# Competition for market share or for market size: oligopolistic equilibria with varying competitive toughness*

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February 14, 2003

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*We are grateful to David Encaoua and to Yves Smeers for their useful comments and suggestions.

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# 1 Introduction

A fundamental problem of oligopoly theory is equilibrium indeterminacy. This indeterminacy is not only of the kind associated with a given equilibrium concept and the possibility of multiple solutions. It is mainly related to the choice of the equilibrium concept itself. This is most simply demonstrated in static models by the choice between quantity competition (usually assimilated to the approach of Cournot, 1838) and price competition (linked to Bertrand's 1883 critique of Cournot). However, this second kind of indeterminacy[1] is not to be seen as the failure of the theory, but rather as reflecting the variety of observed regimes of oligopolistic competition, with varying degree of toughness resulting from firms more or less coordinated behavior. Certainly this variety of regimes cannot be reduced to the dichotomous choice between Cournot and Bertrand.

Moreover each static concept itself (Cournot, Bertrand, or else) may be seen as the reduced form of various industrial situations, involving different dynamic settings and different norms of conduct for the firms. A good example of this multiplicity is the variety of structural and behavioral justifications for Cournot outcome. It may be based for instance on the way production and sales are organised in time. As well illustrated[2] by Kreps and Scheinkman (1983) for a symmetric duopoly, if firms fix their production (or capacity) levels in advance and then compete in prices, the Cournot outcome obtains as the only subgame perfect equilibrium. Cournot outcome may also be associated with particular "facilitating practices" in the selling policy of firms. If, for example, firms are supposed to use best-price policies ("meet-or-release" and "most-favored-customer" clauses) together with advance notification of list price changes and the possibility of discounts below list prices, then Holt and Scheffman (1987) show that Cournot price is the highest equilibrium price[3]. But there are other ways to select the Cournot outcome. As known since Bowley (1924), it can be obtained by assuming appropriate conjectures in the conjectural variation approach, or, as we will see again below, it can be obtained as a particular supply function equilibrium when assuming (as in Grossman, 1981, and Hart, 1982) that firms compete in supply functions. And the latter approach can be explained by having the owners of the firms designing particular types of incentive contracts for their managers as functions of profitability and sales[4].

[1] This kind of indeterminacy is sometimes identified as "the" oligopoly problem (see Vives, 1999, ch.7).

[2] Their result depends on the choice of a rationing rule for the case where demand is rationed. With other rationing rules, other outcomes may be obtained (Osborne and Pitchik, 1986).

[3] This example amounts to suppose that firms compete both in (announced) prices and in quantities, and that the adjustment of announced prices to the "market price" can be represented by the "min-pricing scheme" (which is only responsive downward). If a pricing scheme which is fully responsive (both downward and upward) is used, then only Cournot outcomes remain (see d'Aspremont, Dos Santos Ferreira and Gérard-Varet, 1991).

[4] See Vickers (1985), Fershtman and Judd (1987) and Sklivas (1987) and the discussion in Vives (1999, chap. 7).

The property for a static model of competition to correspond to different dynamic settings and different norms of conducts (including different types of conjectures and commitments) should hold for other competition regimes than the Cournot one. However, we do not pursue this investigation here but, instead, sticking to a static model of competition, we propose a comprehensive concept of oligopolistic equilibrium allowing for a parametrized continuum of competition regimes. This concept remains in the line of Cournot by considering oligopolistic competition as a generalization of monopoly, with firms setting simultaneously both prices and quantities constrained by market demand (which, unlike Cournot, we require to be fully satisfied only at equilibrium), and also constrained by their rivals' anticipated behavior[5] (implying, as in Cournot, the "necessity" that the selling price be the same for all firms when the good is homogeneous). Hence, each firm, while maximizing its profit both in price and in quantity, is supposed to face two constraints, one relative to the size of the market, the other relative to its share of that market. The type of solution (*i.e.* the competition regime) will vary according to the values of the Lagrange multipliers associated with each of these two constraints. The proposed parametrization, interpreted in terms of competitive toughness, is based on these values.

The set of oligopolistic equilibria will be shown to include the Cournot solution at one extreme, when competition is extremely soft, as well as the competitive equilibrium at the other extreme, when competition is extremely tough. In the homogeneous case, the intermediate competititon regimes will be characterized as supply function equilibria, with firms strategy spaces restricted to the set of non-decreasing supply functions. Alternatively, the set of oligopolistic equilibria outcomes will be identified as a selected subset of the outcomes obtained by conjectural variations of a particular type (the compensating ones).

The proposed concept of oligopolistic equilibrium is applicable to an industry supplying a group of differentiated products, the homogeneous product being the limit case corresponding to perfect substitutability. These differentiated products are aggregated into a composite commodity[6], in a similar but more general framework than the popular Dixit-Stiglitz (1977) and Spence (1976a, b) one. Products in such an industry are not necessarily substitutes (perfect complementarity[7] being actually another limit case). This applies to many man-

[5]Cournot (1838) adopted an approach in terms of quantities mainly for computational convenience. For him, each producer acts as a monopolist, maximizing both in price and in quantity, but facing a 'residual' demand. His main behavioral assumptions are that a firm can control the quantity it is able to sell by adjusting its price, while anticipating that the other producers will manage to keep their sales by appropriate price adjustments, and knowing that "the price is necessarily the same" for all producers. Bertrand's (1883) short and informal critique is mainly oriented against this last behavioral assumption (consisting in ignoring the possibility of undercutting and in taking as given the others' saleable quantities), and ends up describing a price war.

[6]See Gorman (1959) and Green (1964, ch.4).

[7]This corresponds to Cournot's complementary monopoly (Sonnenschein, 1968). Sonnenschein uses the expression "complementary monopoly" for Cournotian (price) competition between firms producing perfectly complementary goods. Cournot himself used the expression *concours des producteurs* (producers' "conjunction", ch.IX), by contrast with *concurrence*

ufactured good industries, such as the automobile and electronic industry, if we include maintenance, parts and complementary equipments, as well as to many service industries (*e.g.* hotel bookings and trip organizations in the travel industry). Of course, the present paper does not treat product differentiation in its full generality (horizontal and vertical) and remains in a strict partial equilibrium setting. However, a comprehensive concept of oligopolistic equilibrium, encompassing a large family of competition regimes, can be seen as a building block to be used in a more general, multi-sector setting.

To formalize, in a unified and synthetic form, such a large family of competition regimes, parametrized by their degree of toughness, is also meant to provide a more meaningful and convenient tool to address some relevant economic and policy issues. That the degree of competition may be of importance in studying economic fluctuations was already stressed informally by some authors in the Thirties when concerned by imperfect competition and the trade cycle (*e.g.* Abramovitz, 1938, Kalecki, 1938, Sweezy, 1939). The possibility of studying formally the variation of competitive toughness along the cycle should allow to test some of their claims.[8] More recently, the postulated competition regime appears often to be crucial for the conclusion reached in the industrial organization and international trade literature. Well-known examples are the issues of strategic R&D investment and strategic trade policy (*e.g.* Brander and Spencer, 1983, Eaton and Grossman, 1986), where conclusions (such as the national benefit of subsidies to exports or investment) might be quite sensitive to the assumed competitive toughness. Another example, presently much debated both from an empirical and from a theoretical point of view[9], and analyzed by combining industrial organization models of innovation with endogenous growth theory, is the impact of competition on innovation and growth: whether these are enhanced by the intensity of competition (the "Darwinian view") or by expected monopoly rents (the "Schumpeterian view").

In recent formal models, the variability in the intensity of competition is approached along several dimensions.[10] A classical dimension, associated with Cournot, is specified by the number of competitors, more competitors entailing more intense competition. This dimension can be refined by referring to various concentration indices. Another standard dimension used to parametrize competition in a differentiated product industry is the degree of substitutability between the products. In this paper, we shall focus on still another dimen-

*des producteurs* (producers' competition, ch.VII) in the homogeneous oligopoly case.

[8]Markup countercyclicality has been presented by recent macroeconomic contributions as one of the stylized facts characterizing economic fluctuations. Changes in markups do not necessarily result from variations of competitive toughness, but the idea that the incentives to break tacit collusive agreements tend to move procyclically has for instance been exploited in the framework of repeated games (Rotemberg and Saloner, 1986, Rotemberg and Woodford, 1992).

[9]There is a large recent literature on this issue. See Grossman and Helpman (1991), Aghion, Harris and Vickers (1997), van de Klundert and Smulders (1997), Aghion, Dewatripont and Rey (1999), Aghion, Harris, Howitt and Vickers (2001), Encaoua and Ulph (2001), etc.

