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# Risk times in mission-oriented systems

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**Abstract**

This article assesses risk times in mission-oriented systems with high safety standards. We examine critical times under two safety policies. The first requires that the system's reliability function, known the first failure of the components, must exceed a predetermined reliability level throughout the mission. The second demands that the conditional distribution of the system's lifetime, known the first failure of the components, must be more reliable, in terms of the usual stochastic order, than the original system's lifetime. Our study analyzes the critical times at which both policies remain viable. Our methodology, applicable to multiple failure scenarios, identifies sufficient conditions for the existence of these times. We offer explicit solutions for parallel systems with IID components and a general method for dependent and identically distributed components. The study includes practical examples and introduces a non-parametric estimator for the critical times.

**Keywords:** Coherent systems · Copula · PELCoV · Distorted distributions.

## 1 Introduction

Coherent systems are highly useful in Reliability Theory. In particular, mission-oriented systems are designed to accomplish specific objectives. Such systems comprise multiple components working collaboratively to achieve shared goals within a fixed time, typically called mission time. In each mission, the system can be required to perform different submissions. Hence, each system's component can be activated, depending on whether it is needed for the corresponding submission. Mission-oriented systems can be found in many fields, including military operations, space exploration, servers in cloud computing, unmanned aerial vehicles, and manufacturing, among others; see Winokur and Goldstein (1969), Joshi et al. (2017), Xing and Johnson (2023) and Zandieh et al. (2024).

Redundancy policies are typically implemented to maximize the accomplishment probability of the mission. These policies are usually established during the design stage of a system. They consider the components' quality and associated costs to enhance the system's reliability and increase the probability of mission success. Redundancy policies are rarely modified during the lifetime of the system. Maintenance policies, conversely, are performed in rest periods between consecutive missions. These policies can be periodically settled. In that case, they are called preventive maintenance (PM) policies. After a PM action, the system may enter an intermediate state between its original and most recent condition. This indicates that preventive maintenance is not entirely effective and only partially improves the system's performance, see Guo et al. (2013). Conversely, condition-based maintenance (CBM) is carried out based on the system's state. These maintenance policies consider the collected information related to the system's state to make repairs to the system. Although PM and CBM can be performed simultaneously, CBM arises as a valuable tool to achieve cost-effective maintenance; see Wang (2002), Jardine et al. (2006) and Scarf (2007).

The preceding policies focus on improving systems' reliability to accomplish the mission's objectives. Nevertheless, in scenarios where mission-oriented systems demand high safety requirements or strict cost

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control, establishing a mission abort policy is a highly effective approach. Typically, an abort action includes a rescue procedure that pursues the system's survival. Consequently, an abort action produces an immediate mission failure. Mission abort policies are designed to achieve a balanced system's performance between safety requirements and mission objectives.

In any mission abort policy, it is crucial to carefully select appropriate risk criteria to serve as the basis for mission interruption actions. These risk criteria vary depending on the nature of the mission, the potential hazards associated with it, and the impact of a mission failure. The literature provides a wide range of criteria that can be used to evaluate risks in mission-critical systems. For example, a frequently used criterion is to consider a degradation parameter in the model that characterizes the current state of a system, see Zhao et al. (2021). Once the parameter reaches a predetermined threshold, the mission should be terminated, and a recovery procedure initiated. In this framework, external impacts or shocks represent events that significantly reduce the system's reliability. For instance, environmental factors such as extreme temperatures, humidity, or corrosive substances can cause damage to the system's components. Hence, the safety of the system can be compromised. Levitin and Finkelstein (2018a) proposed a methodology to model and evaluate mission success probability and resilience of systems subject to both internal failures and external shocks. In the field of reliability and risk analysis, there exists a significant number of papers focused on the modelling of shocks, as evidenced by various monographs, see Nakagawa (2007), Finkelstein (2008), Finkelstein and Cha (2013), that are primarily devoted to this topic. According to Levitin and Finkelstein (2018a), there are two primary categories of shock models: cumulative shock models and extreme shock models. In the former class, the collective impact of shocks results in system failure. A common criterion reported is the number of shocks experienced during the mission, see Levitin and Finkelstein (2018b,c), Levitin et al. (2020a,b). In contrast, in the latter class, each shock has a certain probability of causing system failure. Some recent contributions to these topics can be found in Cha et al. (2017), Eryilmaz and Kan (2019), Li et al. (2019), Zhao et al. (2020). Other authors, see Cavalcante et al. (2018), Yang et al. (2019, 2020), use early warning signals as a risk criterion. Early warning signals indicate deviations from the normal operating conditions of one or more components or subsystems, which may result in critical system failure. Typically, early warning signals are indicative of undesired working conditions, such as overheating in a propulsion system or low-pressure conditions in a hydraulic system. Another risk criterion for making abort decisions is considering a system's number of failed components. For example, Myers (2009) proposed a model that assumes homogeneous components with exponential time-to-failure distributions for $k$-out-of-$n$:G systems. Under a numerical approach, this model was extended to systems with heterogeneous components and adaptive abort policy in Levitin et al. (2018) for 1-out-of-$n$:G systems, i.e., parallel systems.

The present article considers mission-oriented systems subject to high safety requirements and studies the risk times associated with two safety policies. In the first policy, we assume that the reliability function associated with the system's lifetime, given that the first failure of the components occurs at time $x$, must be higher than a minimum prefixed level of reliability $q$ during the mission time $t_M$. In the second policy, we require that the conditional distribution of the system's lifetime, given that the first failure of the components occurs at time $x$, must be more reliable than the lifetime of the original system in terms of the usual stochastic order. We will assume that once a component fails, it cannot be repaired during the remainder of the mission. In this framework, our research focuses on analyzing the critical times for the first failure of the components such that both policies are satisfied (simultaneously or not). The proposed methodology in this paper can be easily extended to study the risk times for the second or subsequent failures of the components. This article aims to examine the critical moments at which a system continues to meet safety policies rather than creating a mission abort policy. However, the findings of our study could be utilized, in conjunction with other criteria, to suggest mission abort or maintenance policies.

The rest of the paper is organized as follows: In Section 2, we give some basic definitions and establish the notation followed throughout the paper. In Section 3, we determine some sufficient conditions for the existence of critical times for both safety policies. We provide explicit solutions in the case of parallel systems with independent and identically distributed components. Furthermore, we show a general method to calculate the critical times when the components are dependent and identically distributed. In Section 4, we illustrate the proposed methodology with several examples. In Section 5, we provide a non-parametric estimator for the critical times. Finally, we place the conclusions in Section 6.

## 2 Notation and preliminary results

Let $X$ be a random variable over a probability space, and let $F_X(x) = \Pr(X \le x)$ be its distribution function. Its reliability (or survival) function is $\bar{F}_X(x) = 1 - F_X(x) = \Pr(X > x)$. The left continuous inverse function of $F_X$ is defined as

$$F_X^{-1}(u) = \inf\{x : F_X(x) \ge u\} \text{ for } u \in (0,1).$$

Throughout the paper, the partial derivative of a function $G$ with respect to its $i$th variable is represented by $\partial_i G$. Moreover, $\partial_{i,j} G = \partial_i \partial_j G$ and so on. Whenever we use a partial derivative, an expectation, or a conditional distribution, we will assume that they exist. We say that $G$ is increasing (decreasing) if $G(x) \le (\ge) G(y)$ for all $x \le y$.

In this paper, we will make stochastic comparisons of two random variables in terms of magnitude. In particular, we will use the usual stochastic order, which we will define next.

**Definition 2.1** *Given two non-negative random variables $X$ and $Y$, with reliability functions $\bar{F}_X$ and $\bar{F}_Y$, respectively. We say that $X$ is smaller than $Y$ in the usual stochastic order, denoted by $X \le_{st} Y$, if $\bar{F}_X(t) \le \bar{F}_Y(t)$ for all $t$.*

Next, we recall a definition related to the idea of a positively dependent structure in a random vector. We will refer to this concept as stochastically increasing; see Shaked (1977) and Drouet Mari and Kotz (2001), p. 38.

**Definition 2.2** *Let $(X, Y)$ be a bivariate random vector with an absolutely continuous joint distribution. Then, we say that $Y$ is stochastically increasing (SI) in $X$, denoted by $Y \uparrow_{SI} X$, if*

$$\Pr(Y > y | X = x_1) \le \Pr(Y > y | X = x_2) \text{ for all } y \text{ and all } x_1 < x_2, \tag{2.1}$$

*or equivalently,*

$$\{Y | X = x_1\} \le_{st} \{Y | X = x_2\} \text{ for all } x_1 < x_2.$$

*In the case that the inequality (2.1) holds strictly, we will say that $Y$ is strictly stochastically increasing (SSI) in $X$, denoted by $Y \uparrow_{SSI} X$.*

From Theorem 5.2.10 in Nelsen (2006), p. 196, $Y \uparrow_{SI} X$ holds if and only if $\partial_1 C(u, v)$ is decreasing in $u$ for all $v \in (0,1)$, where $C$ is the copula of the vector $(X, Y)$. A sufficient condition for SI is the property totally positive of order 2, denoted by $TP_2$. Next, we define the generalization of the concept $TP_2$ for the multivariate case, denoted by $MTP_2$, initially investigated in Karlin and Rinott (1980).

**Definition 2.3** *We say that the random vector $(X_1, \ldots, X_n)$ with an absolutely continuous distribution is $MTP_2$ if its probability density function (pdf) $f$ is log-supermodular, i.e., $f$ satisfies*

$$f(\mathbf{x} \vee \mathbf{y}) f(\mathbf{x} \wedge \mathbf{y}) \ge f(\mathbf{x}) f(\mathbf{y})$$

*for all $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, where $\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$.*

If $f$ is twice differentiable, it is supermodular if, and only if $\partial_{i,j} f(\mathbf{x}) \ge 0$ for all $i \ne j$ and for all $\mathbf{x}$ in the support of $f$ (see, e.g., Lemma 2.4 in Badía et al. (2014)). The $MTP_2$ property implies that $X_i \uparrow_{SI} X_j$ for all $i \ne j$ (see Drouet Mari and Kotz , 2001, p. 40).

**Definition 2.4** *Given a random vector $(X_1, \ldots, X_n)$, we say that it is exchangeable (EXC) if it is invariant in law under permutations, that is, the distributions of $(X_1, \ldots, X_n)$ and $(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ coincide for all permutations $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.*

The possible dependence among the components' lifetimes $(X_1, \ldots, X_n)$ will be represented by a survival copula $\widehat{C}$, which allows us to represent the joint reliability (or survival) function as

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) = \widehat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n))$$

for all $x_1, \ldots, x_n$. The main properties of the copula representations can be seen in Nelsen (2006) and Durante and Sempi (2016).

Finally, $X_{1:n}, \ldots, X_{n:n}$ will represent the increasing ordered values of $(X_1, \ldots, X_n)$ and $F_{1:n}, \ldots, F_{n:n}$ their associated distribution functions. We say that $q : [0, 1] \to [0, 1]$ is a distortion function if it is increasing, continuous and satisfies $q(0) = 0$ and $q(1) = 1$. If $X_1, \ldots, X_n$ are identically distributed (ID) with a common distribution function $F$, then $F_{i:n} = q_{i:n}(F)$ for some distortion function $q_{i:n}$, for all $i = 1, \ldots, n$.

# 3 Main results

The paper aims to study the risk times of a mission-oriented system subjected to high safety requirements. To achieve this, we compare the reliability function of a coherent system with $n$ components and the conditional reliability function of the system, given that the first component failure occurs at time $x$. Hence, we must compare the reliability functions

$$\bar{F}_T(t) = \Pr(T > t) \quad \text{and} \quad \bar{F}_x(t) = \Pr(T > t \mid X_{1:n} = x)$$

for $t \geq x \geq 0$, where $T = \phi(X_1, X_2, \ldots, X_n)$ and $X_{1:n} = \min(X_1, \ldots, X_n)$ are the lifetime of a coherent system formed by $n$ components and the lifetime of a series system formed by the same components, respectively. The random variables $X_i$ represent the lifetimes of the components. We assume that the random vector $(X_1, \ldots, X_n)$ has a joint absolutely continuous distribution and that the system does not fail with the first component failure, i.e., $\Pr(T > X_{1:n}) = 1$. Therefore, $\bar{F}_x(t) = 1$ for $0 \leq t \leq x$. In the present paper, we will study the system's lifetime conditioned on the first component failure time, represented here by $X_{1:n}$. However, similar results can be obtained considering the second or third failures (represented by $X_{2:n}$ and $X_{3:n}$, respectively).

