FROM GENERALIZED RIESZ BASES TO SEMI-FRAMES AND BEYOND

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Abstract. The main purpose of this paper is to review the notion of continuous frame in a Hilbert space $\mathcal{H}$ and their various generalizations. We follow essentially two different but related paths. The first one consists in considering continuous families of vectors that are the image of a resolution of the identity through some operator. As for the second, we consider the sesquilinear forms in Hilbert space associated to given continuous families of vectors of $\mathcal{H}$. Along this path, we begin introducing and discussing continuous Riesz bases. Next we turn to semi-frames, both upper and lower. On the way we point the connection between lower semi-frames and metric operators, a familiar tool in the theory of the so-called $\mathcal{PT}$-symmetric quantum mechanics. This panorama includes more recent topics like A-frames (i.e. frames controlled by some densely defined operator A), reproducing pairs and, going beyond Hilbert spaces, bases and frames in a Rigged Hilbert space (typically a space of distributions).

1. Prologue: Frames

In 1952 Duffin and Schaefer, considering some deep questions in nonharmonic Fourier series, were led to weaken Parseval identity for an orthonormal (ON) basis $\{e_n\}_{n\in\mathbb{N}}$ of a separable Hilbert space $\mathcal{H}$

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle f|e_n\rangle|^2, \quad \forall f \in \mathcal{H}$$

and introduced *frames*.

In fact if $\mathcal{H}$ is a separable Hilbert space with inner product $\langle\cdot|\cdot\rangle$, a family of vectors $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a frame in $\mathcal{H}$ if there exist $\alpha, \beta > 0$ such that

$$\alpha\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f|\phi_n\rangle|^2 \leq \beta\|f\|^2, \qquad \forall f \in \mathcal{H}.$$

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This structure allows reconstruction of vectors: if $\{\phi_n\}_{n\in\mathbb{N}}$ is a frame then, for every $f \in \mathcal{H}$

$$f = \sum_{n=1}^{\infty} c_n(f)\phi_n,$$

where $\{c_n(f)\}_{n\in\mathbb{N}}$ is an adequate sequence of complex numbers. However this decomposition is, in general, not unique.

This notion was extended to a continuous environment by Ali, Antoine and Gazeau [1] and Kaiser [19] in the '90.

Let (X, Σ, μ) be a measure space. A *continuous frame* for $\mathcal{H}$ is a function $\phi : x \in X \to \phi_x \in \mathcal{H}$ for which

i) $\forall f \in \mathcal{H}$, $x \to \langle f|\phi_x\rangle$ is a measurable function on X [1]

ii) there exist constants $\alpha, \beta > 0$ such that

$$\alpha\|f\|^2 \leq \int_X |\langle f|\phi_x\rangle|^2 d\mu(x) \leq \beta\|f\|^2, \ \forall f \in \mathcal{H}.$$

Reconstruction sounds as follows: $f = \int_X \xi(x)\phi_x d\mu(x)$ in a sense to be precised. But again we do not get uniqueness of the representation. The non-uniqueness and thus *redundancy* is probably the main reason for the popularity of frames. In fact every $f \in \mathcal{H}$ may have infinitely many representations w.r.to the frame. This is crucial for applications: we can sustain a loss of data without a dramatic loss of information, a very important fact in signal analysis or internet coding etc.

However, in several applications (mainly in Quantum Physics), the notion of frame appeared to be too restrictive. For this reason, several generalizations have been proposed, generalized Riesz bases, semi-frames (upper and lower), and so on. We will overview many of them in the sequel of this paper.

2. Basic aspects

Given an operator A on $\mathcal{H}$, we will denote by $D(A)$ its domain, by $N(A)$ its null space and by $R(A)$ its range. Moreover, if $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert spaces, by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ we will indicate the space of bounded linear operators on $\mathcal{H}$ into $\mathcal{K}$. When $\mathcal{H} = \mathcal{K}$, we will simply write $\mathcal{B}(\mathcal{H})$.

Let (X, Σ, μ) be a measure space with the following properties: X is a locally compact Hausdorff topological space, Σ is the σ-algebra of Borel sets, μ is a positive regular measure on Σ; this in particular implies that $\mu(K) < \infty$ for every K compact subset of X.

[1]Equivalently, if $x \mapsto \phi_x$ is a weakly measurable function on X

Let $\phi : x \in X \mapsto \phi_x \in \mathcal{H}$ be a μ-weakly measurable function; that is, for every $f \in \mathcal{H}$, $x \mapsto \langle f|\phi_x\rangle$ is a μ-measurable complex function. From now on we will not make explicit mention of μ, if no ambiguity can arise.

We put

$$D(\Omega_\phi) = \left\{ f \in \mathcal{H} : \int_X |\langle f|\phi_x\rangle|^2 d\mu < \infty \right\}. \tag{2.1}$$

Then

$$\Omega_\phi(f,g) = \int_X \langle f|\phi_x\rangle \langle \phi_x|g\rangle \, d\mu, \quad f,g \in D(\Omega_\phi) \tag{2.2}$$

is a well-defined nonnegative sesquilinear form with domain $D(\Omega_\phi)$. If $D(\Omega_\phi)$ is dense in $\mathcal{H}$, then Ω_ϕ is a closed positive sesquilinear form and so, by the representation theory of closed positive sesquilinear forms, there exists a unique positive self-adjoint operator S_ϕ in $\mathcal{H}$ such that $D(\Omega_\phi) = D(S_\phi^{1/2})$ and

$$\Omega_\phi(f,g) = \left\langle S_\phi^{1/2} f | S_\phi^{1/2} g \right\rangle, \quad \forall f,g \in D(\Omega_\phi).$$

The operator S_ϕ has been called the *generalized frame operator* [3]. It satisfies

$$D(S_\phi) \subset D(\Omega_\phi) \text{ and } \Omega_\phi(f,g) = \langle S_\phi f|g\rangle, \ \forall f,g \in D(S_\phi). \tag{2.3}$$

We denote by C_ϕ the linear operator $C_\phi : D(C_\phi) \to L^2(X,\mu)$ with $D(C_\phi) = D(\Omega_\phi)$ defined by

$$(C_\phi f)(x) = \langle f|\phi_x\rangle.$$

The operator C_ϕ is called the *analysis operator* of ϕ.

As for Ω_ϕ we have

$$\Omega_\phi(f,g) = \langle C_\phi f|C_\phi g\rangle_2, \quad \forall f,g \in D(\Omega_\phi).$$

Remark 2.1. C_ϕ is a closed operator, but in general is not densely defined, therefore the adjoint operator C_ϕ^* need not exist. The adjoint operator C_ϕ^*, when well-defined, is called the *synthesis operator* of ϕ and we will denote it by T_ϕ.

