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The Teichmüller TQFT volume conjecture for twist knots

La conjecture du volume de la TQFT de Teichmüller pour les nœuds twist

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A R T I C L E   I N F O

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A B S T R A C T

The Teichmüller TQFT, defined by Andersen and Kashaev, gives rise to a quantum invariant of triangulated hyperbolic knot complements; it has an associated volume conjecture, where the hyperbolic volume of the knot appears as a certain asymptotic coefficient.
In this note, we announce a proof of this volume conjecture for all twist knots up to 14 crossings; along the way we explicitly compute the partition function of the Teichmüller TQFT for the whole infinite family of twist knots.
Among other tools, we use an algorithm of Thurston to construct a convenient ideal triangulation of a twist knot complement, as well as the saddle point method for computing limits of complex integrals with parameters.

R É S U M É

La TQFT de Teichmüller, définie par Andersen et Kashaev, produit un invariant quantique des complémentaires de nœuds hyperboliques triangulés ; elle a une conjecture du volume associée, où le volume hyperbolique du nœud apparaît comme un certain coefficient asymptotique.
Dans cette note, nous annonçons une preuve de cette conjecture du volume pour tous les nœuds twist de 14 croisements ou moins ; nous calculons au passage explicitement la TQFT pour l'intégralité de la famille infinie des nœuds twist.
Entre autres outils, nous utilisons un algorithme de Thurston pour construire une triangulation idéale pratique du complémentaire d'un nœud twist, ainsi que la méthode du point selle pour calculer des limites d'intégrales complexes paramétrées.

Version française abrégée

Dans cette note, nous annonçons le résultat suivant et nous esquissons sa preuve (voir la version anglaise pour un bref historique et [3] pour les détails de la preuve).

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Théorème 0.1. *La conjecture du volume de la TQFT de Teichmüller est vérifiée pour tous les nœuds twist à au plus 14 croisements.*

Commençons par clarifier quelques termes de l'énoncé précédent. Dans [1], les auteurs ont défini une TQFT généraliséée à partir de la théorie de Teichmüller quantique, de fonction partition associée qu'on notera $Z_\hbar$ dans la suite (ici $\hbar > 0$ est le paramètre quantique). Pour construire cette *TQFT de Teichmüller*, on munit les 3-variétés de triangulations idéales avec des structures d'angles diédraux sur les arêtes ; on associe ensuite aux tétraèdres des distributions tempérées, pour finalement obtenir les fonctions partition par des intégrales sur des variables d'état (variables qui correspondent aux faces des tétraèdres). Les auteurs ont ensuite proposé une *conjecture du volume* (voir Conjecture 2 ci-après), énonçant que, pour un nœud K de complément hyperbolique dans une 3-variété close M, le volume hyperbolique $\mathrm{Vol}(M \setminus K)$ apparaît comme le facteur de décroissance exponentielle de $Z_\hbar(M \setminus K)$ dans la limite semi-classique $\hbar \to 0^+$.

Les *nœuds twist* sont les nœuds de la forme de la Fig. 2 : K_n dénote le nœud twist à $n+2$ croisements. Pour $n \geqslant 2$, K_n est toujours hyperbolique. Cette famille est ainsi la famille infinie de nœuds hyperboliques la plus simple à étudier et est donc apparue naturellement pour s'attaquer à la conjecture du volume de la TQFT de Teichmüller, qui n'était démontrée jusqu'ici que pour un nombre fini de cas. Précisons maintenant comment nous avons démontré le résultat principal.

Tout d'abord, un algorithme remontant à Thurston nous a permis de construire une *triangulation idéale* du complémentaire d'un nœud twist K_n à partir d'un diagramme de ce nœud K_n (voir Figs. 3 et 4).

Ces triangulations suivent une *forme générale* dont la complexité conceptuelle ne varie pas quand n augmente, ce qui est très pratique pour calculer et étudier l'*asymptotique* d'invariants construits à partir de triangulations.