We assume that the system must satisfy high safety requirements during the mission time, denoted by $t_M$. For example, if we would be interested in the lifetime of the propulsion system of an aeroplane with four engines, which cover the route from New York to Madrid, we could require a minimum reliability level $q = 0.9999$ and a mission time $t_M = 7$ hours (note that the average flight time is around 6 hours and 30 minutes). We could also assume that the aeroplane can fly with only one engine, and therefore the system lifetime is given by $T = X_{4:4} = \max(X_1, X_2, X_3, X_4)$, where $X_i$ represents the lifetime of the $i$th engine. Other systems' structures will be considered later. Hence, a crucial question is how to proceed in case of an eventual failure of an engine at a given time $x \in (0, t_M)$. Here, we must consider if there exists evidence to conclude the mission and the aeroplane should enter emergency mode, landing as soon as possible, or if the working conditions required to $T$ are fulfilled by $(T \mid X_{1:4} = x)$, allowing the aircraft to proceed to its intended destination. In general, there are two main options for the working conditions. The first one is to explore those values of $x$ to assure

$$\Pr(T > t \mid X_{1:n} = x) \geq q \text{ for } t \in [0, t_M]. \tag{3.1}$$

The second option, more restrictive, is to find those values of $x$ which satisfy

$$\Pr(T > t \mid X_{1:n} = x) \geq \Pr(T > t) \text{ for all } t \geq 0, \tag{3.2}$$

that is, $(T \mid X_{1:n} = x) \geq_{st} T$.

In terms of the previous example, inequality (3.2) indicates that the aeroplane which has been flying towards its destination with one less engine since the instant $x$ is more reliable than another aeroplane (with the same characteristics) that is yet to take off towards the same destination.

To solve the cases (3.1) and (3.2), we will use a concept similar to the notion of Probability Equivalent Level of CoVaR-VaR, introduced in Ortega-Jiménez et al. (2022) in the context of Risk Theory.

**Definition 3.1** *Let $(X, Y)$ be a random vector having a copula $C$ such that $Y \uparrow_{SSI} X$ and let assume that $\partial_1 C(s, w)$ is continuous in $s$ for all $w$. Given $w \in (0, 1)$, the Probability Equivalent Level of CoVaR-VaR (PELCoV) at risk level $w$ is defined as the unique risk level $s_w$ for $X$ such that*

$$F^{-1}_{Y|X=F_X^{-1}(s_w)}(w) = F_Y^{-1}(w).$$

Observe that the assumption $Y \uparrow_{SSI} X$ is required to ensure the uniqueness of the PELCoV $s_w$. However, we can relax this condition and consider $Y \uparrow_{SI} X$ instead; in this case, the PELCoV at risk level $w$ is defined as

$$s_w = \inf\{s \in (0,1) : F^{-1}_{Y|X=F_X^{-1}(s)}(w) = F_Y^{-1}(w)\}.$$

Since we will work with reliability functions, we will use 1-PELCoV at risk $1-w$ in the calculations. To simplify the notation, we refer to this value as survival PELCoV at risk $v$, denoted by $u_v = 1 - s_{1-v}$.

To ensure the condition (3.1), and by assuming that $T \uparrow_{SI} X_{1:4}$ and $\Pr(T > t_M) = q$, we could study the *critical time* $x_q \in (0, t_M)$ such that

$$\Pr(T > t_M \mid X_{1:4} = x_q) = \Pr(T > t_M), \tag{3.3}$$

or equivalently, study the survival PELCoV $u_q$ at risk $q$ of the random variable $X_{1:4}$, where $u_q = \bar{F}_{1:4}(x_q)$ and $q = \bar{F}_T(t_M)$. Obviously, the property (3.3) implies

$$\Pr(T > t \mid X_{1:4} = x_q) \geq q \text{ for all } t \in [0, t_M].$$

Furthermore, from the assumption $T \uparrow_{SI} X_{1:4}$, we obtain that

$$\Pr(T > t \mid X_{1:4} = x) \geq \Pr(T > t \mid X_{1:4} = x_q) \geq q \text{ for all } t \in [0, t_M] \tag{3.4}$$

holds for all $x \geq x_q$. Therefore, the value $x_q > 0$ is a *critical time* for the system. Hence, we would consider the suitability of an emergency landing or continue with the established route depending on whether $X_{1:4} = x < x_q$ and other risk criteria recommend it or $X_{1:4} = x \geq x_q$, respectively. The time $x_q$ can be presented as a quantile of the series system lifetime $X_{1:4}$ by calculating $x_q = F_{1:4}^{-1}(1 - u_q)$. Then, the value $u_q$ represents a lower bound of the probability of continuing the flight when the first failure occurs.

We could do the same in the second option considered in (3.2). In this case, we study the minimum time value (if it exits) $x_c \in (0, t_M)$ such that

$$\Pr(T > t \mid X_{1:4} = x_c) \geq \Pr(T > t) \text{ for all } t \geq 0,$$

or equivalently, we study if the infimum value $u_c$ of the survival PELCoVs $u_v$ at risk $v = \bar{F}_T(t)$ for all $t \geq 0$ is a positive value. Because, given $t \geq 0$ we define $v = \bar{F}_T(t)$ and calculate the corresponding survival PELCoV $u_v$ at risk $v$ such that

$$\Pr(T > t \mid X_{1:4} = x) = \Pr(T > t),$$

where $u_v = \bar{F}_{1:4}(x)$. Hence, if $u_c = \inf\{u_v : u_v \text{ is the survival PELCoV at risk } v = \bar{F}_T(t) \text{ for all } t \geq 0\}$ is a positive value, then there exists $x_c \geq 0$ such that $\bar{F}_{1:4}(x_c) = u_c \leq u_v = \bar{F}_{1:4}(x)$. Therefore, $x_c \geq x$ and by using that $T \uparrow_{SI} X_{1:4}$, we have that

$$\Pr(T > t \mid X_{1:4} = x_c) \geq \Pr(T > t \mid X_{1:4} = x) = \Pr(T > t) \tag{3.5}$$

for all $t \geq 0$. Indeed, the condition (3.5) can be extended to all values greater than $x_c$ as follows

$$\Pr(T > t \mid X_{1:4} = x) \geq \Pr(T > t \mid X_{1:4} = x_c) \geq \Pr(T > t) \text{ for all } t \geq 0,$$

holds for all $x \geq x_c$. The value $x_c$ is a *critical time* for the system, and the values $x$ for the first failure of the components such that $x \geq x_c$ assure a better performance of the conditional system $(T \mid X_{1:4} = x)$ than the initial system $T$. Note that, in this case, we cannot assure the equality between $\Pr(T > t \mid X_{1:4} = x_c)$ and $\Pr(T > t)$, we will see some examples in the next section. Of course, the value $x_c$ satisfies the condition (3.1), and also

$$\Pr(T > t \mid X_{1:4} = x) \geq \Pr(T > t) \geq q \text{ for all } t \in [0, t_M]$$

holds for all $x \geq x_c$. Furthermore, we could represent $x_c$ in term of the quantiles of the random variable $X_{1:4}$, as $x_c = F_{1:4}^{-1}(1 - u_c)$. In this case, the value $u_c$ represents the probability of the event $(T \mid X_{1:4} \geq x) \geq_{st} T$.

To solve the considered options in this paper, we will assume that the components are identically distributed (ID), with a common reliability function $\bar{F}$. Then, we will use the representation of the joint reliability function of $(X_{1:n}, T)$ as a distortion function of $\bar{F}$, that is, we will write

$$\Pr(X_{1:n} > x, T > y) = \widehat{D}(\bar{F}(x), \bar{F}(y))$$

for all $x, y \geq 0$, where $\widehat{D} : [0,1] \times [0,1] \to [0,1]$ is a continuous bivariate distribution. The main properties of such distortion representations can be seen in Navarro et al. (2022). In particular, this procedure allows us to obtain a representation in terms of distortions for the conditional reliability function $\Pr(T > t \mid X_{1:n} = x)$. In this sense, we will use the representations obtained in Proposition 3.5 and Proposition 3.6 of Navarro and Buono (2023). In general, if we assume that $T \uparrow_{SI} X_{1:n}$ and the dependence among the components is given by an exchangeable survival copula $\widehat{C}$ such that

$$\Pr(X_1 \geq x_1, X_2 \geq x_2, \dots, X_n \geq x_n) = \widehat{C}(\bar{F}(x_1), \bar{F}(x_2), \dots, \bar{F}(x_n)),$$

then $\Pr(T > t \mid X_{1:n} = x) = H(\bar{F}(x), \bar{F}(t))$ where $H(w_1, w_2)$ is a decreasing function in $w_1 \in (0,1)$ and it is an increasing distortion function in $w_2 \in (0,1)$. Similarly, we can express the survival function of the system lifetime $T$ as $\bar{F}_T(t) = h(\bar{F}(t))$, where $h$ is a strictly increasing distortion function, see Navarro (2022).

In the context of the aeroplane example, we can write

$$\Pr(T > t \mid X_{1:4} = x) = \frac{A_{4|1:4}(x,t)}{\partial_1 \widehat{C}(\bar{F}(x), \bar{F}(x), \bar{F}(x), \bar{F}(x))} = H(\bar{F}(x), \bar{F}(t)), \tag{3.6}$$

for all $0 \leq x < t$ such that $\partial_1 \widehat{C}(\bar{F}(x), \bar{F}(x), \bar{F}(x), \bar{F}(x)) > 0$ and $\lim_{t \to \infty} A_{4|1:4}(x,t) = 0$, where

$$A_{4|1:4}(x,t) = 3\partial_1 \widehat{C}(\bar{F}(x), \bar{F}(x), \bar{F}(x), \bar{F}(t)) - 3\partial_1 \widehat{C}(\bar{F}(x), \bar{F}(x), \bar{F}(t), \bar{F}(t)) + \partial_1 \widehat{C}(\bar{F}(x), \bar{F}(t), \bar{F}(t), \bar{F}(t)).$$

Therefore, to solve the case (3.1) we need to find the value $x_q$ such that

$$\Pr(T > t_M \mid X_{1:n} = x_q) = \Pr(T > t_M),$$

or equivalently, solve in $w_1$ the equation

$$H(w_1, w_2) = q, \tag{3.7}$$

where $w_1 = \bar{F}(x_q)$ and $w_2 = \bar{F}(t_M)$. Similarly, to solve the case (3.2) we need to find the value $x_c$ such that

$$\Pr(T > t \mid X_{1:n} = x_c) \geq \Pr(T > t)$$

for all $t \geq 0$. To do that, we calculate the infimum survival PELCoV $u_v$ at risk $v \in (0,1)$. Given $t \geq 0$ we solve the following equation in $w_1$

$$H(w_1, h^{-1}(w_2)) = w_2, \tag{3.8}$$

where $w_1 = \bar{F}(x)$ and $w_2 = \bar{F}(t)$. Then, we obtain the corresponding survival PELCoV at risk $v = h(w_2) = \bar{F}_T(t)$ by using the survival copula:

$$u_v = \bar{F}_{1:n}(x) = \widehat{C}(w_1, \dots, w_1).$$

Hence, if the value $u_c = \inf\{u_v : \ v \in (0,1)\}$ is positive, then there exists $x_c \geq 0$ such that $\bar{F}_{1:n}(x_c) = u_c$.

The methodology proposed in this paper can deal with different sorts of information to establish a suitable protocol in each case. In general, we can use the information of an auxiliary system $T_1 = \phi_1(X_1, \dots, X_n)$ to predict the reliability of the real system $T = \phi(X_1, \dots, X_n)$ by assuming $\Pr(T_1 < T) = 1$. Note that both systems share the same components. The following result guarantees (as particular case) the existence of a solution in the equations (3.7) and (3.8).

**Proposition 3.2** *If $T_1 = \phi_1(X_1, \dots, X_n)$ and $T = \phi(X_1, \dots, X_n)$ are two coherent systems satisfying that $\Pr(T > t \mid T_1 = x)$ is continuous in $x \geq 0$, $\Pr(T_1 < T) = 1$ and $T \uparrow_{SI} T_1$, then for $t_M > 0$ and $q = \Pr(T > t_M)$ there exists $x_q \in (0, t_M)$ such that*

$$\Pr(T > t \mid T_1 = x) \geq \Pr(T > t_M \mid T_1 = x_q) = q \text{ for } t \in [0, t_M]$$

*holds for all $x \geq x_q$.*

**Proof.** Since

$$\Pr[T > t_M] = \int_0^\infty \Pr[T > t_M | T_1 = x] dF_{T_1}(x)$$

and $\Pr[T > t_M | T_1 = x]$ is non-decreasing in $x$, necessarily

$$\Pr[T > t_M | T_1 = 0] \le \Pr[T > t_M] \le \Pr[T > t_M | T_1 = t_M] = 1.$$

Therefore, as $\Pr[T > t_M | T_1 = x]$ is continuous in $x$, there exists $x_q \in (0, t_M)$ such that $\Pr[T > t_M | T_1 = x_q] = \Pr[T > t_M]$. In particular, as $T \uparrow_{SI} T_1$,

$$\Pr[T > t | T_1 = x] \ge \Pr[T > t_M | T_1 = x_q] = \Pr[T > t_M]$$

for all $x \ge x_q$ and $t \in [0, t_M]$. ■

To apply Proposition 3.2 we need to prove first the condition $T \uparrow_{SI} T_1$. This is not an easy task. In practice, we can check this property by plotting the respective reliability functions (or distortions) as we do in the next section. For the particular case of $k$-out-of-$n$:F systems, we have the following property, which is of independent interest in the sampling theory for order statistics (ordered data).