[10]Boone (2000) and (2001) are attempts to axiomatize a general measure of intensity of competition, incorporating such dimensions.

sion, along which different competition regimes can be ordered according to the competitive toughness of the participants. Usually, this dimension is treated as a binary parameter, indicating either the Cournot or the Bertrand regimes. An exception is the conjectural variations approach, which introduces for each firm a continuous parameter to represent its competitors' conjectured reactivity. Conjectural variations have been criticized for their lack of game-theoretic foundations,[11] although the corresponding outcomes can be isomorphically associated with well-defined supply function equilibria, and for their lack of determinacy (every individually rational outcome being supported by some conjectural variations).[12] The concept of oligopolistic equilibrium proposed here, providing an alternative parametrization of competitive toughness, is based on simpler strategies (price-quantity pairs), leads to a natural narrowing down of the set of outcomes obtained by supply function equilibria, and lends itself to a straightforward extension to the composite good case.

In section 2 we define the concept of oligopolistic equilibrium for an industry producing a single homogeneous good. We then characterize the concept by introducing a measure of competitive toughness, and compare it with already existing alternatives, namely conjectural variations and supply function equilibria. In section 3, we extend the concept to an industry producing a composite commodity, and analyze in more detail the standard case of a CES aggregator. We briefly conclude in section 4.

# 2 A comprehensive concept of oligopolistic equilibrium: the homogeneous good case

We begin by considering the homogeneous good case. We take an industry consisting in $n$ firms ($n > 1$), each firm $i$ producing the same good with a technology that is described by a non-decreasing cost function $C_i$, with $C_i(0) = 0$ and continuously differentiable on $(0, \infty)$.

## 2.1 Definition and characterization

In our approach, neither the price nor the quantity will be privileged as a strategic variable. Each firm $i$ is supposed to choose, simultaneously with its competitors, an *offer* consisting in a pair $(p_i, q_i)$, stating the price $p_i$ at which it accepts to sell a proposed quantity $q_i$. It does so in order to maximize its profit under two feasibility conditions. The first is that its price cannot be higher than the minimum price offered by its competitors, so that $p_i = \underline{P}(p_i, p_{-i}) \equiv \min_j \{p_j\}$. The second is that the total quantity supplied $\overline{Q}(q_i, q_{-i}) = \sum_j q_j$ cannot exceed the aggregate demand $D(\underline{P}(p_i, p_{-i}))$, where $D$ is a decreasing, continuously differentiable function. The two constraints correspond to two different dimensions of competition. The first one is related to competition within the industry, since

[11] See for example Shapiro (1989).
[12] See, e.g., Vives (1999).

it bounds the price that each firm can set, given the prices chosen by its immediate competitors. The second relates to competition with respect to the rest of the economy (which is not explicitly considered in the present partial equilibrium approach): it bounds total supply by market size, as determined by the price level in the industry. Combining these two constraints amounts to define a kinked feasibility frontier, or a kinked demand,[13] restricting firm choices. Also notice that we assume voluntary trading: out of equilibrium, we do not impose that market demand be entirely satisfied.

To be precise, let us introduce the concept of *oligopolistic equilibrium* as a $2n$-tuple $(p^*, q^*)$ which is a Nash equilibrium of the static game defined by having each firm strategies in $\mathbb{R}_+^2$ and payoffs $\Pi_i$ specified as follows:

$$\begin{aligned} \Pi_i(p,q) &= p_i q_i - C_i(q_i), \\ \text{if } p_i &= \underline{P}(p_i, p_{-i}) \text{ and } \overline{Q}(q_i, q_{-i}) \leq D\left(\underline{P}(p_i, p_{-i})\right), \\ \Pi_i(p,q) &= 0, \text{ otherwise,} \end{aligned} \tag{1}$$

and satisfying in addition

$$\overline{Q}(q^*) = D\left(\underline{P}(p^*)\right). \tag{2}$$

This last requirement is meant to exclude rationing of consumers at equilibrium, and accordingly to eliminate irrelevant equilibria.

Let us derive the first order conditions for firm $i$ program. As in Cournot (1838), "it will be convenient to adopt the inverse notation" $D^{-1}$ for inverse demand (which is well-defined by assumption). At equilibrium $(p^*, q^*)$, the program can be equivalently written:

$$\begin{aligned} & \max_{(p_i, q_i) \in \mathbb{R}_+^2} \{p_i q_i - C_i(q_i)\} \\ \text{s.t. } p_i &\leq \min_{j \neq i} \left\{p_j^*\right\} \text{ and} \\ \underline{P}\left(p_i, p_{-i}^*\right) &\leq D^{-1}\left(\overline{Q}\left(q_i, q_{-i}^*\right)\right). \end{aligned} \tag{3}$$

We get the first order conditions[14]:

$$\begin{bmatrix} q_i^* \\ p_i^* - C_i'(q_i^*) \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \nu_i \begin{bmatrix} 1 \\ -1/D'\left(\underline{P}(p^*)\right) \end{bmatrix} \tag{4}$$

for some couple of Kuhn and Tucker multipliers $(\lambda_i, \nu_i) \in \mathbb{R}_+^2 \setminus \{0\}$. These multipliers are respectively associated with each constraint, the one directly generated by the competitors prices and the one generated by the market size. They can be interpreted as the (shadow) costs firm $i$ would accept to bear

[13] Of course, while related to the classical kinked demand model (Sweezy, 1939), the kink does not result here from producer's asymmetric conjectures about the competitors' reactions to upward and downward price changes.

[14] In establishing these conditions, and considering the discontinuity of the derivative of $\underline{P}(p^*)$, we have to use the left-hand derivative $\partial_i^- \underline{P}(p^*) = 1$. Indeed, when the first constraint is binding, the price can only be decreased.

in order to relax each constraint respectively, and hence to ease the pressure coming from its competitors within the industry, or coming from outside the industry. The normalized parameter $\theta_i \equiv \lambda_i / (\lambda_i + \nu_i)$ can thus be interpreted as the *competitive toughness* of firm $i$ within the industry.

Putting aside equilibria with inactive firms (so that $y_i^* > 0$, and hence $p_i^* = \underline{P}(p^*)$, for any $i$), we can express the degree of monopoly of each firm $i$, at equilibrium $(p^*, q^*)$, as a function of $\theta_i$:

$$\frac{p_i^* - C_i'(q_i^*)}{p_i^*} = (1 - \theta_i) \frac{q_i^* / \overline{Q}(q^*)}{-\epsilon D(\underline{P}(p^*))} \equiv \mu_i(\theta_i, \underline{P}(p^*), q^*), \tag{5}$$

where $\epsilon$ is the elasticity operator. We shall not examine here the second order conditions or, more generally, sufficient conditions for existence. They are weaker than conditions ensuring the existence of standard concepts, such as Cournot or the competitive solution, since these solutions are included as particular instances of oligopolistic equilibria. Besides, curvature conditions are somewhat softened by the shape of the set of admissible strategies, which is bounded by a kinked frontier.

Just assuming sufficiency of the first order conditions, we have a system of $n+1$ equations

$$\mu_i(\theta_i, P^*, q^*) = \frac{P^* - C_i'(q_i^*)}{P^*} \ (i = 1, ..., n) \text{ and } \overline{Q}(q^*) = D(P^*) \tag{6}$$

in $n+1$ unknowns $(P^*, q^*)$, parametrized by $\theta = (\theta_1, ..., \theta_n)$ in the set $\Theta^* \subset [0,1]^n$ of the parameter values entailing a solution of the system (6). We thus obtain a parametrization by $\theta$ of the set of oligopolistic equilibria.

Let us establish the relationship between our oligopolistic equilibrium and other concepts.[15] First, recall that in the case of Cournot equilibrium the first order conditions are:

$$\frac{p_i^* - C_i'(q_i^*)}{p_i^*} = \frac{q_i^* / \overline{Q}(q^*)}{-\epsilon D(\underline{P}(p^*))} \equiv \mu_i(0, \underline{P}(p^*), q^*), \text{ for all } i.$$

Thus, the first order conditions for an oligopolistic equilibrium with minimal competitive toughness for all firms correspond to the Cournot equilibrium first order conditions. If the latter conditions are also sufficient, then the Cournot equilibrium can be realized as an oligopolistic equilibrium. Formally, let $q^*$ be a Cournot equilibrium and let $\overline{p}^* = D^{-1}(\sum_i q_i^*)$ be the corresponding price. It is easy to see that $((\overline{p}^*, ..., \overline{p}^*), q^*)$ is an oligopolistic equilibrium. If that were not the case, some firm $i$ would be able, through some choice $(p_i, q_i)$, to get a profit $p_i q_i - C_i(q_i) > \overline{p}^* q_i^* - C_i(q_i^*)$. But the two constraints in (3) require that $\min\{p_i, \overline{p}^*\} \leq D^{-1}\left(q_i + \sum_{j \neq i} q_j^*\right)$ and $p_i \leq \overline{p}^*$. Hence, $D^{-1}\left(q_i + \sum_{j \neq i} q_j^*\right) q_i - C_i(q_i) \geq p_i q_i - C_i(q_i) > D^{-1}(\sum_i q_i^*) q_i^* - C_i(q_i^*)$, contradicting the assumption that $q^*$ is a Cournot equilibrium.

