On Persistence of Some $\mathcal{W}_I$-Endotactic Chemical Reaction Networks

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***Abstract*—Motivated by the concept of the endotactic network, a kind of special geometric structure in chemical reaction networks developed for persistence analysis, we propose a new notion, named $\mathcal{W}_I$-endotactic network. The corresponding network set is a larger class of set than the endotactic network, let alone the weakly reversible network. Based on an energy-like function, we prove that all 1-dimensional mass-action $\mathcal{W}_I$-endotactic networks are persistent. Furthermore, we prove some higher dimensional network systems also to support persistence if the system is composed of a series of 1-dimensional $\mathcal{W}_I$-endotactic networks. We discuss two cases for the subsystems with and without intersecting species, and present the corresponding sufficient conditions to capture persistence. Finally, we use some examples including abstract and real biochemical reaction networks to illustrate our results.**

***Index Terms*—ω-limit point, boundary points, chemical reaction network (CRN), mass-action kinetics, persistence, $\mathcal{W}_I$-endotactic network.**

Nomenclature

$\mathbb{R}^n_{\geq 0}, \mathbb{R}^n_{>0}$:	n-dimensional nonnegative real space, n-dimensional positive real space, respectively.
$\mathbb{Z}^n_{\geq 0}$:	n-dimensional nonnegative integer space.
$x^{y_{\cdot i}}$:	$x^{y_{\cdot i}} \triangleq \prod_{j=1}^n x_j^{y_{ji}}$, where $x, y_{\cdot i} \in \mathbb{R}^n$.
$\mathrm{Ln}(x)$:	$\mathrm{Ln}(x) \triangleq (\ln x_1, \ldots, \ln x_n)^\top$, where $x \in \mathbb{R}^n_{>0}$.
supp x:	Support set, defined by supp $x = \{i \vert x_i \neq 0\}$, $\forall x \in \mathbb{R}^n_{\geq 0}$.
supp^c x:	Complementary set of support set, defined by supp^c $x = \{i \vert x_i = 0\}$, $\forall x \in \mathbb{R}^n_{\geq 0}$.
$\mathbb{0}_n$:	n-dimensional vector with each element to be zero.
0^0:	Result is defined by 1.
$\bigotimes$:	Cartesian product, defined by $A \bigotimes B = \{(a, b) \vert a \in A \text{ and } b \in B\}$.

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I. Introduction

PERSISTENCE is a concept from population processes that can be interpreted as nonextinction [1], [2]. A system of species interactions is said to be persistent if initial present species are not allowed to approach or asymptotically approach extinction. There is a great deal of practical examples that can be modeled by population processes [3], such as animal populations, the spread of infectious diseases, and the evolution of biochemical reaction systems. The species in these examples might correspond to types of animals, infected and noninfected individuals, and chemical reactants and products, respectively. Persistence is, thus, a property of central practical interest [4], [5], [6], [7]. Also, it is of major theoretical interest since from a mathematical perspective persistence means that the ω-limit set of any trajectory starting in the interior of the positive orthant does not intersect the boundary of the positive orthant [8]. In this work, we focus on the persistence of mass-action systems (MASs) with special network structure, i.e., chemical reaction networks (CRNs) with mass-action kinetics.

Mass-action CRNs often give rise to a family of ordinary differential equations that lie at the limits of what we can treat with the current tools of dynamical systems. Although the past 40 years have witnessed the fast development of chemical reaction network theory (CRNT) since the pioneering work of Horn and Jackson [9], [10], and Feinberg (see, e.g., [11], [12]), it still poses a great challenge to determine the qualitative property of the trajectory of MASs, like persistence, etc. The meaning of persistence in CRNT is analogous to that in population processes. Namely, none of the chemical species in a MAS can be completely "used up" in the course of the reaction if every species has a nonzero initial concentration. Mathematically, it measures if any trajectory in a MAS will tend to the boundary of $\mathbb{R}^n_{\geq 0}$. Therefore, for a bounded MAS, studying persistence is equivalent to analyzing whether the boundary points of MAS and limit sets of trajectories have intersections. A famous conjecture around persistence presented by [11] says that any weakly reversible MAS is persistent. A further result is that persistence coupled with complex balancing will guarantee global convergence to equilibria in complex balanced (CB) MASs (an important subnetwork of weakly reversible networks) [4], [13]. Due to the theoretical research value on CRNs and their application value in practical biochemical systems, there has been much

activity aimed at the persistence of CB systems (see, e.g., [3], [5], [14]). Angeli et al. [8] and Anderson [4] proved that if a boundary point is the ω-limit point, then the species set I in which every species has zero concentration must be a semilocking set. So Anderson and his coauthors [3], [4], [5] made a systematic study on the persistence of CB systems by analyzing the structure of semilocking boundaries of MAS. They said that if all semilocking boundary points of each stoichiometric compatibility class are at most discrete [4], this CB MAS is persistent. Also, if for each semilocking set, the codimension of the set of the corresponding boundary points is one relative to stoichiometric compatibility class [3], the CB MAS is persistent. And all CB MASs with a single linkage class are persistent [5]. Craciun [15] stated that each CB MAS is persistent, but this result is still under verification.

The studies related to persistence even go beyond the structure of weakly reversible MASs [8], [16], [17], [18], [19]. Angeli et al. [8], [17] used the relationship between species brought by the form of reactions to exclude the corresponding points of each semilocking set. They reported that the global conservative CRNs with each semilocking set having a conservation relation between some species in it are persistent. In addition, they obtained that the global conservative CRNs with some semilocking sets without local conservation relation in it are persistent if the following two conditions are true: 1) the semilocking sets are dynamically nonemptiable; 2) there are no nested distinct locking sets without local conservation relation in the network. Johnson et al. [20] extended this result by putting forward that bounded networks with all weak dynamical nonemptiable semilocking sets have persistence. Some scholars proposed a class of network larger than the weakly reversible network called endotactic network [16], [19] based on the geometry structure of the reaction network. This kind of network has a strong geometric meaning, and become a focus for persistence study. Pantea [19] proved that any 2 d (this means the stoichiometric subspace of the network is 2) endotactic MAS with bounded trajectories is persistent. Craciun et al. [16] stated that any endotactic MAS with two species is persistent, and Gopalkrishnan et al. [18] further asserted that any strongly endotactic MAS is persistent where the strong endotactic MAS is a special kind of endotactic network.

The abovementioned discussion means that it is very popular to define special topological structures for CRNs to perform persistence study, such as weakly reversible and endotactic networks as two kinds of typical representatives of structure. In line with the same logic, this article also tries to define a kind of new structure of networks, called $\mathcal{W}_I$-endotactic networks, which is based on the notion of endotactic network but induces a larger network set than the latter and let alone the weakly reversible network. We succeed in capturing the persistence of any $1d$ $\mathcal{W}_I$-endotactic MAS, and that of some higher dimensional MASs composed of more than one $1d$ $\mathcal{W}_I$-endotactic subnetworks. We discuss two cases of higher dimensional MASs according to whether there are common species between the intersecting species set of all subnetworks and the nontrivial semilocking set (note that the trivial semilocking set is namely the species set of the network) of the network. The main contributions may be summarized as follows.

1) We prove that all $1d$ $\mathcal{W}_I$-endotactic systems are persistent regardless of the reaction rate constants, and moreover, any MAS composed of a series of $1d$ $\mathcal{W}_I$-endotactic subnetworks or persistent subnetworks without intersecting species among subnetworks is persistent;
2) In the case of a MAS composed of some $1d$ $\mathcal{W}_I$-endotactic subnetworks but with intersecting species among subnetworks, we prove that if there is no common species between the intersecting species set and any nontrivial semilocking set of the network, then the MAS is persistent; also, we give the sufficient condition that the composed MAS is persistent if there are common species between them.

Since $\mathcal{W}_I$-endotactic network induces a larger set than endotactic network and weakly reversible network, the abovementioned results also apply to the latter two kinds of networks. This means that the proposed method may address the issue of persistence for more networks than the previous methods for endotactic networks and weakly reversible networks. Arguably, for the examples in the context, if they are identified by $\mathcal{W}_I$-endotactic network, but not endotactic network, then they cannot be dealt with on persistence analysis using the previous methods.

The rest of this article is organized as follows. Section II introduces some preliminaries about CRN. This is followed by the definition of $\mathcal{W}_I$-endotactic system, and its persistence in the $1d$ case is also shown in Section III. Section IV presents the persistence of higher dimensional $\mathcal{W}_I$-endotactic systems. Finally, Section V concludes this article.

II. Preliminaries

In this section, some fundamental concepts about the CRN [11] and its dynamics [16] are reviewed.

A. Chemical Reaction Networks

Consider a CRN containing n chemical species, denoted by $S_1, S_2, \ldots, S_n$ that take part in r chemical reactions with the ith reaction R_i $(i = 1, \ldots, r)$. Each reaction can be expressed as

$$\sum_{j=1}^{n} v_{ji} S_j \longrightarrow \sum_{j=1}^{n} v'_{ji} S_j \tag{1}$$

where the nonnegative integers $v_{j,i} \in \mathbb{R}^n_{\geq 0}$ and $v'_{j,i} \in \mathbb{R}^n_{\geq 0}$ are stoichiometric coefficients. Organize $v_{\cdot i} = (v_{1,i}, \ldots, v_{n,i})^\top$ and $v'_{\cdot i} = (v'_{1,i}, \ldots, v'_{n,i})^\top$, termed by complexes. Let $\mathcal{S}, \mathcal{C}, \mathcal{R}$ be the set of species, complexes, and reactions, respectively. Then, each CRN can be determined by a triple $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$. $v'_{\cdot i} - v_{\cdot i}$ is called the reaction vector of the ith $(i = 1, \ldots, r)$ reaction $v_{\cdot i} \to v'_{\cdot i}$. The linear space spanned by all reaction vectors is named as the stoichiometric subspace, denoted by $\mathscr{S} = \text{span}\{v'_{\cdot 1} - v_{\cdot 1}, \ldots, v'_{\cdot r} - v_{\cdot r}\}$.

Definition 2.1 (Stoichiometric Compatibility Class): For an $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$, the sets

$$\mathscr{S}(x_0) \triangleq \{x_0 + \xi | \xi \in \mathscr{S}\}, x_0 \in \mathbb{R}^n_{\geq 0}. \tag{2}$$

$\mathscr{S}(x_0) \cap \mathbb{R}^n_{\geq 0}$ and $\mathscr{S}(x_0) \cap \mathbb{R}^n_{>0}$ are called the stoichiometric compatibility classes, nonnegative stoichiometric compatibility classes, and positive stoichiometric compatibility classes of x_0, respectively.

A CRN can be also viewed as a directed graph where vertices represent complexes and directed edges correspond to reactions. A connected component of the graph is called a linkage class. In the following, we shall introduce several networks with special structures.

Definition 2.2 (Reversible and Weakly Reversible CRN): An $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ is called

1) reversible if $v_{\cdot i} \to v'_{\cdot i} \in \mathcal{R}$ whenever $v'_{\cdot i} \to v_{\cdot i} \in \mathcal{R}$;
2) weakly reversible if for any reaction $v_{\cdot i} \to v'_{\cdot i} \in \mathcal{R}$ there exists a series of reactions starting from $v'_{\cdot i}$ and ending with $v_{\cdot i}$, i.e., $v'_{\cdot i} \to v_{\cdot i_1} \in \mathcal{R}, \ldots, v_{\cdot i_m} \to v_{\cdot i} \in \mathcal{R}$, $m < r$.

A reversible CRN is necessarily a weakly reversible CRN, but the converse is not true.

Utilizing the geometric property of CRNs, Craciun et al. [16] defined the concept of endotactic CRN, which serves closely the purpose of addressing the persistence issue. They also stated that the class of endotactic networks is larger than the well-known class of weakly reversible networks. Instead, Gopalkrishnan et al. [18] redefined this concept using algebraic language. Their redefinition begins with defining a partial order relation in $\mathbb{R}^n$.

Definition 2.3 (w-Preorder [18]): Let w be a vector in $\mathbb{R}^n$. The w-Preorder on $\mathbb{R}^n$, denoted by $\leq_w$, is defined by

$$y_1 \leq_w y_2 \iff \langle w,\ y_1\rangle \leq \langle w,\ y_2\rangle \tag{3}$$

where $y_1, y_2 \in \mathbb{R}^n$, and $\langle \cdot, \cdot \rangle$ represents standard inner product. We write $y_1 <_w y_2$ if $\langle w, y_1\rangle < \langle w, y_2\rangle$.

Definition 2.4 ($\leq_w$-maximal, $\leq_w$-minimal Elements [18]): Let $w \in \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$. The element $y_r \in Y$ is said to be a $\leq_w$-maximal element in Y if

$$\langle w, y_r\rangle \geq \langle w, y\rangle \quad \forall\ y \in Y. \tag{4}$$

The element $y_l \in Y$ is called a $\leq_w$-minimal element in Y if

$$\langle w, y_l\rangle \leq \langle w, y\rangle \quad \forall\ y \in Y. \tag{5}$$

When applying this definition to an $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$, a complex is called leftmost relative to w if it is $\leq_w$-minimal in $\mathcal{C}$, and rightmost relative to w if it is $\leq_w$-maximal in $\mathcal{C}$.

Definition 2.5 (Endotactic CRN): An $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ is

1) w-endotactic with respect to a certain $w \in \mathbb{R}^n$ if for the set of all reactants such that the reaction vectors are not orthogonal to w, any rightmost reactant ($\leq_w$-maximal) $v_{\cdot i}$ in this set point to the left, and each leftmost reactant ($\leq_w$-minimal) point to the right. Namely,

$$\begin{aligned} &\langle w, v'_{\cdot i} - v_{\cdot i}\rangle < 0,\ w \text{ is } \leq_w \text{-maximal;} \\ &\langle w, v'_{\cdot i} - v_{\cdot i}\rangle > 0,\ w \text{ is } \leq_w \text{-minimal} \end{aligned} \tag{6}$$

2) an endotactic network if it is w-endotactic with respect to any $w \in \mathbb{R}^n$.