**Theorem 3.3** *If $(X_1, \dots, X_n)$ is absolutely continuous, exchangeable and MTP$_2$, then $(X_{1:n}, \dots, X_{n:n})$ is also MTP$_2$ and $X_{i:n} \uparrow_{SI} X_{j:n}$ for all $i \neq j$.*

**Proof.** If $(X_1, \dots, X_n)$ is exchangeable, then it is straightforward to prove that its joint probability density function $f(x_1, \dots, x_n)$ is symmetric, that is, $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any $\sigma \in \mathcal{P}_n$, where $\mathcal{P}_n$ is the set of all the permutations of the set $\{1, \dots, n\}$. Furthermore, the joint pdf $g$ of the order statistics $(X_{1:n}, \dots, X_{n:n})$ can be written as

$$g(x_1, \dots, x_n) = \sum_{\sigma \in \mathcal{P}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = n!\, f(x_1, \dots, x_n)$$

for $x_1 \le \cdots \le x_n$ (zero elsewhere), where the second equality holds due to the exchangeability condition. Hence, if $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) = (\min(x_1, y_1), \dots, \min(x_n, y_n))$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) = (\max(x_1, y_1), \dots, \max(x_n, y_n))$, then

$$\begin{aligned} g(\boldsymbol{\alpha}) g(\boldsymbol{\beta}) &= [n! f(\alpha_1, \dots, \alpha_n)][n! f(\beta_1, \dots, \beta_n)] \\ &\ge [n! f(x_1, \dots, x_n)][n! f(y_1, \dots, y_n)] \\ &= g(\mathbf{x}) g(\mathbf{y}) \end{aligned}$$

and, therefore, $(X_{1:n}, \dots, X_{n:n})$ is MTP$_2$. As this property also holds for the marginals (see Drouet Mari and Kotz (2001), p. 40), then $(X_{i:n}, X_{j:n})$ is TP$_2$. Furthermore, this implies that $(X_{i:n}, X_{j:n})$ is CI (see Drouet Mari and Kotz (2001), p. 40), which means that $X_{i:n} \uparrow_{SI} X_{j:n}$ holds for all $i \neq j$. ■

As a consequence of Theorem 3.3, the condition (3.4) holds for $k$-out-of-$n$:F systems whether $(X_1, \dots, X_n)$ is MTP$_2$. From Corollary 11 in Sordo et al. (2015), a similar property holds for $T_1 = X_i$ because $T \uparrow_{SI} X_i$ holds for all $i$ and all coherent systems $T$. However, we must note that in practice, the condition $X_i = x$ is not usual. Notice that this condition is not the same as $X_{1:n} = X_i = x$, which implies $X_j \ge x$ for all $j \neq i$. This last property is more reasonable in practice, but we do not know if $(T | X_{1:n} = X_i = x)$ is, in general, *st*-increasing in $x$. Intuitively, we could expect that $X_{i:n} \uparrow_{SI} X_{j:n}$ generally holds. Although, this is not always true. The following counterexample shows a case in which $X_{i:n} \uparrow_{SI} X_{j:n}$ does not hold when we drop out the exchangeability and $MTP_2$ conditions for $(X_1, \dots, X_n)$.

**Example 3.4** *Let us consider the random vector $(U, V)$ with uniform marginal distributions over the interval $(0, 1)$ and the following absolutely continuous (exchangeable) Clayton copula*

$$W(u, v) = \frac{uv}{u + v - uv}.$$

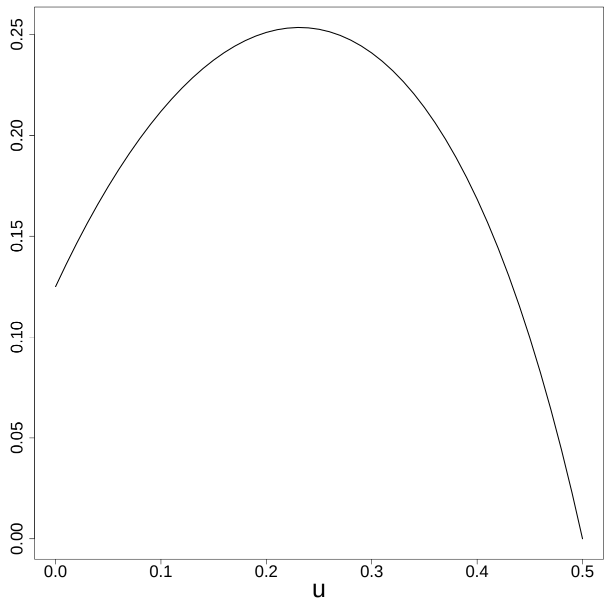


Figure 1: Plot of the function $D_{2|1}(v|u)$, given in Example 3.4, for $u \in (0, v)$ and $v = 0.5$.

*Now we consider the copula generated by the distribution function of $(U, 1-V)$ given by*

$$C(u,v) = \Pr(U \le u, 1-V \le v) = \Pr(U \le u, 1-v \le V) = W(u,1) - W(u,1-v) = \frac{u^2 v}{1-v+uv}$$

*for $u, v \in [0,1]$. It is straightforward to show that $C$ is a non-exchangeable and non $TP_2$ copula.*

*Let us consider now a random vector $(X_1, X_2)$ with a common absolutely continuous marginal $F$, and dependence structure given by the copula $C$. The distortion function for the distribution function of $(X_{2:2}|X_{1:2} = x)$ was obtained in expression (15) of Navarro et al. (2022). Thus,*

$$\Pr(X_{2:2} \le y | X_{1:2} = x) = D_{2|1}(F(y)|F(x)),$$

*for $y \ge x$, where*

$$D_{2|1}(v|u) = \frac{\partial_1 C(u,v) + \partial_2 C(v,u) - \partial_1 C(u,u) - \partial_2 C(u,u)}{\partial_1 C(u,1) + \partial_2 C(1,u) - \partial_1 C(u,u) - \partial_2 C(u,u)} \quad v \ge u > 0,$$

*and where $\partial_i C(u,v)$ denotes the partial derivative of $C$ with respect to its $i$th variable for $i = 1, 2$ evaluated at the point $(u,v)$. A straightforward calculation leads to*

$$D_{2|1}(v|u) = \frac{uv\dfrac{2-2v+uv}{(1-v+uv)^2} + v^2\dfrac{1}{(1-u+uv)^2} - u^2\dfrac{3-2u+u^2}{(1-u+u^2)^2}}{2 - u^2\dfrac{3-2u+u^2}{(1-u+u^2)^2}} \quad v \ge u > 0.$$

*Figure 1 shows the graphic of $D_{2|1}(v|u)$ for $u \in [0,v]$ and $v = 0.5$, as we see $D_{2|1}(v|u)$ is not monotone in $u$ and, therefore, $\Pr(X_{2:2} \le y|X_{1:2} = x)$ is not monotone in $x$ and the SI property does not hold for $(X_{1:2}, X_{2:2})$.*

The following result shows that the condition (3.2) holds for any parallel systems with IID components. First, we provide a necessary result known as monotone form of l'Hôpital's Rule, see Theorem 1.25 in Anderson et al. (1997).

**Lemma 3.5 (Anderson et al. (1997))** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on $(a, b)$. Let $g'(x) \neq 0$ on $(a, b)$. Then, if $f'(x)/g'(x)$ is increasing (decreasing) on $(a, b)$, so are the functions*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \textit{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}. \tag{3.9}$$

*If $f'(x)/g'(x)$ is strictly monotone, then the functions in (3.9) are strictly monotone.*

**Theorem 3.6** *Let $(X_1, \ldots, X_n)$ be an absolutely continuous random vector with IID components and common distribution function $F$. Then*

$$(X_{n:n} \mid X_{1:n} = x) \geq_{st} X_{n:n} \quad \textit{for all} \quad x \geq F^{-1}(1/n) \textit{ and } n \in \mathbb{N} \textit{ with } n \geq 2. \tag{3.10}$$

**Proof.** Let us denote $\bar{F}_{1:n}$ and $\bar{F}_{n:n}$ the survival functions of the random variables $X_{1:n}$ and $X_{n:n}$, respectively. To prove (3.10), we need to find the value $u^\star$ determined as the infimum of the survival PELCoV at risk $v = \bar{F}_{n:n}(t)$. Firstly, given $t \geq 0$ we calculate the survival PELCoV $u_v$ at risk $v = \bar{F}_{n:n}(t)$ such that

$$\bar{F}_{X_{n:n}|X_{1:n}=x}(t) = \bar{F}_{n:n}(t), \tag{3.11}$$

where $u_v = \bar{F}_{1:n}(x)$. Note that $(X_{n:n} \mid X_{1:n} = x) =_{st} (X_{n-1:n-1} \mid X_{1:n-1} > x)$ for all $x \geq 0$, and therefore both random variables have the same survival functions, i.e., $\bar{F}_{X_{n:n}|X_{1:n}=x}(t) = \bar{F}_{X_{n-1:n-1}|X_{1:n-1}>x}(t)$ for all $t \geq 0$. Hence,

$$\begin{aligned}\bar{F}_{X_{n-1:n-1}|X_{1:n-1}>x}(t) =& 1 - \Pr[X_{n-1:n-1} \leq t \mid X_{1:n} > x] = 1 - \frac{\Pr[x < X_1 \leq t, \ldots, x < X_{n-1} \leq t]}{\Pr[X_1 > x, \ldots, X_{n-1} > x]} \\ =& 1 - \left(\frac{\bar{F}(x) - \bar{F}(t)}{\bar{F}(x)}\right)^{n-1} = 1 - \left(1 - \frac{\bar{F}(t)}{\bar{F}(x)}\right)^{n-1}.\end{aligned} \tag{3.12}$$

On the other hand, $\bar{F}_{n:n}(t) = 1 - (1 - \bar{F}(t))^n = v$ and then $\bar{F}(t) = 1 - (1 - v)^{\frac{1}{n}}$. Therefore, from (3.11) and (3.12) we obtain that

$$\bar{F}(x) = \frac{1 - (1 - v)^{\frac{1}{n}}}{1 - (1 - v)^{\frac{1}{n-1}}}.$$

Using that $u_v = \bar{F}_{1:n}(x) = \bar{F}^n(x)$, we finally obtain the expression of the survival PELCoV at risk $v$ as

$$u_v = \left(\frac{1 - (1 - v)^{\frac{1}{n}}}{1 - (1 - v)^{\frac{1}{n-1}}}\right)^n. \tag{3.13}$$

Let us prove that $u_v^n$ is an increasing function in $v$. Firstly, note that $u_v^n = \dfrac{f(v) - f(0)}{g(v) - g(0)}$ where $f, g : [0, 1] \to [0, 1]$ with $f(v) = (1 - v)^{1/n}$ and $g(v) = (1 - v)^{1/(n-1)}$. It is straightforward to show that $f'(x)/g'(x)$ is increasing on $(0, 1)$, then from Lemma 3.5 we have that $u_v^n$ (and therefore $u_v$) is increasing in $v$. Thereby,

$$u^\star = \inf\{u_v : v \in (0, 1)\} = \lim_{v \to 0^+} \left(\frac{1 - (1 - v)^{\frac{1}{n}}}{1 - (1 - v)^{\frac{1}{n-1}}}\right)^n = \left(\frac{n - 1}{n}\right)^n.$$

Since $u^\star > 0$ for all $n \geq 2$, then there exists $x^\star \geq 0$ such that $\bar{F}_{1:n}(x^\star) = u^\star$. Hence, $x^\star = F^{-1}(1/n)$ and taking into account that the independence copula is $\mathrm{MTP}_2$, we have that

$$\bar{F}_{X_{n:n}|X_{1:n}=x^\star}(t) \geq \bar{F}_{X_{n:n}|X_{1:n}=x}(t) = \bar{F}_{n:n}(t)$$

for all $t \geq 0$. The result follows considering $x \geq F^{-1}(1/n)$. ∎

# 4 Examples

Let us illustrate the theoretical results with some examples. We start with the simplest case by assuming independent and identically distributed (IID) components. However, note that $X_{1:4}$ and $T$ are always dependent. To enhance the presentation and legibility of the figures, we will relax the safety conditions of the systems. Thus, we will consider values of the minimum required reliability $q$ lower than those required in real examples.