Proposition 2.2. *If $D(\Omega_\phi) = \mathcal{H}$, then Ω_ϕ is bounded, i.e., there exists $\beta > 0$ such that*

$$\Omega_\phi(f,f) \le \beta \|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.4}$$

Proof. If $D(\Omega_\phi) = \mathcal{H}$, then C_ϕ is a bounded operator; hence, there exists $\beta > 0$ such that

$$\|C_\phi f\|_2^2 \leq \beta \|f\|^2, \quad \forall f \in \mathcal{H}.$$

Thus

$$\Omega_\phi(f,f) = \int_X |\langle f|\phi_x\rangle|^2 d\mu = \|C_\phi f\|_2^2 \leq \beta\|f\|^2, \quad \forall f \in \mathcal{H}.$$

□

Remark 2.3. Just to avoid ambiguities we recall the following definitions: a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is a *Bessel sequence* if there exists a constant $\beta > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f|\phi_n\rangle|^2 \leq \beta\|f\|^2, \quad \forall f \in \mathcal{H}.$$

A sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is a *Riesz basis* if $\{\phi_n\}_{n\in\mathbb{N}}$ is the image of a ON basis $\{e_n\}_{n\in\mathbb{N}}$ under a bounded operator with bounded inverse.

Definition 2.4. The map ϕ is called

(i) *total* or *complete* if $\overline{span}\{\phi_x\} = \mathcal{H}$;
(ii) *independent* if $\int_X u(x)\phi_x d\mu = 0$, for $u \in L^2(X,\mu)$, implies $u = 0$ almost everywhere;
(iii) a *resolution of the identity* if it is total and $\Omega_\phi(f,g) = \langle f|g\rangle$, for every $f, g \in \mathcal{H}$;
(iv) a *Bessel map* if (2.4) holds;
(v) a *continuous frame* if ϕ is Bessel and Ω_ϕ is also bounded from below, i.e., there exist $\alpha, \beta > 0$ such that

$$\alpha\|f\|^2 \leq \Omega_\phi(f,f) \leq \beta\|f\|^2, \quad \forall f \in \mathcal{H};$$

(vi) a *lower semi-frame* if Ω_ϕ is bounded from below (but not necessarily from above);
(vii) an *upper semi-frame* if it is total and Ω_ϕ is bounded from above (but not necessarily from below).

If Ω_ϕ is bounded on $\mathcal{H}$ (i.e., if ϕ is Bessel), then the operator S_ϕ is bounded and

$$\langle S_\phi f|g\rangle = \int_X \langle f|\phi_x\rangle \langle \phi_x|g\rangle \, d\mu, \quad f, g \in \mathcal{H},$$

which we can also write as

$$S_\phi f = \int_X \langle f|\phi_x\rangle \, \phi_x d\mu, \quad f \in \mathcal{H}. \tag{2.5}$$

The operator S_ϕ is then called simply the *frame operator* of ϕ. If ϕ is a resolution of the identity, then S_ϕ is the identity operator I of $\mathcal{H}$. If ϕ is a continuous frame, then S_ϕ has closed range.

Remark 2.5. Of course, there is a close connection between discrete and continuous frames: how can one obtain a discrete frame by taking a discrete sample of a continuous one? This discretization problem is particularly important in signal processing, be it with wavelets or Gabor frames. For more information, see Chapter VII in the textbook of Ali, Antoine and Gazeau [2].

The above examples make it clear that in a more general set-up, i.e. in the unbounded case, the possibility of representing Ω_ϕ is very crucial: indeed, the nature of the function $x \mapsto \langle f|\phi_x\rangle$ closely depends on the properties of the related sesquilinear form. We will come back to this later.

The equality in (2.5) should be read as a Dunford integral with values in $\mathcal{H}$. We discuss briefly this point starting from the simplest situation.

Proposition 2.6. *Let ϕ be a Bessel map. For every function $u \in L^2(X,\mu)$ there exists a unique vector $h \in \mathcal{H}$ such that*

$$\int_X u(x)\, \langle \phi_x|f\rangle \, d\mu = \langle h|f\rangle \,, \quad \forall f \in \mathcal{H}.$$

We put

$$h = \int_X u(x)\phi_x d\mu.$$

Proof. By the assumption,

$$\left| \int_X u(x)\, \langle \phi_x|f\rangle \, d\mu \right| \leq \|u\|_2 \| \langle \phi_x|f\rangle \|_2 \leq \sqrt{B}\|u\|_2\|f\|.$$

By Riesz's Lemma, there exists a unique $h \in \mathcal{H}$ such that

$$\int_X u(x)\, \langle \phi_x|f\rangle \, d\mu = \langle h|f\rangle \,, \quad \forall f \in \mathcal{H}.$$

Then, $u \cdot \phi$ is Dunford integrable and h is its Dunford integral. □

Taking into account the previous proposition, if u is a measurable complex valued function, we say that $\int_X u(x)\phi_x d\mu$ exists in $\mathcal{H}$, if

$$L(f) := \int_X u(x)\, \langle f|\phi_x\rangle \, d\mu, \quad f \in \mathcal{H}$$

defines a bounded linear functional on $\mathcal{H}$.

Let us put

$$D(T_\phi) = \left\{ u \in L^2(X,\mu) : \int_X u(x)\phi_x d\mu \text{ exists in } \mathcal{H} \right\}$$

and

$$T_\phi u = \int_X u(x)\phi_x d\mu, \quad u \in D(T_\phi).$$

If $D(T_\phi)$, the adjoint T_ϕ^* exists and is defined on

$$D(T_\phi^*) = \{g \in \mathcal{H} : \langle g|\phi_.\rangle \in L^2(X,\mu)\}$$

by

$$(T_\phi^* g)(x) = \langle g|\phi_x\rangle, \quad g \in D(T_\phi^*).$$

In other words, $T_\phi^* = C_\phi$.

In particular, under the assumptions of Proposition 2.6, the map

$$T_\phi : u \in L^2(X,\mu) \to \int_X u(x)\phi_x d\mu \in \mathcal{H}$$

is a bounded linear operator with $\|T_\phi\| \leq \sqrt{\beta}$, where $\beta > 0$ is the upper bound in (2.4). The adjoint operator $T_\phi^* = C_\phi$ is also bounded.

Corollary 2.7. *Let ϕ be weakly-measurable on X. Then $\int_X u(x)\phi_x d\mu$ exists for every $u \in L^2(X,\mu)$ if, and only if, ϕ is Bessel.*

The proof is straightforward.