Geometrically, a network is w-endotactic if the projection of this network to the line to which w belongs is endotactic, and a network is endotactic if its projection on any line in phase space is endotactic [16].

Example 1: Consider the following network $\mathcal{N}$ with two species and four reactions

$$\begin{aligned} S_1 &\leftrightharpoons S_2 \\ S_1 + S_2 &\to 2S_2 \to 2S_1 \end{aligned}$$

the network is a $1d$ (1-dimensional) CRN and contains two linkage classes. It is obviously an endotatic network as for any line L not orthogonal to its stoichiometric subspace, all the reactions with the leftmost (rightmost) reactants of the projected network point to right (left).

The geometric representation of this network is shown in Fig. 1. A red point in the figure denotes a reactant complex and the green point is the complex that is only in the set of the product complex. Also, Fig. 1 presents the endotactic structure of above network: (b) and (c) project network $\mathcal{N}$ into four different representative lines L_1, L_2, L_3, L_4. Red plus and green plus signs indicate the corresponding projected reactant and product complex, respectively. And orange vectors are the projected reactions with the leftmost or the rightmost reactant complex. It is clear that the projected reactions with a leftmost and rightmost complex point to each other. Thus, this network $\mathcal{N}$ is an endotactic network.

But for the subnetwork $\mathcal{N}_1$

$$\begin{aligned} S_1 &\to S_2 \\ 2S_2 &\to 2S_1. \end{aligned}$$

By projecting the subnetwork $\mathcal{N}_1$ to the horizontal line containing $w = (1, 0)^\top$, as can be seen in Fig. 1, the leftmost reactant complex $2S_2$ points to right and the rightmost reactant complex S_1 points to left. Hence, this network is at least a w-endotactic network where $w = (1, 0)^\top$. We further project it into the line L_5 containing another w, and the projection is shown to be not endotactic. Fig. 1(d) expounds the details.

It is not easy to check whether a network is endotactic or not due to the arbitrariness of w. [16] presented the sufficient and necessary conditions for a 2-species endotactic network while [21] developed a computation algorithm for endotactic network checking.

B. Mass-Action Systems

The dynamics of a CRN system capturing the change of concentration of every species S_j $(j = 1, \ldots, n)$, identified by x_j, is given once the reaction rate is specified as a function $V : \mathbb{R}^n_{\geq 0} \mapsto \mathbb{R}^r_{\geq 0}$ of $x = (x_1, \ldots, x_n)^\top$. The most frequently-used model to specify the reaction rate is mass-action kinetics, under which the reaction rate $V_i(x)$ of the ith reaction $v_{\cdot i} \to v'_{\cdot i}$ is evaluated by

$$V_i(x) = k_i x^{v_{\cdot i}} \triangleq k_i \prod_{j=1}^{n} x_j^{v_{ji}} \tag{7}$$

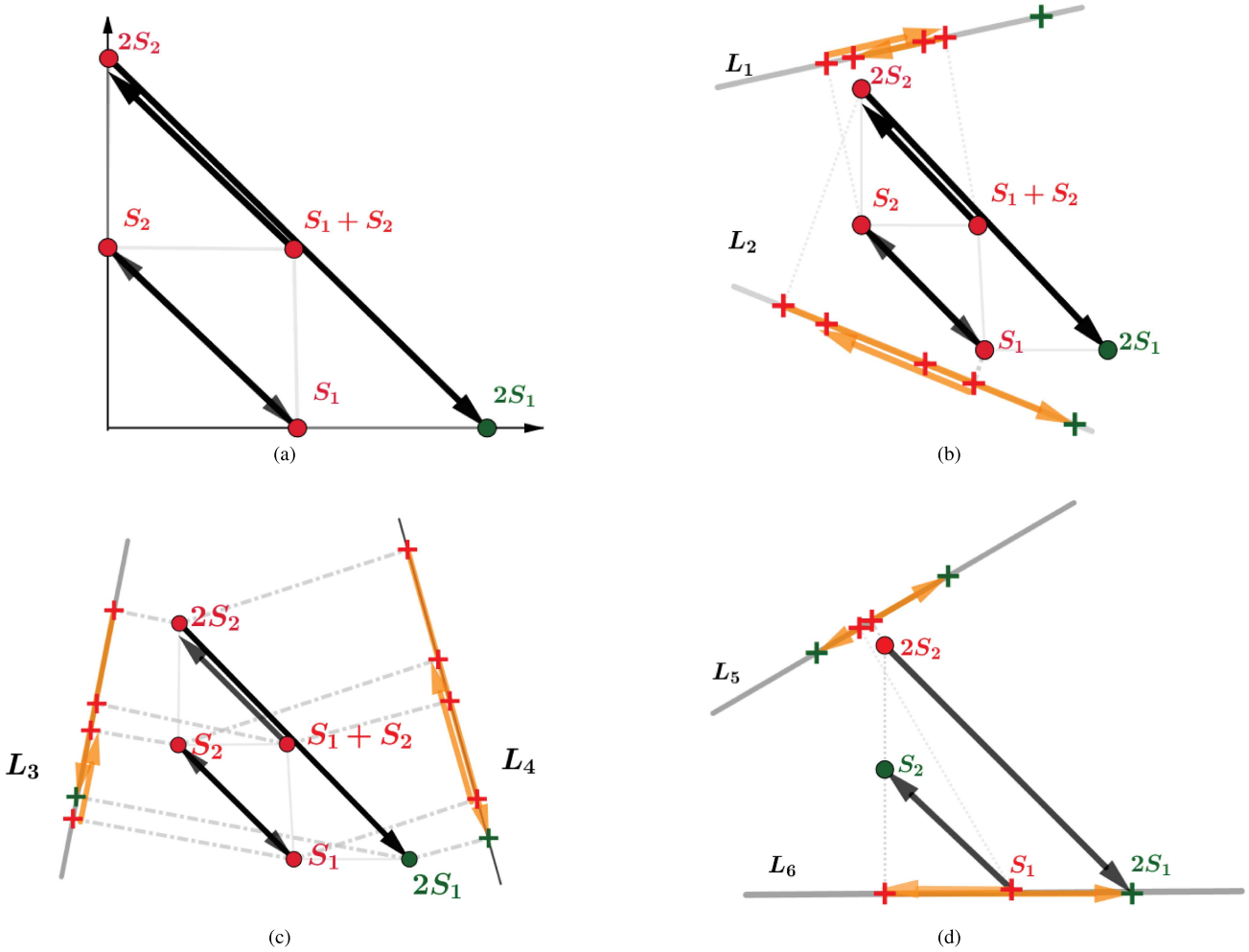

Fig. 1. (a) Complexes, reactions, and the reactant complexes polytope of the network $\mathcal{N}$ in Example 1 in the phase space. (b) and (c) Directions of leftmost and rightmost reactions of each projected network of $\mathcal{N}$ on the four different representative lines L_1, L_2, L_3, L_4. (d) Projects the subnetwork $\mathcal{N}_1$ into L_5 and L_6.

where $k_i \in \mathbb{R}_{>0}$ is the reaction rate coefficient.

Definition 2.6 (MAS): An $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ equipped with mass-action kinetics is called a MAS, which is usually represented by the quaternary $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$.

The dynamical equation of an $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ can be written as

$$\frac{dx}{dt} \triangleq \dot{x}(t) = \sum_{i=1}^{r} k_i x^{v_{\cdot i}} (v'_{\cdot i} - v_{\cdot i}). \tag{8}$$

Definition 2.7 (Equilibrium): For an $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$, a concentration vector $x^* \in \mathbb{R}^n_{>0}$ is called an equilibrium if $\dot{x}(t) = 0$ at $x = x^*$, and a complex balanced (CB) equilibrium if

$$\sum_{\{v_{\cdot i} = C\}} k_i (x^*)^{v_{\cdot i}} = \sum_{\{v'_{\cdot i} = C\}} k_i (x^*)^{v_{\cdot i}} \qquad \forall\, C \in \mathcal{C}. \tag{9}$$

A CB equilibrium must be an equilibrium, but not vice versa. A MAS admits a (CB) equilibrium is called (complex) balanced MAS. If there exists a CB equilibrium in a MAS, any other equilibrium (if exists) in this MAS is also a CB equilibrium [10].

Persistence is an important property of a MAS. In a chemical reaction system, persistence means that none of the concentrations of species can tend to zero if they are not zero at the beginning of reactions. Mathematically, it is defined as follows.

Definition 2.8 (Persistence): An $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ described by (8) is called persistent if any forward trajectory $x(t) \in \mathbb{R}^n_{\geq 0}$ with positive initial condition $x(0) \in \mathbb{R}^n_{>0}$ satisfies

$$\liminf_{t \to \infty} x_j(t) > 0 \quad \text{for all } j \in \{1, \ldots, n\}.$$

The abovementioned definition works for all cases, including the cases of bounded trajectory and unbounded trajectory. In the case of bounded trajectory, the definition may be reduced to describe its ω-limit point.

Definition 2.9 (Persistence for Bounded Trajectory): For an $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ with bounded trajectory, it is persistent if

$$\omega(x(0)) \cap \partial \mathbb{R}^n_{\geq 0} = \varnothing \qquad \forall\, x(0) \in \mathbb{R}^n_{>0} \tag{10}$$

where ω-limit set $\omega(x(0))$ is the set of ω-limit points for the trajectory $x(t)$ with positive initial condition $x(0) \in \mathbb{R}^n_{>0}$ at a

certain time subseries $\{t_N\}$, defined as

$$\omega(x(0)) := \{x \in \mathbb{R}^n_{\geq 0} \mid x(t_N) \to x, \text{ for some sequence } t_N \to \infty\}. \tag{11}$$

Remark 2.10: Given a boundary point $\bar{x}$, let I represent the set of species with 0 concentration in $\bar{x}$. We define the set of boundary points with the same I as

$$L_I = \left\{x \in \mathbb{R}^n_{\geq 0} | x_i = 0 \text{ for } S_i \in I;\ x_i > 0 \text{ for } S_i \notin I\right\}.$$

Thus, L_I is one boundary of $\mathbb{R}^n_{\geq 0}$ and

$$\partial\mathbb{R}^n_{\geq 0} = \bigcup_{I \subset \mathcal{S}} L_I.$$

For each stoichiometric compatibility class $\mathscr{S}_{x_0}$, the boundary of $\mathscr{S}_{x_0}$ is denoted as $F_I = \mathscr{S}(x_0) \cap L_I$.

The following concept plays an important role in characterizing the persistence of a MAS.

Definition 2.11 (Semilocking set and Locking Set [3], [8]): Consider an $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$. A nonempty set $I \subseteq \mathcal{S}$ is said to be a semilocking set or siphon if one species of I is in a product complex $v'_{\cdot i}$, there must exist a species of I in the reactant complex $v_{\cdot i}$. If for any reaction $v_{\cdot i} \to v'_{\cdot i}$ there exists an $S_j \in I$ in reactant complex $v_{\cdot i}$, then $I \subseteq \mathcal{S}$ is called a locking set.

Angeli et al. [8] also reported a necessary and sufficient condition to suggest a semilocking set, that is, as follows.

Proposition 2.12: A nonempty set $I \subseteq \mathcal{S}$ in an $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ is a semilocking set if and only if the face F_I is forward invariable for the dynamics (8).

The physical explanation of this proposition is easily caught as the concentrations of species in I are always zero and every reaction induced by the species in I will stop once the trajectory enters into F_I.

Based on the concepts of semilocking set and endotactic network, some checkable criteria have been reported for persistence, as stated in Section I. In this article, we continue to follow this project, but define another kind of network and analyze the persistence of these networks by using the combination.

III. $\mathcal{W}_I$-Endotactic Networks and Persistence in Case of $1d$ Stoichiometric Subspace

In this section, we will define $\mathcal{W}_I$-endotactic network and prove that all $1d$ $\mathcal{W}_I$-endotactic networks are persistent.

A. $\mathcal{W}_I$-Endotactic Network

We first give the definition of $\mathcal{W}_I$-endotactic network.

Definition 3.1: Given a reaction network $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$, let I represent a semilocking set attached to this network. Define $w_I \in \{0,1\}^{\mathcal{S}}$ as

$$(w_I)_j = \begin{cases} 1, & j \in I \\ 0, & j \notin I \end{cases} \tag{12}$$

and $\mathcal{W}_I = \{w_I | I \text{ is any semilocking set attached to} \mathcal{N}\}$, then $\mathcal{N}$ is called $\mathcal{W}_I$-endotactic if for every $w_I \in \mathcal{W}_I$ this network is w_I-endotactic.

The comparison of this definition to Definition 2.5 might reveal that an endotatic network must be $\mathcal{W}_I$-endotactic. However, the converse does not hold. An example of $\mathcal{W}_I$-endotactic network is given as follows.

Example 2: For a network like

$$2S_1 \to S_1 + 2S_2$$
$$S_1 + 4S_2 \to 3S_1$$

it can be seen from Fig. 2(a) that this network is not endotactic since its projection onto the straight line L_1 fails to support the endotactic property. However, we can easily verify it to be $\mathcal{W}_I$-endotactic. The entire semilocking sets attached to this network are $\{S_1\}$ and $\{S_1, S_2\}$, then we get $w_{\{S_1\}} = (1,0)$ and $w_{\{S_1,S_2\}} = (1,1)$. Also from Fig. 2, the projected networks onto the lines where $w_{\{S_1\}}$ and $w_{\{S_1,S_2\}}$ lie, respectively, both show to be w_I-endotactic. So this network is $\mathcal{W}_I$-endotactic.