[15] For a synthetic presentation of these concepts, see Vives (1999).

Taking, for some firm $i$, the other extreme, that is, a maximal competitive toughness ($\theta_i = 1$), we see from (5) that firm $i$ equalizes marginal cost to price, i.e. that it has a competitive behavior. As an implication, if a competitive equilibrium exists, the competitive outcome can be obtained as an oligopolistic equilibrium with maximal toughness in competition for all firms, provided the first order conditions are sufficient, assuming cost convexity. As far as the Bertrand symmetric solution is concerned, it is covered by the same argument under the assumption of identical linear cost functions: $C_i(q_i) = cq_i$ for any $i$.

In order to cover the intermediate cases, we may start by referring to the well-known "conjectural variations" approach (Bowley, 1924): each firm $i$ when choosing its quantity is supposed to make a specified type of conjecture concerning the reaction of the other firms to any of its deviations. Following the presentation in Dixit (1986), a sufficient specification[16] consists in introducing conjectural variation terms $r_i = \sum_{j \neq i} \partial q_j / \partial q_i$ for each $i$. Looking at the corresponding first order conditions, one obtains:

$$\frac{P^* - C_i'(q_i^*)}{P^*} = (1 + r_i) \frac{q_i^* / \overline{Q}(q^*)}{-\epsilon D(P^*)}, \tag{7}$$

giving the same characterization as (5) with $-r_i = \theta_i$. In other words, the set of oligopolistic equilibria outcomes appears as a selected subset of the outcomes obtained by conjectural variations when restricting these variations to be compensating, i.e. $r_i$ to be in the interval[17] $[-1, 0]$, for every $i$. The concept of oligopolistic equilibrium thus provides some game-theoretic foundation for the concept of conjectural variations, since the conjectural variation terms (within the relevant class) can be identified to the parametrization of the equilibria of a fully specified game.

This foundation is even more simple and direct than the foundation provided by the concept of supply function equilibrium to which we now turn.

## 2.2 Relation to the concept of supply function equilibrium

A supply function equilibrium (Grossman, 1981 and Hart, 1982)[18] is a Nash equilibrium of a game where each firm $i$ strategies are *supply functions*, namely functions $S_i$ associating with every price $p_i$ in $[0, \infty)$ a quantity $q_i = S_i(p_i)$. The functions $S_i$ may be restricted to some admissible set $\mathbf{S}$. For any $n$-tuple $S$ of supply functions in $\mathbf{S}^n$, the price $\hat{P}(S)$ is said to be well-defined if it is nonnegative and uniquely solves the equation

$$\sum_{j=1}^{n} S_j(P) = D(P). \tag{8}$$

[16] Dixit considers $r_i$ as a function of both $q_i$ and $\sum_{j \neq i} q_j$.

[17] Matching variations ($r_i > 0$) are excluded, and in particular the collusive solution.

[18] See also Singh and Vives (1984).

The corresponding payoffs are defined by:

$$\begin{aligned}\Pi_i^s(S_i, S_{-i}) &= \hat{P}(S_i, S_{-i}) S_i\left(\hat{P}(S_i, S_{-i})\right) - C_i\left(S_i\left(\hat{P}(S_i, S_{-i})\right)\right), \quad (9)\\ &\quad \text{if } \hat{P} \text{ is well-defined}\\ \Pi_i^s(S_i, S_{-i}) &= 0, \text{ otherwise.}\end{aligned}$$

Observe that at an equilibrium $S^*$, for any firm $i$, maximizing $\Pi_i^s\left(S_i, S_{-i}^*\right)$ amounts to select $P^*$ in

$$\arg\max_{P\in\mathbb{R}_+}\left\{PD^s\left(P, S_{-i}^*\right) - C_i\left(D^s\left(P, S_{-i}^*\right)\right)\right\}, \quad (10)$$

with $D^s\left(P, S_{-i}^*\right) = \max\left\{D(P) - \sum_{j\neq i} S_j^*(P), 0\right\}$ the residual demand function. In fact firm $i$ could choose any supply function $S_i$ for which $S_i(P) = D^s\left(P, S_{-i}^*\right)$ has the unique solution $P^*$.

The multiplicity of supply function equilibria is well-known. In order to compare this concept with our own, we shall[19] restrict strategies to the set $\mathbf{S}_+$ of *non-decreasing supply functions*. We then have the following characterization:

**Proposition 1** *For any supply function equilibrium $S^* \in \mathbf{S}_+^n$, there is an oligopolistic equilibrium $(p^*, q^*)$ such that $q^* = S^*\left(\hat{P}(S^*)\right)$ and, for any $j$, $p_j^* = \hat{P}(S^*)$. Conversely, for any oligopolistic equilibrium $(p^*, q^*)$, there is a supply function equilibrium $S^* \in \mathbf{S}_+^n$ such that $S^*\left(\underline{P}(p^*)\right) = q^*$.*

***Proof.*** *Let $S^* \in \mathbf{S}_+^n$ be a supply function equilibrium. We show that $(p^*, q^*)$ with $q^* = S^*\left(\hat{P}(S^*)\right)$ and, for any $j$, $p_j^* = \hat{P}(S^*) = \underline{P}(p^*)$, is an oligopolistic equilibrium. Using, for any $i$, the fact that the residual demand $D^s\left(p_i, S_{-i}^*\right)$ is decreasing and that the profit $p_i q_i - C_i(q_i)$ is increasing in $p_i$ we get, by (10), that $(p_i^*, q_i^*)$ maximizes $p_i q_i - C_i(q_i)$ on*

$$\Delta_i \equiv \left\{(p_i, q_i) \in \mathbb{R}_+^2 \mid q_i \leq D^s\left(p_i, S_{-i}^*\right)\right\}.$$

*For $(p^*, q^*)$ to be an oligopolistic equilibrium, $(p_i^*, q_i^*)$ should maximize $p_i q_i - C_i(q_i)$ on*

$$\hat{\Delta}_i \equiv \left\{(p_i, q_i) \in \mathbb{R}_+^2 \mid p_i \leq \underline{P}(p^*), q_i \leq \max\left\{D(p_i) - \sum_{j\neq i} q_j^*, 0\right\}\right\},$$

*for every $i$. Since, by construction, $(p^*, q^*) \in \hat{\Delta}_i$ and $\hat{\Delta}_i \subset \Delta_i$, this is clearly true.*

*To prove the converse, let us suppose that $(p^*, q^*)$ is an oligopolistic equilibrium. We may construct an associated supply function equilibrium by using simple supply functions, namely by letting, for every $i$, $S_i^*(p_i) = q_i^*$ if $p_i \leq p_i^*$, and $S_i^*(p_i) = \infty$ otherwise. Clearly, the solution to (10) cannot be larger than $\underline{P}(p^*)$, hence any profitable deviation by some firm $i$ from $S_i^*$ must determine a price below $\underline{P}(p^*)$ and thus constitute a deviation with respect to the oligopolistic equilibrium. The result follows.* ■

[19] As Delgado and Moreno (2000) do, but assuming identical firms.

We see that, with the restrictions imposed on the admissible class of supply functions, the outcomes corresponding to the two sets of equilibria coincide, including the Cournot and the competitive outcomes.