Definition 2.8. We denote by $\mathcal{H}^\infty$ the set of all weakly measurable functions $\phi : x \mapsto \phi_x \in \mathcal{H}$ for which there exists $\gamma_\phi > 0$ such that

$$\left\| \int_X u(x)\phi_x d\mu \right\|^2 \leq \gamma_\phi \int_X |u(x)|^2 d\mu, \tag{2.6}$$

for every $u \in L^2(X,\mu)$.

Lemma 2.9. *The following statements are equivalent.*

(i) *$\phi \in \mathcal{H}^\infty$.*
(ii) *The analysis operator C_ϕ is everywhere defined and bounded.*
(iii) *The synthesis operator $T_\phi \in \mathcal{B}(L^2(X,\mu),\mathcal{H})$, that is, T_ϕ is well-defined and bounded on $L^2(X,\mu)$.*
(iv) *ϕ is a Bessel map.*

Proof. (i)⇒(ii): If (i) holds and $u \in L^2(X,\mu)$, we have

$$\left|\int_X u(x)\langle\phi_x|f\rangle\, d\mu\right| = \left|\left\langle \int_X u(x)\phi_x d\mu | f\right\rangle\right|$$
$$\leq \left\|\int_X u(x)\phi_x d\mu\right\| \|f\| \leq \sqrt{\gamma_\phi}\|u\|_2\|f\|.$$

These inequalities imply that the function $x \in X \mapsto \langle f|\phi_x\rangle \in \mathbb{C}$ is in $L^2(X,\mu)$, because the integral on the lhs is finite for every $u \in L^2(X,\mu)$ and that

$$\|\langle f|\phi_.\rangle\|_2 = \sup_{\|u\|_2=1}\left|\int_X u(x)\langle\phi_x|f\rangle\, d\mu\right| \leq \sqrt{\gamma_\phi}\|f\|. \tag{2.7}$$

This proves that C_ϕ is bounded.
(ii)⇔(iii) follows from $T_\phi = C_\phi^*$.
(iii)⇒(iv): the proof is similar to (i)⇒(ii).
(iv)⇒(i): If ϕ is Bessel, then $\left|\int_X u(x)\langle\phi_x|f\rangle\, d\mu\right|$ is well defined for every $u \in L^2(X,\mu)$ and

$$\left|\int_X u(x)\langle\phi_x|f\rangle\, d\mu\right| \leq \sqrt{\gamma_\phi}\|u\|_2\|f\|.$$

This implies that (2.6) holds for every $u \in L^2(X,\mu)$. □

Example 2.10. Let ϕ be a resolution of the identity. Then $\phi \in \mathcal{H}^\infty$. In particular, the Parseval equality implies that T_ϕ is a unitary operator from $L^2(X,\mu)$ onto $\mathcal{H}$. Morerover, let B be a bounded operator in $\mathcal{H}$. Put $\xi_x = B\phi_x$, $x \in X$. Then the function ξ belongs to $\mathcal{H}^\infty$. Indeed, if $u \in L^2(X,\mu)$

$$\left\|\int_X u(x)\xi_x d\mu\right\| = \sup_{\|f\|=1}\left|\int_X u(x)\langle\xi_x|f\rangle\, d\mu\right|$$
$$= \sup_{\|f\|=1}\left|\int_X u(x)\langle B\phi_x|f\rangle\, d\mu\right| = \sup_{\|f\|=1}\left|\int_X u(x)\langle\phi_x|B^*f\rangle\, d\mu\right|$$
$$\leq \left\|\int_X u(x)\phi_x d\mu\right\| \sup_{\|f\|=1}\|B^*f\| = \|B\|\left\|\int_X u(x)\phi_x d\mu\right\|$$
$$\leq \sqrt{\gamma_\phi}\|B\|\left(\int_X |u(x)|^2 d\mu\right)^{1/2}.$$

In $\mathcal{H}^\infty$ we introduce the following norm

$$\|\phi\|_\infty := \sup\left\{\left\|\int_X u(x)\phi_x d\mu\right\|,\ \|u\|_2 \leq 1\right\}.$$

In other words,

$$\|\phi\|_\infty = \|T_\phi\|;$$

that is, we endow the space of Bessel maps with the topology induced by the operator norm of $\mathcal{B}(L^2(X,\mu),\mathcal{H})$ (see also [15]).

Proposition 2.11. *Assume that $L^2(X,\mu)$ possesses a resolution of the identity $j : x \in X \to j_x \in L^2(X,\mu)$. Then $\mathcal{H}^\infty[\|\cdot\|_\infty]$ is a Banach space.*

Proof. The map $\phi \in \mathcal{H}^\infty \to T_\phi \in \mathcal{B}(L^2(X,\mu),\mathcal{H})$ is, by definition isometric and, therefore, injective. If $A \in \mathcal{B}(L^2(X,\mu),\mathcal{H})$, define $\phi_x = Aj_x$. By the definition itself, $T_\phi = A$. Indeed, if $u \in L^2(X,\mu)$ (see also Proposition 2.6),

$$T_\phi u = \int_X u(x)\phi_x d\mu = \int_X u(x)Aj_x d\mu = Au.$$

Therefore the map $\phi \in \mathcal{H}^\infty \to T_\phi \in \mathcal{B}(L^2(X,\mu),\mathcal{H})$ is an isomorphism of Banach spaces and consequently $\mathcal{H}^\infty$ is complete. □

Remark 2.12. We show how to construct a resolution of the identity in $L^2(X,\mu)$, provided that this space is separable. In this case, in fact, $L^2(X,\mu)$ possesses a countable ON basis $\{w_n\}_{n\in\mathbb{N}}$. Without loss of generality, we may suppose that each w_n is a continuous function. Moreover we assume that the evaluation functional

$$L_x(f) := \sum_{n=1}^{\infty} \langle f|w_n\rangle\, w_n(x).$$

is bounded in $L^2(X,\mu)$, for almost all $x \in X$. Then, for almost all $x \in X$, there exists $j_x \in L^2(X,\mu)$, such that

$$\langle f|j_x\rangle := \sum_{n=1}^{\infty} \langle f|w_n\rangle\, w_n(x).$$

It is clear that j is total and by the previous equation, it is a resolution of the identity of $L^2(X,\mu)$.

We have,

$$\int_X \langle f|j_x\rangle \langle j_x|g\rangle\, d\mu = \sum_{n=1}^{\infty} \langle f|w_n\rangle \langle w_n|g\rangle = \langle f|g\rangle\,.$$

3. The image under a linear operator of a resolution of the identity

Let ζ be a resolution of the identity for $\mathcal{H}$. Let us denote by $\mathcal{D}$ the linear span of $\{\zeta_x\}_{x\in X}$. $\mathcal{D}$ is a dense subspace of $\mathcal{H}$. Indeed, if $\langle f|\zeta_x\rangle = 0$ for almost every $x \in X$, then $f = 0$.