Nous pouvons ensuite calculer la *fonction partition de la TQFT de Teichmüller* pour les complémentaires des nœuds twist. Soit $n \geqslant 5$ un entier et X_n la triangulation idéale construite pour $S^3 \setminus K_n$. Pour tout $\hbar > 0$ et toute structure d'angle α sur X_n, on a

$$|Z_\hbar(X_n, \alpha)| = \left| \int_{\mathbb{R}} J_{S^3,K_n}(\hbar, x)\, \mathrm{e}^{-\frac{1}{\sqrt{\hbar}} x \lambda_{X_n}(\alpha)} \mathrm{d}x \right|,$$

où J_{S^3,K_n} est une fonction explicitement connue ne dépendant que du nœud K_n, et λ_{X_n} est une combinaison linéaire explicite d'angles diédraux. Nous ne détaillons pas ici les valeurs exactes de J_{S^3,K_n} et λ_{X_n} afin de simplifier la lecture, mais nous renvoyons le lecteur au Théorème 4.1 (1) ci-après. Pour démontrer l'égalité précédente, nous avons remarqué des simplifications dans les intégrales d'état, conséquences des *équations d'équilibrage* sur les angles de la triangulation X_n. Nous ne doutons pas que la forme particulièrement pratique de X_n a aidé à ces simplifications.

Une fois la fonction J_{S^3,K_n} connue, on peut démontrer la Conjecture 2 pour de nouveaux nœuds. Cette conjecture avait été prouvée pour les deux premiers nœuds twist hyperboliques 4_1 et 5_2 dans [1], puis pour le nœud 6_1 dans [2]. En se fondant sur des idées de [1] et notamment la *méthode du point selle*, nous pouvons finalement vérifier numériquement la Conjecture 2 pour tous les K_n avec $n \leqslant 12$.

Remarquons que notre méthode de vérification est *indépendante du nombre de croisements* du nœud twist et que sa réussite ne dépend que de la puissance de calcul de l'ordinateur avec lequel nous faisons les vérifications numériques. Ainsi, la valeur 12 de n où nous nous sommes arrêtés pourrait certainement être remplacée par un entier arbitrairement plus grand.

1. Introduction

The *volume conjecture* of Kashaev and Murakami–Murakami is perhaps the most studied conjecture in quantum topology currently (see the original papers [4,7] and [6,8] for a survey); it states that a certain asymptotic of the colored Jones polynomials of a hyperbolic knot yields the hyperbolic volume of this knot. As such, it hints at a deep connection between quantum topology and classical geometry.

The *Teichmüller TQFT* is a quantum invariant defined in [1] by Andersen–Kashaev that one can understand as an infinite-dimensional version of the colored Jones polynomials. Taking its roots in quantum Teichmüller theory and making use of Faddeev's quantum dilogarithm, this infinite-dimensional TQFT is constructed with state integrals on tempered distributions from the data of a 3-dimensional triangulation with angles. Andersen and Kashaev conjectured that the hyperbolic volume of a knot complement appears in the Teichmüller TQFT as well.

Conjecture 1. *[1, Conjecture 1] Let M be a closed oriented 3-manifold and $K \subset M$ a knot whose complement is hyperbolic. Then the partition function of the Teichmüller TQFT associated with (M, K) can be computed from a function $J_{M,K}$ independent of the triangulation on M, and the hyperbolic volume* $\mathrm{Vol}(M \setminus K)$ *appears as an asymptotic exponential decrease rate.*

We refer to Conjecture 2 in the note for a more detailed statement. This *volume conjecture* was proven for the first two hyperbolic knots $(S^3, 4_1)$, $(S^3, 5_2)$ in [1] and the next one $(S^3, 6_1)$ in [2]. This conjecture can also be generalized to the case of non-balanced shape structures (see Section 2 for the terminology and [5] for the generalization).

In this note we announce further proof of the conjecture for several new examples, all part of the infinite family of *twist knots*:

Theorem 1.1 *(Main theorem). For the n-th twist knot K_n in S^3, the function J_{S^3,K_n} can be computed explicitly. Furthermore, the asymptotic part of the conjecture is checked numerically for all $n \leqslant 12$.*

We refer to Theorem 4.1 for more details. The newness of our approach resides in considering an infinite family of new examples globally, so that patterns are more easily discerned than for the previous isolated examples. Notably, we compute general triangulations for the twist knots, see Theorem 2.1, and we use them to prove the Main Theorem.