B. Persistence of $1d$ $\mathcal{W}_I$-Endotactic System.

A $\mathcal{W}_I$-endotactic network $(\mathcal{S}, \mathcal{C}, \mathcal{R})$ is a $1d$ $\mathcal{W}_I$-endotactic network if $\dim\mathscr{S} = 1$, which is our main concern in this section. Let b denote the base of the stoichiometric subspace $\mathscr{S}$, $b_j = 0$ means that the concentration of the corresponding species S_j always remains unchanged with the occurrence of the reactions, i.e., $x_j(t) = x_j(0) > 0$ for all t. Therefore, these species may not need to be considered in the persistence analysis. Therefore, for the sake of simplicity, in the $1d$ system studied in this section, we agree that the only basis vector b does not contain zero elements, namely, $b_j \neq 0$ for all species S_j.

Note that the persistence analysis using means of the ω-limit set requests that the trajectories in the MAS are bounded. The following properties for $1d$ MASs will help our following analysis.

Lemma 3.2: Consider a $1d$ $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$. Let $x(0) \in \mathbb{R}^n_{\geq 0}$ be any initial point of the system. Then, for arbitrary trajectory $x(t) \in \mathbb{R}^n_{\geq 0}$ originating from $x(0)$, if the concentration of one species tends to infinity as time runs to infinity, so do the concentrations of other species S_j with $b_j \neq 0$ where b is the basis of stoichiometric subspace of $\mathcal{M}$.

Proof: From $\dim\mathscr{S} = 1$, let $b \in \mathbb{R}^n$ be a group of a base in this $1d$ MAS, then there exists a real function $a(t)$ satisfying $x(t) - x(0) = a(t)b$. Also, denote the positive/negative terms index set of b by P_b/N_b. Without loss of generality, suppose $l \in P_b$ and $\lim_{t\to\infty} x_l(t) = \infty$, which means $\lim_{t\to\infty} a(t) = \infty$, so $\forall\, j \in P_b$ we have $\lim_{t\to\infty} x_j(t) = \infty$ while $\forall\, j \in N_b$ we have $\lim_{t\to\infty} x_j(t) = -\infty$. The latter is apparently not true. Thus, we obtain $N_b = \varnothing$. Therefore, either P_b or N_b must be empty under the given conditions. The result is thus true. ■

Then, we can catch the boundedness of trajectories in $1d$ $w_{\mathcal{S}}$-endotactic networks, where $w_{\mathcal{S}}$ follows (12) and is, thus, n-dimensional satisfying $w_{\mathcal{S}} = (1, \ldots, 1)^\top$.

Lemma 3.3: Any trajectory in an $1d$ $w_{\mathcal{S}}$-endotactic MAS $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ is bounded.

Proof: We use the proof by contradiction to reach the result. Assume a trajectory $x(t)$ in the $1d$ $w_{\mathcal{S}}$-endotactic MAS $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ is unbounded as time runs to infinity. From Lemma 3.2, we know that the concentration $x_j(t)$ of every species X_j

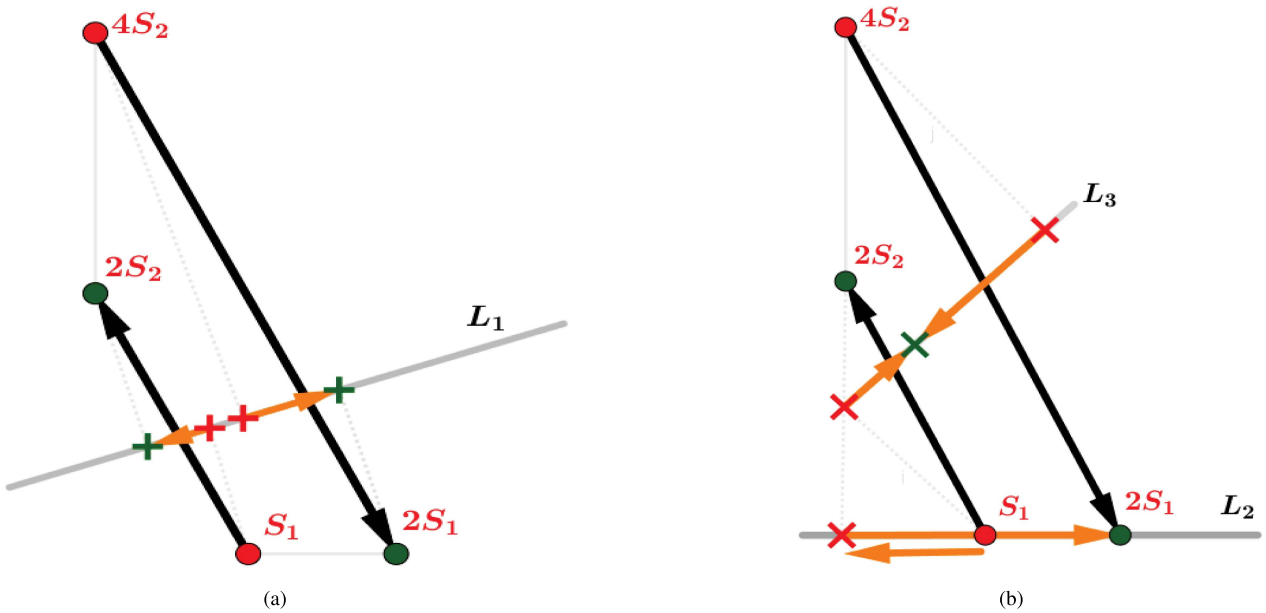

Fig. 2. Schematic diagram of network in Example 2, and the corresponding projections on some straight lines. (a) Graph of network structure and its projected network on L_1. (b) Graph of network structure and its projected networks on L_2 and L_3.

with $b_j \neq 0$ subject to $\lim_{t\to\infty} x_j(t) = \infty,\ \forall\, S_j \in \mathcal{S}$. Using the same variables as used in the proof of Lemma 3.2, like $x(0)$, b, $a(t)$, etc., and setting all nonzero entries in b to be positive for simplicity but without loss of generality, we get

$$\lim_{t\to\infty} \frac{x_j(t)}{x_1(t)} = \lim_{t\to\infty} \frac{x_j(0) + a(t)b_j}{x_1(0) + a(t)b_1} = \frac{b_j}{b_1} > 0 \quad \forall\, S_j \in \mathcal{S}.$$

This means $\exists\, T_1, \beta_j, \bar{\beta}_j > 0$ such that when $t > T_1$, there is

$$\beta_j \leq \frac{x_j(t)}{x_1(t)} \leq \bar{\beta}_j \quad \forall\, S_j \in \mathcal{S}.$$

i.e.,

$$\beta^{v_{\cdot i}} x_1^{\langle v_{\cdot i}, w_{\mathcal{S}}\rangle} \leq x^{v_{\cdot i}} \leq \bar{\beta}^{v_{\cdot i}} x_1^{\langle v_{\cdot i}, w_{\mathcal{S}}\rangle} \quad \forall\, v_{\cdot i} \in \mathcal{C} \tag{13}$$

where $\beta = (\beta_1, \ldots, \beta_n)^\top$, $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_n)^\top$, $\mathcal{S}$ is naturally a semilocking set, and $w_{\mathcal{S}} = (1, \ldots, 1)^\top$ following Definition 3.1.

Further define the set composed of all $\leq_{w_{\mathcal{S}}}$-maximal reactant complexes by $\mathcal{C}_r$, i.e., $\forall\, v_{\cdot l} \in \mathcal{C}_r$ and $\forall\, v_{\cdot i} \in \mathcal{C}\backslash\mathcal{C}_r$, there is $\langle v_{\cdot l}, w_{\mathcal{S}}\rangle > \langle v_{\cdot i}, w_{\mathcal{S}}\rangle$. Then, utilizing the explosiveness of the exponential function, and the fact that $\beta^{v_{\cdot l}}$ and $\bar{\beta}^{v_{\cdot i}}$ are constants, we get $\exists\, T_2 > 0$ such that when $t > T_2$ there is

$$\bar{\beta}^{v_{\cdot i}} x_1^{\langle v_{\cdot i}, w_{\mathcal{S}}\rangle} < \beta^{v_{\cdot l}} x_1^{\langle v_{\cdot l}, w_{\mathcal{S}}\rangle}.$$

Combining the abovementioned inequality with (13) yields that when $t > \max\{T_1, T_2\}$, we have

$$x^{v_{\cdot i}} \leq \bar{\beta}^{v_{\cdot i}} x_1^{\langle v_{\cdot i}, w_{\mathcal{S}}\rangle} < \beta^{v_{\cdot l}} x_1^{\langle v_{\cdot l}, w_{\mathcal{S}}\rangle} \leq x^{v_{\cdot l}}$$

for $\forall\, v_{\cdot l} \in \mathcal{C}_r, v_{\cdot i} \in \mathcal{C}\backslash\mathcal{C}_r$. For the same reasons as above, there must exist $T_3 > 0$ such that when $t > \max\{T_1, T_2, T_3\}$, we have

$$k_l x^{v_{\cdot l}} > k_i x^{v_{\cdot i}} \quad \forall\, v_{\cdot l} \in \mathcal{C}_r, v_{\cdot i} \in \mathcal{C}\backslash\mathcal{C}_r$$

and $\exists\, T_4 > 0$ such that when $t > \max\{T_1, T_2, T_3, T_4\}$ we further have

$$\sum_{v_{\cdot i}\in\mathcal{C}_r} k_i x^{v_{\cdot i}} > \sum_{v_{\cdot i}\in\mathcal{C}\backslash\mathcal{C}_r} k_i x^{v_{\cdot i}}. \tag{14}$$

Note that the network under consideration is $1d$ $w_{\mathcal{S}}$-endotactic, and also $w_{\mathcal{S}}$ is not orthogonal to $\mathscr{S}$ (due to $\langle w_{\mathcal{S}}, b\rangle \neq 0$), from Definition 2.5, we, thus, get

$$\langle v'_{\cdot i} - v_{\cdot i}, w_{\mathcal{S}}\rangle < 0 \quad \forall\, v_{\cdot i} \in \mathcal{C}_r$$

which combines (15) to yield $m_i < 0$.

The abovementioned results are used to rewrite the dynamics of the network, which gives

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} k_i x^{v_{\cdot i}}(v'_{\cdot i} - v_{\cdot i}) \\ &= \sum_{\{i|v_{\cdot i}\notin\mathcal{C}_r\}} k_i x^{v_{\cdot i}} m_i b + \sum_{\{i|v_{\cdot i}\in\mathcal{C}_r\}} k_i x^{v_{\cdot i}} m_i b \\ &= \sum_{\{j|m_i>0\}} k_i x^{v_{\cdot i}} m_i b - \sum_{\{i|m_i<0\}} k_i x^{v_{\cdot i}} |m_i| b.\end{aligned}$$

Again, by using the explosiveness of the exponential function and (14), we obtain that $\exists\, T_4 > 0$ satisfying when $t > \max\{T_1, T_2, T_3, T_4\}$, every entry of $\sum_{\{j|m_i<0\}} k_i x^{v_{\cdot i}} |m_i| b$ is larger than that of $\sum_{\{i|m_i>0\}} k_i x^{v_{\cdot i}} m_i b$. As a result, there is $\dot{x}_j(t) < 0$ if $t > \max\{T_1, T_2, T_3, T_4\}, \forall\, S_j$. This is contrary to the assumption that $x(t)$ is unbounded as $t \to \infty$. Therefore, the result holds. ■

Remarkably, the result given in [4] implies that we only need to consider semilocking boundary points in the subsequent

analysis as each boundary point cannot be the ω-limit point of each trajectory for arbitrary chemical reaction MAS.

Proposition 3.4: Each semilocking boundary point in a $1d$ $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ is a boundary equilibrium point.

Proof: In a $1d$ $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$, any reaction vector may be expressed as

$$v'_{\cdot i} - v_{\cdot i} = m_i b \qquad \forall\, i = 1, \ldots, r \tag{15}$$

where $b \in \mathbb{R}^n \setminus \{\mathbb{0}_n\}$ is a base in $\mathscr{S}$ and $m_i \in \mathbb{Z}\setminus\{0\}$ is the corresponding coefficient. Let $\bar{x} \in \mathbb{R}^n_{\geq 0}$ represent a semilocking boundary point in the $1d$ MAS under consideration, i.e., $I = \text{supp}^c\, \bar{x}$ is a semilocking set, then any reaction $v_{\cdot i} \to v'_{\cdot i}$ in $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ does not support supp $v_{\cdot i} \subseteq$ supp $\bar{x}$ and supp $v'_{\cdot i} \nsubseteq$ supp $\bar{x}$.

Assume that there is a reaction $v_{\cdot l} \to v'_{\cdot l}$ in $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ supporting supp $v_{\cdot l} \subseteq$ supp $\bar{x}$ and supp $v'_{\cdot l} \subseteq$ supp $\bar{x}$. Then, $\forall\, j \in I$ we have $v'_{jl} - v_{jl} = 0$, which together with (15) suggests $b_j = 0$. This result conversely shows that there is no change on the concentration of species S_j, $\forall\, j \in I$, within all reactions in $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$. We may naturally ignore these species in persistence analysis since they are requested to have nonzero concentrations in the beginning and will not affect the persistence property of the system. Hence, we only need to be concerned with those species whose concentrations will change within reactions. We suppose all species concentrations will vary during the reaction process, i.e., every entry of b is not zero. This means that among all reactions it is impossible to have a reaction $v_{\cdot l} \to v'_{\cdot l}$ supporting supp $v_{\cdot l} \subseteq$ supp $\bar{x}$ and supp $v'_{\cdot l} \subseteq$ supp $\bar{x}$.