Let us consider a linear operator $U : \mathcal{D} \to \mathcal{H}$. We examine the behaviour of the map ϕ where $\phi_x = U\zeta_x$, $x \in X$; we will write, for short, $\phi = U\zeta$. What we discuss here is well-known, but we rewrite it for completeness.

Remark 3.1. In the discussion below the following properties of a closed densely defined operator A play an important role:

(i) $R(A) = \mathcal{H}$ if, and only if A^* has a bounded inverse;
(ii) $R(A^*) = \mathcal{H}$ if, and only if A has a bounded inverse.

See [29, Ch.VII, §5; p. 205]).

3.1. **Bounded U.** Suppose first that U is bounded on $\mathcal{D}$. Then we extend U to a bounded operator in $\mathcal{H}$ denoted by the same symbol.

Theorem 3.2 ([14, Theorem 5.4.4])**.** *Given a resolution of the identity ζ, the continuous frames of $\mathcal{H}$ are exactly the maps $\{U\zeta\}$ where U is a bounded surjective operator in $\mathcal{H}$.*

Proof. If $\{U\zeta\}$ is a continuous frame, then there exist $\alpha, \beta > 0$ such that

$$\alpha \|f\|^2 \leq \int_X |\langle f|U\zeta_x\rangle|^2 \leq \beta\|f\|^2, \quad \forall f \in \mathcal{H}$$

that is

$$\alpha\|f\|^2 \leq \|U^* f\|^2 \leq \beta\|f\|^2.$$

This implies that U^* is injective and its range is closed. Then by [24, Theorem 4.15], $R(U) = \mathcal{H}$.

Conversely, suppose that U is bounded and surjective, then by Remark 3.1, U^* has a bounded inverse. Therefore, there exists $\alpha > 0$ such that

$$\alpha\|f\|^2 \leq \|U^* f\|^2 \leq \beta\|f\|^2,$$

for every $f \in \mathcal{H}$. □

Let us examine some particular choiches of U. Let ζ be a resolution of the identity and $\phi = U\zeta$.

(a) If U is unitary, i.e., $U^*U = UU^* = I$, then ϕ is a resolution of the identity of $\mathcal{H}$; conversely, if ζ is a fixed resolution of the identity and ϕ is an arbitrary resolution of the identity, then there exists a unitary operator U such that $U\zeta_x = \phi_x$, $\forall x \in X$.
(b) If U is isometric (but not unitary), i.e., $U^*U = I$, but $UU^* =: P$ is a projection different from I, then ϕ is a resolution of the identity of $P\mathcal{H}$.

If $f = \int_X \langle f|\zeta_x\rangle \zeta_x d\mu$ then

$$Uf = \int_X \langle f|\zeta_x\rangle U\zeta_x d\mu = \int_X \langle f|\zeta_x\rangle \phi_x d\mu$$

and every vector $h \in R(U)$ has a representation of this type: indeed, if $h \in R(U)$, then there exists a unique $f \in \mathcal{H} : h = Uf$. If $f = \int_X \langle f|\zeta_x\rangle \zeta_x d\mu$, then

$$h = Uf = \int_X \langle f|\zeta_x\rangle U\zeta_x d\mu = \int_X \langle f|\zeta_x\rangle \phi_x d\mu.$$

Thus ϕ is a resolution of the identity of $R(U)$. It is worth reminding that

$$R(U) = \{h \in \mathcal{H} : \; Ph = h\} \;\text{ where }\; P = UU^*.$$

In this case we have $\|U^* f\| \leq \|f\|$, but U^* is not injective, hence ϕ is not a continuous frame.

(c) U is bounded, with bounded inverse U^{-1}. In this case $\phi = U\zeta$ is a continuous Riesz basis.

3.2. **Continuous Riesz bases.** In the discrete case, if $\{\phi_n\}_{n\in\mathbb{N}}$ is a Riesz basis then the following properies hold.

- *Reconstruction formula*: for every $f \in \mathcal{H}$ there exists a unique sequence of numbers $\{c_n\}_{n\in\mathbb{N}}$ such that

$$f = \sum_{n=1}^{\infty} c_n \phi_n,$$

i.e., Riesz bases are *Schauder bases*;
- every Riesz basis $\{\phi_n\}_{n\in\mathbb{N}}$ has a *dual* basis $\{\psi_n\}_{n\in\mathbb{N}}$ which is Riesz too;
- $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ are *biorthogonal*: $\langle \phi_k|\psi_h\rangle = \delta_{k,h}$, where $\delta_{k,h}$ is the Kronecker symbol.

Our next goal is to study weakly measurable maps that act as Riesz bases of $\mathcal{H}$. In particular we will reformulate a Paley-Wiener Theorem for sequences [23, §86].

If ζ is a resolution of the identity, by the definition itself, it follows that

$$\int_X \langle f|\zeta_x\rangle \langle \zeta_x|g\rangle d\mu = \langle f|g\rangle, \quad \forall f, g \in \mathcal{H},$$

which can also be read as

$$\int_X \langle f|\zeta_x\rangle \zeta_x d\mu = f, \quad \forall f \in \mathcal{H}.$$

Moreover, if $y \in X$

$$\langle \zeta_y|g\rangle = \int_X \langle \zeta_y|\zeta_x\rangle \langle \zeta_x|g\rangle \, d\mu.$$

Hence, $\langle \zeta_y|\zeta_x\rangle \, d\mu$ can be read as the Dirac measure on X centered at the point $y \in X$.