**Example 4.1** *Let us assume that we have a parallel system with four components and lifetime $T = X_{4:4}$ (e.g. an aeroplane with four engines that can fly with only one of them). Let us also assume that the components are IID having a common continuous reliability function $\bar{F}$. Then, the system reliability function is*

$$\bar{F}_T(t) = 4\bar{F}(t) - 6\bar{F}^2(t) + 4\bar{F}^3(t) - \bar{F}^4(t) = h(\bar{F}(t))$$

*for $t \geq 0$, where $h(v) = 4v - 6v^2 + 4v^3 - v^4$ for $v \in [0,1]$ is a distortion function (see, e.g., Navarro, 2022). The plot of the function $h(v)$ is represented in Figure 2, left (black line). Let us assume a minimum required reliability $q = 0.976031$ for the mission period $[0, t_M]$. By solving $h(v) = 0.976031$, we get $v = 0.606529$, i.e., the prescribed reliability holds from time zero until the quantile $F^{-1}(1 - 0.606529)$. For example, if the components are exponentially distributed with mean 20 hours, then $\bar{F}_T(t) \geq 0.976031$ for $t \in [F^{-1}(0), F^{-1}(0.39347)] = [0, 10]$. The plot of the system reliability function can be seen in Figure 2, right (black line).*

*From (3.12) we can write*

$$\Pr(T > t \mid X_{1:4} = x) = 1 - \left(1 - \frac{\bar{F}(t)}{\bar{F}(x)}\right)^3 = H(\bar{F}(x), \bar{F}(t)) \tag{4.1}$$

*for $t \geq x$ (one elsewhere), where*

$$H(u,v) = \begin{cases} 1 - \left(1 - \dfrac{u}{v}\right)^3 & 0 \leq v \leq u; \\ 1 & u \leq v \leq 1. \end{cases}$$

*Expression (4.1) shows that the conditional reliability function $\Pr(T > t \mid X_{1:4} = x)$ is increasing in $x$, that is, $T \uparrow_{SI} X_{1:4}$. This property is also a consequence of Theorem 3.3 since the product copula is $MTP_2$. The distortion functions $H(u,v)$ obtained for different values of $u$ are plotted in Figure 2, left. The case $u = 1$ is an extreme value where the first component failure occurs at time $t = 0$. Next, we consider two possible scenarios for the safety conditions.*

***Case I:*** *$\Pr(T > t | X_{1:4} = x) \geq q$ for $t \in [0, t_M]$.*

*Firstly, from the results included in the preceding section, we can determine the critical time $x_q$ for the first failure of the components, from which the system still satisfies the safety conditions. To determine the critical time $x_q$, we calculate the survival PELCoV at risk $q = 0.976031$ using the expression in (3.13). Hence, we have*

$$u_q = \left(\frac{1 - (1-q)^{\frac{1}{4}}}{1 - (1-q)^{\frac{1}{3}}}\right)^4 = 0.527572.$$

*Since the value $x_q$ satisfies that $\bar{F}_{1:4}(x_q) = 0.527572$, then $x_q = F^{-1}(1 - u_q^{1/4})$. In the case of exponentially distributed components with mean 20, we have that $x_q = 3.19735$. Therefore, we can ensure the safety conditions when the first failure occurs at any time $x \geq x_q$. However, the system's reliability might be under 0.976031 when $x \in [0, x_q)$. To show this fact, we plot in Figure 2 (right) the reliability functions of the system when the first failure of the components occurs at times $x = 1, 2 < x_q$ (red curves) and $x = 6, 8 > x_q$ (blue curves). In contrast to the blue curves, the red curves do not satisfy the safety conditions within the interval $[0, 10]$.*

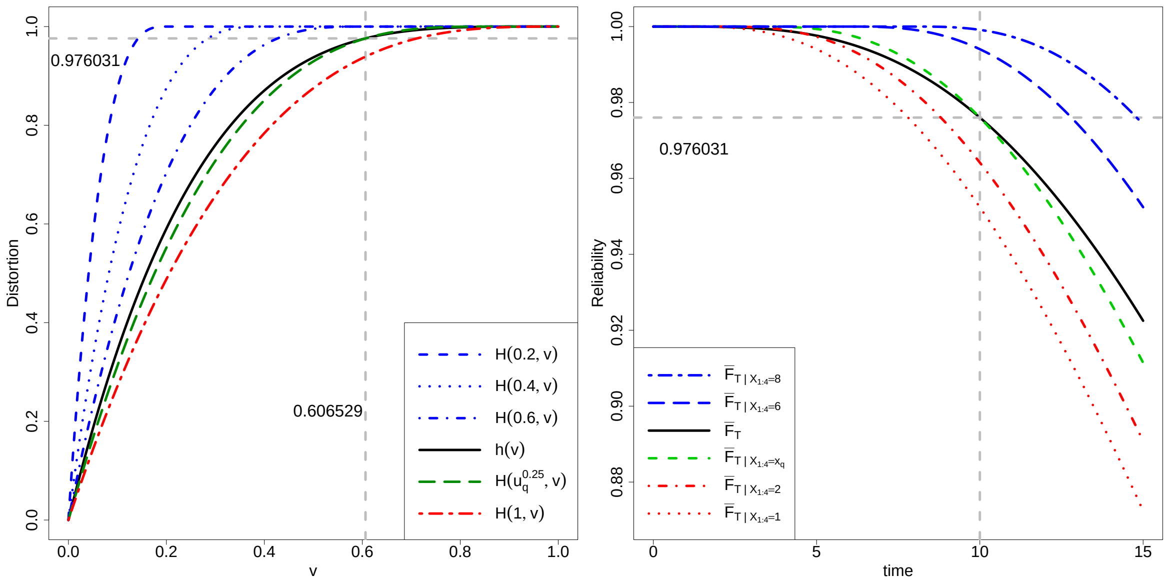


Figure 2: Distortion (left) and reliability (right) functions of the system in Example 4.1 when the components are IID with a common exponential distribution with mean 20 hours. The black curves stand for the original system, and the rest for the system given that the first failure of the components occurs at different times.

*Additionally, the probability that the system does not satisfy safety conditions is* $F_{1:4}(x_q) = 1 - u_q = 0.472428$*. This means that in* $47.24\%$ *of initial failures, we cannot guarantee that the prescribed condition in Case I will be met. Hence, we recommend following the security protocol established by the relevant authority.*

*On the other hand, it is possible to calculate the probability of accomplishing the mission given that the first failure of the components occurs after the critical time* $x_q$*.*

$$\Pr(T > t_M | X_{1:4} \geq x_q) = 1 - \left(1 - \frac{\bar{F}(t_M)}{\bar{F}(x_q)}\right)^4 = 0.9930893.$$

*Note that there is an increase in reliability with respect to the initial probability of accomplishing the mission.*

***Case II:*** $\Pr(T > t | X_{1:4} = x) \geq \Pr(T > t)$ *for all* $t \geq 0$.

*This case can be solved directly from Theorem 3.6. The critical time for case II is given by* $x_c = F^{-1}(1/4)$*, which in the considered exponential model leads to* $x_c = 5.753641$ *hours. Therefore,*

$$\Pr(T > t | X_{1:4} = x) \geq \Pr(T > t)$$

*for all* $t \geq 0$ *and* $x \geq 5.753641$*, i.e.,* $(T \mid X_{1:4} = x) \geq_{st} T$ *for all* $x \geq 5.753641$*. However, this comparison does not hold in the hazard rate order because the ratio of the distortion functions given by* $h(v)/H(\bar{F}(x_c), v)$ *or, equivalently, the ratio of the reliability functions* $\bar{F}_T(t)/\bar{F}_{T|X_{1:4}=x_c}(t)$ *are not monotonic, see Figure 3.*

*We calculate the probability that the system does not satisfy the security condition given in Case II.*

$$\Pr(X_{1:4} \leq x_c) = 1 - \bar{F}_{1:4}(x_c) = 1 - \bar{F}(x_c)^4 = 1 - 0.75^4 = 0.6835938.$$

*Thus, the safety conditions do not hold in the* $68.36\%$ *of the first failures of the components. To improve this percentage, we need to increase the mean of the exponential distribution, i.e., enhancing the quality of the components.*

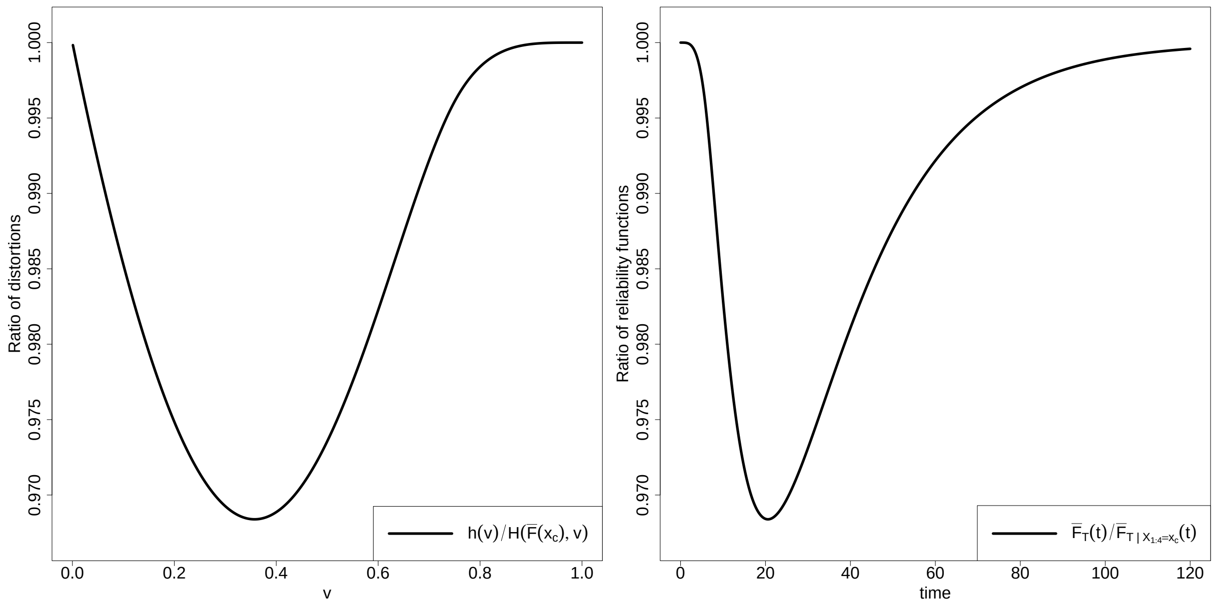


Figure 3: Ratio of distortion (left) and reliability (right) functions of the systems in Example 4.1 when $x_c = 5.753641$ and the components are IID with a common exponential distribution with mean 20 hours.

*Finally, we calculate the probability of accomplishing the mission given that the first failure of the components has exceeded the critical time $x_c$*

$$\Pr(T > t_M | X_{1:4} > x_c) = 1 - \left(1 - \frac{\bar{F}(t_M)}{\bar{F}(x_c)}\right)^4 = 0.998661.$$

Although the independent assumption for the component lifetimes is a common procedure, in several practical situations, this assumption might be unrealistic since the components share the same environment or load. Moreover, in the present analysis, the usual positive dependence among the components' lifetimes will lead to higher risks and the possible failure of the system requirements. This fact is shown in the following example.