A similar approach can be used to prove the conjecture for various hyperbolic knots in lens spaces [9].

The rest of this note is organized as follows: in Section 2 we review preliminaries and we show the construction of particularly convenient triangulations in Theorem 2.1; in Section 3, we recall the definition of the Teichmüller TQFT and we state its associated volume conjecture; finally, in Section 4, we state the main result, Theorem 4.1, and give ideas of its proof.

2. Topological preliminaries

2.1. Triangulations

We mostly follow the notations of [1]. We consider triangulations of 3-dimensional manifolds, with *ordered* tetrahedra, i.e. with vertices numbered 0, 1, 2, 3 for each tetrahedron. We *orient the edges* of T according to the order on the vertices. We denote $\epsilon(T)$ the *sign* of the tetrahedron T, defined to be $\epsilon(T) = +1$ if the edges $\overrightarrow{01}$, $\overrightarrow{02}$, $\overrightarrow{03}$ induce a right-handed coordinate system, and $\epsilon(T) = -1$ otherwise.

A *triangulation* $X = (T_1, \ldots, T_N, \sim)$ is the data of N distinct tetrahedra $T_1, \ldots, T_N$ and an equivalence relation $\sim$ first defined on the faces by pairing, and the only gluing that respects the vertex order, which is also induced on edges and vertices by the combined identifications. We call M_X the (pseudo-) 3-manifold $M_X = T_1 \sqcup \ldots \sqcup T_N / \sim$ obtained by quotient.

We denote by X^i (for $i = 0, \ldots, 3$) the set of i-cells of X after identification by $\sim$. In this paper, we always consider that every face is paired with an other face by $\sim$, thus X^2 is always of cardinal $2N$. By a slight abuse of notation we also call T_i the 3-cell inside the tetrahedron T_i, so that $X^3 = \{T_1, \ldots, T_N\}$.

An *ideal triangulation* X contains ideal tetrahedra (which are homeomorphic to closed balls minus 4 boundary vertices), and in this case the quotient space minus its vertices $M_X \setminus X^0$ is an open manifold. In this case, we will denote $M = M_X \setminus X^0$ and say that the open manifold M admits the ideal triangulation X.

A (one-vertex) *H-triangulation* is a triangulation Y with compact tetrahedra (which are homeomorphic to closed balls) so that $M = M_Y$ is a closed manifold and Y^0 is a singleton, with one particular edge in Y^1 distinguished; this edge will represent a knot K (up to ambient isotopy) in the closed manifold M, and we will write that Y is an *H-triangulation of* (M, K).

Finally, for $i = 0, 1, 2, 3$, we define by $x_i \colon X^3 \to X^2$ the map such that $x_i(T)$ is the equivalence class of the face of T opposed to its vertex i.

Fig. 1 displays two possible ways of representing an ideal triangulation of the open complement of the figure-eight knot $M = S^3 \setminus 4_1$, with one positive and one negative tetrahedron. Here $X^3 = \{T_+, T_-\}$, $X^2 = \{A, B, C, D\}$, $X^1 = \{\rightarrow, \twoheadrightarrow\}$ and X^0 is a singleton. On the left, the tetrahedra are drawn as usual and all the cells are named; on the right, we represent each tetrahedron by a "comb" with four spikes numbered 0, 1, 2, 3 from left to right, we join the spike i of T to the spike j of T' if $x_i(T) = x_j(T')$, and we add a $+$ or $-$ next to each tetrahedron according to its sign.

2.2. Angle structures

For a given triangulation $X = (T_1, \ldots, T_N, \sim)$, we denote by $\mathcal{S}_X$ the set of *shape structures on* X, defined as

$$\mathcal{S}_X = \left\{ \alpha = \left(\alpha_1^1, \alpha_2^1, \alpha_3^1, \ldots, \alpha_1^N, \alpha_2^N, \alpha_3^N\right) \in \mathbb{R}^{3N} \,\middle|\, \forall (j,k) \in \{1,2,3\} \times \{1,\ldots,N\}, \alpha_j^k \in (0;\pi), \sum_{j=1}^{3} \alpha_j^k = \pi \right\}.$$

An angle α_j^k represents the value of a dihedral angle on the edge $\overrightarrow{0j}$ and its opposite edge in the tetrahedron T_k. If a particular shape structure $\alpha \in \mathcal{S}_X$ is fixed, we define the associated maps $\alpha_j \colon X^3 \to (0;\pi)$ for $j = 1, 2, 3$ by $\alpha_j(T_k) = \alpha_j^k$ for $k \in \{1, \ldots, N\}$.