Theorem 3.3. *Let ζ be a resolution of the identity in $\mathcal{H}$ and ϕ a weakly measurable function with $\phi \in \mathcal{H}^\infty$. If $\|\phi - \zeta\|_\infty < 1$, then*

(i) there exists a bounded operator B with bounded inverse such that $\phi_x = B\zeta_x$ for almost every $x \in X$;
(ii) there exists a weakly measurable map ψ such that for every $f \in \mathcal{H}$ the following equalities hold

$$f = \int_X \langle f|\psi_x\rangle \phi_x d\mu; \quad f = \int_X \langle f|\phi_x\rangle \psi_x d\mu.$$

Proof. (i): Let $f \in \mathcal{H}$. Put $\theta = \|\phi - \zeta\|_\infty$. We define

$$Kf := \int_X \langle f|\zeta_x\rangle (\zeta_x - \phi_x) d\mu.$$

We have

$$\left\| \int_X \langle f|\zeta_x\rangle (\zeta_x - \phi_x) d\mu \right\|^2 \le \|\phi - \zeta\|^2 \int_X |\langle f|\zeta_x\rangle|^2 d\mu = \theta^2 \|f\|^2.$$

Thus K is bounded and $\|K\| \le \theta < 1$. Hence, the operator $B = I - K$ possesses a bounded inverse. Finally, $B\zeta_y = \phi_y$, for every $y \in X$. Indeed,

$$\begin{aligned}\langle B\zeta_y|g\rangle = \langle \zeta_y|g\rangle - \langle K\zeta_y|g\rangle &= \langle \zeta_y|g\rangle - \int_X \langle \zeta_y|\zeta_x\rangle \langle (\zeta_x - \phi_x)|g\rangle \, d\mu \\ &= \int_X \langle \zeta_y|\zeta_x\rangle \langle \phi_x|g\rangle \, d\mu = \langle \phi_y|g\rangle, \quad \forall g \in \mathcal{H};\end{aligned}$$

the last equality follows from the fact that $B\zeta_x = \phi_x$ for every $x \in X$ and

$$\int_X \langle \zeta_y|\zeta_x\rangle \langle \phi_x|g\rangle \, d\mu = \int_X \langle \zeta_y|\zeta_x\rangle \langle \zeta_x|B^*g\rangle \, d\mu = \langle \zeta_y|B^*g\rangle = \langle \phi_y|g\rangle.$$

(ii): Define $\psi_x = (B^{-1})^*\phi_x$, $x \in X$. Then, from

$$Bf = \int_X \langle f|\zeta_x\rangle B\zeta_x d\mu$$

it follows that

$$f = BB^{-1}f = \int_X \left\langle B^{-1}f|\zeta_x \right\rangle B\zeta_x d\mu = \int_X \langle f|\psi_x\rangle \phi_x d\mu, \quad \forall f \in \mathcal{H}$$

and

$$f = (B^{-1}B)^* f = \int_X \langle f|B\zeta_x\rangle (B^{-1})^* \zeta_x d\mu = \int_X \langle f|\phi_x\rangle \psi_x d\mu, \quad \forall f \in \mathcal{H}.$$

□

This leads us to introduce the following

Definition 3.4. *Let ζ be a resolution of the identity. The weakly measurable function $\phi : X \to \mathcal{H}$ is a* continuous Riesz basis *if there exists a bounded linear operator U in $\mathcal{H}$ with bounded inverse such that $\phi = U\zeta$.*

As in the discrete case, one can prove that a Bessel map ϕ is a continuous Riesz basis if, and only if, it is complete and independent (Definition 2.4; see also [8]). This applies of course also to resolutions of the identity.

4. Lower semi-frames and A-frames

So far we have considered the case where the sesquilinear form Ω_ϕ associated to a weakly continuous map is bounded from above and from below, that is, a continuous frame. Other situations are however of some interest: the first is when Ω_ϕ is bounded from below (lower semi-frames) and the second one is when the boundedness from below is, so to say *mediated* by some linear operator A.

4.1. **Lower semi-frames and metric operators.** Lower semi-frames, mentioned above in Definition 2.4 (vi), lead to a somewhat unexpected connection with the so-called metric operators, that is, strictly positive self-adjoint operators, i.e. $G > 0$ or $\langle Gf|f\rangle \geq 0$ for every $f \in D(G)$ and $\langle Gf|f\rangle = 0$ if and only if $f = 0$. Such metric operators are, for instance, encountered in the so-called $\mathcal{PT}$-symmetric quantum mechanics, where non-self-adjoint Hamiltonians play a systematic role [6]. The correspondence has been analyzed in [3], to which we refer for more details.

On one hand, let S_ϕ be, as before, the generalized frame operator. Then $S_\phi^{-1/2} : \mathcal{H} \to \mathcal{H}$ and $\psi = S_\phi^{-1/2}\phi$ is a Parseval frame for $\mathcal{H}$. As a matter of fact, $S_\phi^{-1/2} : \mathcal{H} \to \mathcal{H}$ is a metric operator. But there might be other metric operators G that do the same job, namely that $\chi = G\phi$ is a Parseval frame for $\mathcal{H}$.

Conversely, given a closed densely defined unbounded operator A, the operator $R_A = I + A^*A$ is an unbounded metric one, with domain $D(A^*A)$. Let $\mathcal{H}_1$ denote $D(G_1^{1/2})$ with a norm equivalent to the graph norm and $\mathcal{H}_{-1} = \mathcal{H}(G_1)^\times$, the conjugate dual of $\mathcal{H}_1$. Thus we get the triplet

$$\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} = {\mathcal{H}_1}^\times. \tag{4.1}$$

Actually, the triplet (4.1) is the central part of the discrete scale of Hilbert spaces V_{G_1} built on the powers of $G_1^{1/2}$. This means that $V_{G_1} := \{\mathcal{H}_n, n \in \mathbb{Z}\}$, where $\mathcal{H}_n = D(G_1^{n/2}), n \in \mathbb{N}$, with a norm equivalent to the graph norm, and $\mathcal{H}_{-n} = \mathcal{H}_n^\times$:

$$\ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \ldots \tag{4.2}$$

Moreover, one may add the end spaces of the scale, namely,

$$\mathcal{H}_\infty(G_1) := \bigcap_{n\in\mathbb{Z}} \mathcal{H}_n, \qquad \mathcal{H}_{-\infty}(G_1) := \bigcup_{n\in\mathbb{Z}} \mathcal{H}_n. \tag{4.3}$$

In this way, we get a genuine rigged Hilbert Space (RHS):

$$\mathcal{H}_\infty(G_1) \subset \mathcal{H} \subset \mathcal{H}_{-\infty}(G_1). \tag{4.4}$$

(see Section 6 for a definition of RHS). Of course, as before, every $\chi_n = G_1^{n/2}\phi$ is a lower semi-frame, and even a Parseval frame for $\mathcal{H}$.

In conclusion, lower semi-frames allow to recover the formalism based on metric operators that we have developed for the theory of quasi-Hermitian operators, in particular non-self-adjoint Hamiltonians. See [6] for more details.

4.2. *A*-**frames.** The following notion has already been introduced and studied in [10, 11] (there, it was called weak A-frame). Let us consider the form Ω_ϕ defined in (2.2).