**Example 4.2** *Let us consider the same system as in Example 4.1, but now the lifetime of the components $(X_1, X_2, X_3, X_4)$ have the following Farlie-Gumbel-Morgenstern (FGM) survival copula*

$$\widehat{C}_\alpha(u_1, u_2, u_3, u_4) = u_1 u_2 u_3 u_4 + \alpha(u_1 - u_1^2)(u_2 - u_2^2)(u_3 - u_3^2)(u_4 - u_4^2) \tag{4.2}$$

*for $u_1, u_2, u_3, u_4 \in [0, 1]$ and $\alpha \in [-1, 1]$. This copula induces a weak dependence on the components' lifetimes. It is positive for $\alpha > 0$ (the most usual case in practice) and negative for $\alpha < 0$. The independent case studied in Example 4.1 is obtained when $\alpha = 0$ (product copula).*

*To obtain the conditional distribution of the parallel system $T = X_{4:4}$ given that the first failure of the components occurs at time $x$, we calculate the following partial derivative*

$$\partial_1 \widehat{C}_\alpha(u_1, u_2, u_3, u_4) = u_2 u_3 u_4 + \alpha(1 - 2u_1)(u_2 - u_2^2)(u_3 - u_3^2)(u_4 - u_4^2).$$

*Hence, from (3.6), we get*

$$\Pr(T > t \mid X_{1:4} = x) = \frac{A_{4|1:4}(x, t)}{\bar{F}^3(x) + \alpha(1 - 2\bar{F}(x))\bar{F}^3(x)F^3(x)} = H_\alpha(\bar{F}(x), \bar{F}(t)),$$

*for all* $0 \le x < t$*, where*

$$\begin{aligned} A_{4|1:4}(x,t) &= 3\bar{F}^2(x)\bar{F}(t) + 3\alpha(1-2\bar{F}(x))\bar{F}^2(x)F^2(x)\bar{F}(t)F(t) \\ &\quad - 3\bar{F}(x)\bar{F}^2(t) - 3\alpha(1-2\bar{F}(x))\bar{F}(x)F(x)\bar{F}^2(t)F^2(t) \\ &\quad + \bar{F}^3(t) + \alpha(1-2\bar{F}(x))\bar{F}^3(t)F^3(t), \end{aligned}$$

*with* $\lim_{t\to+\infty} A_{4|1:4}(x,t) = 0$ *for all* $x \ge 0$ *and*

$$H_\alpha(u,v) = \begin{cases} \dfrac{3vu^2 - 3v^2u + v^3 + 3\alpha(1-2u)(u-u^2)^2(v-v^2)}{u^3 + \alpha(1-2u)(u-u^2)^3} \\ \quad + \dfrac{\alpha(1-2u)(v-v^2)^3 - 3\alpha(1-2u)(u-u^2)(v-v^2)^2}{u^3+\alpha(1-2u)(u-u^2)^3} & 0 \le v < u, \\ 1 & u \le v \le 1. \end{cases}$$

*Similarly, the system reliability function* $\bar{F}_T(t) = \Pr(X_{4:4} > t)$ *can be written as* $\bar{F}_T(t) = h(\bar{F}(t))$ *with*

$$h(v) = 4v - 6\widehat{C}(v,v,1,1) + 4\widehat{C}(v,v,v,1) - \widehat{C}(v,v,v,v)$$

*for* $v \in [0,1]$ *and any exchangeable survival copula* $\widehat{C}$*. In particular, for the FGM survival copula, we get*

$$h_\alpha(v) = 4v - 6v^2 + 4v^3 - v^4 - \alpha(v-v^2)^4$$

*for* $v \in [0,1]$*. Note that the distortion* $h_\alpha(v)$ *is decreasing in* $\alpha$*, as expected, that is, the worse reliability is obtained with the maximum positive dependence for this copula, i.e., with* $\alpha = 1$*. The distortion functions* $H_\alpha(u,v)$ *and* $h_\alpha(v)$ *are plotted in Figure 4 (left) for different values of* $u$ *and* $\alpha = 1$*. As we can see, these plots of* $H_1(u,v)$ *are decreasing in* $u$*, this suggests that* $\Pr(T > t \mid X_{1:4} = x)$ *is increasing in* $x$ *for all* $t \ge 0$*, or equivalently,* $T \uparrow_{SI} X_{1:4}$*. To prove that, we study the sign of* $\partial_{i,j} c_\alpha(u_1,u_2,u_3,u_4)$ *where* $c_\alpha$ *is the probability density function of the copula* $\widehat{C}_\alpha$ *and* $i \ne j$*.The pdf* $c_\alpha$ *of the FGM copula is*

$$c_\alpha(u_1,u_2,u_3,u_4) = 1 + \alpha(1-2u_1)(1-2u_2)(1-2u_3)(1-2u_4).$$

*Hence,*

$$\partial_{1,2}(u_1,u_2,u_3,u_4) = 4\alpha(1-2u_3)(1-2u_4) \ge 0$$

*for all* $\alpha \ge 0$*. We obtain similar results for the other second partial derivatives because* $c_\alpha$ *is exchangeable. Therefore, the random vector* $(X_1,X_2,X_3,X_4)$ *is MTP*$_2$ *for* $\alpha \ge 0$*. Hence, from Theorem 3.3, we conclude that* $T \uparrow_{SI} X_{i:4}$ *holds for* $i = 1,2,3$ *and* $\alpha \ge 0$*. Next, we study two possible scenarios for the safety conditions under the assumption* $\alpha = 1$*.*

***Case I:*** $\Pr(T > t|X_{1:4} = x) \ge q$ *for all* $t \in [0, t_M]$*.*

*Let us consider the same reliability for the components as in Example 4.1 (to compare the results). Firstly, we verify if the system satisfies the safety conditions, i.e.,* $\bar{F}_T(t_M) \ge q$ *with* $q = 0.976031$*. Recall that the mission time in this example is set at* $t_M = 10$ *hours.*

$$\bar{F}_T(t_M) = h_1(\bar{F}(t_M)) = 0.9727875 < q.$$

*Therefore, the system does not satisfy the established safety conditions, as expected, because the positive dependence among the components reduces the reliability of a parallel system. For illustrative purposes, we change the minimum reliability required and set it at* $q = 0.9727875$*.*

*We search for the value* $x_q$ *such that* $\bar{F}_{T|X_{1:4}=x_q}(t_M) = \bar{F}_T(t_M) = q$*. Hence, we solve numerically the following equation*

$$H_1(\bar{F}(x_q), h_1^{-1}(q)) = q, \tag{4.3}$$

*obtaining that* $\bar{F}(x_q) = 0.8587113$*. Note that the distortion* $h_1$ *is continuous and strictly increasing, then there exists the inverse of* $h_1$*, denoted by* $h_1^{-1}$*.*

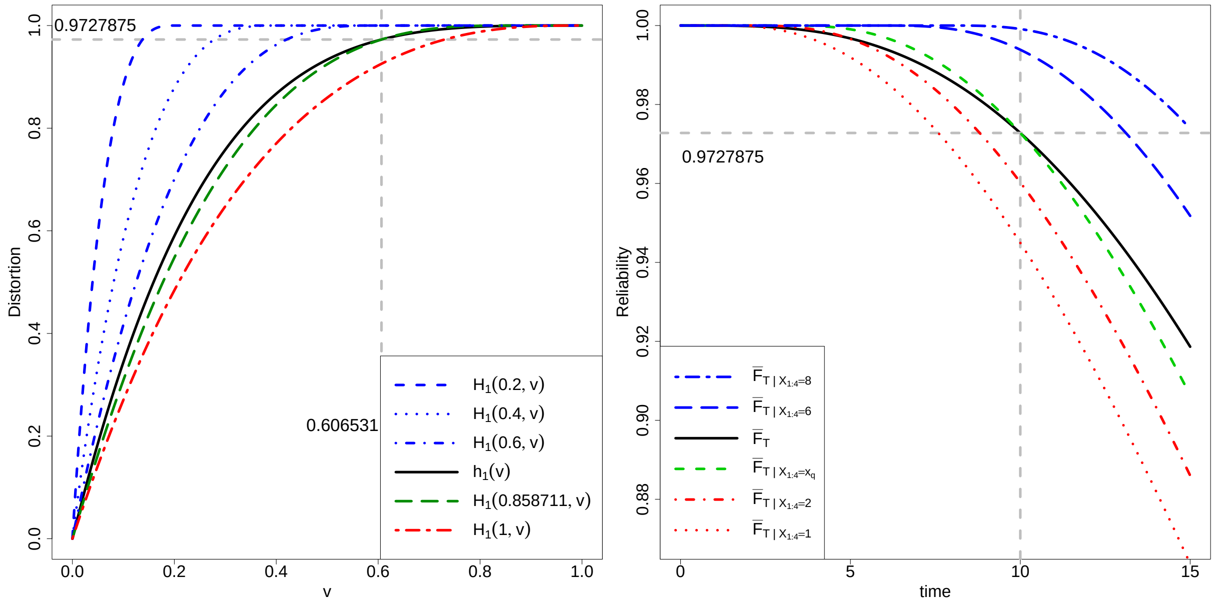


Figure 4: Distortion (left) and reliability (right) functions of the system in Example 4.2 when the components are ID with an FGM survival copula ($\alpha = 1$) and common exponential distribution with mean 20 hours. The black curves stand for the original system, and the rest for the system given that the first failure of the components occurs at different times.

*Assuming that the components are exponentially distributed with a mean of* 20 *hours, then* $x_q = 3.046451$ *hours. Thereby, we can ensure that* $\Pr(T > t|X_{1:4} = x) \geq q$ *for all* $t \in [0, t_M]$ *and* $x \geq 3.046451$*. In Figure 4 (right), we plot the reliability functions for the system* $\bar{F}_T$ *(black) and for the conditional systems* $\bar{F}_{T|X_{1:4}=x}$ *with* $x = x_q = 3.046451$ *(green curve),* $x = 6, 8$ *(blue curves) and* $x = 1, 2$ *(red curves). We observe that the green and blue curves satisfy the safety conditions of case I, and the red curves do not.*

***Case II:*** $\Pr(T > t|X_{1:4} = x) \geq \Pr(T > t)$ *for all* $t \geq 0$*.*

*To determine if there exists a value of* $x \geq 0$ *that satisfies the conditions of case II, we need to examine the infimum of the survival PELCoVs* $u_v$ *at risk* $v$ *for all* $v \in (0, 1)$*. First, we calculate* $u_v = \bar{F}_{1:4}(x) = \widehat{C}_1(\bar{F}(x), \bar{F}(x), \bar{F}(x), \bar{F}(x))$*, where* $\bar{F}(x)$ *is the solution of the equation:*

$$H_1(\bar{F}(x), h_1^{-1}(v)) = v$$

*for* $v \in (0, 1)$*. We obtain the solution of the above equation by using numerical methods. We have represented in Figure 5 (left) the plot of the survival PELCoVs* $u_v$ *for all* $v \in (0, 1)$*. The infimum of the survival PELCoVs is obtained as* $\lim_{v \to 0^+} u_v = 0.287847 > 0$*. Note that* $u_v$ *is not defined for* $v = 0$*. Then, there exists the critical time* $x_c$ *for case II and it can be calculated from the equation*

$$\bar{F}_{1:4}(x_c) = \widehat{C}_1(\bar{F}(x_c), \bar{F}(x_c), \bar{F}(x_c), \bar{F}(x_c)) = 0.287847.$$

*Thus, we obtain that* $\bar{F}(x_c) = 0.731523$*, and using that the components are exponentially distributed, we have that* $x_c = -20 \log 0.731523 = 6.252539$ *hours. The system satisfies the safety conditions for case II when the first failure of the components occurs at any time* $x \geq 6.252539$ *hours.*

*Finally, we show in Table 1 a comparison of the results obtained in Example 4.1 and Example 4.2. Observe that the critical time for case I is larger for independent components than for components with positive dependence (FGM copula with* $\alpha = 1$*) because we have reduced the minimum required reliability of the system and* $T \uparrow_{SI} X_{1:4}$*. As a result, the probability of satisfying the conditions in case I is higher in Example 4.2 than Example 4.1. However, the critical time for case II is larger for components with the FGM*

| | | Case I | | | | Case II | |
|---|---|---|---|---|---|---|---|
| | Dependence structure | $t_M$ | $q$ | $x_q$ | $\bar{F}_{1:4}(x_q)$ | $x_c$ | $\bar{F}_{1:4}(x_c)$ |
| Example 4.1 | Independence | 10 | 0.9760310 | 3.197350 | 52.76% | 5.753641 | 31.64% |
| Example 4.2 | FGM Copula ($\alpha = 1$) | 10 | 0.9727875 | 3.046451 | 54.40% | 6.252539 | 28.78% |

Table 1: Comparison of the results for a parallel system with 4 components based on their dependence structure.