Let (X, α) be a triangulation with a shape structure as before. We denote by $\omega_{X,\alpha} \colon X^1 \to \mathbb{R}$ the associated *weight function*, which sends a quotient edge $e \in X^1$ to the sum of angles α_j^k corresponding to tetrahedral edges that are pre-images of e by $\sim$. For example, if we denote by $\alpha = (\alpha_1^+, \alpha_2^+, \alpha_3^+, \alpha_1^-, \alpha_2^-, \alpha_3^-)$ a shape structure on the triangulation X of Fig. 1, then $\omega_{X,\alpha}(\rightarrow) = 2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^-$.

In Conjecture 2 (2), we will use the notation $\omega_{Y,\alpha}(K)$ to denote the weight function of the edge K (representing the knot K with a slight abuse of notation) in an H-triangulation Y of (M, K) with a shape structure $\alpha \in \mathcal{S}_Y$.

One can also consider the closure $\overline{\mathcal{S}_X}$ where the α_j^k are taken in $[0;\pi]$ instead. The definitions of the maps α_j and $\omega_{X,\alpha}$ can immediately be extended.

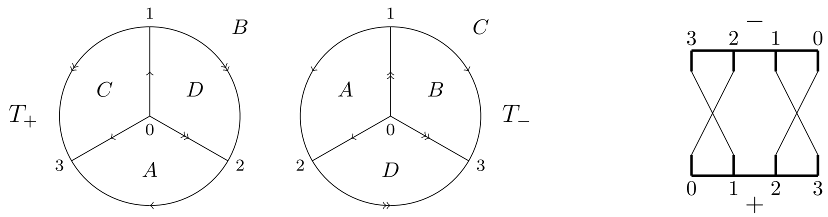

Fig. 1. Two representations of an ideal triangulation of the knot complement $S^3 \setminus 4_1$. Deux représentations d'une triangulation idéale du complémentaire de nœud $S^3 \setminus 4_1$.

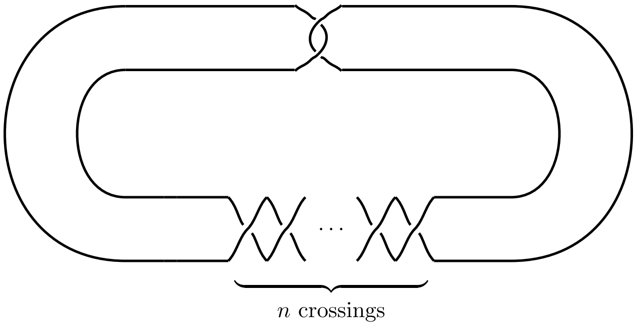

Fig. 2. The twist knot K_n. Le nœud twist K_n.

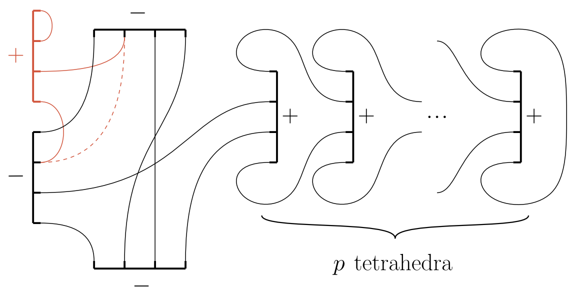

Fig. 3. An H-triangulation for (S^3, K_n) and an ideal triangulation for $S^3 \setminus K_n$ for odd $n \geqslant 5$, with $p = \frac{n-3}{2}$. Une H-triangulation de (S^3, K_n) et une triangulation idéale de $S^3 \setminus K_n$ pour $n \geqslant 5$ impair, avec $p = \frac{n-3}{2}$.