Definition 4.1. *Let A be a densely defined operator in $\mathcal{H}$. A continuous A-frame for $\mathcal{H}$ is a function $\phi : x \in X \to \phi_x \in \mathcal{H}$ such that for all $f \in D(A^*)$, the map $x \to \langle f|\phi_x\rangle$ is a measurable function on X and*

$$\alpha\|A^*f\|^2 \leq \Omega_\phi(f,f) < \infty,$$

for every $f \in D(A^*)$ and some $\alpha > 0$.

In [10] the following two results are shown.

Proposition 4.2. *Let A be a densely defined operator in $\mathcal{H}$ and ϕ a continuous A-frame, then $D(A^*) \subset D(\Omega_\phi)$. Moreover, if A is closable, then Ω_ϕ is densely defined.*

However, in general $D(A^*) \subsetneq D(\Omega_\phi)$.

Corollary 4.3. *Let A be a closable and densely defined operator in $\mathcal{H}$ and ϕ a continuous A-frame. Then the synthesis operator C_ϕ^* is closed.*

Remark 4.4. If A a is closable and densely defined operator and ϕ is a continuous A-frame, the sesquilinear form Ω_ϕ is *densely defined*, nonnegative and closed. Then there exists the generalized frame operator S_ϕ of ϕ defined as in (2.3), see [20, p. 279]. The map $g \mapsto \int_X \langle f|\phi_x\rangle \langle \phi_x|g\rangle \, d\mu$ is bounded on $D(\Omega_\phi)$ w.r.to the norm of $\mathcal{H}$. By definition of continuous A-frame, $u(x) = \langle f|\phi_x\rangle$ is a measurable complex valued function on X for all $f \in D(A^*)$. Moreover, the analysis operator C_ϕ is closed and densely defined and one has

$$\begin{aligned} \alpha\|A^*f\|^2 \quad &\leq \Omega_\phi(f,f) = \|C_\phi f\|_2^2 = \left\| S_\phi^{1/2} f \right\|^2, \\ &\qquad \forall f \in D(A^*) \subset D(\Omega_\phi) = D(S_\phi^{1/2}). \end{aligned}$$

Corollary 4.5. *Let A be a closable, densely defined operator and ϕ a continuous A-frame for $\mathcal{H}$. Then the generalized frame operator S_ϕ of ϕ is self-adjoint.*

If ϕ is a continuous A-frame, the generalized frame operator is not invertible on $\mathcal{H}$, in general, but under further hypothesis it can be inverted on a subspace of $\mathcal{H}$.

$$\alpha\|A^*f\|^2 \leq \Omega_\phi(f,f) = \langle S_\phi f|f\rangle, \quad \forall f \in D(A^*) \subset D(\Omega_\phi).$$

Let P_M indicate the projection on M.

As an aside, we recall the following

Lemma 4.6. [13] *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Let $W : D(W) \subset \mathcal{K} \to \mathcal{H}$ a closed, densely defined operator with closed range $R(W)$. Then, there exists a unique $W^\dagger \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that*

$$N(W^\dagger) = R(W)^\perp, \ \overline{R(W^\dagger)} = N(W)^\perp, \ WW^\dagger f = f, \quad f \in R(W).$$

The operator $W^\dagger$ is called the *pseudo-inverse* of the operator W.

Proposition 4.7. *Let A be a closable, densely defined operator such that $\mathcal{R}(A) = \mathcal{H}$. If ϕ is a continuous A-frame, then the sesquilinear form Ω_ϕ is bounded from below on $D(A^*)$.*

Proof. Let A be a closable and densely defined, surjective operator, then, by the previous Lemma, there exists $A^\dagger \in \mathcal{B}(\mathcal{H})$ such that $AA^\dagger = I$, hence $(A^\dagger)^*A^* \subset I$ and in particular we have:

$$D((A^\dagger)^*A^*) = D(A^*),$$

since $(A^\dagger)^* \in \mathfrak{B}(\mathcal{H})$ and

$$\|f\| = \|(A^\dagger)^* A^* f\| \leq \|A^\dagger\| \|A^* f\|, \quad \forall f \in D(A^*),$$

then

$$\|A^\dagger\|^{-1} \|f\| \leq \|A^* f\|, \quad \forall f \in D(A^*).$$

If ϕ is a continuous A-frame, then the sesquilinear form Ω_ϕ is semi-bounded on $D(A^*)$:

$$\alpha \|A^\dagger\|^{-2} \|f\|^2 \leq \alpha \|A^* f\|^2 \leq \Omega_\phi(f,f), \quad \forall f \in D(A^*) \subset D(\Omega_\phi).$$

□

Remark 4.8. Note that ϕ is not a lower semi-frame in general; indeed the inequality holds true only on $D(A^*)$ and possibly $D(A^*) \subsetneq \mathcal{H}$.

Corollary 4.9. *Let $A \in \mathcal{B}(\mathcal{H})$. If ϕ is a continuous A-frame, then ϕ is a lower semi-frame for $\mathcal{H}$ and there exists an upper semi-frame ψ dual to ϕ, that is, one has, in the weak sense,*

$$f = \int_X \langle f|\phi_x\rangle \, \psi_x d\mu, \quad \forall f \in D(\Omega_\phi).$$

Proof. It suffices to use Proposition 4.7 and [5, Proposition 2.1, ii)]. □

Under suitable hypotheses, we can weakly decompose the domain of A^* by means of a continuous weak A-frame.

Theorem 4.10. [10, Theorem 3.16] *Let A be a closed densely defined operator with $\mathcal{R}(A) = \mathcal{H}$ and $A^\dagger$ the pseudo-inverse of A. Let ϕ be a continuous weak A-frame and ψ a Bessel weak A-dual of ϕ. Then, the function ϑ with $\vartheta_x := (A^\dagger)^* \psi_x \in \mathcal{H}$, for every $x \in X$, is Bessel and every $u \in D(A^*)$ can be weakly decomposed as follows*

$$\langle h|u\rangle = \int_X \langle h|\vartheta_x\rangle \, \langle \phi_x|u\rangle \, d\mu(x) \qquad \forall h \in \mathcal{H}, u \in D(A^*).$$

By Theorem 4.10,

$$\langle h|u\rangle = \left\langle AA^\dagger h|u\right\rangle = \textstyle\int_X \langle A^\dagger h|\psi_x\rangle \, \langle \phi_x|u\rangle \, d\mu(x) = \left\langle MA^\dagger h|Bu\right\rangle_2, \quad \forall h \in \mathcal{H}, u \in D(A^*)$$

with B a closable densely defined operator which is a restriction of C_ϕ and $M \in \mathcal{B}(\mathcal{H}, L^2(X,\mu))$.

5. Riesz and Generalized Riesz functions

In the discrete case [9], generalized Riesz bases were introduced in [21, 22]. A continuous variant sounds as follow.