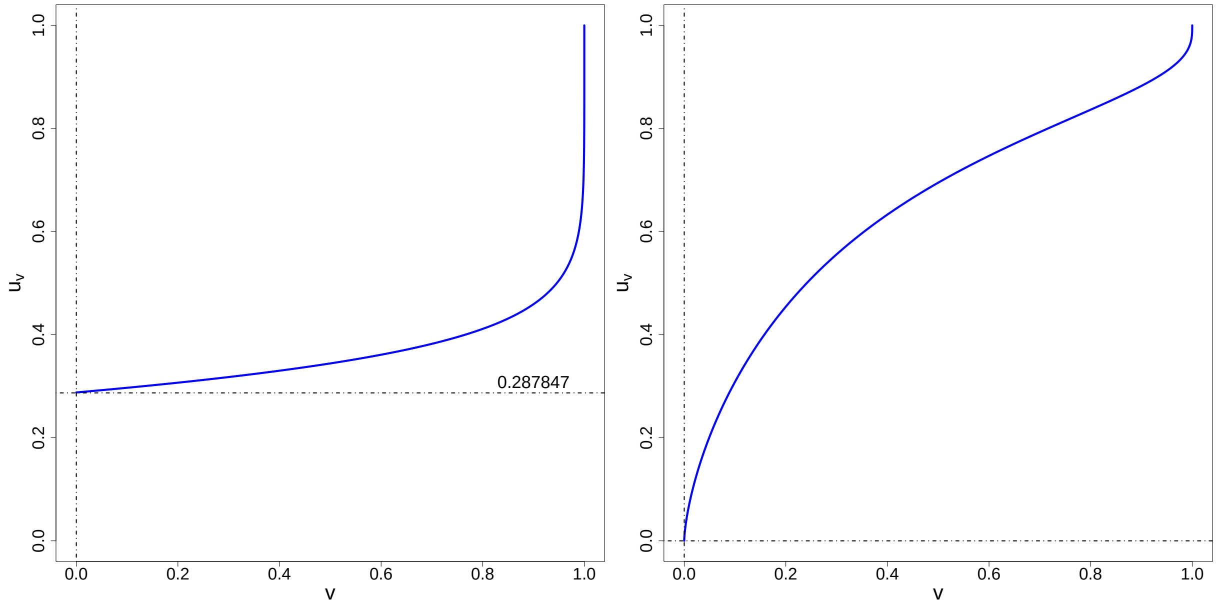


Figure 5: Survival PELCoV $u_v$ at risk $v$ of the random variable $X_{1:4}$ in Example 4.2 (left) and Example 4.3 (right).

*copula than for independent components. Therefore, the probability of satisfying the conditions in case II is higher in Example 4.1 than Example 4.2. Note that the reliability of a series system increases when the components have a positive dependence among them. In both examples, case I has a smaller critical time than case II, as expected, because case II has stronger safety requirements than case I.*

**Example 4.3** *We consider an unmanned aerial vehicle (UAV), commonly known as a drone, which participates in the search for survivors after a natural disaster such as an earthquake, a tsunami or a flooding. The use of these autonomous devices is especially relevant when the transportation systems, telecommunication systems and basic services are seriously damaged; see Ruiz-Estrada and Ndoma (2019) and Dukkanci et al. (2023) for a complete review of the use of UAVs in social logistics. In particular, we consider a quadcopter, i.e., a drone with four propellers. One interesting feature of many quadcopters is the ability to fly with only two of the four propellers. This capability is usually incorporated to improve safety and stability in case one or more motors fail during flight.*

*Let us assume that $X_i$ represents the lifetime of the ith propeller for $i = 1, 2, 3, 4$ and that the system's lifetime is $T = X_{3:4}$ because the system fails when the third failure occurs. Let us also assume that there is a positive dependence among the propellers because they share the same weather conditions. We will use a Clayton copula to model that dependence. Hence, in the case of ID components, we have that*

$$P[X_1 > t, X_2 > t, X_3 > t, X_4 > t] = \widehat{C}_\theta(\bar{F}(t), \bar{F}(t), \bar{F}(t), \bar{F}(t))$$

*where*

$$\widehat{C}_\theta(u_1, u_2, u_3, u_4) = (-3 + u_1^{-\theta} + u_2^{-\theta} + u_3^{-\theta} + u_4^{-\theta})^{-1/\theta} \tag{4.4}$$

*for $u_1, u_2, u_3, u_4 \in (0,1]$ and $\theta > 0$. This copula can be used only to model positive dependence (see Nelsen , 2006, p. 152 for further properties). The system reliability function is*

$$\bar{F}_T(t) = 6\widehat{C}_\theta(\bar{F}(t), \bar{F}(t), 1, 1) - 8\widehat{C}_\theta(\bar{F}(t), \bar{F}(t), \bar{F}(t), 1) + 3\widehat{C}_\theta(\bar{F}(t), \bar{F}(t), \bar{F}(t), \bar{F}(t)) = h_\theta(\bar{F}(t))$$

*for $t \geq 0$, where $h_\theta(v) = 6(-1+2v^{-\theta})^{-1/\theta} - 8(-2+3v^{-\theta})^{-1/\theta} + 3(-3+4v^{-\theta})^{-1/\theta}$ for all $v \in (0,1]$. Let us consider $\theta = 2$, and exponentially distributed components with mean 17.914 hours. If the mission time is fixed to be $t_M = 4$ hours due to battery limitations and the minimum reliability required is $q = 0.9$, then it is clear that $\Pr[T \geq t] \geq q$ for all $t \in [0, t_M]$ because $\bar{F}_T(t_M) = 0.9000045$. The plots of the distortion function $h_2(v)$ and the system reliability function $\bar{F}_T(t)$ are displayed in Figure 6 (left and right, respectively).*

*From Proposition 3.4 in Navarro and Buono (2023), we calculate the conditional reliability function of the system given that the first failure of the components occurs at time $x$.*

$$\Pr(T > t \mid X_{1:4} = x) = \frac{3\partial_1\widehat{C}_2(\bar{F}(x), \bar{F}(x), \bar{F}(t), \bar{F}(t)) - 2\partial_1\widehat{C}_2(\bar{F}(x), \bar{F}(t), \bar{F}(t), \bar{F}(t))}{\partial_1\widehat{C}_2(\bar{F}(x), \bar{F}(x), \bar{F}(x), \bar{F}(x))} \tag{4.5}$$

*for all $x < t$ such that $\partial_1\widehat{C}_2(\bar{F}(x), \bar{F}(x), \bar{F}(x), \bar{F}(x)) > 0$ and $\lim_{v\to 0^+} 3\partial_1\widehat{C}_2(u,u,v,v) - 2\partial_1\widehat{C}_2(u,v,v,v) = 0$. After some calculations, we have that*

$$3\partial_1\widehat{C}_2(u,u,v,v) - 2\partial_1\widehat{C}_2(u,v,v,v) = v^3\left(\frac{3\sqrt{-3u^2v^2+2v^2+2u^2}}{(-2v^2+u^2(-2+3v^2))^2} - \frac{2\sqrt{-3u^2v^2+v^2+3u^2}}{(v^2-3u^2(-1+v^2))^2}\right)$$

*and, hence, $\lim_{v\to 0^+} 3\partial_1\widehat{C}_2(u,u,v,v) - 2\partial_1\widehat{C}_2(u,v,v,v) = 0$ holds. Therefore, from (4.5) we have that $\Pr(T > t|X_{1:4} = x) = H_2(\bar{F}(x), \bar{F}(t))$, where*

$$H_2(u,v) = \begin{cases} \left(-3+\dfrac{4}{u^2}\right)^{3/2}\left(\dfrac{3}{\left(-3+\dfrac{2}{u^2}+\dfrac{2}{v^2}\right)^{3/2}} - \dfrac{2}{\left(-3+\dfrac{1}{u^2}+\dfrac{3}{v^2}\right)^{3/2}}\right) & 0 < v \leq u, \\ 1 & u \leq v \leq 1. \end{cases}$$

*To prove that $T \uparrow_{SI} X_{1:4}$, i.e., $H_2(u,v)$ is decreasing in $u$ for all $v \in (0,1)$, we need to show that the Clayton copula in (4.4) is $MTP_2$. Firstly, note that $\widehat{C}_\theta$ is an Archimedean copula for $\theta > 0$ with $\psi_\theta(x) = (1+x)^{-1/\theta}$ the corresponding copula generator. From Theorem 2.11 in Müller and Scarsini (2005), we have that $\widehat{C}_\theta$ is $MTP_2$ if, and only if $\frac{d^4}{dx^4}\psi_\theta(x)$ is log-convex.*

*After some calculations, we get that*

$$\frac{d^2}{dx^2}log(\Psi_\theta(x)) = \frac{4\theta+1}{\theta(1+x)^2} \geq 0,$$

*where $\Psi_\theta(x) = \frac{d^4}{dx^4}\psi_\theta(x)$ and therefore $\widehat{C}_\theta$ is $MTP_2$ for all $\theta > 0$. The plots of $H_2(u,v)$ for different values of $u$ are shown in Figure 6 (left). As in the previous example, we study two possible scenarios for the safety conditions:*

***Case I:*** $\Pr(T > t|X_{1:4} = x) \geq q$ *for all $t \in [0, t_M]$.*

*Firstly, we calculate the value $x_q$ as the solution of the following equation:*

$$H_2(\bar{F}(x_q), h_2^{-1}(q)) = q \tag{4.6}$$

*We solve the equation (4.6) by using numerical methods, obtaining $\bar{F}(x_q) = 0.88252285$. Therefore, the critical time $x_q$, in the case of exponentially distributed components with mean 17.914 hours, is given by*

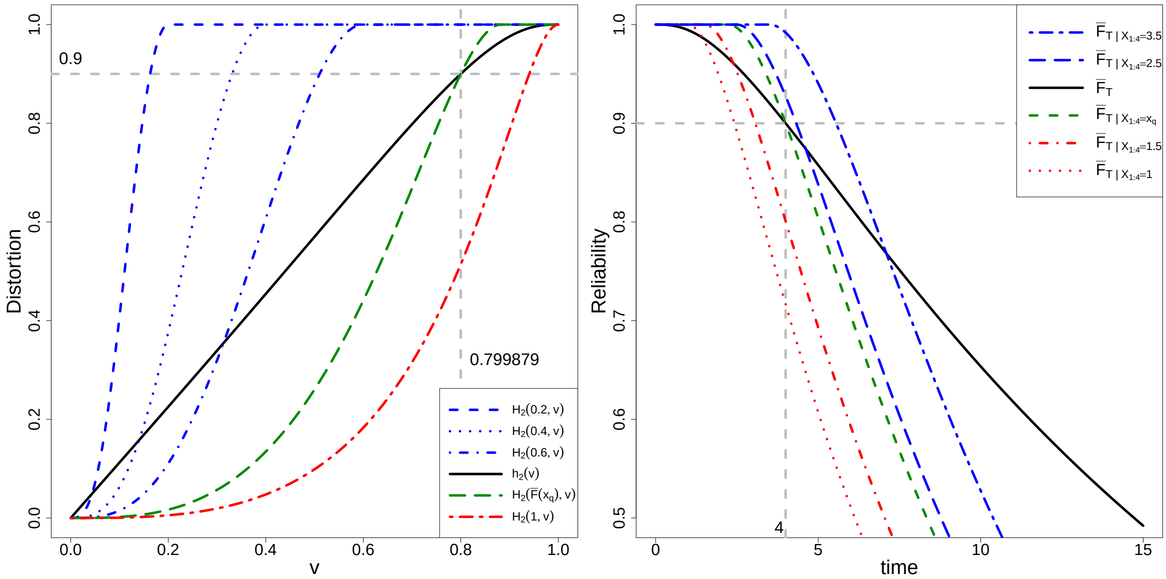


Figure 6: Distortion (left) and reliability (right) functions of the system in Example 4.3 when the components are ID with a Clayton copula and common exponential distribution with mean 17.914 hours. The black curves stand for the original system and the rest for the system given that the first failure of the components occurs at different times.