We finally define $\mathcal{A}_X = \left\{\alpha \in \mathcal{S}_X \;\middle|\; \forall e \in X^1,\ \omega_{X,\alpha}(e) = 2\pi\right\}$ the set of *balanced shape structures on* X, or *angle structures on* X.

2.3. Hyperbolic knot complements

We denote by K_n the unoriented twist knot with n half-twists and $n+2$ crossings, whose diagram is drawn in Fig. 2.

Theorem 2.1. *For every $n \geqslant 5$ odd (respectively for every $n \geqslant 6$ even), the triangulations X_n and Y_n represented in Fig. 3 (respectively in Fig. 4) are an ideal triangulation of $S^3 \setminus K_n$ and an H-triangulation of (S^3, K_n) respectively.*

Figs. 3 and 4 display an H-triangulation for (S^3, K_n), and the corresponding ideal triangulation for $S^3 \setminus K_n$ is obtained by replacing the upper left red tetrahedron (glued to itself) by the dotted line (note that we did not write the numbers 0, 1, 2, 3 of the vertices). Theorem 2.1 is proven by applying an algorithm of Thurston to construct a polyhedral decomposition of S^3 where the knot K_n is one of the edges, starting from a diagram of K_n; along the way we apply some combinatorial tricks to reduce the number of edges and we choose a convenient triangulation of the polyhedra. Once we have the H-triangulation for (S^3, K_n), we can collapse both the edge representing the knot K_n and its underlying tetrahedron to obtain an ideal triangulation of $S^3 \setminus K_n$.

Recall that a knot K in a closed 3-manifold M is *hyperbolic* if $M \setminus K$ admits a complete hyperbolic structure; the associated *volume* is denoted $\mathrm{Vol}(M \setminus K)$. Note that, for $n \geqslant 2$, the twist knot K_n is hyperbolic.

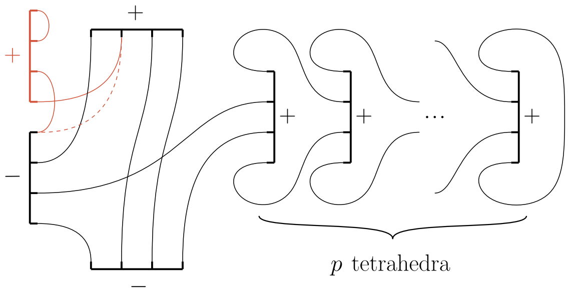

Fig. 4. An H-triangulation for (S^3, K_n) and an ideal triangulation for $S^3 \setminus K_n$ for even $n \geqslant 6$, with $p = \frac{n-2}{2}$. Une H-triangulation de (S^3, K_n) et une triangulation idéale de $S^3 \setminus K_n$ pour $n \geqslant 6$ pair, avec $p = \frac{n-2}{2}$.

3. The Teichmüller TQFT

3.1. Definition

Recall that for $\hbar > 0$ and $\mathsf{b} \in \mathbb{C}^*$ such that $(\mathsf{b} + \mathsf{b}^{-1})\sqrt{\hbar} = 1$, *Faddeev's quantum dilogarithm* Φ_b is the meromorphic function defined by

$$\Phi_\mathsf{b}(x) = \exp\left(\frac{1}{4}\int\limits_{z\in\mathbb{R}+\mathrm{i}0^+} \frac{\mathrm{e}^{-2\mathrm{i}xz}\mathrm{d}z}{\sinh(\mathsf{b}z)\sinh(\mathsf{b}^{-1}z)z}\right),$$

and which depends only on $\hbar = \frac{1}{(\mathsf{b}+\mathsf{b}^{-1})^2}$. Note that the poles and zeroes of Φ_b are all outside of the strip $\mathbb{R} + \mathrm{i}\left(\frac{-1}{2\sqrt{\hbar}}; \frac{1}{2\sqrt{\hbar}}\right)$.

