_0 + (\omega^2 - 1)\bar{q}_0 = \alpha_3(x; q_0), \\ \alpha_3(x; \omega q_0) &= (\omega^2 - \omega)q_0 + (\omega - \omega^2)\bar{q}_0 = \alpha_1(x; q_0). \end{aligned}$$

It follows that the equations for f_2 and f_3 are obtained by replacing q_0 by ωq_0 and $\omega^2 q_0$, respectively, in the equation for f_1 . Moreover, the normalization conditions (A.9) are invariant under these replacements. Thus,

$$f_1(x, k; \omega q_0) = f_2(x, k; q_0), \quad f_1(x, k; \omega^2 q_0) = f_3(x, k; q_0). \quad (\text{A.11})$$

We conclude that it is enough to consider one of the three equations. Indeed, if the inequality (A.7) holds for either $j = 1$, $j = 2$, or $j = 3$ for any choice of the complex-valued compactly supported initial data, then it automatically holds for $j = 1$, $j = 2$, and $j = 3$.

Let us consider the equation (A.10) for f_3 . Fix $k < 0$ and let

$$g(x) := \frac{f_3(x, k)}{f_3(x, 0)}.$$

We know that $f_3(x, 0) = e^{-\int_{+\infty}^x \alpha_3(x') dx'}$, and hence it follows that g satisfies the equation

$$g''' + 3p_1 g'' + 3p_2 g' = k^3 g, \quad (\text{A.12})$$

where

$$p_1(x) := -\alpha_3(x) = (1 - \omega)q_0 + (1 - \omega^2)\bar{q}_0, \quad (\text{A.13})$$

$$p_2(x) := 3|q_0|^2 - 3\omega q_0^2 - 3\omega^2 \bar{q}_0^2 - \omega q_0' - \omega^2 \bar{q}_0'. \quad (\text{A.14})$$

We next solve this equation using the method of variation of constants. Let $x_0 \in \mathbb{R}$ be a point to the right of the support of q_0 . Let $y_1(x) \equiv 1$, $y_2(x)$, $y_3(x)$, denote the three linearly independent solutions of the equation

$$y''' + 3p_1 y'' + 3p_2 y' = 0 \quad (\text{A.15})$$

satisfying the initial conditions

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0, \quad y_1''(x_0) = 0, \quad (\text{A.16a})$$

$$y_2(x_0) = 0, \quad y_2'(x_0) = 1, \quad y_2''(x_0) = 0, \quad (\text{A.16b})$$

$$y_3(x_0) = 0, \quad y_3'(x_0) = 0, \quad y_3''(x_0) = 1. \quad (\text{A.16c})$$

Note that $g(x) = f_3(x, k) = e^{xk} > 0$ for all $x \geq x_0$. We seek $c_j(x)$, $j = 1, 2, 3$, such that

$$g(x) = \sum_{j=1}^3 c_j(x) y_j(x). \quad (\text{A.17})$$

Assume that

$$\sum_{j=1}^3 c_j'(x) y_j(x) = 0, \quad \sum_{j=1}^3 c_j'(x) y_j'(x) = 0, \quad \sum_{j=1}^3 c_j'(x) y_j''(x) = k^3 g(x). \quad (\text{A.18})$$

Repeated differentiation of (A.17) gives

$$g'(x) = \sum_{j=1}^3 c_j(x) y_j'(x), \quad g''(x) = \sum_{j=1}^3 c_j(x) y_j''(x),$$

and

$$g''' = \sum_{j=1}^3 c_j' y_j'' + \sum_{j=1}^3 c_j y_j''' = k^3 g - \sum_{j=1}^3 c_j (3p_1 y_j'' + 3p_2 y_j') = k^3 g - (3p_1 g'' + 3p_2 g'),$$

showing that g indeed satisfies (A.12). The equations (A.18) can be written as

$$\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \\ c_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ k^3 g \end{pmatrix}$$

and by Cramer's rule the solution of this linear system is

$$c_1' = \frac{\det \begin{pmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ k^3 g & y_2'' & y_3'' \end{pmatrix}}{W(x)}, \quad c_2' = \frac{\det \begin{pmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & k^3 g & y_3'' \end{pmatrix}}{W(x)}, \quad c_3' = \frac{\det \begin{pmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & k^3 g \end{pmatrix}}{W(x)},$$

where

$$W(x) := \det \begin{pmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{pmatrix}.$$

