On Lie-holomorphs and biderivations of Leibniz algebras

Manuel Mancini

Dipartimento di Matematica e Informatica

Università degli Studi di Palermo, Italy

Institut de Recherche en Mathématique et Physique
Université catholique de Louvain, Belgium

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The notion of holomorph

- In Grp: Let G be a group. The *holomorph* of G is

$$\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G)$$

with operation

$$(g, \varphi) \cdot (h, \varphi') = (g\varphi(h), \varphi \circ \varphi').$$

- In Lie: Let L be a Lie algebra over a field $\mathbb{F}$. The *holomorph* of L is

$$\mathrm{Hol}(L) = L \rtimes \mathrm{Der}(L)$$

with Lie bracket

$$[(x, d), (y, d')] = ([x, y] + d(y) - d'(x), [d, d']),$$

where $[d, d'] = d \circ d' - d' \circ d$.

Leibniz algebras

Definition

A *right Leibniz algebra* over a field $\mathbb{F}$ is a vector space L over $\mathbb{F}$ endowed with a bilinear map (called *bracket* or *commutator*) $[-,-] : L \times L \to L$ which satisfies the *right Leibniz identity*

$$[[x,y],z] = [[x,z],y] + [x,[y,z]], \quad \forall x,y,z \in L.$$

- Every Lie algebra is a Leibniz algebra
- Every Leibniz algebra with a skew-symmetric bracket is a Lie algebra.
- The *Leibniz kernel* of L

$$\mathrm{Leib}(L) = \langle [x,x] \mid x \in L \rangle$$

is the minimal two-sided ideal such that $L/\mathrm{Leib}(L)$ is a Lie algebra.

- The left and the right center of a Leibniz algebra L are

$$\mathrm{Z_l}(L) = \{x \in L \mid [x, L] = 0\} \text{ and } \mathrm{Z_r}(L) = \{x \in L \mid [L, x] = 0\}.$$

- The *center* of L is

$$\mathrm{Z}(L) = \mathrm{Z_l}(L) \cap \mathrm{Z_r}(L).$$

- The *Lie-center* of a Leibniz algebra L is

$$\mathrm{Z_{Lie}}(L) = \{x \in L \mid [x, y] + [y, x] = 0, \forall y \in L\}.$$

- $\mathrm{Z_{Lie}}(L)$ is a characteristic ideal of L containing $Z(L)$.
- $\mathrm{Z_{Lie}}(L) = L$ if and only if L is a Lie algebra.

Derivations and anti-derivations

- A *derivation* of L is a linear map $d\colon L \to L$ such that

$$d([x,y]) = [d(x),y] + [x,d(y)]\,, \quad \forall x,y \in L.$$

- The right adjoint maps $\mathrm{ad}_x = [-,x]$ are derivations of L.
- An *anti-derivation* of L is a linear map $D\colon L \to L$ such that

$$D([x,y]) = [D(x),y] - [D(y),x], \quad \forall x,y \in L.$$

- The left adjoint maps $\mathrm{Ad}_x = [x,-]$ are anti-derivations of L.

Biderivations of Leibniz algebras

Definition (J.-L. Loday, 1993)

A *biderivation* of L is a pair

$$(d, D) \in \mathrm{Der}(L) \times \mathrm{ADer}(L)$$

such that

$$[x, d(y)] = [x, D(y)], \quad \forall x, y \in L.$$

- The set $\mathrm{Bider}(L)$ is a Leibniz algebra with bracket

$$[(d, D), (d', D')] = ([d, d'], [D, d']).$$

- There is a Leibniz algebra homomorphism

$$L \to \mathrm{Bider}(L) \colon x \mapsto (-\operatorname{ad}_x, \operatorname{Ad}_x).$$

- The pair $(-\operatorname{ad}_x, \operatorname{Ad}_x)$ is called *inner biderivation* of L.

The holomorph(s) of a Leibniz algebra

1. **K. Boyle, K. C. Misra and E. Stitzinger, 2020:** Let L be a left Leibniz algebra. The holomorph of L is $L \times \mathrm{Der}(L)$ endowed with the bilinear operation

$$[(x,d),(y,d')] = ([x,y] + d(y), [\mathrm{Ad}_x, d'] + [d,d']).$$

Remark

- The bilinear operation above defines a Leibniz algebra.
- Different results for complete Lie algebras extend naturally to complete Leibniz algebras.
- L does not embed as an ideal of its holomorph.
- If L is a Lie algebra, the construction does not recover the classical holomorph $\mathrm{Hol}(L) = L \rtimes \mathrm{Der}(L)$.