Definition 5.1. A weakly measurable function $\phi : X \to \mathcal{H}$ is called a *generalized Riesz map* if there exists a resolution of the identity ζ in $\mathcal{H}$ and a densely defined closed operator W in $\mathcal{H}$ with densely defined inverse, such that

$$\zeta_x \in D(W) \cap D((W^{-1})^*) \text{ and } W\zeta_x = \phi_x, \text{ for almost every } x \in X.$$

For short, the couple (ζ, W) will be called a *constructing pair* for ϕ.

As before, we can define the sesquilinear form Ω_ϕ by (2.1) and (2.2). If $D(\Omega_\phi)$ is dense in $\mathcal{H}$, then we can apply Kato's theorem on closed nonnegative forms and introduce as before the operator S_ϕ.

Proposition 5.2. *Let ϕ be a generalized Riesz map, with constructing pair (ζ, W). Then $D(\Omega_\phi) = D(W_0^*)$, where W_0 denotes the restriction of W to the linear span D_ζ of the resolution of the identity ζ and $D(\Omega_\phi)$ is dense in $\mathcal{H}$.*

Proof. Suppose that $h \in D(W_0^*)$. Then, $h \in D(\Omega_\phi)$:

$$\int_X |\langle h|\phi_x\rangle|^2 = \int_X |\langle h|W\zeta_x\rangle|^2 d\mu = \int_X |\langle W_0^* h|\zeta_x\rangle|^2 d\mu = \|W_0^* h\|^2 < \infty.$$

On the other hand, if $h \in D(\Omega_\phi)$, then $\langle h|\phi_\cdot\rangle \in L^2(X, \mu)$. Hence we can define $g := \int_X \langle h|\phi_x\rangle \zeta_x d\mu$. Then

$$\langle h|W\zeta_x\rangle = \langle h|\phi_x\rangle = \langle g|\zeta_x\rangle, \quad \mu - a.e.$$

indeed both $g = \int_X \langle g|\zeta_x\rangle \zeta_x d\mu$ and $g = \int_X \langle h|\phi_x\rangle \zeta_x d\mu$, then $\int_X (\langle g|\zeta_x\rangle - \langle h|\phi_x\rangle)\zeta_x d\mu = 0$; hence, since ζ is independent, $\langle g|\zeta_x\rangle - \langle h|\phi_x\rangle = 0$, μ-a.e. . This equality extends obviously to D_ζ; then $h \in D(W_0^*)$ and $W_0^* h = g = \int_X \langle h|\phi_x\rangle \zeta_x d\mu$. Furthermore, $D(\Omega_\phi)$ is dense in $\mathcal{H}$ because $D(\Omega_\phi) = D(W_0^*)$ and $W_0 = W_{\restriction D_\zeta}$ is a closable densely defined operator (D_ζ is dense since ζ is total). □

Now we introduce the following

Definition 5.3. *Let ϕ and ψ be weakly measurable functions and $\mathcal{D}$ be a dense domain containing the (essential) images of ϕ anf ψ and contained in $D(\Omega_\phi) \cap D(\Omega_\psi)$. The functions ϕ and ψ are* biorthogonal continuous $\mathcal{D}$-bases *if*

$$\int_X \langle f|\phi_x\rangle \langle \psi_x|g\rangle \, d\mu = \int_X \langle f|\psi_x\rangle \langle \phi_x|g\rangle \, d\mu = \langle f|g\rangle, \quad \forall f, g \in \mathcal{D}. \tag{5.1}$$

This equality turns out to be equivalent to ϕ and ψ being generalized Riesz maps as the following discussion which generalizes [9, Theorem III.1] shows. Again an important role is played by the forms Ω_ϕ and Ω_ψ and by the operators S_ϕ, S_ψ which describe them (by Kato's theorem), respectively. Let R_ϕ, R_ψ denote the restrictions of $S_\phi^{1/2}$, $S_\psi^{1/2}$, respectively. Then, under the assumption made so far, one proves the equivalence of the following statements:

(i) ϕ and ψ are biorthogonal continuous $\mathcal{D}$-bases,

(ii) ϕ and ψ are generalized Riesz maps with constructing pairs $(\overline{R}_\phi, \zeta)$, $(\overline{R}_\psi, \zeta)$, respectively, where ζ is a resolution of the identity contained in $\mathcal{D} \cap D(S_\phi) \cap D(S_\psi)$.

5.1. **Reproducing pairs.** If ϕ, ψ are weakly measurable functions, it is natural to introduce the following two sesquilinear forms:

$$\Omega_{\phi,\psi}(x, y) = \int_X \langle f|\phi_x\rangle \langle \psi_x|g\rangle \, d\mu,$$

and

$$\Omega_{\psi,\phi}(x, y) = \int_X \langle f|\psi_x\rangle \langle \phi_x|g\rangle \, d\mu$$

for all f, g for which these make sense.

In principle, these forms are neither semi-bounded nor sectorial. Thus, Kato's representation theorems cannot be applied. However several variants of these celebrated theorems have been proposed (we refer to [16, 17], where the notion of *solvable form* has been introduced and studied, and for a rather complete bibliography on this topic). We will not consider this aspect here, but simply some particular case.

The couple of weakly measurable functions (ϕ, ψ) is a *reproducing pair* if the form

$$\Omega_{\phi,\psi}(f, g) = \int_X \langle f|\phi_x\rangle \langle \psi_x|g\rangle \, d\mu$$

is represented by a bounded operator $S_{\phi,\psi}$ with bounded inverse; i.e.,

$$\langle S_{\phi,\psi} f|g\rangle = \int_X \langle f|\phi_x\rangle \langle \psi_x|g\rangle \, d\mu.$$

This notion was introduced in [26] and studied in several papers [**?**, 27] and also in [4, 7] for the continuous version.

The relevant point of this approach is again the reconstruction formula

$$f = \int_X \left\langle f|((S_{\phi,\psi})^*)^{-1}\phi_x\right\rangle \psi_x d\mu = \int_X \left\langle f|((S_{\psi,\phi})^*)^{-1}\psi_x\right\rangle \phi_x d\mu,$$

to be understood, of course, in weak sense.

6. Beyond Hilbert space

For several applications, it is not enough to stay in the world of functions, but one has to allow distributions. In a certain sense it is natural to try and generalize what we have discussed so far in this larger world and the natural arena is that of *Gel'fand triplets* or *rigged Hilbert spaces*. Attempts in this direction (Riesz-like bases, Distribution frames) have in fact been considered in [12] and [28]. A rigged Hilbert space (RHS) consists in fact of a triplet of spaces

$$\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[t^\times]$$

where $\mathcal{D}$ is a locally convex space with topology t; $\mathcal{D}^\times$ is the conjugate dual of $\mathcal{D}$, with the strong dual topology $t^\times$; $t^\times \preceq t_{\|\cdot\|} \preceq t$, $\mathcal{D} \equiv \mathcal{D}^{\times\times}$ (i.e. $\mathcal{D}$ is reflexive). Here $\hookrightarrow$ stands for continuous and dense inclusion.