Based on the above obtained results, every reaction $v_{\cdot i} \to v'_{\cdot i}$ in a $1d$ MAS does not support supp $v_{\cdot i} \subseteq$ supp $\bar{x}$. The network can only include reactions satisfying supp $v_{\cdot i} \nsubseteq$ supp $\bar{x}$. As a result, I is a locking set as well as a semilocking set. Therefore, $\bar{x}$ is a boundary equilibrium point. ■

Also, for a $1d$ MAS with positive equilibria, the following function proposed in [22] is dissipative.

Lemma 3.5 (see[22]): For a $1d$ $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ modeled by (8) admitting an positive equilibrium, the function defined by

$$f(x) = \int_0^{\gamma(x)} \ln \tilde{u}(y^\dagger(x) + \tau b) d\tau \tag{16}$$

is dissipative along all trajectories, i.e., $\dot{f}(x) \leq 0$. In the above $f(x)$, $\tilde{u}$ is the nonzero solution of $h(x, u) = 0$ and $h(x, u)$ is defined by

$$h(x,u) = \sum_{\{i|m_i>0\}} (k_i x^{v_{\cdot i}}) \left(\sum_{j=0}^{m_i-1} u^j \right) + \sum_{\{i|m_i<0\}} (k_i x^{v_{\cdot i}}) \left(- \sum_{j=m_i}^{-1} u^j \right). \tag{17}$$

$b \in \mathbb{R}^n \setminus \{\mathbb{0}_n\}$ is a base of the stoichiometric subspace $\mathscr{S}$. $y^\dagger(x) \in \mathscr{C}^2(\mathbb{R}^n_{>0}; \mathbb{R}^n_{>0})$ is the unique solution with respect to x of $J(y) = 0$ in each positive stoichiometric compatibility class $\mathscr{S}(x) \cap \mathbb{R}^n_{>0}$ where $J(y) : \mathscr{S}(x) \cap \mathbb{R}^n_{>0} \to \mathbb{R}$ is defined as

$$J(y) = \begin{cases} \prod_{j\in P_b} y_j - \prod_{j\in N_b} y_j, & P_b, N_b \neq \emptyset \\ \prod_{j\in P_b} y_j - 1, & P_b \neq \emptyset, N_b = \emptyset \\ \prod_{j\in N_b} y_j - 1, & N_b \neq \emptyset, P_b = \emptyset \end{cases}$$

and P_b, N_b share the same meaning as those in the proof of Lemma 3.2. $\gamma(x) \in \mathscr{C}^2(\mathbb{R}^n_{>0}; \mathbb{R})$, which together with $y^\dagger(x)$, is constrained by

$$x = y^\dagger(x) + \gamma(x) b \quad \text{and} \quad \gamma(x + \delta b) = \gamma(x) + \delta \qquad \forall \delta \in \mathbb{R}.$$

Remark 3.6: Note that Proposition 3.5 holds on the premise of the existence of a positive equilibrium in the $1d$ MAS. The role is to ensure the terms in (17) neither $\{i|m_i > 0\}$ nor $\{i|m_i < 0\}$ is empty [22], which will further guarantee that there exists a unique $\tilde{u} \in \mathscr{C}^2$ supporting $h(x, \tilde{u}(x)) = 0$. Clearly, $1d$ $\mathcal{W}_I$ endotactic networks admit the result that neither $\{i|m_i > 0\}$ nor $\{i|m_i < 0\}$ is empty. Hence, for a $1d$ $\mathcal{W}_I$ endotactic network, $\tilde{u} \in \mathscr{C}^2$ is unique to support $h(x, \tilde{u}(x)) = 0$, and the function $f(x)$ given in (16) is dissipation.

Proposition 3.4 tells us that in the case of $1d$ MASs, the concerning semilocking boundary points are all boundary equilibria. We, thus, use the above f to derive that each boundary equilibrium is not an ω-limit point of some trajectory for the $1d$ $\mathcal{W}_I$-endotactic MAS. Before giving the main theorem, we first introduce two important lemmas.

Lemma 3.7: For a MAS $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ given by (8) and $f \in \mathscr{C}^1$ defined as (16), if $\bar{x}$ is the local maximal point for $f(x)$, namely, $f(\bar{x})$ or $\lim_{x\to\bar{x}} f(x)$ reaches the local maximum, then $\bar{x}$ is not an ω-limit point.

Proof: We also use rebuttals of evidence to conduct proof. Assume that $\bar{x}$ is an ω-limit point of some trajectory with an initial point $x(0) \in \mathbb{R}^n_{>0} \cap \mathscr{S}(\bar{x})$. Namely, there exists a time series $\{t_N\}$ such that $\bar{x} \in \omega(x(0))$. Also, $\forall\, \epsilon > 0$, $\exists$ a moment $t_\epsilon < t_N \to +\infty$ such that $x(t_\epsilon) \in N_\epsilon(\bar{x})$, where $N_\epsilon(\bar{x})$ is a ϵ-neighborhood of $\bar{x}$. Note that $\dot{f}(x) \leq 0$, so it is impossible that $f(\bar{x})$ ($f(x)$ can be defined at $x = \bar{x}$) or $\lim_{x\to\bar{x}} f(x)$ ($f(x)$ cannot be defined at $x = \bar{x}$) is local maximal. Therefore, $\bar{x}$ is not an ω-limit point. ■

Lemma 3.8: For a $1d$ $\mathcal{W}_I$-endotactic MAS $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$, let $\bar{x} \in \mathbb{R}^n_{\geq 0}$ represent any semilocking boundary point, and $x_0 \in \mathscr{S}(\bar{x}) \cap \mathbb{R}^n_{>0}$ represent any initial point in this MAS. Without loss of generality, assume that $I \triangleq$ supp $\bar{x}$ belongs to the positive terms index set of the base b in $\mathscr{S}$. Then, the function f, given by (16), is constrained by $\lim_{\delta\to 0^+} \tilde{u}(\bar{x} + \delta b) < 1$.

Proof: Based on Proposition 3.5, the positive function $\tilde{u}(x)$ emerging in the solution $f(x)$ of (16) should satisfy $h(x, \tilde{u}(x)) = 0$, i.e., (17) becomes

$$\sum_{\{i|m_i>0\}} (k_i x^{v_{\cdot i}}) \left(\sum_{j=0}^{m_i-1} \tilde{u}^j(x) \right) = \sum_{\{i|m_i<0\}} (k_i x^{v_{\cdot i}}) \left(\sum_{j=m_i}^{-1} \tilde{u}^j(x) \right). \tag{18}$$

It is clear that $I \triangleq \text{supp}^c\ \bar{x}$ is a semilocking set. Without loss of generality, I is set as $\{S_1, \ldots, S_{n_1}\}$ in the following proof, where $1 \le n_1 \le n$, then $\bar{x} = \left(\underbrace{0, \ldots, 0}_{n_1}, \bar{x}_{n_1+1}, \ldots, \bar{x}_n\right)^\top$ and $w_I = \left(\underbrace{1, \ldots, 1}_{n_1}, 0, \ldots, 0\right)^\top$ according to (3.1). Also we define a set $\mathcal{C}_l$ of $\le_{w_I}$-minimal reactant complexes, i.e., $\langle v_{\cdot l} - v_{\cdot i}, w_I\rangle \le 0,\ \forall\ v_{\cdot i} \in \mathcal{C}$ and $\forall\ v_{\cdot l} \in \mathcal{C}_l$.

Since $I \triangleq \text{supp}^c\ \bar{x}$ belongs to the positive terms' index set of the base b, $\exists\ \bar{a} > 0$ such that $\bar{x} = x(0) - \bar{a}b$. For those $x(t) \in \mathscr{S}(\bar{x})$ near $\bar{x}$, there exists a real function $a(t)$ such that $x(t) = x(0) - a(t)b$. We, thus, get

$$\lim_{a(t)\to\bar{a}^-} \frac{x_j(t)}{x_1(t)} = \frac{b_j}{b_1} \quad \forall\ S_j \in I$$

$$\lim_{a(t)\to\bar{a}^-} x_j = \bar{x}_j \quad \forall\ S_j \in \mathcal{S}\backslash I.$$

The first equation means that $\exists\ \tau_1, \theta_j, \bar{\theta}_j > 0$ such that when $\bar{a} - a(t) < \tau_1$ there exists

$$\theta_j \le \frac{x_j(t)}{x_1(t)} \le \bar{\theta}_j \qquad \forall\ S_j \in I.$$

Therefore, for any reaction $v_{\cdot} \to v'_{\cdot i}$ in $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ we have

$$\theta x_1^{\langle v_{\cdot i}, w_I\rangle} \le \prod_{S_j\in I} x_j^{v_{ji}} \le \bar{\theta} x_1^{\langle v_{\cdot i}, w_I\rangle}$$

where $\theta = \prod_{S_j\in I} \theta_j^{v_{ji}}$ and $\bar{\theta} = \prod_{S_j\in I} \bar{\theta}_j^{v_{ji}}$ are two positive constants. Utilizing the property of the exponential function and $\mathcal{C}_l$ being the set of $\le_{w_I}$-minimal reactant complexes, we know that $\forall\ v_{\cdot l} \in \mathcal{C}_l,\ v_{\cdot i} \notin \mathcal{C}_l,\ \exists\ \tau_2 > 0$ such that when $\bar{a} - a(t) < \min\{\tau_1, \tau_2\}$, i.e., $x_1 < \min\{x_1(0) - (\bar{a} - \tau_1)b_1, x_1(0) - (\bar{a} - \tau_2)b_1\}$, there is

$$\prod_{S_j\in I} x_j^{v_{jl}} \ge \theta x_1^{\langle v_{\cdot l}, w_I\rangle} >> \bar{\theta} x_1^{\langle v_{\cdot i}, w_I\rangle} \ge \prod_{S_j\in I} x_j^{v_{ji}}. \tag{19}$$

This, together with the facts that x_j $(S_j \notin I)$ is positively bounded and every reaction rate coefficient is a positive constant, means that $\exists\ \tau_3 > 0$ such that when $\bar{a} - a(t) < \min\{\tau_1, \tau_2, \tau_3\}$, the inequality $k_l x^{v_{\cdot l}} > k_i x^{v_{\cdot i}}$ holds. For the same reason, $\exists\ \tau_4 > 0$ supporting

$$\sum_{\{l|v_{\cdot l}\in\mathcal{C}_l\}} k_l x^{v_{\cdot l}} >> \sum_{\{i|v_{\cdot i}\notin\mathcal{C}_l\}} k_i x^{v_{\cdot i}} \tag{20}$$

when $\bar{a} - a(t) < \min\{\tau_1, \tau_2, \tau_3, \tau_4\}$.

Again, since $I \triangleq \text{supp}^c\ \bar{x}$ belongs to the positive terms index set of the base b, w_I is not orthogonal to $\mathscr{S}$, i.e., w_I not orthogonal to any reaction vector $v'_{\cdot i} - v_{\cdot i}$. We then get

$$\langle v'_{\cdot i} - v_{\cdot i}, w_I\rangle > 0 \qquad \forall\ v_{\cdot i} \in \mathcal{C}_l$$

which is further combined with $v'_{\cdot i} - v_{\cdot i} = m_i b$ to yield $m_i > 0,\ \forall\ v_{\cdot i} \in \mathcal{C}_l$. Therefore, (20) may be rewritten as

$$\sum_{\{i|m_i>0\}} k_i x^{v_{\cdot i}} > \sum_{\{i|m_i<0\}} k_i x^{v_{\cdot i}}. \tag{21}$$

Therefore, in order to make (18) hold, there must exist $\tilde{u}(x) < 1$ when $x \in \mathbb{R}^n_{>0}$ and $\bar{a} - a(t)$ is enough small. This implies $\lim_{\delta\to 0^+} \tilde{u}(\bar{x} + \delta b) < 1$. ■

From the proof of the abovementioned lemma, we can obtain the following proposition which will be used in Section IV.

Proposition 3.9: Consider a generalized $1d$ $\mathcal{W}_I$-endotactic system $\tilde{\mathcal{M}} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{k})$, where every reaction rate constant $\tilde{k}_i(t)$ for $v_{\cdot i} \to v'_{\cdot i}$, $i = 1, \ldots, r$, is no longer invariant but a bounded function about t, i.e., satisfying $M_d^i < \tilde{k}_i(t) < M_u^i$ with $M_d^i > 0$ and $M_u^i > 0$. Furthermore, without loss of generality, we suppose $I \triangleq \text{supp}^c\ \bar{x}$ belongs to the positive terms index set of the base b, and x and $\bar{x}$ are close enough. Then, for arbitrary bounded $\tilde{k}_i(t) > 0$ we have

$$\sum_{\{i|m_i>0\}} \tilde{k}_i(t) x^{v_{\cdot i}} > \sum_{\{i|m_i<0\}} \tilde{k}_i(t) x^{v_{\cdot i}}. \tag{22}$$

Proof: Since I is a semilocking set of $\tilde{\mathcal{M}}$ and the reaction rate constants do not influence the structure of the network, (19) holds when the trajectory is close enough to each boundary point $\bar{x}$ with $\text{supp}^c\bar{x} = I$. Because $\tilde{k}_i(t) > 0$ is bounded for all reactions, namely, $M_d^i < \tilde{k}_i(t) < M_u^i$, there also exists a small enough $\tau > 0$ supporting $M_d^l x^{v_{\cdot l}} > M_u^i x^{v_{\cdot i}}$ for $v_{\cdot l} \in \mathcal{C}_l$ and $v_{\cdot i} \notin \mathcal{C}_l$ where $\bar{a} - a(t) < \tau$. Therefore, $k_l(t) x^{v_{\cdot l}} > \tilde{k}_i(t) x^{v_{\cdot i}}$. Further from the same proof of Lemma 3.8, we can obtain that (22) holds. ■

Finally, we present our main result as follows.