Let X be a triangulation and $\alpha \in \mathcal{A}_X$ an angle structure. The *partition function of the Teichmüller TQFT* of (X, α) at $\hbar > 0$ is defined as:

$$Z_\hbar(X,\alpha) := \int\limits_{\mathbf{x}\in\mathbb{R}^{X^2}} \mathrm{d}\mathbf{x} \prod_{T\in X^3} \frac{\delta\left(x_0(T) - x_1(T) + x_2(T)\right)\mathrm{e}^{2\pi\mathrm{i}\epsilon(T)x_0(T)(x_3(T)-x_2(T))}\mathrm{e}^{\frac{\alpha_3(T)(x_3(T)-x_2(T))}{\hbar^{1/2}}}}{\Phi_\mathsf{b}\left((x_3(T)-x_2(T)) - \frac{\mathrm{i}}{2\pi\hbar^{1/2}}\epsilon(T)(\alpha_2(T)+\alpha_3(T))\right)^{\epsilon(T)}}.$$

At first glance, it may not be obvious that this multi-integral is well defined or convergent. We refer to [1] for technical details.

Moreover, in [1], the authors prove that the module of the partition function is topologically invariant, in the sense that $|Z_\hbar(X,\alpha)| = |Z_\hbar(X',\alpha')|$ when X, X' are two triangulations related by a Pachner $3-2$ move, and $\alpha' \in \mathcal{A}_{X'}$ is the angle structure obtained from $\alpha \in \mathcal{A}_X$ under this Pachner move.

3.2. The volume conjecture

From now on, we will write $z \overset{\star}{=} w$ if $z, w \in \mathbb{C}$ have the same module. Let us recall the general statement of the volume conjecture for the Teichmüller TQFT, due to Andersen–Kashaev.

Conjecture 2. *[1, Conjecture 1] Let M be a closed oriented 3-manifold and $K \subset M$ a knot whose complement is hyperbolic. The following statements hold:*

(1) for every ideal triangulation X of $M \setminus K$, there exists a degree-one real polynomial λ_X of dihedral angles of X and a smooth function $J_{M,K} : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{C}$ such that, for all angle structures $\alpha \in \mathcal{A}_X$ and all $\hbar > 0$,

$$Z_\hbar(X,\alpha) \overset{\star}{=} \int\limits_{\mathbb{R}} J_{M,K}(\hbar, x)\,\mathrm{e}^{-\frac{1}{\sqrt{\hbar}}x\lambda_X(\alpha)}\mathrm{d}x;$$

(2) for every one-vertex H-triangulation Y of the pair (M, K), for every $\hbar > 0$ and for every $\tau \in \overline{\mathcal{S}_Y}$ such that $\omega_{Y,\tau}$ vanishes on K and is equal to 2π on every other edge, one has

$$\lim_{\alpha\to\tau,\alpha\in\mathcal{S}_Y} \Phi_\mathsf{b}\left(\frac{\pi - \omega_{Y,\alpha}(K)}{2\pi\mathrm{i}\sqrt{\hbar}}\right) Z_\hbar(Y,\alpha) \overset{\star}{=} J_{M,K}(\hbar, 0);$$

(3) the hyperbolic volume of the complement of K in M is obtained as the following limit:

$$\lim_{\hbar\to 0^+} 2\pi\hbar \log|J_{M,K}(\hbar,0)| = -\mathrm{Vol}(M\backslash K).$$

Note that thanks to the invariance properties proved in [1], one needs only to prove parts (1) and (2) of Conjecture 2 for a specific ideal triangulation X or H-trangulation Y, as we will do for the twist knots.

4. Main result

Conjecture 2 was proven for the cases $(S^3, 4_1)$, $(S^3, 5_2)$ in [1] and for $(S^3, 6_1)$ in [2]. Note that for this last case, part (3) of Conjecture 2 was checked numerically. The main result of this paper is the following:

Theorem 4.1. *The following statements hold.*

(1) Part (1) of Conjecture 2 is satisfied for all twist knots K_n such that $n \geqslant 5$. More precisely:

(i) if n is odd, let $p = \frac{n-3}{2}$. We take the ideal triangulation $X = X_n = (T_1, \dots, T_p, T_U, T_V, T_W, \sim)$ of $S^3 \setminus K_n$ described in Fig. 3, the angle polynomial $\lambda_X : \alpha \mapsto -3\alpha_1^U + 2\alpha_1^V - \alpha_3^V - \alpha_1^W + 2\pi$, and the smooth map

$$J_{S^3,K_n} : (\hbar, x) \mapsto \int \mathrm{d}\mathbf{y}\, e^{2i\pi \mathbf{y}^\intercal Q \mathbf{y}} e^{2i\pi x(x-y_U-y_W)} e^{\frac{1}{\sqrt{\hbar}}\mathbf{y}^\intercal \mathcal{W}} \frac{\Phi_b(y_U)\,\Phi_b(y_U+x)\,\Phi_b(y_W)}{\Phi_b(y_1)\dots\Phi_b(y_p)},$$

with

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \\ y_U \\ y_W \end{bmatrix}, \quad \mathcal{W} = \begin{bmatrix} -2p\pi \\ \vdots \\ -2\pi\left(kp - \frac{k(k-1)}{2}\right) \\ \vdots \\ -p(p+1)\pi \\ (p^2+p+1)\pi \\ \pi \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 1 & \cdots & 1 & -1 & 0 \\ 1 & 2 & \cdots & 2 & -2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & \cdots & p & -p & 0 \\ -1 & -2 & \cdots & -p & p & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix};$$

(ii) if n is even, let $p = \frac{n-2}{2}$; we take the ideal triangulation $X = X_n = (T_1, \dots, T_p, T_U, T_V, T_W, \sim)$ of $S^3 \setminus K_n$ described in Fig. 4, the angle polynomial $\lambda_X : \alpha \mapsto \alpha_1^U - 2\alpha_1^V - \alpha_3^V - \alpha_1^W + 2\pi$, and the smooth map

$$J_{S^3,K_n} : (\hbar, x) \mapsto \int \mathrm{d}\mathbf{y}\, e^{2i\pi \mathbf{y}^\intercal Q \mathbf{y}} e^{2i\pi x(y_U-y_W-x)} e^{\frac{1}{\sqrt{\hbar}}\mathbf{y}^\intercal \mathcal{W}} \frac{\Phi_b(x-y_U)\,\Phi_b(y_W)}{\Phi_b(y_1)\dots\Phi_b(y_p)\,\Phi_b(y_U)},$$

with

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \\ y_U \\ y_W \end{bmatrix}, \quad \mathcal{W} = \begin{bmatrix} -2p\pi \\ \vdots \\ -2\pi\left(kp - \frac{k(k-1)}{2}\right) \\ \vdots \\ -p(p+1)\pi \\ -(p^2+p+3)\pi \\ \pi \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 2 & \cdots & 2 & 2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & \cdots & p & p & 0 \\ 1 & 2 & \cdots & p & p+1 & -\frac{1}{2} \\ 0 & 0 & \cdots & 0 & -\frac{1}{2} & 0 \end{bmatrix}.$$

(2) Part (2) of Conjecture 2 holds for all twist knots K_n with $n \geqslant 5$ with the H-triangulation Y_n of Figs. 3 and 4.

(3) Part (3) of Conjecture 2 is checked numerically for all twist knots K_n up to $n = 12$.

Let us give a sketch of the proof. We first compute the partition function $Z_\hbar$ on the triangulation X_n using the definitions; then we simplify the multi-integral, starting with removing the Dirac delta functions and then applying the balancing equations on the dihedral angles satisfied by any angle structure $\alpha \in \mathcal{A}_{X_n}$. This gives us part (1) of Conjecture 2 and the explicit values of λ_{X_n} and J_{S^3,K_n} for any n.

The computations in part (2) are similar, but it is important to notice that the balancing angle equations for Y_n slightly differ from those of X_n.

Finally, for part (3), we follow the techniques of [1,2] and we employ the saddle point method to equate the limit $\lim_{\hbar \to 0^+} 2\pi\hbar \log|J_{S^3,K_n}(\hbar, 0)|$ with the extremum of a potential function constructed from J_{S^3,K_n}. We find and compute this extremum on *Mathematica* numerically, and we check, for each separate n, that it coincides with the value of $-\mathrm{Vol}(S^3 \setminus K_n)$ taken from *Snappy*. This process can be applied for any n without difference, and we decided to stop at $n = 12$. We expect part (3) of Conjecture 2 to hold similarly for every n.

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