Thus

$$g(x) = \sum_{j=1}^3 c_j(x) y_j(x) = \sum_{j=1}^3 C_j y_j(x) + \int_{x_0}^x \frac{k^3 g(t)}{W(t)} \det \begin{pmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \end{pmatrix} dt,$$

where the constants $\{C_j\}_1^3$ are determined by the initial conditions

$$g(x_0) = e^{kx_0} > 0, \quad g'(x_0) = ke^{kx_0} < 0, \quad g''(x_0) = k^2 e^{kx_0} > 0. \quad (\text{A.19})$$

Using that $y_1 \equiv 1$, we find

$$W(x) = \det \begin{pmatrix} y_2'(x) & y_3'(x) \\ y_2''(x) & y_3''(x) \end{pmatrix}$$

and

$$g(x) = C_1 + C_2 y_2(x) + C_3 y_3(x) + \int_{x_0}^x \frac{k^3 g(t)}{W(t)} \det \begin{pmatrix} 1 & y_2(x) & y_3(x) \\ 1 & y_2(t) & y_3(t) \\ 0 & y_2'(t) & y_3'(t) \end{pmatrix} dt. \quad (\text{A.20})$$

Since

$$\begin{aligned} \frac{d}{dx} \int_{x_0}^x \frac{k^3 g(t)}{W(t)} \det \begin{pmatrix} 1 & y_2(x) & y_3(x) \\ 1 & y_2(t) & y_3(t) \\ 0 & y_2'(t) & y_3'(t) \end{pmatrix} dt &= \int_{x_0}^x \frac{k^3 g(t)}{W(t)} \det \begin{pmatrix} 0 & y_2'(x) & y_3'(x) \\ 1 & y_2(t) & y_3(t) \\ 0 & y_2'(t) & y_3'(t) \end{pmatrix} dt, \\ \frac{d^2}{dx^2} \int_{x_0}^x \frac{k^3 g(t)}{W(t)} \det \begin{pmatrix} 1 & y_2(x) & y_3(x) \\ 1 & y_2(t) & y_3(t) \\ 0 & y_2'(t) & y_3'(t) \end{pmatrix} dt &= \int_{x_0}^x \frac{k^3 g(t)}{W(t)} \det \begin{pmatrix} 0 & y_2''(x) & y_3''(x) \\ 1 & y_2(t) & y_3(t) \\ 0 & y_2'(t) & y_3'(t) \end{pmatrix} dt, \end{aligned}$$

we find that

$$C_1 = g(x_0) > 0, \quad C_2 = g'(x_0) < 0, \quad C_3 = g''(x_0) > 0.$$

Abel's identity implies that

$$W(x) = C e^{-\int_{x_0}^x 3p_1(t) dt}$$

for some constant C ; evaluation at $x = x_0$ gives $C = 1$, and hence

$$W(x) = e^{-\int_{x_0}^x 3p_1(t) dt} > 0.$$

We will prove that $g'(x) \leq 0$ for all $x \leq x_0$. Using (A.20), we get

$$g'(x) = C_2 y_2'(x) + C_3 y_3'(x) - \int_{x_0}^x k^3 g(t) e^{\int_{x_0}^t 3p_1(t') dt'} \det \begin{pmatrix} y_2'(x) & y_3'(x) \\ y_2'(t) & y_3'(t) \end{pmatrix} dt \quad (\text{A.21})$$

Recall that y_2' and y_3' are two solutions of $u'' + 3p_1 u' + 3p_2 u = 0$. We seek to transform this equation to a simpler form by introducing v by

$$u(x) = e^{F(x)} v(x),$$

where F is some function yet to be determined. Letting $f = F'$, we find that v satisfies

$$v'' + (2f + 3p_1)v' + (f^2 + f' + 3fp_1 + 3p_2)v = 0. \quad (\text{A.22})$$

More explicitly, the coefficient of v can be written as

$$f^2 + f' + 3fp_1 + 3p_2 = f^2 + f' + 3f((1-\omega)q_0 + (1-\omega^2)\bar{q}_0) + 9|q_0|^2 - 9\omega q_0^2 - 9\omega^2 \bar{q}_0^2 - 3\omega q_0' - 3\omega^2 \bar{q}_0'.$$