To establish more precisely the relation, we may consider the first order condition to firm $i$ program (10) at equilibrium, while restricting to differentiable supply functions in $\mathbf{S}_+$. Denoting $q_i^* = D^s\left(P^*, S_{-i}^*\right)$ and $\overline{Q}\left(q^*\right) = \sum_{i=1}^n q_i^*$, we get:

$$q_i^* + \left(D'\left(P^*\right) - \sum_{j \neq i} S_j^{*\prime}\left(P^*\right)\right)\left(P^* - C_i'\left(q_i^*\right)\right) = 0, \tag{11}$$

which is equivalent to

$$\frac{P^* - C_i'\left(q_i^*\right)}{P^*} = \frac{q_i^*/\overline{Q}\left(q^*\right)}{\sum_{j \neq i}\left(q_j^*/\overline{Q}\left(q^*\right)\right)\epsilon S_j^*\left(P^*\right) + \left[-\epsilon D\left(P^*\right)\right]} = \mu_i\left(\theta_i, P^*, q^*\right), \tag{12}$$

with $\mu_i$ as defined in (5), taking

$$\theta_i = \frac{\sum_{j \neq i}\left(q_j^*/\overline{Q}\left(q^*\right)\right)\epsilon S_j^*\left(P^*\right)}{\sum_{j \neq i}\left(q_j^*/\overline{Q}\left(q^*\right)\right)\epsilon S_j^*\left(P^*\right) + \left[-\epsilon D\left(P^*\right)\right]}. \tag{13}$$

Not surprisingly, the Cournot solution corresponds to an elasticity $\epsilon S_j^*\left(P^*\right)$ of the supply functions equal to 0, for all $j$, and the competitive solution to $\epsilon S_j^*\left(P^*\right) = \infty$, for at least two $j$'s. Notice also that varying the elasticities of the supply functions in the relevant class allows us to cover the whole range of admissible values of the $\theta_i$. The competitive toughness of firm $i$, as measured by $\theta_i$ at the oligopolistic equilibrium, is seen to correspond positively to a measure of the "reactivity of the other firms" as anticipated by firm $i$ at the supply function equilibrium. Although the elasticity of a supply function chosen by any firm is indifferent from the point of view of the firm itself (for which only the price-quantity pair actually implemented matters), it is crucial in shaping the anticipations of its competitors.

## 2.3 An example

In order to illustrate the concept of oligopolistic equilibrium, and visualize the set of equilibrium outcomes as parametrized by $\theta \in \Theta^*$, we refer to a simple (but non-symmetric) example of homogeneous duopoly. We suppose that each firm $i$ has a constant marginal cost $c_i$ (by convention, $c_1 \leq c_2$), and that consumers allocate to the sector a constant budget, normalized to one at the aggregate level, so that $D\left(P\right) = 1/P$.

From condition (5), we see that, at equilibrium $(p^*, q^*)$, with $p_1^* = p_2^* = P^*$,

$$\frac{P^* - c_1}{P^*} = \left(1 - \theta_1\right)\frac{q_1^*}{q_1^* + q_2^*},$$

so that firm 1 market share is

$$\frac{q_1^*}{q_1^* + q_2^*} = \frac{1}{1-\theta_1}\frac{P^* - c_1}{P^*}. \tag{14}$$

Since the right-hand side is increasing in both $P^*$ and $\theta_1$, we get a family of increasing curves in the space $(P^*, q_1^*/(q_1^* + q_2^*))$ parametrized by $\theta_1$, the higher curves corresponding to larger values of the parameter. The same market share can also be expressed by referring to firm 2 first order condition:

$$\frac{q_1^*}{q_1^* + q_2^*} = 1 - \frac{1}{1-\theta_2}\frac{P^* - c_2}{P^*}, \tag{15}$$

which defines a family of decreasing curves in the same space, parametrized by $\theta_2$, the higher curves corresponding to smaller values of the parameter.

The two first order conditions together imply that

$$\begin{aligned} c_2 &\leq P^* \leq c_1 + c_2, \text{ and} \\ 1 - \frac{c_1}{P^*} &\leq \frac{q_1^*}{q_1^* + q_2^*} \leq \frac{c_2}{P^*}. \end{aligned}$$

The lower and upper bounds on $P^*$ appear to be the Bertrand and Cournot equilibrium prices. Also, the lower and upper bounds on firm 1 market share correspond respectively to $\theta_1 = 0$ and $\theta_2 = 0$.

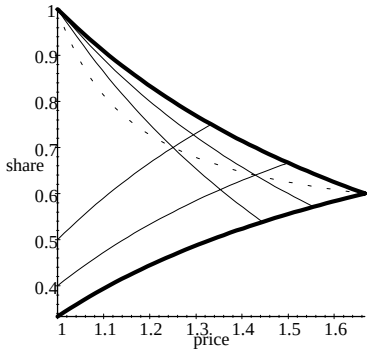


Figure 1

Figure 1 shows, for parameter values $c_1 = 2/3$ and $c_2 = 1$, the set of equilibrium outcomes in the market price-market share space $(P^*, q_1^*/(q_1^* + q_2^*))$ as the region bounded by the vertical axis and the two thick curves, the lower one corresponding to $\theta_1 = 0$, the upper one to $\theta_2 = 0$. With each point in this region we associate a unique pair $(\theta_1, \theta_2) \in [0,1]^2$. The increasing solid curves (including the thick one) are three elements of the family of curves stemming from firm 1 first order condition (14) (with $\theta_1 = 1/3$, $1/6$, 0, respectively, from left to right). Similarly, the decreasing solid curves are three elements, corresponding to the same values of $\theta_2$, of the family of curves stemming from firm 2

first order condition (15). Fundamentals (demand and cost functions) are kept constant all along.

Consider now the (standard) case of uniform competitive toughness, *i.e.* with equal parameter values: $\theta_1 = \theta_2 = \bar{\theta}$. This is given by the "dotted" curve, which links the Cournot outcome (determined at the point of intersection of the two "thick" curves with $\bar{\theta} = 0$) to the Bertrand outcome (associated with the highest market share and the highest value of $\bar{\theta}$). We observe that, as $\bar{\theta}$ increases, the market expands and splits more and more asymmetrically in favor of the firm with a cost advantage. Considering the effects on profits, some additional properties[20] can be deduced in this case of uniform competitive toughness:

- the least efficient firm loses as competitive toughness increases (both its market share and the price decrease);
- if the cost advantage of the most efficient firm is large enough, its profit increases as competitive toughness increases (a higher market share more than compensates a lower price);
- if costs are equal for both firms (implying equal market shares), profits decrease as competitive toughness increases.

In the case where competitive toughness is asymmetric, effects are more intricate. If the value of $\theta_i$ is kept unchanged and $\theta_j$, $j \neq i$, increases, both firm $i$ market share and the price fall, implying a lower profit for firm $i$. However, an increase in the value of $\theta_i$ while keeping $\theta_j$, $j \neq i$, constant, is associated with a larger market share for firm $i$ but with a lower price, so that an increase in its profit is not guaranteed.

# 3 Extension to the composite good case

The purpose now is to extend the concept of oligopolistic equilibrium to a situation of product differentiation. More precisely, we consider an industry producing a composite commodity, and consisting in $n$ firms ($n > 1$), each firm $i$ producing a single component of the composite product with a technology that is described by a cost function $C_i$ with the same properties as in the preceding section.

## 3.1 Demand to the industry and demand to the firm

We suppose that any basket $x \in \mathbb{R}^n_+$ of industry products, viewed as elements of a composite good, can be aggregated into a quantity $Q(x)$ of that good. The aggregator $Q$ is assumed to be a twice continuously differentiable, increasing,

[20] These properties correspond to properties (b), (c) and (d) in Definition 2.2 of Boone (2001), where the "intensity of competition" parameter is supposed to be the same for all firms. Cournot and Bertrand regimes, but not the intermediate cases, are considered in an example. In Boone (2000), the parameter measuring "competitive pressure" is differentiated but his examples concentrate on comparative statics when the fundamentals vary.

strongly quasi-concave[21] function, which is homogeneous of degree 1. A well-known example of this class is the CES aggregator, introduced by Dixit and Stiglitz (1977) and Spence (1976a,b).

Any demander $h$ for the composite good, whether a household or a downstream firm, minimizes at prices $p = (p_1, ..., p_n)$ the cost $\sum_i p_i x_{hi}$ of a basket $x_h = (x_{h1}, ..., x_{hn})$ under the constraint $Q(x_h) \geq X_h$, where $X_h$ is an argument of its utility or production function. The cost function, associating the minimal expenditure with the price vector $p$ and the quantity $X_h$, can be written as $P(p) X_h$, where $P$ is a price aggregator with the same properties as $Q$. There are two important limit cases for the aggregators, strictly speaking outside the framework of this section (because of loss of regularity and differentiability). The first one is the homogeneous commodity case, already analyzed, where the quantity and price aggregators can be respectively defined as $\overline{Q}(x) = \sum_i x_i$ and $\underline{P}(p) = \min_i \{p_i\}$. The second one is the Leontief case of perfect complements, where the quantity and price aggregators are respectively given by $\underline{Q}(x) = \min_i \{x_i\}$ and $\overline{P}(p) = \sum_i p_i$.

We shall in the following use the first order condition for cost minimization $p_i = P(p) \partial_i Q(x_h)$ (where $\partial_i$ denotes the $i$-th partial derivative operator) and, by Shephard's lemma, the dual condition $x_{hi} = Q(x_h) \partial_i P(p)$. By the latter condition, and through aggregation over all $h$'s, we obtain multiplicative separability of the aggregate *demand to firm* $i$, $d_i(p) = \partial_i P(p) \sum_h Q(x_h)$, allowing identification of two components: the *market share*

$$\frac{d_i(p)}{\sum_h Q(x_h)} = \partial_i P(p), \tag{16}$$

and the *market size*, as determined by the *demand to the industry* $D$

$$\sum_h Q(x_h) = D(P(p)), \tag{17}$$

where $D$ is assumed to have the same properties as in the previous section.