1. **N. P. Souris, 2025:** Let L be a (right) Leibniz algebra. The *Lie-holomorph* of L is the semidirect product

$$\mathrm{hol}_{\mathrm{Lie}}(L) = L \rtimes \mathrm{Der}_{\mathrm{Lie}}(L),$$

where

$$\mathrm{Der}_{\mathrm{Lie}}(L) = \{d \in \mathrm{Der}(L) \mid \mathrm{Im}(d) \subseteq \mathrm{Z}_{\mathrm{Lie}}(L)\}$$

endowed with the bilinear operation

$$[(x, d), (y, d')] = ([x, y] + d(y) - d'(x), [d, d']).$$

Remark

- $\mathrm{hol}_{\mathrm{Lie}}(L)$ is a Leibniz algebra.
- L can be embedded as an ideal into $\mathrm{hol}_{\mathrm{Lie}}(L)$ via $\iota_1 \colon x \mapsto (x, 0)$.
- If L is a Lie algebra, then $\mathrm{Der}_{\mathrm{Lie}}(L) = \mathrm{Der}(L)$ and $\mathrm{hol}_{\mathrm{Lie}}(L) = \mathrm{Hol}(L)$.

The semi-abelian context

- Semi-abelian categories were introduced in 2002 by G. Janelidze, L. Márki and W. Tholen in order to provide a categorical setting which would capture algebraic properties of groups, rings and algebras.
- Let $\mathcal{C}$ be a semi-abelian category, i.e., $\mathcal{C}$ is pointed, Barr-exact, Bourn-protomodular with finite coproducts.
- Let B, X be objects of $\mathcal{C}$. A *split extension* of B by X is a diagram

$$0 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$

in $\mathcal{C}$ such that $\alpha \circ \beta = 1_B$ and (X, k) is a kernel of α.

- For any object X of $\mathcal{C}$, we consider the functor

$$\mathrm{SplExt}(-, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$$

  where, for every object B of $\mathcal{C}$, $\mathrm{SplExt}(B, X)$ is the set of isomorphism classes of split extensions of B by X.
- There is a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X).$$

Action representable categories

Definition (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ is representable. This means that there exists an object $[X]$ of $\mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]).$$

- Prototype examples: Grp, Lie and any abelian category.
- In Grp: $[X] = \mathrm{Aut}(X)$.
- In Lie: $[X] = \mathrm{Der}(X)$.
- In any abelian category: $[X] = 0$.

Remark

Action representability is equivalent to the condition that, for every object X of $\mathcal{C}$, the category $\mathsf{SplExt}(X)$ of split extensions with kernel X has a terminal object of the form

$$0 \longrightarrow X \longrightarrow X \rtimes [X] \underset{\longleftarrow}{\overset{\longrightarrow}{}} [X] \longrightarrow 0,$$

called the *split extension classifier*.

Remark

In Grp and Lie, we have $\mathrm{Hol}(X) = X \rtimes [X]$.

- The notion of action representability has proven to be quite restrictive.

Theorem (X. García Martínez, M. Tsishyn, T. Van der Linden, C. Vienne, 2021)

Let $\mathcal{V}$ be a variety of non-associative algebras over an infinite field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$. *If $\mathcal{V}$ is action representable, then $\mathcal{V} = \mathsf{AbAlg}$ or $\mathcal{V} = \mathsf{Lie}$.*

- More recently, G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of semi-abelian categories.

Weakly action representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ admits a weak representation. This means there exists an object $T = T(X)$ of $\mathcal{C}$ and a natural monomorphism of functors

$$\tau \colon \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, T).$$

- T is called *weak representing object* of X;
- A morphism $(\varphi \colon B \to T) \in \mathrm{Im}(\tau_B)$ is called *acting morphism*.

Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is a natural isomorphism.