Theorem 3.10: Each $1d$ $\mathcal{W}_I$-endotactic MAS $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ is persistent.

Proof: Let $\bar{x} \in \mathbb{R}^n_{\ge 0}$ be any semilocking boundary point in the MAS under consideration. Define $I \triangleq \text{supp}^c\ \bar{x}$ and w_I following (3.1), so I is a semilocking set and the network in the study is a $1d$ $\mathcal{W}_I$-endotactic network, in which the trajectory originating any initial point is naturally bounded, as suggested in Lemma 3.3. We, thus, can apply Lemma 3.7 to analyze persistence for this MAS. The function $f(x)$ given in (16) is a solution of its Lyapunov function PDEs.

For any $x \in \mathscr{S}(\bar{x}) \cap \mathbb{R}^n_{>0}$ near $\bar{x}$, there must exist $\delta > 0$ such that $x = \bar{x} + \delta b$, where b is the base of $\mathscr{S}$, and is set with its positive terms index set including I. For convenient usage, we denote the functions in (16) evaluated at $x \to \bar{x}$ by

$$\tilde{u}(\bar{x}) = \lim_{\delta\to 0^+} \tilde{u}(\bar{x} + \delta b) \text{ and } f(\bar{x}) = \lim_{\delta\to 0^+} f(\bar{x} + \delta b).$$

Then, we have

$$\begin{aligned}
&\lim_{\delta\to 0^+} \frac{f(\bar{x} + \delta b) - f(\bar{x})}{\delta} \\
&= \lim_{\delta\to 0^+} \frac{1}{\delta} \int_{\gamma(\bar{x})}^{\gamma(\bar{x}+\delta b)} \ln \tilde{u}(y^\dagger(\bar{x}) + \tau b) d\tau \\
&= \lim_{\delta\to 0^+} \frac{1}{\delta} \int_{\gamma(\bar{x})}^{\gamma(\bar{x})+\delta} \ln \tilde{u}(y^\dagger(\bar{x}) + \tau b) d\tau \\
&= \ln \tilde{u}(y^\dagger(\bar{x}) + \gamma(\bar{x}) b) \\
&= \ln \tilde{u}(\bar{x}).
\end{aligned}$$

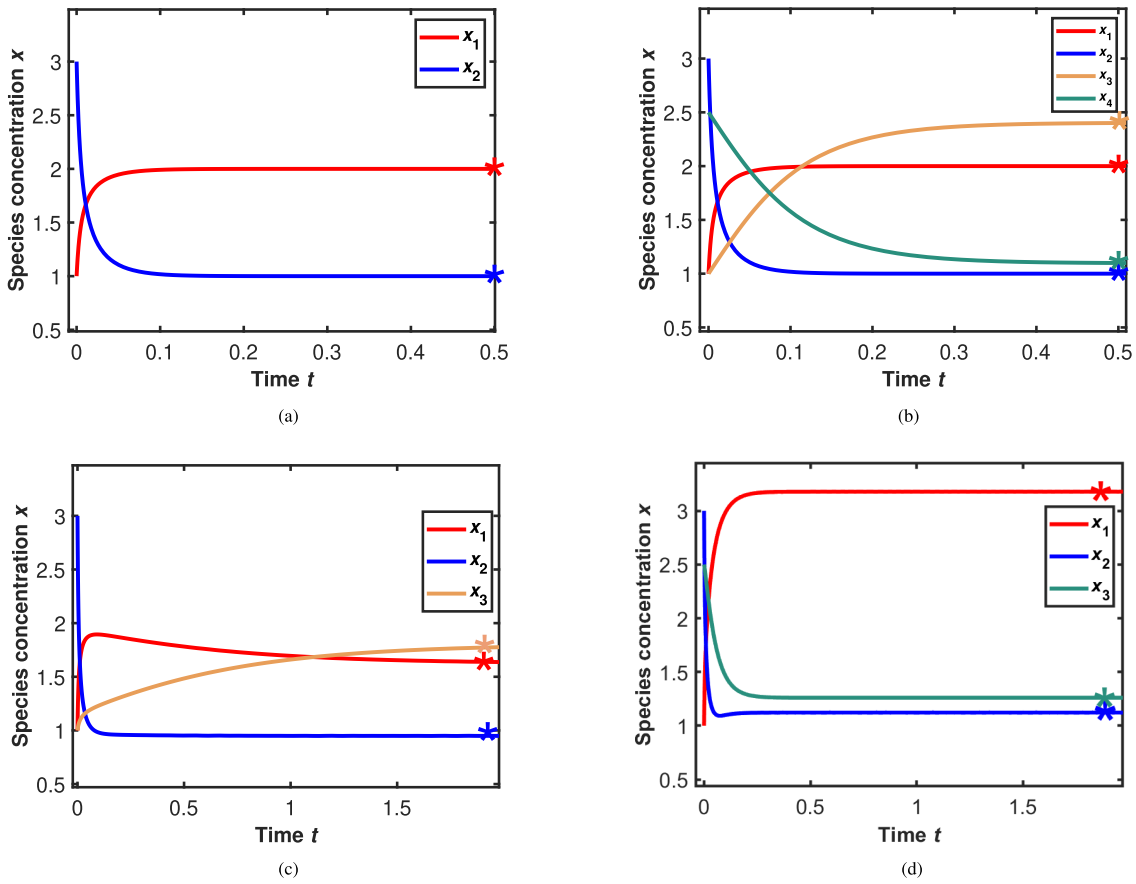

Fig. 3. (a), (b), (c), (d) According to the combination method described in Section IV, the combined system shows persistence regardless of whether the subsystems have intersecting species or not. (a) Evolution of the concentration of species in the system of Example 2. (b) Evolution of the concentration of species in the system $\mathcal{M}$ of Example 4. (c) Evolution of the concentration of species in the system $\mathcal{M}$ of Example 5. (d) Evolution of the concentration of species in the system $\mathcal{M}$ of Example 8.

From Lemma 3.8 that says $0 < \tilde{u}(\bar{x}) < 1$, we get

$$\lim_{\delta\to 0^+} \frac{f(\bar{x}+\delta b) - f(\bar{x})}{\delta} = \lim_{\delta\to 0^+} \frac{f(x) - f(\bar{x})}{\delta} < 0.$$

Hence, $\bar{x}$ is a local maximum point for the function $f(x)$. Further based on Lemma 3.7, we obtain the result that each $1d$ $\mathcal{W}_I$-endotactic MAS $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ is persistent. ■

Example 3: We use Example 2 to exhibit an application of Theorem 3.10. Assume that k_1, $k_2 > 0$ are the reaction rate constants of the first and second reactions in this network, respectively. As said in Example 2, the network is $\mathcal{W}_I$-endotactic. Also, from the stoichiometric matrix

$$\Gamma = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$$

the network is, thus, a $1d$ $\mathcal{W}_I$-endotactic network. Furthermore, the application of Theorem 3.10 yields that this system is persistent. Fig. 3(a) shows the evolution of each species concentration starting from $(1, 3)^\top$.

IV. Persistence of Some Higher Dimensional $\mathcal{W}_I$-Endotactic MASs

This section focuses on the persistence of higher dimensional $\mathcal{W}_I$-endotactic MASs by utilizing the method of combination.

Consider an $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ as a combination of some $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)} = (\mathcal{S}^{(p)}, \mathcal{C}^{(p)}, \mathcal{R}^{(p)}, \boldsymbol{k}^{(p)})$, $p = 1, \ldots, \ell$. For the pth subsystem $\mathcal{M}^{(p)}$, it contains n_p species and r_p reactions. $v_{.i^{(p)}}$ and $v'_{.i^{(p)}}$ are the reactant and product complex of the ith reaction of system $\mathcal{M}^{(p)}$, respectively. The system $\mathcal{M} = \{\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k}\}$ composed of n species and r reactions can be expressed by

$$\mathcal{S} = \bigcup_{p=1}^{\ell} \mathcal{S}^{(p)};\ \mathcal{C} = \bigcup_{p=1}^{\ell} \mathcal{C}^{(p)} = \bigcup_{p=1}^{\ell} \bigcup_{i=1}^{r_p} \{v_{.i}^{(p)}, v'^{(p)}_{.i}\}$$

$$\mathcal{R} = \bigcup_{p=1}^{\ell} \mathcal{R}^{(p)} = \bigcup_{p=1}^{\ell} \bigcup_{i=1}^{r_p} \{v_{.i}^{(p)} \to v'^{(p)}_{.i}\}$$

$$\boldsymbol{k} = \bigcup_{p=1}^{\ell} \boldsymbol{k}^{(p)} = \bigcup_{p=1}^{\ell} \bigcup_{i=1}^{r_p} \{k_i^{(p)}\}$$

where

$$v_{ji}^{(p)} = \begin{cases} v_{ji^{(p)}}, \text{ if } S_j \in \mathcal{S}^{(p)} \\ 0, \text{ if } S_j \notin \mathcal{S}^{(p)} \end{cases}; v'^{(p)}_{ji} = \begin{cases} v'_{ji^{(p)}}, \text{ if } S_j \in \mathcal{S}^{(p)} \\ 0, \text{ if } S_j \notin \mathcal{S}^{(p)}. \end{cases} \tag{23}$$

In order to analyze the persistence of the combined system $\mathcal{M}$, we need to consider the relation between semilocking sets of $\mathcal{M}$ and $\mathcal{M}^{(p)}$.

Lemma 4.1: For an $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ composed of some subsystems $\mathcal{M}^{(p)} = (\mathcal{S}^{(p)}, \mathcal{C}^{(p)}, \mathcal{R}^{(p)}, \boldsymbol{k}^{(p)})$, $p = 1, \ldots, \ell$, each semilocking set of $\mathcal{M}$ is a union of the semilocking sets of subsystems $\mathcal{M}^{(p)}$.

Proof: Suppose I is a semilocking set of $\mathcal{M}$ that is not the union of semilocking sets of subsystems $\mathcal{M}^{(p)}$. Namely, there exists some subsystem $\mathcal{M}^{(p)}$, such that $I^{(p)} \triangleq I \cap \mathcal{S}^{(p)}$ is not a semilocking set of the subsystem $\mathcal{M}^{(p)}$. In this case, at least one reaction $v_{\cdot i}^{(p)} \to v'^{(p)}_{\cdot i}$ that satisfies

$$\text{supp} v'^{(p)}_{\cdot i} \cap I^{(p)} \neq \emptyset \text{ and } \text{supp} v_{\cdot i}{}^{(p)} \cap I^{(p)} = \emptyset$$

can be found in subsystem $\mathcal{M}^{(p)}$. This reaction is also in system $\mathcal{M}$ and $I^{(p)} \subset I$, so $\text{supp} v'^{(p)}_{\cdot i} \cap I \neq \emptyset$. Further from the definition of $I^{(p)}$, we have $(I - I^{(p)}) \cap \mathcal{S}^{(p)} = \emptyset$. Thus, $\text{supp} v_{\cdot i}{}^{(p)} \cap I = \emptyset$. This contradicts the fact that I is a semilocking set. Thus, we can conclude our result. ■

The following theorem guarantees that the combined system is still a $\mathcal{W}_I$-endotactic system.

Theorem 4.2: An $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ composed of some $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)}, p = 1, \ldots, \ell$, is also a $\mathcal{W}_I$-endotactic system.

Proof: Let I denote any semilocking set of the system $\mathcal{M}$, now we research the reactions with $\leq_{w_I}$-maximum and $\leq_{w_I}$-minimal reactant complex.

Each I can be written as $I = \bigcup_{p \in \{p_1, \ldots, p_2\}} I^{(p)}$ where $I^{(p)} = I \cap \mathcal{S}^{(p)}$ and $\{p_1, \ldots, p_2\}$ denotes the set of p such that $I \cap \mathcal{S}^{(p)} \neq \emptyset$. From the Lemma 4.1, each $I^{(p)}$ is a semilocking set of subsystem $\mathcal{M}^{(p)}$. Let P denote the set $\{p_1, \ldots, p_2\}$, the corresponding w_I of semilocking set I in $\mathcal{M}$ and and $w_{I^{(p)}}, p \in P$ of semilocking set $I^{(p)}$ in $\mathcal{M}^{(p)}$ are defined as (12) and $w_{I^{(p)}} = \mathbb{0}_{n^{(p)}}$ for $p \notin P$. Thus, all the reactions $v_{\cdot i}^{(p)} \to v'^{(p)}_{\cdot i}$, $p \notin P$ are orthogonal to w_I. So we just need to consider the reactions in $\mathcal{M}^{(p)}$ where $p \in P$. $Y^{(p)}$ denotes the reactant complex of the subsystem $\mathcal{M}^{(p)}$ and $Y \in \bigcup_{p \in P} Y^{(p)}$. Further from

$$\langle w_I, v_{\cdot i}^{(p)} \rangle = \langle w_{I^{(p)}}, v_{\cdot i^{(p)}} \rangle.$$

we can obtain the rightmost leftmost) reactants for w_I of system $\mathcal{M}$ is must be rightmost (leftmost) reactants for some $w_I^{(p)}$ in subsystem $\mathcal{M}^{(p)}$. Let $\mathcal{C}_r$ and $\mathcal{C}_l$ denote $\leq_{w_I}$-maximal reactants and $\leq_{w_I}$-minimal reactants, respectively, then

$$\langle w_I, v'^{(p)}_{\cdot i} - v_{\cdot i}^{(p)} \rangle = \langle w_{I^{(p)}}, v'_{\cdot i^{(p)}} - v_{\cdot i^{(p)}} \rangle < 0 \quad \forall v_{\cdot i}^{(p)} \in \mathcal{C}_r \tag{24}$$

Also

$$\langle w_I, v'^{(p)}_{\cdot i} - v_{\cdot i}^{(p)} \rangle = \langle w_{I^{(p)}}, v'_{\cdot i^{(p)}} - v_{\cdot i^{(p)}} \rangle > 0 \quad \forall v_{\cdot i}^{(p)} \in \mathcal{C}_l. \tag{25}$$

The inequality of (24), (25) is derived from the fact that all subsystems $\mathcal{M}^{(p)}$ are $\mathcal{W}_I$-endotactic systems. (24) and (25) reveal that $\mathcal{M}$ is also a $\mathcal{W}_I$-endotactic system. ■

The following three sections provide the persistence of several types of higher dimensional $\mathcal{W}_I$-endotactic systems through different types of combinations of subsystems.