$x_q = -17.914 \log 0.88252285 = 2.238723$ *hours. Thereby, we can ensure the safety conditions of case I when the first failure of the components occurs at* $x \geq 2.238723$*. On the other hand, the required safety conditions may not be satisfied when the first failure occurs at any time in the interval* $[0, 2.238723)$*. This situation is displayed in the plots shown in Figure 6 (right). The blue curves represent the reliability of the system, given that the first failure occurs at times* $x = 2.5$ *and* $x = 3.5$*. The red curves are obtained by choosing* $x = 1$ *and* $x = 1.5$*.*

*We can provide the probability of the system no longer complying with the established safety conditions after the first component failure. Using the value of* $\bar{F}_{1:4}(x_q) = \widehat{C}_2(\bar{F}(x_q), \bar{F}(x_q), \bar{F}(x_q), \bar{F}(x_q)) = 0.684258$*, we can calculate* $\Pr(X_{1:4} \leq x_q) = 1 - 0.684258 = 0.315742$*. This means that in* $31.57\%$ *of first failures, we will not be able to guarantee the prescribed safety conditions. Note that this probability can be reduced by using components with higher quality.*

*Furthermore, it is possible to calculate the probability of accomplishing the mission given that the first failure of the components occurs after the critical time* $x_q$*.*

$$\Pr(T > t_M | X_{1:4} \geq x_q) = \frac{6\widehat{C}_2(\bar{F}(x_q), \bar{F}(x_q), \bar{F}(t_M), \bar{F}(t_M)) - 8\widehat{C}_2(\bar{F}(x_q), \bar{F}(t_M), \bar{F}(t_M), \bar{F}(t_M))}{\widehat{C}_2(\bar{F}(x_q), \bar{F}(x_q), \bar{F}(x_q), \bar{F}(x_q))} + \frac{3\widehat{C}_2(\bar{F}(t_M), \bar{F}(t_M), \bar{F}(t_M), \bar{F}(t_M))}{\widehat{C}_2(\bar{F}(x_q), \bar{F}(x_q), \bar{F}(x_q), \bar{F}(x_q))} = 0.992958.$$

***Case II:*** $\Pr(T > t | X_{1:4} = x) \geq \Pr(T > t)$ *for all* $t \geq 0$*.*

*Studying the requirements in case II is equivalent to checking if* $H_2(u, v) \geq h_2(v)$ *holds for all* $v \in [0, 1]$ *and for some values of* $u$*. However, if we define the function* $d_2(u, v) = H_2(u, v) - h_2(v)$*, then it is straightforward to see that* $\lim_{v \to 0^+} \frac{\partial}{\partial v} d_2(u, v) = (-9 - 18\sqrt{2} + 16\sqrt{3})/6 < 0$ *for all* $u \in (0, 1)$*. Therefore, for all* $u \in (0, 1)$ *there exists* $\varepsilon > 0$ *such that* $H_2(u, v) < h_2(v)$ *for all* $v \in (0, \varepsilon)$*. This means that the condition* $\Pr(T > t | X_{1:4} = x) \geq \Pr(T > t)$ *does not hold for all* $t \geq 0$ *and any value* $x$*. Observe the curves* $\bar{F}_{T|X_{1:4}=x}(t)$ *for different values of* $x$*, in Figure 6 (right), which intersect with the system reliability function* $\bar{F}_T(t)$*.*

*We have calculated numerically the survival PELCoV at risk* $v = h_2(\bar{F}(t))$ *for all* $t \geq 0$*, represented in*

*Figure 5 (right). As we can see, the infimum of the survival PELCoVs is equal to zero. Hence, as we have proved above, the requirements in case II do not hold.*

*Finally, note that* $\Pr(T > t|X_{1:4} = x) \geq \Pr(T > t)$ *for all* $t \in [0, t_M]$ *and* $x \geq x_q$*, see the green and black curves in Figure 6 (left).*

# 5 A non-parametric estimator of the critical time at risk level $q$

As discussed above, the critical time $x_q$ is linked to studying the survival PELCoV $u_q = \bar{F}_{1:4}(x_q)$. This survival PELCoV represents the threshold level of the first component failure from which the lifetime of the system given the first failure dominates the unconditional system lifetime. As seen in Section 3, the survival PELCoV at level $q$ can be uniquely determined by the PELCoV at level $1-q$ as $u_q = 1 - s_{1-q}$. Therefore, following the approach of Ortega-Jiménez et al. (2022), this section is devoted to obtaining a non-parametric estimator of the PELCoV at risk level $v$.

In this section, we consider a random vector $(X, Y)$ with copula $A$ and absolutely continuous strictly increasing marginal distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$ and strictly increasing conditional distribution $F_{Y|X=x}(\cdot)$, for all $x$, on their respective supports. Under this scenario, as seen in Ortega-Jiménez et al. (2022), if $Y \uparrow_{SSI} X$ then the PELCoV $s_v$ is determined by the copula $A$ as $\partial_1 A(s_v, v) = v$. Note that, under the framework established throughout the paper, $Y$ represents the system lifetime $T$ and $X$ the first component failure $X_{1:n}$. Let us remark that the copula $A$ models the dependence structure between $T$ and $X_{1:n}$ and it shouldn't be confused with the copula among the lifetime of the components of a system. We must also remark that this representation is different from the distortion representation used in Section 3. Here, the joint distribution of $X_{1:n}$ and $T$ can be written using $A$ as

$$\Pr(X_{1:n} \leq x, T \leq y) = A(F_{1:n}(x), F_T(y)).$$

Both $F_{1:n}$ and $F_T$ have distortion representations from the distribution functions of the components, see Navarro (2022).

## 5.1 Background on the non-parametric estimator of Markov Kernels

Since the critical point is determined by the copula $A$, the non-parametric estimator of the critical point relies on an estimator of the copula. To define the estimator, it is important to recall the notions of Markov kernels and weak conditional convergence. The interested reader is referred to Kasper et al. (2021) for further discussion on these notions. Note that, for every metric space $(S, d)$, the Borel $\sigma$-field on $S$ is denoted by $\mathcal{B}(S)$. A Markov kernel from $\mathbb{R}$ to $\mathbb{R}$ is a mapping $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that for every fixed $E \in \mathcal{B}(\mathbb{R})$ the mapping $x \mapsto K(x, E)$ is (Borel-)measurable and for every fixed $x \in \mathbb{R}$ the mapping $E \mapsto K(x, E)$ is a probability measure. Considering two real-valued random variables $X, Y$ we say that a Markov kernel $K$ is a regular conditional distribution of $Y$ given $X = x$ if $K(x, E) = \Pr(Y \in E \mid X = x)$ holds almost surely for every $E \in \mathcal{B}(\mathbb{R})$. If $C$ is a copula, then we will denote the Markov kernel of $C$ as $K_C(\cdot)$ and consider it as a map $K_C : [0, 1] \times \mathcal{B}([0, 1]) \to [0, 1]$. Note that, if copulas are differentiable, the identity $\partial_1 C(u, v) = K_C(u, [0, v])$ holds. We can now recall the notion of weak conditional convergence.

**Definition 5.1 (Weakly conditional convergence)** *Suppose that* $C, C_1, C_2, \ldots$ *are copulas and let* $K_C$*,* $K_{C_1}, K_{C_2}, \ldots$ *be the corresponding Markov kernels. We will say that* $(C_m)_{m\in\mathbb{N}}$ *converges weakly conditional to* $C$ *if for* $\lambda$*-almost every* $x \in [0, 1]$ *we have that the sequence* $(K_{C_m}(x, \cdot))_{m\in\mathbb{N}}$ *of probability measures on* $\mathcal{B}([0, 1])$ *converges weakly to the probability measure* $K_C(x, \cdot)$*. In the latter case, we will write* $C_m \xrightarrow{wcc} C$ *(where 'wcc' stands for 'weak conditional convergence').*

For a copula $C$ and $\gamma \in (0, 1)$, we consider the map $\delta^{=}_{\gamma,C} : [0, 1] \to [0, 1]$ given by $\delta^{=}_{\gamma,C}(t) := K_C(\gamma, [0, t])$. The following continuity result by Fuchs and Trutschnig (2020) provides an estimator of $\partial_1 C(\gamma, F_Y(y))$.

**Theorem 5.2 (Fuchs and Trutschnig (2020))** *Consider a sequence of copulas* $(C_m)_{m\in\mathbb{N}}$ *converging weakly conditional to* $C$ *and a sequence of continuous univariate distribution functions* $(F_{Y,m})_{m\in\mathbb{N}}$ *converging weakly* *to $F_Y$. Then there exists a set $\Lambda \in \mathcal{B}((0,1))$ with $\lambda(\Lambda) = 1$ such that for every $\gamma \in \Lambda$*

$$\lim_{m\to\infty} \left(\delta^{=}_{\gamma,c_m} \circ F_{Y,m}\right)(y) = \left(\delta^{=}_{\gamma,C} \circ F_Y\right)(y) \tag{5.1}$$

*for every $y \in \mathbb{R}$ with $K_C(\gamma, \{F_Y(y)\}) = 0$. Additionally, if either (a) $F_Y$ is strictly increasing or (b) the map $v \mapsto K_C(\gamma, [0, v])$ is continuous, then Equation (5.1) holds for all but at most countable infinitely many $y \in \mathbb{R}$ (interpreting the expressions as conditional distribution functions on $\mathbb{R}$ we hence have weak convergence).*

## 5.2 Empirical estimator of the PELCoV

It is important to remark that, given a sequence of copulas converging weakly conditional to $A$, the convergence of the Markov Kernels is not uniform. Although Egoroff's Theorem states that, under pointwise convergence almost everywhere, the uniform convergence holds everywhere except on some subset of arbitrarily small measure. Note that given a set $E$, we denote by $E^{\mathrm{C}}$ the complement of $E$.

**Theorem 5.3 (Egoroff's Theorem)** *Suppose that $\mu$ is a finite measure, and that $f_m, f : X \to \mathbb{C}$ are measurable. If $f_m \to f$ pointwise a.e., then for every $\varepsilon > 0$ there exists a measurable $E \subseteq X$ such that $\lambda(E^{\mathrm{C}}) < \varepsilon$, and $f_m$ converges uniformly to $f$ on $E$, i.e., $\lim_{m\to\infty} \sup_{x\in E} |f(x) - f_m(x)| = 0$.*

Note that if $Y \uparrow_{SSI} X$, since $u \mapsto \partial_1 A(u, v)$ is strictly decreasing, $s_v = \lambda(U)$ where

$$U = \{u : \partial_1 A(u, v) > v\}.$$

Therefore, considering a sequence of copulas $(A_m)_{m\in\mathbb{N}}$ converging weakly conditional to $A$ and a sequence of continuous univariate distribution functions $(F_{Y,m})_{m\in\mathbb{N}}$ converging weakly to $F_Y$, we can define the non-empirical estimator of the PELCoV on the following form:

$$s_{m,v} = \lambda(U_m), \quad \text{with} \quad U_{m,v} = \{u : \left(\delta^{=}_{u,A_m} \circ F_{Y,m}\right)(F_Y^{-1}(v)) \geq v\}.$$

The consistency of the estimator follows in the following result.

**Proposition 5.4 (Consistency of the estimator)** *Consider a sequence of copulas $(A_m)_{m\in\mathbb{N}}$ converging weakly conditional to $A$ and a sequence of continuous univariate distribution functions $(F_{Y,m})_{m\in\mathbb{N}}$ converging weakly to $F_Y$. Let us assume that $A$ is absolutely continuous, $\partial_1 A(u, v)$ is strictly decreasing in $u$ for all $v$ and $F_Y$ is strictly increasing. Let us define $U_{m,v} = \{u : \left(\delta^{=}_{u,A_m} \circ F_{Y,m}\right)(F_Y^{-1}(v)) \geq v\}$. Then*

$$\lim_{m\to\infty} \lambda(U_{m,v}) = s_v.$$

**Proof.** By Theorem 5.2, fixed $y = F_Y^{-1}(v) \in \mathbb{R}$, there exists a set $\Lambda \in \mathcal{B}((0,1))$ with $\lambda(\Lambda) = 1$ such that for every $\gamma \in \Lambda$, $\lim_{m\to\infty} \left(\delta^{=}_{\gamma,A_m} \circ F_{Y,m}\right)(y) = \left(\delta^{=}_{\gamma,A} \circ F_Y\right)(y)$, that is $\left(\delta^{=}_{\gamma,A_m} \circ F_{Y,m}\right)(y) \to \left(\delta^{=}_{\gamma,A} \circ F_Y\right)(y)$ a.e.

Fixed $\varepsilon > 0$, by Theorem 5.3, there exists a measurable $E \subseteq X$ such that $\lambda(E^{\mathrm{C}}) < \frac{\varepsilon}{2}$, and we have $\lim_{m\to\infty} \sup_{\gamma\in E} \left|\left(\delta^{=}_{\gamma,A} \circ F_Y\right)(y) - \left(\delta^{=}_{\gamma,A_m} \circ F_{Y,m}\right)(y)\right| = 0$. Now, let us consider

$$\eta = \max\left\{\left(\delta^{=}_{s_v-\frac{\epsilon}{2},A} \circ F_Y\right)(y) - F_Y(y), F_Y(y) - \left(\delta^{=}_{s_v+\frac{\epsilon}{2},A} \circ F_Y\right)(y)\right\}.$$

Then, there exists $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$ and for all $\gamma \in E$,

$$\left|\left(\delta^{=}_{\gamma,A} \circ F_Y\right)(y) - \left(\delta^{=}_{\gamma,A_m} \circ F_{Y,m}\right)(y)\right| \leq \eta$$

and, in particular, since $\partial_1 A(u, v) = \delta^{=}_{u,A}(v)$ is strictly decreasing in $u$:

- For all $\gamma < u_v - \frac{\epsilon}{2}$, $\left(\delta^{=}_{\gamma,A_m} \circ F_{Y,m}\right)(y) < F_Y(y)$.
- For all $\gamma > u_v - \frac{\epsilon}{2}$, $\left(\delta^{=}_{\gamma,A_m} \circ F_{Y,m}\right)(y) > F_Y(y)$.