## 3.2 Definition and characterization of the extended concept

We can now generalize the concept of oligopolistic equilibrium. Using the first order condition for cost minimization, we get an extended version of the first constraint in (1), imposing that firm $i$ do not set a relative price $p_i/P(p)$ higher than the common marginal utility (or productivity) $\partial_i Q(q)$ of the $i$-th good. The second constraint remains the same.

[21] By strong quasi-concavity of a utility or production function, we mean strict quasi-concavity, together with regularity of its bordered Hessian. It entails twice continuous differentiability of the cost function, and continuous differentiability of the demand function (see Barten and Böhm, 1982).

An *oligopolistic equilibrium* may accordingly be defined as a $2n$-tuple $(p^*, q^*)$ which is a Nash equilibrium of the game with the following payoff for firm $i$

$$\begin{aligned} \Pi_i(p,q) &= p_i q_i - C_i(q_i), \\ \text{if } p_i &\leq \partial_i Q\left(q_i, q_{-i}^*\right) P\left(p_i, p_{-i}^*\right) \text{ and } Q\left(q_i, q_{-i}^*\right) \leq D\left(P\left(p_i, p_{-i}^*\right)\right), \\ \Pi_i(p,q) &= 0, \text{ otherwise,} \end{aligned} \tag{18}$$

satisfying in addition

$$Q(q^*) = D(P(p^*)). \tag{19}$$

Notice that, in the homogeneous case, $\partial_i \overline{Q}\left(q_i, q_{-i}^*\right) = 1$, and $\underline{P}(p) = \min_j \{p_j\}$, so that this definition indeed generalizes the one established for the homogeneous oligopoly case.

As already suggested, the first inequality may be viewed as a constraint imposed on *market share* $q_i/Q(q)$, and the second one as a constraint imposed on *market size* $Q(q)$. Also, taking advantage of duality, we can rewrite the two conditions in (18) in two alternative forms, as constraining either the quantity aggregate $Q(q)$

$$q_i/\partial_i P\left(p_i, p_{-i}^*\right) \leq Q\left(q_i, q_{-i}^*\right) \leq D\left(P\left(p_i, p_{-i}^*\right)\right), \tag{20}$$

or the price index $P(p)$

$$p_i/\partial_i Q\left(q_i, q_{-i}^*\right) \leq P\left(p_i, p_{-i}^*\right) \leq D^{-1}\left(Q\left(q_i, q_{-i}^*\right)\right). \tag{21}$$

Clearly, market size can be equivalently constrained either by referring to the demand function, or to its inverse. It is also equivalent to bound each relative quantity by the corresponding market share resulting from the selected prices, or to bound each relative price by the corresponding marginal utility (or productivity) resulting from the selected quantities. In equilibrium, equation (19) implies that both inequalities are in each case satisfied as equalities.[22]

The first order necessary conditions for the maximization of $p_i q_i - C_i(q_i)$ under the two alternative pairs of constraints, (20) and (21), are respectively:

$$\begin{aligned} \begin{bmatrix} q_i^* \\ p_i^* - C_i'(q_i^*) \end{bmatrix} &= \frac{\lambda_i'}{\partial_i P(p^*)} \begin{bmatrix} -q_i^* \partial_{ii}^2 P(p^*)/\partial_i P(p^*) \\ 1 - \partial_i P(p^*) \partial_i Q(q^*) \end{bmatrix} \\ &\quad + \nu_i' \begin{bmatrix} -D'(P(p^*)) \partial_i P(p^*) \\ \partial_i Q(q^*) \end{bmatrix}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \begin{bmatrix} q_i^* \\ p_i^* - C_i'(q_i^*) \end{bmatrix} &= \frac{\lambda_i}{\partial_i Q(q^*)} \begin{bmatrix} 1 - \partial_i Q(q^*) \partial_i P(p^*) \\ -p_i^* \partial_{ii}^2 Q(q^*)/\partial_i Q(q^*) \end{bmatrix} \\ &\quad + \nu_i \begin{bmatrix} \partial_i P(p^*) \\ -\partial_i Q(q^*)/D'(P(p^*)) \end{bmatrix}. \end{aligned} \tag{23}$$

[22] Indeed, as $Q$ is increasing, the strict inequality $q_i^* < \partial_i P(p^*) Q(q^*) = \partial_i P(p^*) D(P(p^*))$, for some $i$, implies the contradiction: $Q(q^*) < Q(\partial P(p^*) D(P(p^*))) = D(P(p^*)) Q(\partial P(p^*)) = D(P(p^*))$.

for some couples of Kuhn and Tucker multipliers $\left(\lambda_i', \nu_i'\right), (\lambda_i, \nu_i) \in \mathbb{R}_+^2 \setminus \{0\}$. We shall again use the normalization $\theta_i' = \lambda_i' / \left(\lambda_i' + \nu_i'\right)$ (resp. $\theta_i = \lambda_i / \left(\lambda_i + \nu_i\right)$), taking the relative shadow cost associated with the market share constraint of firm $i$ to measure the *competitive toughness* of firm $i$ within the industry, at equilibrium $(p^*, q^*)$.

Denote by $s\left(p_i / P\left(p\right)\right)$ the elasticity of substitution of good $i$ for the composite commodity, which is the absolute value of the elasticity of $q_i / Q\left(q\right)$ with respect to $p_i / P\left(p\right)$. The elasticity of $\partial_i P\left(p\right)$ with respect to its $i$-th argument can then be written $\epsilon_i \partial_i P\left(p\right) = -s\left(p_i / P\left(p\right)\right)\left[1 - \epsilon_i P\left(p\right)\right]$. Using the simplifying notations $s_i^* \equiv s\left(p_i^* / P\left(p^*\right)\right)$, $\sigma^* \equiv -\epsilon D\left(P\left(p^*\right)\right)$, and $\alpha_i^* \equiv \epsilon_i P\left(p^*\right)$, conditions (22) and (23) can now be expressed in terms of the degree of monopoly $\left(p_i^* - C_i'\left(q_i^*\right)\right) / p_i^* = \mu_i^*$, giving respectively

$$\mu_i^* = \mu_i\left(\theta', p^*\right) \equiv \frac{\theta_i'\left(1 - \alpha_i^*\right) + \left(1 - \theta_i'\right)\alpha_i^*}{\theta_i'\left(1 - \alpha_i^*\right) s_i^* + \left(1 - \theta_i'\right)\alpha_i^* \sigma^*}, \tag{24}$$

and

$$\mu_i^* = \mu_i\left(\theta, p^*\right) \equiv \frac{1}{\theta_i\left(1 - \alpha_i^*\right) + \left(1 - \theta_i\right)\alpha_i^*}\left(\frac{\theta_i\left(1 - \alpha_i^*\right)}{s_i^*} + \frac{\left(1 - \theta_i\right)\alpha_i^*}{\sigma^*}\right), \tag{25}$$

for every $i$. Notice that a one-to-one relationship between the two indices of competition toughness of firm $i$ can be established for $s_i^* \neq \sigma^*$ from these equations:

$$\frac{1/\theta_i' - 1}{s_i^*} = \frac{1/\theta_i - 1}{\sigma^*}. \tag{26}$$

As illustrated in Figure 2 (drawn in the $(q_i, p_i)$ space) the individual firm outcome is generically a corner solution at the kink of the feasibility frontier, represented by the solid curve.[23] This frontier has two branches, corresponding to the two constraints on market share and market size, with elasticities at the kink equal to $-1/s_i^*$ and $-1/\sigma^*$, respectively (so that the relative position of the two branches depends upon the relative values of $s_i^*$ and $\sigma^*$).[24] The dotted isoprofit curve (with an elasticity equal to $-\mu_i^*$ at the point of intersection of the three curves) is tangent to neither of the two branches. But the gradient of the profit function at the kink (the direction of which is represented by the dotted line) belongs to the normal cone (bounded by the two dashed lines) generated at the same point by the two constraints.

[23] The data underlying this figure are those of a symmetric duopoly in which firms produce at zero fixed cost and constant marginal cost $c = .75$, and face an isoelastic demand to the industry (with $\sigma = 2$). The aggregator $Q$ is CES, with $s = 25$. The represented symmetric equilibrium is $(p_i, q_i) = (1, 1)$, for $i = 1, 2$.

[24] The two branches of the kinked feasibility frontier are local approximations (at the kink) of Chamberlin's (1933) demand curves $dd'$ (with other prices kept constant when varying $p_i$) and $DD'$ (with other prices in the industry varying proportionally with $p_i$).