A. CRNs Composed of a Series of $1d$ $\mathcal{W}_I$-Endotactic CRNs With Nonintersecting Species

This section focuses on higher dimensional $\mathcal{W}_I$-endotactic system $\mathcal{M}$ composed of some $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)}$ but with no-intersecting species between any two subsystems, i.e., $\mathcal{S}^{(p_1)} \cap \mathcal{S}^{(p_2)} = \emptyset$ for any $p_1, p_2 = 1, \ldots, \ell$. In this case, $n = \sum_{p=1}^{\ell} n_p$, $r = \sum_{p=1}^{\ell} r_p$, and the dynamics follow

$$\dot{x} = \sum_{p=1}^{\ell} \sum_{i=1}^{r_p} k_i^{(p)} x^{v_{\cdot i}^{(p)}} (v'^{(p)}_{\cdot i} - v_{\cdot i}^{(p)}) = \bigotimes_{p=1}^{\ell} \dot{x}^{(p)}$$

where x and $x^{(p)}$ denote the state of systems $\mathcal{M}$ and $\mathcal{M}^{(p)}$, respectively. We present an example to illustrate this case.

Example 4: Consider an $\mathcal{M}$ taking

$$\mathcal{M}: \begin{array}{l} 2S_1 \xrightarrow{k_1} S_1 + 2S_2, \\ S_1 + 4S_2 \xrightarrow{k_2} 3S_1 \\ 2S_3 \xrightarrow{k_3} S_3 + S_4 \\ 2S_4 + S_3 \xrightarrow{k_4} 3S_3. \end{array} \quad \begin{array}{l} \mathcal{M}^{(1)}: \quad 2S_1 \xrightarrow{k_1} S_1 + 2S_2 \\ \qquad\quad S_1 + 4S_2 \xrightarrow{k_2} 3S_1 \\ \\ \mathcal{M}^{(2)}: \quad 2S_3 \xrightarrow{k_3} S_3 + S_4 \\ \qquad\quad 2S_4 + S_3 \xrightarrow{k_4} 3S_3 \end{array}$$

which is $2d$ with $\mathscr{S} = \text{span}\{(-1, 2, 0, 0)^\top, (0, 0, -1, 1)^\top\}$. It can be clearly thought to be composed of $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$. Moreover, these two subsystems are both $1d$ $\mathcal{W}_I$-endotactic networks and there are no intersecting species between them.

For MASs like Example 4, we have the following

Theorem 4.3: For an $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$, if it is composed of a series of $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)}$ $(p = 1, \ldots, \ell)$ with nonintersecting species among them, then $\mathcal{M}$ is persistent.

Proof: The result derives obviously from Theorem 3.10 and the fact that each $\dot{x}_j$ of $\mathcal{M}$ is the same as that in $\mathcal{M}^{(p)}$s. ■

The direct application of Theorem 4.3 to Example 4 reveals that $\mathcal{M}$ is persistent. Fig. 3(b) displays the trajectory of each species starting from $(1, 3, 1, 2.5)^\top$. Actually Theorem 4.3 can be further extended to any MAS consisting of some persistent subsystems without intersecting species among them.

Corollary 4.4: A MAS is persistent if it consists of some persistent subsystems and there is no intersecting species between any two subsystems.

B. CRNs Composed of Some $1d$ $\mathcal{W}_I$-Endotactic CRNS With Intersecting Species Just in Nonsemilocking Set

We continue to consider the composed $\mathcal{W}_I$-endotactic networks but with intersecting species among $1d$ $\mathcal{W}_I$-endotactic subsystems, which means that for some $p_1, p_2 = 1, \ldots, \ell$, $\mathcal{S}^{(p_1)} \cap \mathcal{S}^{(p_2)} \neq \emptyset$. Let $\mathcal{S}_c$ denote the set of intersecting species that appear in more than one subsystem. We begin with a relatively simple case of $\mathcal{S}_c \cap I = \emptyset$ to discuss persistence of the

composed $\mathcal{W}_I$-endotactic MASs, where $I \subsetneqq \mathcal{S}$ represents any nontrivial semilocking set, i.e., no intersecting species appearing in any nontrivial semilocking set. The following example exhibits the current case under consideration.

Example 5: An $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ is composed of two $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ following

$$\left.\begin{array}{c|c} \mathcal{M}: & \mathcal{M}^{(1)}: \\ 2S_1 \xrightarrow{k_1} S_1 + 2S_2, & 2S_1 \xrightarrow{k_1} S_1 + 2S_2, \\ S_1 + 4S_2 \xrightarrow{k_2} 3S_1, & S_1 + 4S_2 \xrightarrow{k_2} 3S_1, \\ 2S_3 \xrightarrow{k_3} S_3 + S_2, & \mathcal{M}^{(2)}: \\ 2S_2 + S_3 \xrightarrow{k_4} 3S_3. & 2S_3 \xrightarrow{k_3} S_3 + S_2, \\ & 2S_2 + S_3 \xrightarrow{k_4} 3S_3. \end{array}\right\} \mathcal{S}_c = \{S_2\}.$$

The set of intersecting species between two subsystems is $\mathcal{S}_c = \{S_2\}$. There include three nontrivial semilocking sets $I_1 = \{S_1\}, I_2 = \{S_3\}$ and $I_3 = \{S_1, S_3\}$. We can see that $\mathcal{S}_c \cap I_j = \emptyset$ for $j = 1, 2, 3$.

The abovementioned example is obviously very different from Example 4 with the latter being a decoupled system. Note that the dynamical evolution of species in the semilocking set is the key to the study of persistence while the dynamics of species in nonsemilocking sets can be ignored. We are, thus, only concerned with the evolution of part species of a MAS under study. For this purpose, we define a reduced version of MAS as follows.

Definition 5.5: For an $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ described by (8), $\tilde{\mathcal{M}} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{K})$ is a reduced system of $\mathcal{M}$ if $\tilde{\mathcal{M}}$ just considers the dynamics of part species in $\mathcal{S}$, i.e., $\tilde{\mathcal{S}} \subset \mathcal{S}$. From the view of $\tilde{\mathcal{M}}$, the reaction rate of each reaction $\tilde{v}_{\cdot i} \xrightarrow{\tilde{k}_i(t)} \tilde{v}'_{\cdot i}$ (corresponding to reaction $v_{\cdot i} \xrightarrow{k_i} v'_{\cdot i}$ in $\mathcal{M}$) is written as

$$\tilde{k}_i(t) X^{\tilde{v}_{\cdot i}} = k_i x^{v_{\cdot i}}(t) \tag{26}$$

where $\tilde{k}_i(t) = k_i \prod_{S_j \notin \tilde{\mathcal{S}}} x_j^{v_{ji}}(t)$, and X represents the state of $\tilde{\mathcal{M}}$.

For the semilocking set $I_3 = \{S_1, S_3\}$ in Example 5, we set it as the part species set to induce a reduced system $\tilde{\mathcal{M}}$, whose dynamics is

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1^2 + 2k_2 x_1 x_2^4, \\ \dot{x}_3 &= -k_3 x_3^2 + 2k_4 x_2^2 x_3. \end{aligned} \iff \begin{aligned} \dot{x}_1 &= -k_1 x_1^2 + 2\tilde{k}_2(t) x_1 \\ \dot{x}_3 &= -k_3 x_3^2 + 2\tilde{k}_4(t) x_3. \end{aligned}$$

The abovementioned dynamics can be seen as induced by the following generalized MAS:

$$2S_1 \xrightarrow{\tilde{k}_1(t)} S_1, S_1 \xrightarrow{\tilde{k}_2(t)} 3S_1$$

$$2S_3 \xrightarrow{\tilde{k}_3(t)} S_3, S_3 \xrightarrow{\tilde{k}_4(t)} 3S_3.$$

A look at this generalized MAS might suggest $\tilde{\mathcal{M}}$ consists of two decoupled subsystems. Hence, it is possible to utilize the previous result, such as Corollary 4.4, to analyze its persistence. Also, since only the species in semilocking set needs to be cared for persistence analysis, the strategy of generating the reduced system from maximum semilocking set(s) of $\mathcal{M}$ may provide a solution to catch the persistence of MASs like Example 5. We affirm this interference through the theorem as follows.

Theorem 4.6: Consider a conservative system $\mathcal{M}$ composed of some $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)}, p = 1, \ldots, \ell$, with nonempty intersecting species set $\mathcal{S}_c$ among subsystems. Suppose for each semilocking set $I \subsetneqq \mathcal{S}$ there is $I \cap \mathcal{S}_c = \emptyset$. Then, $\mathcal{M}$ is persistent.

Proof: The system $\mathcal{M}$ is conservative, so all the trajectories of $\mathcal{M}$ are bounded. At the same time, conservation ensures that the trajectory from the positive starting point will not tend to the origin. Let $n^{(p^{nc})}$ denote the number of the species that only exist in the subsystem $\mathcal{M}^{(p)}$ and n_c is the number of intersecting species in more than one subsystem. Thus, $\sum_{p=1}^{\ell} n^{(p^{nc})} + n_c = n$. Now we reorder the species set $\mathcal{S}$ as $\mathcal{S} = \bigotimes_{p=1}^{\ell} \mathcal{S}^{(p^{nc})} \bigotimes \mathcal{S}_c$ where $\mathcal{S}^{(p^{nc})}$ is the set of the species only in $\mathcal{M}^{(p)}$.

1) If $n_c = n$, any species is the intersecting species, namely, $\mathcal{S}_c = \mathcal{S}$. As each intersecting species does not exist in a semilocking set except $\mathcal{S}$, $\mathcal{M}$ only has one semilocking set $\mathcal{S}$ and one semilocking boundary point $\mathbb{0}_n$. Further from the conservation, $\mathbb{0}_n$ is not an ω-limit point for any trajectory with a positive starting point. Thus, $\mathcal{M}$ is persistent.
2) If $0 < n_c < n$, we consider a reduced system $\tilde{\mathcal{M}} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{K})$ with

$$\tilde{\mathcal{S}} = \bigotimes_{p=1}^{\ell} \mathcal{S}^{(p^{nc})}, \quad \tilde{\mathcal{C}} = \bigotimes_{p=1}^{\ell} \bigotimes_{i=1}^{r_p} \left\{ v_{\cdot i}^{(p^{nc})}, v'^{(p^{nc})}_{\cdot i} \right\}$$

$$\tilde{\mathcal{R}} = \bigotimes_{p=1}^{\ell} \bigotimes_{i=1}^{r_p} \left\{ v_{\cdot i}^{(p^{nc})} \to v'^{(p^{nc})}_{\cdot i} \right\}$$

$$\tilde{K} = \bigotimes_{p=1}^{\ell} \bigotimes_{i=1}^{r_p} \tilde{k}_{\cdot i}^{(p)}(t) = \bigotimes_{p=1}^{\ell} \bigotimes_{i=1}^{r_p} k_{\cdot i}^{(p)} \prod_{j=1}^{n_c} x_j^{v_{\cdot i}^{(p_c)}}(t)$$

where $v_{\cdot i}^{(p^{nc})}, v'^{(p^{nc})}_{\cdot i}$ are $v_{\cdot i}^{(p)}, v'^{(p)}_{\cdot i}$ restricted to the set $\mathcal{S}^{(p^{nc})}$ and $v_{\cdot i}^{(p_c)}$ is $v_{\cdot i}^{(p)}$ restricted to the set $\mathcal{S}_c$. Let $X = \bigotimes_{p=1}^{\ell} \bigotimes_{j=1}^{n^{(p^{nc})}} x_j^{(p^{nc})}$ denote the state of system $\tilde{\mathcal{M}}$. Because any two $\mathcal{S}^{(p_1^{nc})} \cap \mathcal{S}^{(p_1^{nc})} = \emptyset$, $p_1, p_2 = 1, \ldots, \ell$, $\tilde{\mathcal{M}}$ is composed of $1d$ $\tilde{\mathcal{M}}^{(p)} = (\tilde{\mathcal{S}}^{(p)}, \tilde{\mathcal{C}}^{(p)}, \tilde{\mathcal{R}}^{(p)}, \tilde{K}^{(p)})$ with nonintersecting species where $\tilde{\mathcal{M}}^{(p)}$ is a reduced system of $\mathcal{M}^{(p)}$. Further from each $I \subsetneqq \mathcal{S}$ contained in $\tilde{\mathcal{S}}$, we can obtain that $\tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}^{(p)}$ are all $\mathcal{W}_I$-endotactic generalized MASs. And the dynamical equation of $\tilde{\mathcal{M}}$ is $\dot{X} = \bigotimes_{p=1}^{\ell} \dot{X}^{(p)}$ where $\dot{X}^{(p)}$ denote the state of system $\tilde{\mathcal{M}}^{(p)}$.