Therefore, for $m \geq m_0$, $s_v - \frac{\epsilon}{2} < \lambda(U_m \cap E) < s_v + \frac{\epsilon}{2}$. Since $\lambda(U_m \cap E) = \lambda(U_m) - \lambda(E^{\mathrm{C}} \cap U_m)$ and $0 \leq \lambda(E^{\mathrm{C}} \cap U_m) \leq \frac{\epsilon}{2}$, $s_v - \epsilon < s_v - \frac{\epsilon}{2} < \lambda(U_m) < s_v + \epsilon$ and the assertion follows. ■

Following Fuchs and Trutschnig (2020), in order to build our estimator we consider checkerboard aggregations instead of empirical copulas. Note that, as discussed in Fuchs and Trutschnig (2020) and Junker et al. (2021), as the estimated curves require conditioning on events with zero probability, it is not possible to provide the estimator by simply plugging in the empirical copula $\hat{E}_m$, but the problem can be overcome aggregating it. Therefore, let $\mathcal{CB}_N(C)$ be the $N$-checkerboard approximation of $A \in \mathcal{C}$ with $N(m) := \lfloor \sqrt{m} \rfloor$, which roughly speaking, consist on two-dimensional histograms. As seen in Theorem 6.1. in Fuchs and Trutschnig (2020), $\left(\mathcal{CB}_{N(m)}\left(\hat{E}_m\right)\right)_{m\in\mathbb{N}}$ converges weakly conditional to A with probability one. Hence, considering Proposition 5.4, this leads to a consistent estimator for the PELCoV.

**Corollary 5.5** *Suppose that $(X_1, Y_1), (X_2, Y_2), \ldots$ is a random sample from a random vector $(X, Y)$ with copula $A$ for which $\partial_1 A(u, v)$ is continuous in $v$. Let us assume that $F_Y$ is strictly increasing and $Y \uparrow_{SSI} X$. Then, for $v \in (0, 1)$, considering $U_{m,v} = \{u : (\delta^{=}_{u,\mathcal{CB}_{N(m)}(\hat{E}_m)} \circ F_{Y,m})(F_Y^{-1}(v)) \geq v\}$, it is verified that $\lim_{m\to\infty} \lambda(U_{m,v}) = s_v$.*

Note that, as remarked above and, since the survival PELCoV at level $q$ is given by $u_q = 1 - s_{1-q}$, we consider the estimator $1 - \lambda(U_{m,1-q})$.

### 5.3 Numerical simulations

We illustrate the performance of our estimator with some numerical simulations from the examples described in Section 4. The following simulations have been carried out considering random samples of the vector $(X_{1:n}, T)$, which have been obtained from random samples of the lifetimes of the components taking into account the corresponding copula. Note that we only estimate the survival PELCoV $u_q$ at level $q$. Since $x_q = F_{1:4}^{-1}(1 - u_q)$, the corresponding critical times $x_q$ considered in case I, can be obtained considering the empirical quantile of $X_{1:n}$ and the PELCoV estimation.

Let us remark that the examples refer to frameworks with $T \uparrow_{SI} X_{1:n}$. Although, as in all the examples $\Pr(T > t \mid X_{1:4} = x)$ increases in $x$ for $t > x$, the stochastic monotonicity is strict, assuming $T > X_{1:n}$ and since this is verified by the data. Therefore, we consider the estimator provided in Corollary 5.5. Note that, in order to compute the mass distribution of the empirical checkerboard copula, the R package *cort* (see Laverny (2020)) is employed to build the estimator.

| $q$ | 0.9 | 0.95 | 0.976031 | 0.99 |
|---|---|---|---|---|
| $u_q$ | 0.44500 | 0.48519 | 0.52757 | 0.57696 |
| $n = 10^3$ | 0.40734 (0.06009) | 0.45294 (0.06027) | 0.46309 (0.06379) | 0.45431 (0.06368) |
| $n = 10^4$ | 0.45183 (0.02561) | 0.47540 (0.02507) | 0.50592 (0.02612) | 0.52286 (0.02571) |
| $n = 10^5$ | 0.44743 (0.01272) | 0.48624 (0.01361) | 0.52294 (0.01376) | 0.55566 (0.01422) |
| $n = 10^6$ | 0.44596 (0.00472) | 0.48431 (0.00535) | 0.52653 (0.00545) | 0.56897 (0.00577) |
| $n = 10^7$ | 0.44493 (0.00218) | 0.48482 (0.00226) | 0.52718 (0.00225) | 0.57540 (0.00253) |

Table 2: Estimations (mean and sd) of the survival PELCoVs $u_q$ at level $q$ over 1000 replications in the simulation considering a parallel system with 4 independent components.

Following the approach of Section 4, we consider a parallel system with 4 independent components and exponentially distributed lifetimes with mean 20. We set the usual risk levels $q = 0.9, 0.95, 0.99$ jointly with the level $q = 0.9760310$ considered in Example 4.1. Note that, by (3.13) we know that the survival PELCoV at risk $q$ for a parallel system of $n$ independent components is given by

$$u_q = \left(\frac{1 - (1 - q)^{\frac{1}{n}}}{1 - (1 - q)^{\frac{1}{n-1}}}\right)^n.$$

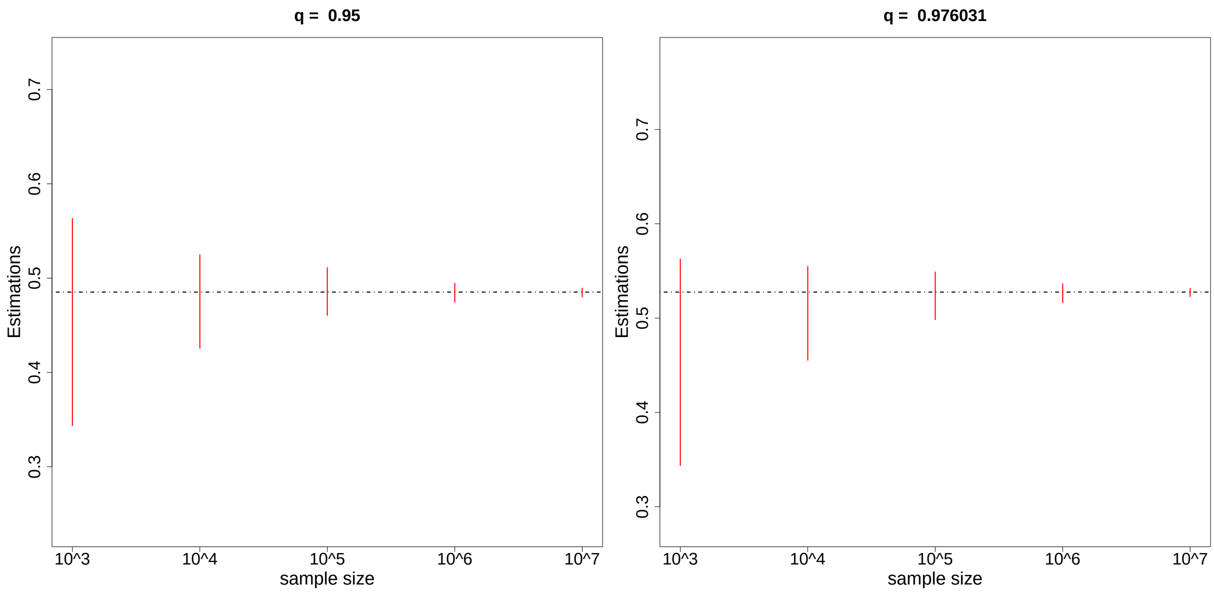


Figure 7: Confident intervals at 95% of survival PELCoVs at risk level $q = 0.95$ (left) and $q = 0.976031$ (right) jointly with the theoretical values (dashed line) for Example 4.1.

We have calculated the mean and standard deviation of 1000 replications for each sample size $n = 10^k$ with $k = 3, 4, 5, 6$ and 7. The results are displayed in Table 2. Figure 7 shows the confidence intervals at 95% for the survival PELCoVs estimated jointly with the theoretical values obtained from the expression (3.13). The error decreases as the sample size increases.

| $q$ | 0.9 | 0.95 | 0.9727875 | 0.99 |
|---|---|---|---|---|
| $u_q$ | 0.45905 | 0.50515 | 0.54395 | 0.60405 |
| $n = 10^3$ | 0.41531 (0.06033) | 0.45831 (0.06060) | 0.47872 (0.06333) | 0.47653 (0.06480) |
| $n = 10^4$ | 0.46059 (0.02399) | 0.49003 (0.02489) | 0.52268 (0.02507) | 0.53969 (0.02578) |
| $n = 10^5$ | 0.45937 (0.01270) | 0.50332 (0.01248) | 0.53725 (0.01395) | 0.57906 (0.01438) |
| $n = 10^6$ | 0.45949 (0.00514) | 0.50347 (0.00492) | 0.54078 (0.00501) | 0.59510 (0.00573) |
| $n = 10^7$ | 0.45885 (0.00210) | 0.50457 (0.00221) | 0.54265 (0.00225) | 0.60202 (0.00248) |

Table 3: Estimations (mean and sd) of the survival PELCoVs $u_q$ at level $q$ over 1000 replications in the simulation considering a parallel system with 4 components with lifetimes linked by an FGM survival copula of parameter $\alpha = 1$.

Following now the approach of Example 4.2, we consider a parallel system with 4 exponentially distributed components with mean 20 and whose lifetimes $(X_1, X_2, X_3, X_4)$ are linked with a Farlie-Gumbel-Morgenstern survival copula with parameter $\alpha = 1$, provided in (4.2). Again, we set the risk levels $q = 0.9, 0.95, 0.99$ together with the level $q = 0.9727875$ in Example 4.2. Theoretical values have been obtained by solving numerically the equation (4.3). The results are given in Table 3.

Finally, according to the framework described in Example 4.3, we consider a 3-out-of-4:F system formed by 4 exponentially distributed components with mean 17.914. The lifetimes of the components are linked by a Clayton copula with parameter $\theta = 2$, provided in (4.4). Analogously as above, theoretical values have been obtained by solving numerically the equation (4.6). We consider risk levels $q = 0.9, 0.95, 0.99$, where $q = 0.9$ is the risk level considered in Example 4.3. The results are shown in Table 4.

| $q$ | 0.9 | 0.95 | 0.99 |
|---|---|---|---|
| $u_q$ | 0.68426 | 0.742661 | 0.837536 |
| $n = 10^3$ | 0.63359 (0.03024) | 0.69975 (0.03437) | 0.74069 (0.02770) |
| $n = 10^4$ | 0.67592 (0.01154) | 0.71958 (0.01209) | 0.78921 (0.01213) |
| $n = 10^5$ | 0.68047 (0.00529) | 0.73903 (0.00557) | 0.82221 (0.00578) |
| $n = 10^6$ | 0.68347 (0.00205) | 0.74024 (0.00216) | 0.83151 (0.00226) |
| $n = 10^7$ | 0.68374 (0.00089) | 0.74189 (0.00090) | 0.83652 (0.00090) |

Table 4: Estimations (mean and sd) of the survival PELCoVs $u_q$ at level $q$ over 1000 replications in the simulation, considering the system in Example 4.3 when the components are ID and dependent with a Clayton copula of parameter $\theta = 2$.

# 6 Conclusions

In the field of mission-oriented systems subject to high safety requirements, this paper presented a new methodology to study risk times considering coherent systems with homogeneous and possibly dependent components. Given that the first failure of the components is known, we have studied the critical times from which the conditional system lifetime satisfies two different safety policies. Under this framework, sufficient conditions were provided for critical times in both safety policies, and a systematic methodology was used to calculate the critical times numerically. Furthermore, the explicit expressions of such critical times were determined for the particular case of parallel systems with IID components. Illustrative examples of coherent systems with dependent components were also provided.

In order to enable a data-driven procedure for the aforementioned analysis of the critical times, a non-parametric estimator was developed. We revisit the examples considered from a theoretical approach, now using numerical simulations, to illustrate the convergence of the estimator (see Tables 2-4). Note that the suggested estimator is just a first step. From a practical perspective, there are still some limitations. Firstly, it is not always possible to guarantee the positive dependence condition between $T$ and $X_{1:n}$ (we have provided a counterexample). Secondly, note that the estimator is constructed using checkerboard copulas with a uniform grid. Therefore, as can be intuitively expected, Tables 2-4 show that the convergence is slower for risk values near $q = 1$. Determining alternative estimators and studying the convergence rate remains an open question of future research.

The methodology proposed in this paper is based on the notion of Probability Equivalent Level of CoVaR-VaR (PELCoV), which is first defined in the context of Risk Theory. An extension of the methodology can be provided to study critical times when the second or subsequent failures of the components are known. Hence, more elaborate risk criteria can be built for alternative policies. Even though the primary goal of this article is not to create a mission abort policy, the findings of our study can be used, in conjunction with other criteria, to suggest mission abort policies.

The approach followed in this paper is based on the assumption of homogeneous components. While this assumption may be too restrictive in certain applications, it provides a solid foundation for future research. The main task for future studies is to extend these results to mission-oriented systems with heterogeneous components, which would significantly broaden the applicability and impact of the methodology.

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