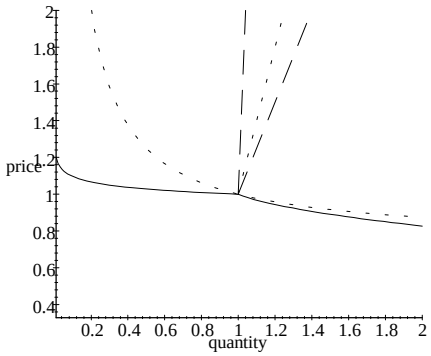


Figure 2

Coming back to equations (24) and (25), expressing first order conditions, we see that the degree of monopoly varies with $\theta_i'$ (resp. $\theta_i$) between

$$\mu_i\left(1,p^*\right)=\frac{1}{s_i^*}, \tag{27}$$

the reciprocal of the elasticity of substitution, when the competitive toughness of firm $i$ is maximal ($\theta_i'=\theta_i=1$), and

$$\mu_i\left(0,p^*\right)=\frac{1}{\sigma^*}, \tag{28}$$

the reciprocal of the elasticity of demand to the industry, when the competitive toughness of firm $i$ is minimal ($\theta_i'=\theta_i=0$). The former extreme value $1/s_i^*$ is the degree of monopoly prevailing when firm $i$ market share $\alpha_i^*$ becomes negligible, *i.e.* in *monopolistic competition* (in Chamberlin's "large group"). The latter extreme value $1/\sigma^*$ is the *collusive* degree of monopoly, corresponding to joint profit maximization.[25] More generally, the equilibrium degree of monopoly is a mean of these two extreme values, the harmonic mean in (24), and the arithmetic mean in (25). When calculating these means, the (monopolistically) competitive value $1/s_i^*$ is weighted by the product of the competitive toughness of firm $i$ and of the market share of its competitors, so that it can alternatively be approached either by increasing competitive toughness of firm $i$ or by reducing its market share. This is just an extension of what we know of the limit homogeneous case ($s_i^*=\infty$), in which a zero degree of monopoly may result either of extreme competitive toughness *à la* Bertrand, or of the vanishing market share of any element of a continuum of perfectly competitive firms.

[25] Joint profit is equal to $\sum_{i=1}^{n}\left(p_iq_i-C_i\left(q_i\right)\right)=P\left(p\right)Q\left(q\right)-\sum_{i=1}^{n}C_i\left(q_i\right)$, and can be maximized in two stages, first in terms of $P$ and $q$, under the constraint $Q\left(q\right)=D\left(P\right)$, and then in terms of $p$, using demander's optimizing conditions. In the first stage, we obtain as first order conditions that, for any $i$, $\left(1-1/\sigma\right)P\partial_iQ\left(q\right)=C_i'\left(q_i\right)$. At the second stage we use $\partial_iQ\left(q\right)=p_i/P\left(p\right)$, and finally get: $\left(p_i-C_i'\left(q_i\right)\right)/p_i=1/\sigma$.

### 3.3 Comparison with quantity and price equilibria

If we now consider the standard, and often used, quantity and price equilibrium concepts, usually referred to as Cournot and Bertrand equilibrium respectively, we can see that they result from restricting the strategy space of each firm to one decision variable only. This amounts to reducing to a single equality the constraints in (21), for the quantity equilibrium, and in (20), for the price equilibrium.

More precisely, a *quantity equilibrium* is a quantity vector $q^*$ such that, for every firm $i$,

$$q_i^* \in \arg\max_{q_i \in \mathbb{R}_+} \partial_i Q\left(q_i, q_{-i}^*\right) D^{-1}\left(Q\left(q_i, q_{-i}^*\right)\right) q_i - C_i\left(q_i\right). \tag{29}$$

A simple calculation yields the first order condition:

$$\frac{p_i^* - C_i'\left(q_i^*\right)}{p_i^*} = \frac{1 - \alpha_i^*}{s_i^*} + \frac{\alpha_i^*}{\sigma^*}, \tag{30}$$

with $p_i^* = \partial_i Q\left(q^*\right) D^{-1}\left(Q\left(q^*\right)\right)$ and $\alpha_i^* = \epsilon_i Q\left(q^*\right) = \epsilon_i P\left(p^*\right)$, for every $i$. Looking at (25), we observe that it coincides[26] with the first order condition for profit maximization in an oligopolistic equilibrium when $\theta_i = 1/2$. This amounts to give the same weight to the two constraints as expressed in (21). In particular, whenever the elasticity of intra-industry substitution tends to infinity, the degree of monopoly of any firm tends to its Cournot value. Observe also that, if $q^*$ is a quantity equilibrium, then $(p^*, q^*)$ is an oligopolistic equilibrium. Otherwise $p_i q_i - C_i\left(q_i\right) > p^* q^* - C_i\left(q_i^*\right)$ for some $(p_i, q_i)$ such that $p_i \leq \partial_i Q\left(q_i, q_{-i}^*\right) P\left(p_i, p_{-i}^*\right)$ and $P\left(p_i, p_{-i}^*\right) \leq D^{-1}\left(Q\left(q_i, q_{-i}^*\right)\right)$, implying $p_i \leq \partial_i Q\left(q_i, q_{-i}^*\right) D^{-1}\left(Q\left(q_i, q_{-i}^*\right)\right)$, and we get

$$\partial_i Q\left(q_i, q_{-i}^*\right) D^{-1}\left(Q\left(q_i, q_{-i}^*\right)\right) q_i - C_i\left(q_i\right) > \partial_i Q\left(q^*\right) D^{-1}\left(Q\left(q^*\right)\right) q_i^* - C_i\left(q_i^*\right),$$

a contradiction.

Similarly, a *price equilibrium* is a price vector $p^*$ such that, for every firm $i$,

$$p_i^* \in \arg\max_{p_i \in \mathbb{R}_+} p_i \partial_i P\left(p_i, p_{-i}^*\right) D\left(P\left(p_i, p_{-i}^*\right)\right) - C_i\left(\partial_i P\left(p_i, p_{-i}^*\right) D\left(P\left(p_i, p_{-i}^*\right)\right)\right). \tag{31}$$

A straightforward calculation leads to the first order condition:

$$\frac{p_i^* - C_i'\left(q_i^*\right)}{p_i^*} = \frac{1}{\left(1 - \alpha_i^*\right) s_i^* + \alpha_i^* \sigma^*}, \tag{32}$$

with $q_i^* = \partial_i P\left(p^*\right) D\left(P\left(p^*\right)\right)$. By referring to (24), we see that it coincides with the first order condition for profit maximization in an oligopolistic equilibrium

[26] Both in this case and in the case of a price equilibrium, the coincidence of the first order conditions does not imply that the conditions for existence of an equilibrium coincide. These will generally be weaker for the corresponding oligopolistic equilibrium.

when $\theta_i' = 1/2$, which amounts to give the same weight to each of the two constraints as expressed in (20). Observe that whenever the elasticity of intra-industry substitution tends to infinity the degree of monopoly tends to zero, and thus, the price equilibrium tends to the corresponding Bertrand homogeneous product equilibrium. Also, with constant marginal costs $c_i \geq 0$, an argument by contradiction shows that $(p^*, q^*)$ is an oligopolistic equilibrium whenever $p^*$ is a price equilibrium, since $(p_i - c_i) q_i > (p_i^* - c_i) q^*$ for some $(p_i, q_i)$ such that $q_i \leq \partial_i P\left(p_i, p_{-i}^*\right) Q\left(q_i, q_{-i}^*\right) \leq \partial_i P\left(p_i, p_{-i}^*\right) D\left(P\left(p_i, p_{-i}^*\right)\right)$ would imply

$$(p_i - c_i)\, \partial_i P\left(p_i, p_{-i}^*\right) D\left(P\left(p_i, p_{-i}^*\right)\right) > (p_i^* - c_i)\, q_i^*,$$

a contradiction. However, this is not generally true for any cost structure.

### 3.4 An example: The CES case with identical linear costs

In order to illustrate the different kinds of oligopolistic equilibria, we now examine the standard example of the (symmetric) CES quantity aggregator (and the associated price aggregator):

$$Q(q) = \left(\sum_j q_j^{(s-1)/s}\right)^{s/(s-1)} \text{ and } P(p) = \left(\sum_j p_j^{1-s}\right)^{1/(1-s)}, \tag{33}$$

with $s > 0$. Assuming for simplicity an isoelastic demand $aP^{-\sigma}$ to the industry (with $a > 0$ and $\sigma > 0$), the demand addressed to firm $i$ takes the explicit form:

$$d_i(p) = \partial_i P(p)\, a\, (P(p))^{-\sigma}, \text{ with } \partial_i P(p) = (p_i / P(p))^{-s}. \tag{34}$$

We also assume the same positive constant marginal cost $c$ for all firms.