Suppose some semilocking boundary point $\bar{x}$ of $\mathcal{M}$, namely, $I \triangleq \text{supp}^c \bar{x}$ is a semilocking set, is an ω-limit point of some trajectory $x(t)$ starting from a positive point. Now we analyze the sign of $\dot{x}_j(t), j \in I$ where $x(t)$ in a small enough τ-neighbourhood of $\bar{x}$ by using the reduced system $\tilde{\mathcal{M}}$. As each $I \subsetneqq S$ is contained in $\tilde{S}$, x_j is in the state variable X of $\tilde{\mathcal{M}}$. Also, we can obtain that I is a semilocking set in $\tilde{\mathcal{M}}$ and it can be denoted as $I = \bigotimes_{p=1}^{\ell} I^{(p)}, p = 1, \ldots, \ell$ where $I^{(p)} = I \cap \tilde{\mathcal{S}}^{(p)}$ is a

semilocking set of the subsystem $\tilde{\mathcal{M}}^{(p)}$. $\mathcal{M}$ is a conservative system with the concentration of each species is bounded. When $x(t)$ in te ϵ-neighborhood of $\bar{x}$, for each $S_j \notin I$, $S_j(t)$ is bigger than a positive constant and also bounded. Thus, each $\tilde{k}_i(t)$ in the reduced system satisfies

$$M_d^i < \tilde{k}_i(t) < M_u^i$$

where M_d^i, M_u^i are all positive constants. Without loss of generality, we suppose $I^{(p)} \triangleq \text{supp}^c \bar{x}$ belongs to the positive terms index set of the base $b^{(p)}$, Proposition 3.9 reveals that for $p = 1, \ldots, \ell$, there exists a small enough τ_1 such that

$$\sum_{\{i|m_i^{(p)}>0\}} \left(\tilde{k}_i(t) {X^{(p)}}^{v_{\cdot i}}\right) > \sum_{\{i|m_i^{(p)}<0\}} (\tilde{k}_i(t) {X^{(p)}}^{v_{\cdot i}}). \tag{27}$$

Thus, when $x(t)$ enters into the τ_1-neighborhood of $\bar{x}$ there exists

$$\begin{aligned} \dot{x}_j^{(p)}(t) = & \sum_{\{i|m_i^{(p)}>0\}} (\tilde{k}_i(t) {X^{(p)}}^{v_{\cdot i}} m_i^{(p)} b_j^{(p)}) \\ & - \sum_{\{i|m_i^{(p)}<0\}} (\tilde{k}_i(t) {X^{(p)}}^{v_{\cdot i}} m_i^{(p)} b_j^{(p)}) \\ > & \ 0 \quad \forall S_j^{(p)} \in I. \end{aligned} \tag{28}$$

From the definition of the reduced system, the dynamics of species $S_j^{(p)}$ denoted as $\dot{x}_j^{(p)}$ for $S_j^{(p)} \in I$ is exactly its dynamics in $\mathcal{M}$.

If $\bar{x}$ is an ω-limit point of some trajectory $x(t)$ of $\mathcal{M}$, then for arbitrary small enough $\tau > 0$, we can find some t_1 such that $x(t_1) \in \mathbb{R}^n_{>0}$ is in the τ-neighborhood of $\bar{x}$. But when $\tau < \tau_1$, there exists $\dot{x}_j > 0$ for any $S_j \in I$. This means that the trajectory will be far away from $\bar{x}$. This contradicts the assumption that $\bar{x}$ is an ω-limit point of some trajectory with a positive starting point. Thus, we can conclude the persistence of $\mathcal{M}$. ■

Let us go back to Example 5. It is obvious that all the conditions in the abovementioned theorem hold, and the system is, thus, persistent. Fig. 3(c) shows the trajectory of each species starting from $(1, 3, 1)^\top$.

Remark 4.7: Note that for any coupled MAS $\mathcal{M}$ as defined in Theorem 4.6, as long as every coupled species among $1d$ $\mathcal{W}_I$-endotactic subsystems is not in any nontrivial semilocking set of $\mathcal{M}$, the persistence of every subsystem readily derives from $\mathcal{M}$. Also, the system is requested to be conservative in this theorem. The main purpose is to request each trajectory of the system to be bounded. Then, we can use the ω-limit set to get the persistence of the system. Note that this is just a sufficient condition. There are obviously nonconservative systems that have bounded trajectories. It may be a point of our future study.

The abovementioned combination pattern can be further extended to combine more persistent MASs from $1d$ $\mathcal{W}_I$-endotactic networks.

Proposition 4.8: For a conservative system $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ composed of some $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)} = (\mathcal{S}^{(p)}, \mathcal{C}^{(p)}, \mathcal{R}^{(p)}, \boldsymbol{k}^{(p)})$, $p = 1, \ldots, \ell$, and any other form of subsystems $\mathcal{M}^{(q)} = (\mathcal{S}^{(q)}, \mathcal{C}^{(q)}, \mathcal{R}^{(q)}, \boldsymbol{k}^{(q)}), q = \ell+1, \ldots, \tilde{\ell}$, but constrained by $\mathcal{S}^{(q)} \subseteq \bigcup_{p=1}^{\ell} \mathcal{S}^{(p)}$, if $\mathcal{S}_c$ denotes the nonempty set of intersecting species of all subsystems and satisfies $\mathcal{S}_c \cap I = \emptyset$ for any nontrivial semilocking set I of the network, then $\mathcal{M}$ is persistent.

Proof: The system $\mathcal{M}$ is a conservative system, so each trajectory cannot tend to the origin, i.e., the origin cannot be the ω-limit point of any trajectory of $\mathcal{M}$ starting from a positive point. Also, the concentration of each species is bounded.

Now we consider the semilocking boundary point $\bar{x} \neq \mathbb{0}_n$ where $I = \text{supp}^c \bar{x}$ is a semiloking set. Each species in $\mathcal{S}^{(q)}, q = \ell+1, \ldots, \tilde{\ell}$ is contained in $\mathcal{S}_c$, from $\mathcal{S}_c \cap I = \emptyset$, we can obtain that all species in $\mathcal{S}^{(q)}$ are not in semilocking set I. Thus, $I \subset \bigcup_{p=1}^{\ell} \mathcal{S}^{(p)}$. And each species in I only exists in one subsystem. By using the reduced system of $\mathcal{M}$ and the same method in Theorem 4.6, we also obtain that $x_j > 0$ for each $j \in I$. Thus, the corresponding boundary point $\bar{x}$ cannot be the ω-limit point of each trajectory starting from a positive point. So we conclude the persistence of system $\mathcal{M}$. ■

Example 6: Consider the following system $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$, it is composed of two $1d$ $\mathcal{W}_I$-endotactic systems $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ and a system $\mathcal{M}^{(3)}$ that converges to its boundary

$$\left.\begin{array}{c|c} \mathcal{M}: & \mathcal{M}^{(1)}: \\ 2S_1 + S_4 \xrightarrow{k_1} S_1 + 2S_2, & 2S_1 + S_4 \xrightarrow{k_1} S_1 + 2S_2, \\ S_1 + 4S_2 \xrightarrow{k_2} 3S_1 + 2S_4, & S_1 + 4S_2 \xrightarrow{k_2} 3S_1 + 2S_4, \\ 2S_3 \xrightarrow{k_3} S_3 + S_2, & \mathcal{M}^{(2)}: \\ 2S_2 + S_3 \xrightarrow{k_4} 3S_3, & 2S_3 \xrightarrow{k_3} S_3 + S_2, \\ 2S_2 \xrightarrow{k_5} S_3. & 2S_2 + S_3 \xrightarrow{k_4} 3S_3, \\ & \mathcal{M}^{(3)}: \\ & 2S_2 \xrightarrow{k_5} S_3. \end{array}\right\} \mathcal{S}_c = \{S_2, S_3\}.$$

The nontrivial semilocking sets $I \subsetneqq \mathcal{S}$ of the system $\mathcal{M}$ are $I_1 = \{S_1\}, I_2 = \{S_1, S_4\}$, respectively. And the intersection of $\mathcal{S}_c$ and I_1, I_2 are all empty sets. Utilizing Theorem 4.6, we can only obtain that the coupled system $\mathcal{M}^{(1)} \cup \mathcal{M}^{(2)}$ is persistent. But Proposition 4.8 further demonstrates that the persistence of the newly combined system $\mathcal{M}$ also holds, despite adding a subsystem $\mathcal{M}^{(3)}$ that converges to the boundary.

C. CRNs Composed of Some $1d$ $\mathcal{W}_I$-Endotactic CRNs With Intersecting Species in Semilocking Set.

We further consider the previous coupled MASs with coupled species in semilocking set. We also give an example to illustrate the current case.

Example 7: An $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, k)$ follows

$$\left.\begin{array}{c|c} \mathcal{M}: & \mathcal{M}^{(1)}: \\ 2S_1 \xrightarrow{k_1} S_1 + 2S_2, & 2S_1 \xrightarrow{k_1} S_1 + 2S_2, \\ S_1 + 4S_2 \xrightarrow{k_2} 3S_1, & S_1 + 4S_2 \xrightarrow{k_2} 3S_1, \\ 2S_1 \xrightarrow{k_3} S_1 + S_3, & \mathcal{M}^{(2)}: \\ S_1 + 2S_3 \xrightarrow{k_4} 3S_1. & 2S_1 \xrightarrow{k_3} S_1 + S_3, \\ & S_1 + 2S_3 \xrightarrow{k_4} 3S_1. \end{array}\right\} \mathcal{S}_c = \{S_1\}$$

where $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ are two subsystems and $\mathcal{S}_c = \{S_1\}$ is the set of intersecting species. The semilocking sets of the system $\mathcal{M}$ are $I_1 = \{S_1\}, I_2 = \{S_1, S_2\}, I_3 = \{S_1, S_3\}, I_4 = \{S_1, S_2, S_3\}$. It is obvious that $\mathcal{S}_c \cap I_i \neq \emptyset$. We need to find new conditions to support persistence for this class of MASs.

Lemma 4.9: Consider a conservative system $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$. It is composed of some $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)}, p = 1, \ldots, \ell$. Let $\mathcal{S}_c$ denote the set of all intersecting species. If for some semilocking set $I \subsetneqq \mathcal{S}$, $I = \{S_{j_1}\}$ where $S_{j_1} \in \mathcal{S}_c$, each semilocking boundary point $\bar{x}$ where $I = \text{supp}^c\bar{x}$ is a semilocking set cannot be an ω-limit point of each trajectory with positive starting point of $\mathcal{M}$ and $\mathcal{S}_c$ shares the same meaning as that in Theorem 4.6.

Proof: We suppose that there exists a boundary point $\bar{x} \neq \mathbb{0}_n$ with $\text{supp}^c\bar{x} = I$ is an ω-limit point for some trajectory $x(t)$ starting from the positive point. Then, for each small enough $\tau > 0$, we can find some t_1 such that $x(t_1)$ is in the τ-neighborhood of $\bar{x}$. Now we consider the dynamics of species S_{j_1} denoted as $\dot{x}_{j_1}$. Because each species is bounded, the reduced system $\tilde{\mathcal{M}} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{K})$ of the system $\mathcal{M}$ is described as Definition 4.5 where $\tilde{\mathcal{S}} = S_{j_1}$ and $\tilde{K}_i, i = 1, \ldots, \tilde{r}$ in $\tilde{K}$ is

$$\tilde{K}_i = \tilde{k}_i(t) = k_i \prod_{j \neq j_1} x_j(t)^{v_{ji}}.$$

We can obtain that the reduced system $\tilde{\mathcal{M}}$ is a single-species generalized MAS and each $\tilde{K}_i$ is bounded.

Each reaction $\tilde{R}_i$ of $\tilde{\mathcal{R}}$ is reduced by R_i in some subsystem $\mathcal{M}^{(p)}$. Suppose there are $\tilde{\ell}$ subsystems containing S_j, then the reduced system $\tilde{\mathcal{M}}$ is composed of $\tilde{\mathcal{M}}^{(\tilde{p})}, \tilde{p} = 1, \ldots, \tilde{\ell}$. Each reduced subsystem $\tilde{\mathcal{M}}^{(\tilde{p})}$ is a generalized $1d$ $\mathcal{W}_I$-endotactic system. Thus, similar to the proof of Theorem 4.6, we can get that there exists $\dot{x}_{j_1}^{(p)} > 0$ when $x(t)$ enters into the $\tau^{(\tilde{p})}$-neighborhood of $\bar{x}$ for some small enough constant $\tau^{(\tilde{p})} > 0$ by using Proposition 3.9. The dynamics of x_{j_1} can be written as

$$\dot{x}_{j_1} = \sum_{p=1}^{\tilde{\ell}} \dot{x}_{j_1}^{(\tilde{p})} \tag{29}$$

and $\dot{x}_{j_1}$ of the reduced system $\tilde{\mathcal{M}}$ is equal to that in $\mathcal{M}$. Thus, choosing $\tau = \min\{\tau^{(\tilde{p})}\}$, we obtain $\dot{x}_j(t) > 0$ when the trajectory $x(t)$ enters into τ-neighborhood of boundary point $\bar{x}$. This is a contradiction to the assumption that $\bar{x}$ is an ω-limit point of some trajectory $x(t)$ starting from the positive point. Thus, we conclude our result. ■

Now we consider the case I is not a single species set but any two species in I are not in one reaction.

Lemma 4.10: $\mathcal{M}$ is a MAS composed of some $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)}, p = 1, \ldots, \ell$. If for some semilocking set $I = \{S_{j_1}, \ldots, S_{j_m}\} \subsetneqq \mathcal{S}$ where any two species in I do not participate in one reaction R_i of $\mathcal{M}$, then the boundary points $\bar{x}$ with $\text{supp}^c\bar{x} = I$ cannot be an ω-limit point of arbitrary trajectory starting from the positive point.