To cancel the terms involving the derivative q_0' , we choose

$$f = 3\omega q_0 + 3\omega^2 \bar{q}_0.$$

With this choice, we find somewhat remarkably that the coefficient of v in (A.22) vanishes identically, which means that v satisfies

$$v'' + (2f + 3p_1)v' = 0, \quad \text{i.e.} \quad v'' - 3(\omega^2 q_0 + \omega \bar{q}_0)v' = 0.$$

It follows that

$$v(x) = C \int_{x_0}^x e^{3 \int_{x_0}^{x'} (\omega^2 q_0 + \omega \bar{q}_0) dt} dx' + D,$$

where C and D are integration constants, and so

$$y' = e^{F(x)} v = C e^{\int_{x_0}^x 3(\omega q_0 + \omega^2 \bar{q}_0) dt} \int_{x_0}^x e^{3 \int_{x_0}^{x'} (\omega^2 q_0 + \omega \bar{q}_0) dt} dx' + D e^{\int_{x_0}^x 3(\omega q_0 + \omega^2 \bar{q}_0) dt}.$$

In particular, since x_0 lies to the right of the support of q_0 , $y'(x_0) = D$ and $y''(x_0) = C$. Using (A.16b) and (A.16c), we obtain the following explicit expressions for y'_2 and y'_3 :

$$y'_2(x) = e^{\int_{x_0}^x 3(\omega q_0 + \omega^2 \bar{q}_0) dt},$$

$$y'_3(x) = e^{\int_{x_0}^x 3(\omega q_0 + \omega^2 \bar{q}_0) dt} \int_{x_0}^x e^{3 \int_{x_0}^{x'} (\omega^2 q_0 + \omega \bar{q}_0) dt} dx'.$$

Clearly, $y'_2(x) > 0$ for all $x \in \mathbb{R}$, $y'_3(x) < 0$ for $x < x_0$, and $y'_3(x) > 0$ for $x > x_0$. Furthermore, we obtain

$$\det \begin{pmatrix} y'_2(x) & y'_3(x) \\ y'_2(t) & y'_3(t) \end{pmatrix} = e^{\int_{x_0}^x 3(\omega q_0 + \omega^2 \bar{q}_0) dt'} e^{\int_{x_0}^t 3(\omega q_0 + \omega^2 \bar{q}_0) dt'} \int_x^t e^{3 \int_{x_0}^{x'} (\omega^2 q_0 + \omega \bar{q}_0) dt'} dx',$$

which is clearly > 0 for $x < t$ (and < 0 for $x > t$). By (A.21), this shows that $g'(x) < 0$ whenever $x \leq x_0$. Since $g(x) = e^{xk} > 0$ for $x \geq x_0$, we conclude that $g(x) > 0$ (and hence also $f_3(x, k) > 0$) for all $x \in \mathbb{R}$. This proves (A.7) for $j = 3$ and completes the proof of (A.3), and hence also of (A.1).

Let us now consider the general case of $q_0 \in \mathcal{S}(\mathbb{R})$ where q_0 is not necessarily compactly supported. Let $\eta \in C_c^\infty$ be a cutoff function satisfying $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$, and for $j \geq 1$, let $\eta_j(x) = \eta(x/j)$. Since $q_0 \in \mathcal{S}(\mathbb{R})$, $(\eta_j q_0)_{j \geq 1}$ is a sequence of smooth functions with compact support converging to q_0 in $\mathcal{S}(\mathbb{R})$ as $j \rightarrow \infty$. As in [6, Lemma 4.5], we have $r_2(k; \eta_j q_0) \rightarrow r_2(k; q_0)$ as $j \rightarrow \infty$. The inequality in (A.4) therefore follows from (A.3). This completes the proof of (A.2).

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