We examine in this context the two extreme types of symmetric oligopolistic equilibria, characterized by the maximal and minimal admissible values of competitive toughness (measured by $\bar{\theta}$ or $\bar{\theta}'$ for all $i$). The simpler case arises when $\bar{\theta} = \bar{\theta}' = 1$, which is only possible if $s > 1$ (if $s \leq 1$, the least upper bound of $\bar{\theta}$ (or $\bar{\theta}'$) is the value[27] would imply a degree of monopoly equal to 1). Taking $s > 1$ and $\bar{\theta} = \bar{\theta}' = 1$, we get an equilibrium degree of monopoly equal to $1/s$, which corresponds to the one obtained by Dixit and Stiglitz *exactly* and not just as an acceptable *approximation*[28] for large enough $n$.

The type of competition which corresponds to the precise Dixit-Stiglitz outcome can be related to Bertrand. Indeed, in Bertrand's argument (for $s = \infty$)

[27] The least upper bound of admissible $\bar{\theta}$ tends to 0 as $s$ tends to 0, making it more and more inconvenient (impossible in the limit) to use this measure of competitive toughness to parametrize oligopolistic equilibria. In the case of strong complementarity, it is preferable to use $\bar{\theta}'$, the least upper bound of which tends to $(\sigma - 1)/(\sigma + n - 2)$ as $s$ tends to 0.

[28] This well illustrates that the Dixit-Stiglitz monopolistic competition equilibrium is not a price equilibrium. Both coincide only in the limit situation of an infinite number of firms (see Yang and Heijdra, 1993, and d'Aspremont *et al.*, 1996).

the market size is immaterial: no price higher than marginal cost can be sustained, since any firm would otherwise have the possibility of slightly undercutting such price to catch the whole demand, whatever its level. In the original Dixit-Stiglitz approach market size remains immaterial for the price outcome because each firm is assumed to take the aggregate price $P$ (hence market size) as independent of its own decision, whereas in our approach the same is still true, whatever $s \in (1, \infty)$, because of a zero relative shadow cost of the constraint on market size ($\bar{\theta} = \bar{\theta}' = 1$).

Existence of a symmetric oligopolistic equilibrium can be easily shown in this case.[29] The other extreme case $\bar{\theta} = \bar{\theta}' = 0$ leads to an equilibrium degree of monopoly equal to the collusive value $1/\sigma$ (if $\sigma > 1$), but necessary second order conditions for local profit maximization may not be verified. For instance, as shown in the Appendix, the collusive solution is enforceable as an oligopolistic equilibrium only if $s$ belongs to some interval $\left[\underline{\underline{s}}(\sigma, n), \overline{s}(\sigma, n)\right]$. A sufficient condition is: $s \in [\underline{s}(\sigma, n), \overline{s}(\sigma, n)] \supset [1, \sigma]$ (with $\underline{s}(\sigma, n) > \underline{\underline{s}}(\sigma, n)$). When $\sigma$ tends to 1, all these intervals degenerate into $\{1\}$. When $n$ increases, both necessary and sufficient conditions become stricter, and $\underline{\underline{s}}(\sigma, \infty) = \sigma - \sqrt{\sigma(\sigma - 1)}$, $\underline{s}(\sigma, \infty) = 1$, $\overline{s}(\sigma, \infty) = \sigma + \sqrt{\sigma(\sigma - 1)}$.

The collusive solution is thus enforceable as an oligopolistic equilibrium for intermediate values of $s$. As $s$ becomes large or small, one must move from the collusive value $1/\sigma$ of the degree of monopoly towards $1/s$ in order to obtain existence of an oligopolistic equilibrium. It seems natural to associate the name of Cournot with the degree of monopoly, say $\mu^C(s)$, which is enforceable and closest to its collusive value, the one which results from the minimum competitive toughness compatible with an oligopolistic equilibrium, say $\bar{\theta}^C(s)$ (or $\bar{\theta}'^C(s)$). Indeed, as $s$ tends to infinity or to zero, $\mu^C(s)$ tends to the values calculated by Cournot, $\mu^C(\infty) = 1/n\sigma$ for the homogeneous oligopoly case, and $\mu^C(0) = n/\sigma$ for the complementary monopoly case. Also, we obtain for these limit cases a minimum admissible competitive toughness $\bar{\theta}^C(\infty) = 1/2$ (resp. $\bar{\theta}'^C(0) = 1/2$), corresponding to the quantity (resp. price) equilibrium degree of monopoly.

To illustrate, let us consider a duopoly example, with $c = 1$ and $\sigma = 5/2$. In Figure 3, we represent in the $x$-axis a normalized index of complementarity $1/(1+s)$ within the industry (so that 0 corresponds to perfect substitutability and 1 to perfect complementarity), and the degree of monopoly in the $y$-axis.

[29] For each firm, the revenue function along the market share branch of the feasibility frontier is concave when $s > 1$ (see Appendix), entailing concavity of the profit function.

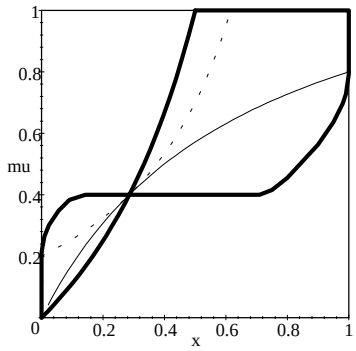


Figure 3

Equilibrium degrees of monopoly correspond to the region inside the thick curves. The thick convex curve starting at the origin from the Bertrand solution, represents the degree of monopoly $1/s$ resulting from $\bar{\theta} = \bar{\theta}' = 1$, and may be called *Bertrand frontier*. It should be distinguished from the thin concave curve, also starting from the Bertrand solution, which represents the *price equilibrium* degree of monopoly $(2/(s+\sigma))$. At the other extreme of this thin curve, we find the Cournot complementary monopoly solution. The thick curve which switches from concave to convex, and links the two Cournot solutions (for homogeneous duopoly at its left extremity and for complementary monopoly at its right extremity) may be called *Cournot frontier*. It represents the degree of monopoly $\mu^C(s)$ which is enforceable and closest to the collusive value $1/\sigma$ (attained along its horizontal middle part). This curve should similarly be distinguished from the dotted convex curve also starting from the Cournot homogeneous duopoly solution, and representing the *quantity equilibrium* degree of monopoly $((1/s + 1/\sigma)/2)$. Notice that the Bertrand frontier lies below the Cournot frontier for $s > \sigma$, and above this frontier[30] for $s < \sigma$.

The last two figures represent (inside the thick curves) the sets of values of $\bar{\theta}$ and $\bar{\theta}'$ for competitive toughness which lead to oligopolistic equilibria in this example. The thin solid (resp. dotted) curve represents the values of $\bar{\theta}$ and $\bar{\theta}'$ which are associated with the price (resp. quantity) equilibrium solution.

[30] We observe that the collusive degree of monopoly is in this case smaller than any other enforceable degree of monopoly. This generalizes Cournot's observation for the limit case of perfect complementarity: "An association of monopolists, working for their own interest, in this instance will also work for the interest of consumers, which is exactly the opposite of what happens with competing producers" (Cournot [1838], 1960, p. 103).

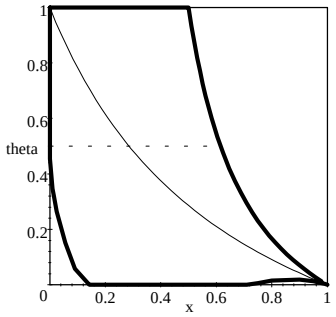


Figure 4

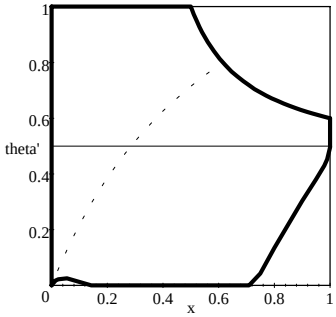


Figure 5

# 4 Conclusion

The classical problem in oligopoly theory, that of choosing between quantity competition and price competition, has been reconsidered in a larger perspective. It has been imbedded in a more general question, that of selecting some equilibrium out of a multiplicity of oligopolistic equilibria. However the indeterminacy we have authorized is limited, and allows to separate two distinct issues.

First, there is the issue of identifying, among possible strategic combinations, those that are enforceable as equilibria, considering the primitive properties of the economy. As an example, when substitutability inside an industry is too weak or too strong, the collusive solution is not enforceable as an equilibrium (a well-known fact, since Cournot). But this ceases to be true, when substitutability belongs to some interval which is bounded away from zero and infinity.