Proof: We consider the reduced system $\tilde{\mathcal{M}}$ with $\tilde{S} = I$. Then, the reduced system $\tilde{M}$ is composed of m single species subsystems with nonintersecting species. If some S_j in I is in $\mathcal{S}_c$, from Lemma 4.9, we can obtain that $\dot{x}_j > 0$ when x approaches to $\bar{x}$. Otherwise the S_j only exists in one $1d$ $\mathcal{W}_I$-endotactic network, thus, also $\dot{x}_j > 0$ when x approaches to $\bar{x}$. Therefore, we can conclude our result. ■

The following lemma is devoted to the case of semilocking set I where species in I can interact with each other.

Lemma 4.11: Consider an $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ composed of some $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)}, p = 1, \ldots, \ell$. Let $I = \{S_{j_1}, S_{j_2}\} \subsetneqq \mathcal{S}$ be a semilocking set of $\mathcal{M}$ where $S_{j_1} \in \mathcal{S}_c, S_{j_2} \notin \mathcal{S}_c$ and the boundary points $\bar{x}$ with $\text{supp}^c\bar{x} = I$. If S_{j_1}, S_{j_2} both exist in some subsystem $\mathcal{M}^{(p_1)}$, let $b^{(p_1)}$ denote the basis of stoichiometric subspace $\mathscr{S}^{(p_1)}$ satisfying $b_{j_1}^{(p_1)} \cdot b_{j_2}^{(p_1)} \leq 0$. Then, the boundary point $\bar{x}$ with $I = \text{supp}^c\bar{x}$ cannot be an ω-limit point of arbitrary trajectory starting from a positive point.

Proof: If there is no $\mathcal{M}^{(p_1)}$ such that S_{j_1}, S_{j_2} are both within $\mathcal{M}^{(p_1)}$, then the result can be derived directly from Lemma 4.10. Now we consider the case that there exists the subsystem $\mathcal{M}^{(p_1)}$ containing both S_{j_1} and S_{j_2}. As $S_{j_1} \in \mathcal{S}_c, S_{j_2} \notin \mathcal{S}_c, S_{j_1}$ is contained in more than one subsystems and S_{j_2} is only in $\mathcal{M}^{(p_1)}$.

1) If $b_{j_1}^{(p_1)} = 0$, the reaction in $\mathcal{M}^{(p)}$ takes place without changing the concentration of S_{j_1}. So in this case, S_{j_2} does not affect the concentration of S_{j_1}. The concentration of S_{j_1} does not tend to zero from proof of Lemma 4.9.
2) If $b_{j_2}^{(p_1)} = 0$, this means the concentration x_{j_2} is always the same, namely, $\dot{x}_{j_2} = 0$. Thus, each trajectory starting from a positive point will not tends to $\bar{x}$ as $\bar{x}_{j_2} = 0$.
3) If $b_{j_1}^{(p_1)} \cdot b_{j_2}^{(p_1)} < 0$ and we assume that $\bar{x}$ with $\text{supp}^c\bar{x} = I$ is an ω-limit point of some trajectory $x(t)$ starting from a positive point. We consider the reduced system $\tilde{\mathcal{M}}$ with $\tilde{\mathcal{S}} = \{S_{j_1}, S_{j_2}\}$. The dynamics of the reduced system is

$$\begin{aligned} \dot{x}_{j_1} &= \sum_{p \neq p_1} (\dot{x}_{j_1})^{(p)} + (\dot{x}_{j_1})^{(p_1)} \\ \dot{x}_{j_2} &= (\dot{x}_{j_2})^{(p_1)} \end{aligned} \tag{30}$$

where $(\dot{x}_{j_1})^{(p)}$ denotes the evolution of x_{j_1} brought by the reactions of subsystem $\mathcal{M}^{(p)}$ (If S_{j_1} not in some $\mathcal{M}^{(p)}$, $(\dot{x}_{j_1})^{(p)} = 0$). The first part of the first equation of (30) denoted as $\sum_{p \neq p_1} (\dot{x}_{j_1})^{(p)}$ is the dynamics of the system composed of the subsystem containing S_{j_1} except p_1. From Lemma 4.9, we can deduce that there exists a small enough $\tau > 0$ such that $\sum_{p \neq p_1} (\dot{x}_{j_1})^{(p)} > 0$ once the trajectory enters into the τ-neighborhood of $\bar{x}$. $\bar{x}$ is an ω-limit point of the trajectory $x(t)$, so for the $\epsilon > 0$, we can find $x(t_\epsilon)$ in this ϵ-neighborhood of $\bar{x}$. And $x(t_\epsilon) + b = \bar{x}$ where $b = \sum_{p=1}^{\ell} \delta_p b^{(p)}$ is a linear combination of $b^{(p)}$ and $b_{j_1} < 0, b_{j_2} < 0$. S_{j_2} is only in $\mathcal{M}^{(p_1)}$, so

$$\delta_{p_1} b_{j_2}^{(p_1)} < 0, \text{ but } \delta_{p_1} b_{j_1}^{(p_1)} > 0.$$

Because $\bar{x}$ is an ω-limit point, the trajectory $x(t)$ will further move from $x(t_\epsilon)$ to $\bar{x}$. Therefore, the concentration of species S_{j_1}, S_{j_2} denoted as x_{j_1}, x_{j_2} are all

decreasing. The decreasing of x_{j_2} is only due to the reactions in $\mathcal{M}^{(p_1)}$, thus, the reaction of $\mathcal{M}^{(p_1)}$ results in an increase of x_{j_1} from $\delta_{p_1} b_{j_1}^{(p_1)} > 0$. From this, the decreasing of x_{j_1} is due to the reaction in $\mathcal{M}^{(p)}, p \neq p_1$, namely, $\sum_{p\neq p_1}(\dot{x}_{j_1}(t))^{(p)} < 0$. It is obviously contradiction to the fact that $\sum_{p\neq p_1}(\dot{x}_{j_1})^{(p)} > 0$ for all $x(t)$ in the ϵ-neighborhood of $\bar{x}$. Thus, the $\bar{x}$ cannot be an ω-limit point for any trajectory with the positive starting point. ■

From the abovementioned three lemmas, we can derive the persistence of some kind of MAS $\mathcal{M}$.

Theorem 4.12: $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \boldsymbol{k})$ is a conservative MAS, and it is composed of some $1d$ $\mathcal{W}_I$-endotactic subsystems $\mathcal{M}^{(p)}, p = 1, \ldots, \ell$. Let $\mathcal{S}_c$ denote the nonempty set of intersecting species of all subsystems. For each semilocking set $I \subsetneqq \mathcal{S}$, if I is one of three forms in Lemmas 4.9, 4.10, 4.11, $\mathcal{M}$ is persistent.

Proof: From Lemmas 4.9, 4.10, and 4.11, each boundary point $\bar{x}$ where $I = \text{supp}^c\bar{x}$ is a semilocking set cannot be an ω-limit point of each trajectory starting from a positive point.

Example 8: Now we use results derived in this subsection to check the persistence of the system in Example 7.

1) For semilocking set I_1, each boundary point $\bar{x}$ with $\text{supp}^c\bar{x} = I_1$ cannot be the ω-limit point of each trajectory with the positive starting point from Lemma 4.9.
2) For semilocking set I_2, I_3, the corresponding boundary points of I_2, I_3 cannot be ω-limit points from Lemma 4.11.
3) For semilocking set I_4, the corresponding boundary point of I_4 is the origin. It also cannot be the ω-limit point as the system $\mathcal{M}$ is conservative.

From the Theorem 4.12, the system $\mathcal{M}$ is persistent. Fig. 3(d) shows the evolution of the species concentration of $\mathcal{M}$.

Remark 4.13: From the proof of the persistence of systems introduced in this article, for each boundary F_I, we can find a proper threshold value such that $\dot{x}_j > 0$ for some $S_j \in I$ once the concentration is below this threshold value (in other words, the trajectory will move toward the direction away from the boundary). Thus, by using these thresholds, we can easily find a compact set such that any trajectory from a positive initial point eventually converges to this compact set. Therefore, we can generalize the persistence of these systems to permanence directly.

D. Applications to Some Real Biochemical Reaction Networks.

In this section, we will use the proposed results to derive the persistence of biochemical reaction systems. These systems are all composed of $1d$ $\mathcal{W}_I$-endotactic networks but some are not endotactic networks that are studied before, like the first model of Examples 9 and 10.

Example 9: Now we consider the PAK-1 network, which plays a vital role in reducing the expression of proteins involved in cancer proliferation and progression such as E-cadherin, CXCR-4 [23].

$$\left.\begin{array}{c|c} \mathcal{M}: & \mathcal{M}^{(1)}: \\ 2E \xrightarrow{k_1} E + EP, & 2E \xrightarrow{k_1} E + EP, \\ E \xleftarrow{k_2} EP \underset{k_3}{\overset{k_4}{\rightleftharpoons}} EPP. & EP \xrightarrow{k_2} E, \\ & \mathcal{M}^{(2)}: \\ & EP \underset{k_3}{\overset{k_4}{\rightleftharpoons}} EPP \end{array}\right\} \mathcal{S}_c = \{EP\}.$$

We can see both $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ are $1d$ $\mathcal{W}_I$-endotactic systems. The intersecting species EP does not belong to any of the semilocking sets except the species set $\mathcal{S} = \{E, EP, EPP\}$. Thus, the system $\mathcal{M}$ is persistent from Theorem 4.6.

If the reaction vector of each reaction remains unchanged but the reaction rates of the reactions in $\mathcal{M}$ are $k_1 x_E^2, k_2 x_{EP}^2, k_3 x_{EP}^2$ and $k_4 x_{EPP} x_{EP}$, respectively. Then, the PAK-1 system can be expressed by

$$\mathcal{M}: \begin{array}{c} 2E \xrightarrow{k_1} E + EP, \\ E + EP \xleftarrow{k_2} 2EP \underset{k_4}{\overset{k_3}{\rightleftharpoons}} EPP + EP. \end{array}$$

It is composed of the following two subsystems:

$$\begin{aligned} \mathcal{M}^{(1)}: \quad & 2E \xrightarrow{k_1} E + EP, \\ & 2EP \xrightarrow{k_2} E + EP \\ \mathcal{M}^{(2)}: \quad & 2EP \underset{k_4}{\overset{k_3}{\rightleftharpoons}} EPP + EP. \end{aligned}$$

In this case, the set of intersection species is also $\mathcal{S}_c = \{EP\}$. It is contained in the semilocking set $\{E, EP\}$ with $E \notin \mathcal{S}_c$ and $EP \in \mathcal{S}_c$. Also, E, EP satisfy $b_E^{(1)} \cdot b_{EP}^{(1)} < 0$ where $b^{(1)} = (-1, 1, 0)^\top$. Lemma 4.11 and Theorem 4.12 reveal that the second model of PAK-1 is persistent too. Of course, this version is also an endotactic network, thus its persistence can be achieved by the existing results in [16].

Example 10: The following system is an autocatalytic system that can be applied in several interdisciplinary fields, such as a high-density model for decision-making processes in swarms of agents and ants [24], [26]

$$\mathcal{M}^{(1)}: \quad \begin{array}{ccc} & & 2S_1, \\ & \overset{k_5}{\nearrow} & \\ S_1 + S_2 & & \\ & \underset{k_6}{\searrow} & \\ & & 2S_2, \end{array}$$

$$S_1 \underset{k_2}{\overset{k_1}{\rightleftharpoons}} S_2,\ 2S_1 + S_2 \xrightarrow{k_3} 3S_1,\ S_1 + 2S_2 \xrightarrow{k_4} 3S_2. \tag{31}$$

This system is a $1d$ $\mathcal{W}_I$-endotactic system. The following system is a nonweakly-reversible system with deficiency 1 and also a $1d$ $\mathcal{W}_I$-endotactic system

$$\begin{aligned} \mathcal{M}^{(2)}: & S_1 + S_3 \underset{k_2}{\overset{k_7}{\rightleftharpoons}} 2S_3 \\ & mS_1 + 2S_3 \xrightarrow{k_9} (m+1)S_1 + S_3. \end{aligned} \tag{32}$$

The system $\mathcal{M}^{(2)}$ corresponds to a slight modification of motif F of [25]. When we combine subsystems $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$, the intersecting species set $\mathcal{S}_c = \{S_1\}$ does not belong to any nontrivial semilocking set, so the combined system $\mathcal{M}$ is also persistent from Theorem 4.6. This result is very important for biochemical networks, as the purpose of synthetic biology is to combine submodules without affecting the properties of submodules.

V. Conclusion

This article defines a class of new networks called $\mathcal{W}_I$-endotactic networks, which include common weakly reversible networks, endotactic networks of special geometric structure, etc., to reach the persistence of CRNs. Through the energy function available for all of $1d$ networks, we prove any $1d$ $\mathcal{W}_I$-endotactic network is of bounded trajectory and has persistence. Based on these, we construct higher dimensional networks that are composed of a series of $1d$ $\mathcal{W}_I$-endotactic networks and prove them to be also of $\mathcal{W}_I$-endotactic structure. Furthermore, we discuss the persistence of the combined higher dimensional $\mathcal{W}_I$-endotactic networks according to two cases for subnetworks with and without intersecting species. Some sufficient conditions are developed for each case. The results, on the one hand, enrich greatly the networks to support persistence; on the other hand, they suggest a challenging way how to combine without destroying the original properties of subnetworks in practical applications. For example, in synthetic biology, biologists often need to use a modular approach when faced with the design of complex biochemical systems. The ideal state is that when these modules are combined, their properties will not change.

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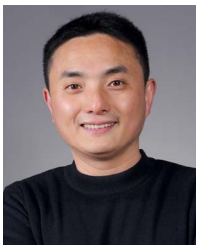

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