Associative algebras and Leibniz algebras

- **G. Janelidze, 2022.** The variety Assoc is weakly action representable. A weak representing object of X is the associative algebra of *bimultipliers* of X.

$$\mathrm{Bim}(X) = \{(f*-, -*f) \in \mathrm{End}(X)\times\mathrm{End}(X)^{\mathrm{op}} \mid f*(xy) = (f*x)y,$$

$$(xy)*f = x(y*f), x(f*y) = (x*f)y,\ \forall x, y \in X\}.$$

- **A. S. Cigoli, M. M., G. Metere, 2023.** The category Leib is weakly action representable. A weak representing object of X is the Leibniz algebra $\mathrm{Bider}(X)$ of *biderivations* of X.
- A homomorphism $\varphi\colon B \to \mathrm{Bider}(X)\colon b \mapsto (-r_b, l_b)$ is an acting morphism if and only if

$$l_b \circ (l_{b'} - r_{b'}) = 0.$$

- There is no canonical action of $\mathrm{Bider}(L)$ on L.
- In other words, the semidirect product $L \rtimes \mathrm{Bider}(L)$ is not in general a Leibniz algebra.

Remark (G. La Rosa, M. M., 2025)

There is a canonical split extension

$$0 \longrightarrow L \longrightarrow \mathrm{hol}_{\mathrm{Lie}}(L) \rightleftarrows \mathrm{Der}_{\mathrm{Lie}}(L) \longrightarrow 0 \tag{1}$$

Thus, $\mathrm{Der}_{\mathrm{Lie}}(L) \cong \mathrm{hol}_{\mathrm{Lie}}(L)/L$, and the acting morphism associated with (1) is the monomorphism

$$\mathrm{Der}_{\mathrm{Lie}}(L) \rightarrowtail \mathrm{Bider}(L) \colon d \mapsto (-d, -d),$$

which gives a canonical embedding of $\mathrm{Der}_{\mathrm{Lie}}(L)$ into $\mathrm{Bider}(L)$.

A connection between $\mathrm{Bider}(L)$ and $\mathrm{hol}_{\mathrm{Lie}}(L)$

Theorem (G. La Rosa, M. M., 2025)

Let L be a Leibniz algebra and let $d \in \mathrm{Der}(L)$. TFAE:

(i) $d \in \mathrm{Der}_{\mathrm{Lie}}(L)$.

(ii) $d \in \mathrm{ADer}(L)$.

(iii) $(d, d) \in \mathrm{Bider}(L)$.

Moreover, we have

$$\mathrm{Der}_{\mathrm{Lie}}(L) = \mathrm{Der}(L) \cap \mathrm{ADer}(L) = \{d \in \mathrm{Der}(L) \mid (d, d) \in \mathrm{Bider}(L)\}.$$

Remark

$\mathrm{Der}_{\mathrm{Lie}}(L)$ is the largest subalgebra S of $\mathrm{Bider}(L)$ such that the semidirect product $L \rtimes S$ is a Leibniz algebra.

Classification in low-dimension

Theorem

Let L be a non-Lie Leibniz algebra with $\dim_{\mathbb{F}} L \leq 3$. Then, the Lie-holomorph $\mathrm{hol}_{\mathrm{Lie}}(L)$ is isomorphic to one of the following Leibniz algebras.

$\dim_{\mathbb{F}} \mathrm{hol}_{\mathrm{Lie}}(L)$	***Lie-holomorph***
2	L_B
3	L_1, $L_6\left(\frac{1}{4}\right)$, L_8, $L_9(\alpha)$, $L_{10}(\alpha)$, L_{11}
4	$\mathfrak{R}_5$, $\mathfrak{L}_{39}$
5	$\mathrm{hol}_{\mathrm{Lie}}(L_2(\alpha))$, $\mathrm{hol}_{\mathrm{Lie}}(L_4)$, $\mathrm{hol}_{\mathrm{Lie}}(L_5(\alpha))$, $\mathrm{hol}_{\mathrm{Lie}}(L_6(\beta))$, $\beta \neq \frac{1}{4}$
6	$\mathrm{hol}_{\mathrm{Lie}}(L_3)$, $\mathrm{hol}_{\mathrm{Lie}}\left(L_6\left(\frac{1}{4}\right)\right)$

Future directions

- Investigate the structure of the Lie-holomorph of solvable, nilpotent, semisimple, and other classes of Leibniz algebras.
- Define the notion of holomorph for other varieties of non-associative algebras (Assoc, CAssoc, Alt, JJord, Pois,...) by using the *universal strict general actor* or the *external weak actor*.

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