The other issue is to select among the different oligopolistic equilibria. The static model analyzed in this paper is not meant to endogenously reach an equilibrium selection. This would require the specification of a precise organization of production and sales within the industry, as well as the description of current practices and standard conjectures. Here selection is purely exogenous, based on a continuous parametrization of competitive toughness, extending the usual Cournot-Bertrand dichotomy. In that respect, the objective is to provide a tool for comparative statics in oligopoly models with the property that, corresponding to each profile of parameter values, there is a well-defined equilibrium concept.

Of course, many issues remain to be treated. Existence has been only touched upon, except as concerns symmetric equilibria with the lowest possible competitive toughness, for a class of examples (the CES aggregator with linear cost functions) analyzed in the Appendix. The existence problem would deserve more development, in particular to explore the role of the kink in the producer's feasibility frontier in relieving curvature conditions. Another issue would be to extend the equilibrium concept allowing for other types of product differentiation. The first case to be examined in that perspective should be the

linear demand case.

# 5 Appendix

In this appendix, we want to establish conditions for the collusive solution to be enforceable as a symmetric oligopolistic equilibrium in our example of a CES aggregator. We shall consider potential deviations by firm $i$, along its feasibility frontier, from a symmetric configuration of strategies for which $\theta_j = 0$ for every $j$ (assuming $\sigma > 1$, so that the degree of monopoly is smaller than 1). As the cost function is assumed to be increasing and linear, we concentrate on the revenue function $p_i q_i$, which should be increasing and concave in $q_i$ (at least locally), either with $p_i = \phi(q_i)$, solving the equation for *market share* $q_i/Q(q_i, q_{-i}) = (p_i/P(p_i, p_{-i}))^{-s}$, or with $p_i = \psi(q_i)$, given by the equality for *market size* $Q(q_i, q_{-i}) = a(P(p_i, p_{-i}))^{-\sigma}$, according to the nature of the admissible deviations.

We first compute

$$\epsilon\phi(q_i) = -\frac{1}{s}\frac{1-\epsilon_i Q(q)}{1-\epsilon_i P(p)} = -\frac{1}{s}, \tag{35}$$

since $\epsilon_i P(p) = (p_i/P(p))^{1-s} = (q_i/Q(q))^{1-1/s} = \epsilon_i Q(q)$ when $p_i = \phi(q_i)$. Also,

$$\epsilon\psi(q_i) = -\frac{1}{\sigma}\frac{\epsilon_i Q(q)}{\epsilon_i P(p)}, \tag{36}$$

which is equal to $-1/\sigma$ at the kink of the feasibility frontier. Thus, the revenue function $\phi(q_i)\, q_i$ is non-increasing if $s \leq 1$, so that we may restrict our analysis in this case to downward deviations in $q_i$, along the market size branch of the frontier, with $p_i = \psi(q_i)$ (since $1/\sigma < 1/s$).

The marginal revenue $\psi(q_i)[1 + \epsilon\psi(q_i)]$ is non-increasing if

$$\frac{\epsilon\psi(q_i)}{1+\epsilon\psi(q_i)}\left[1 + \epsilon\psi(q_i) + \epsilon^2\psi(q_i)\right] \leq 0, \tag{37}$$

where

$$\epsilon^2\psi(q_i) = \left(1-\frac{1}{s}\right)\left[1 - \left(1-\frac{s}{\sigma}\right)\epsilon_i Q(q) - \frac{s}{\sigma}\frac{\epsilon_i Q(q)}{\epsilon_i P(p)}\right], \tag{38}$$

hence

$$\begin{aligned} & 1 + \epsilon\psi(q_i) + \epsilon^2\psi(q_i) \\ = \quad & \left(\frac{1}{s}-1\right)\left[\left(1-\frac{s}{\sigma}\right)\epsilon_i Q(q) - (1-s)\right] + s\left[1 - \frac{1}{\sigma}\frac{\epsilon_i Q(q)}{\epsilon_i P(p)}\right]. \end{aligned} \tag{39}$$

We conclude that marginal revenue, whenever positive ($\epsilon_i Q(q)/\sigma\epsilon_i P(p) < 1$), is decreasing if

$$\left(\frac{1}{s}-1\right)\left[\left(1-\frac{s}{\sigma}\right)\epsilon_i Q(q) - (1-s)\right] \geq 0. \tag{40}$$

This expression is increasing in $\epsilon_i Q(q)$, itself decreasing in $q_i$. As we are interested in downward quantity deviations by firm $i$ from a symmetric configuration, we use $\epsilon_i Q(q) \geq 1/n$, and obtain the following sufficient condition for concavity in $q_i$ of the profit function $(\psi(q_i) - c) q_i$, given symmetry of other firms' strategies, hence for enforceability of the collusive solution as an oligopolistic equilibrium:

$$\left(1 - \frac{s}{\sigma}\right) \frac{1}{n} \geq 1 - s, \tag{41}$$

and finally

$$\underline{s}(\sigma, n) \equiv 1 - \frac{\sigma - 1}{n\sigma - 1} \leq s \leq 1. \tag{42}$$

Let us now consider the case $s > 1$, in which the revenue function $\phi(q_i) q_i$ is concave. Moreover, for $q_i/Q(q) \leq (p_i/P(p))^{-s}$, that is, if the constraint on market share is satisfied:

$$\frac{\epsilon_i Q(q)}{\epsilon_i P(p)} = \left[\frac{q_i/Q(q)}{(p_i/P(p))^{-s}}\right]^{1-1/s} \leq 1. \tag{43}$$

Consequently, by (36) the revenue function $\psi(q_i) q_i$ is increasing and, by (38),

$$\epsilon^2 \psi(q_i) \geq \left(1 - \frac{1}{s}\right)\left(1 - \frac{s}{\sigma}\right)[1 - \epsilon_i Q(q)], \tag{44}$$

so that this revenue function is also concave, by (37), for $s \leq \sigma$. Under this condition, $\partial(\phi(q_i) q_i) \leq \partial(\psi(q_i) q_i) = c$ at the kink, so that the profit $(\phi(q_i) - c) q_i$ is decreasing in $q_i$, forbidding upward deviations along the market share branch of the feasibility frontier. Hence, $1 < s \leq \sigma$ is an alternative sufficient condition for the collusive solution to be enforceable as an oligopolistic equilibrium.

In the opposite case ($s > \sigma$), the profit $(\phi(q_i) - c) q_i$ is concave and locally increasing in $q_i$, so that upward deviations along the market size branch of the frontier are alone to be considered. But, by (36), (37), (43), (44), and since $\epsilon_i Q(q)$ is increasing in $q_i$,

$$1 + \epsilon\psi(q_i) + \epsilon^2 \psi(q_i) \geq -\frac{n-1}{n\sigma s}\left[s^2 - \left(\sigma + \frac{n\sigma - 1}{n-1}\right) s + \sigma\right] \geq 0, \tag{45}$$

which is sufficient to ensure concavity of the revenue function $\psi(q_i) q_i$, and enforceability of the collusive solution as an oligopolistic equilibrium, provided

$$\sigma < s \leq \frac{1}{2}\left(\sigma + \frac{n\sigma - 1}{n-1}\right) + \sqrt{\frac{1}{4}\left(\sigma + \frac{n\sigma - 1}{n-1}\right)^2 - \sigma} \equiv \overline{s}(\sigma, n). \tag{46}$$

The interval $(\underline{s}(\sigma, n), \overline{s}(\sigma, n))$ clearly includes the interval $[1, \sigma]$, and shrinks as $\sigma$ decreases (with $\underline{s}(1, n) = 1 = \overline{s}(1, n)$) and $n$ increases (with $\underline{s}(\sigma, \infty) = 1$ and $\overline{s}(\sigma, \infty) = \sigma + \sqrt{\sigma(\sigma - 1)}$).

To conclude, the *sufficient* condition for the collusive solution to be enforceable as an oligopolistic equilibrium is

$$\underline{s}(\sigma, n) \leq s \leq \overline{s}(\sigma, n) \tag{47}$$

with $\sigma > 1$, and the corresponding *necessary* (local) second order condition (using $\epsilon_i Q(q) = \epsilon_i P(p) = 1/n$ and (45)) is

$$\underline{\underline{s}}(\sigma, n) \equiv \frac{1}{2}\left(\sigma + \frac{n\sigma - 1}{n-1}\right) - \sqrt{\frac{1}{4}\left(\sigma + \frac{n\sigma - 1}{n-1}\right)^2 - \sigma} \leq s \leq \overline{s}(\sigma, n), \tag{48}$$

with $\sigma > 1$, $\underline{\underline{s}}(1, n) = 1$ and $\underline{\underline{s}}(\sigma, \infty) = \sigma - \sqrt{\sigma(\sigma - 1)} < 1 = \underline{s}(\sigma, \infty)$.

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