The main motivation for considering this case comes from the problem of generalized eigenvalues of self-adjoint operators. Let us illustrate it with an elementary example.

Example 6.1. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing C^∞-functions on the real line. Consider the operator $-\frac{d^2}{dx^2}$ acting on $\mathcal{S}(\mathbb{R})$. It is very well known that the eigenvalue problem $-\frac{d^2}{dx^2}f - \mu f = 0$ has no solution in $\mathcal{S}(\mathbb{R})$ or in $L^2(\mathbb{R})$. On the other hand, the family of functions $\{\omega_\lambda; \lambda \in \mathbb{R}\}$ with

$$\omega_\lambda(x) = \frac{1}{\sqrt{2\pi}} e^{i\lambda x}, \ x \in \mathbb{R}$$

is contained in the conjugate dual $\mathcal{S}^\times(\mathbb{R})$, that is, the space of (conjugate) tempered distributions. They are *generalized eigenvectors*, with eigenvalues λ^2.

The properties of the Fourier transform may be interpreted as *completeness* of this set of eigenvectors, in the sense that every $f \in \mathcal{S}(\mathbb{R})$ can be expanded (reconstructed) in terms of the ω_λ's:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\lambda) e^{i\lambda x} d\lambda,$$

with *components*

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\lambda x} dx = \langle f|\omega_\lambda\rangle,$$

$$f(x) = \int_{\mathbb{R}} \langle f|\omega_\lambda\rangle \omega_\lambda d\lambda.$$

Then the equality $\|f\| = \|\hat{f}\|$ (Plancherel) can be interpreted as orthogonality (Parseval).

This is nothing but a realization of the Gel'fand-Maurin theorem which reads as follows.

Theorem 6.2. [18] *Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space with $\mathcal{D}[t]$ a nuclear space and $\mathcal{D}^\times$ conjugate dual.*

Suppose that $A : \mathcal{D}[t] \to \mathcal{D}[t]$ is a continuous linear operator which is essentially self-adjoint on $\mathcal{D}$; then A has a complete family of generalized eigenvectors, i.e., there exists a family $\{\Phi_\lambda; \lambda \in \mathbb{R}\}$, $\Phi_\lambda \in \mathcal{D}^\times$, such that

$$\langle Af|\Phi_\lambda\rangle = \lambda \langle f|\Phi_\lambda\rangle, \quad \forall f \in \mathcal{D},$$

i.e., A extends to $\hat{A} : \mathcal{D}^\times \to \mathcal{D}^\times$ and

$$\hat{A}\Phi_\lambda = \lambda\Phi_\lambda.$$

The word *complete* means that, in weak sense, one can write

$$f = \int_{\mathbb{R}} \langle f|\Phi_\lambda\rangle \Phi_\lambda d\lambda, \quad \forall f \in \mathcal{D}$$

and that Parseval equality

$$\|f\|^2 = \int_X |\langle f|\Phi_\lambda\rangle|^2 d\mu$$

holds. A distribution map with these properties is called a *Gel'fand distribution basis* [28].

Starting from these considerations, the notions of frame, Riesz basis, etc. have been extended to the environment of rigged Hilbert spaces in [28].

Suppose that $\phi : X \to \mathcal{D}^\times$ is a weakly measurable function and that, for every $f \in \mathcal{D}$,

$$\int_X |\langle f|\phi_x\rangle|^2 d\mu < \infty.$$

In this case we say that ϕ is a Bessel distribution map.

In contrast with the case of Hilbert spaces, this condition does not imply that there exists $B > 0$ such that

$$\int_X |\langle f|\phi_x\rangle|^2 d\mu \le B\|f\|^2, \quad \forall f \in \mathcal{D}.$$

If the above inequality holds, we say that ϕ is a bounded Bessel distribution map. Then

$$\Omega_\phi(f, g) = \int_X \langle f|\phi_x\rangle \langle \phi_x|g\rangle d\mu, \quad f, g \in \mathcal{D}$$

is a nonnegative bounded sesquilinear form on $\mathcal{D} \times \mathcal{D}$ and extends to $\mathcal{H} \times \mathcal{H}$.

Let ζ be a Gel'fand distribution basis and $M : \mathcal{D}^\times \to \mathcal{D}^\times$ a weakly continuous linear map.

Put $\omega_x := M\zeta_x$. Then, ω is a Bessel distribution map.

Following the properties of $M^\times : \mathcal{D} \to \mathcal{D}$ we can describe shortly the possible situations as follows.

. ω is a **bounded Bessel map** if, and only if, $M^\times$ is bounded;
. ω is a **distribution frame** if $M^\times$ is bounded,$(M^\times)^{-1}$ exists, is continuous, but it is not everywhere defined;
. ω is a **Riesz distribution basis** if it is a distribution frame and $M^\times \mathcal{D} = \mathcal{D}$.

Example 6.3. We sketch here the construction of a Gel'fand distribution basis in a quite simple situation. Let $K = [a, b] \subset \mathbb{R}$ and consider $L^2(K)$. Let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(K)$, with Lebesgue measure. Let us suppose that every u_n is a continuous function and that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly essentially bounded, i.e., $|u_n(x)| \leq M$, for almost every $x \in K$ and every $n \in \mathbb{N}$.

Consider now a sequence of positive numbers $\{\alpha_n\}_{n\in\mathbb{N}}$ with $\{\alpha_n^{-1}\}_{n\in\mathbb{N}} \in \ell^2$ and define

$$\mathcal{D} = \left\{ f \in L^2(K) : \sum_{n\in\mathbb{N}} \alpha_n^{2k} |\left\langle f|u_n \right\rangle|^2 < \infty, \, \forall k \in \mathbb{N} \right\}.$$

Let us endow $\mathcal{D}$ with the topology t given by the family of seminorms

$$p_k(f) := \left(\sum_{n\in\mathbb{N}} \alpha_n^{2k} |\left\langle f|u_n \right\rangle|^2 \right)^{1/2}.$$

Notice that $p_0(f) = \|f\|$.

With this topology $\mathcal{D}[\mathsf{t}]$ is a Fréchet reflexive space. For every $x \in K$ define a conjugate linear functional ϕ_x on $\mathcal{D}$ by

$$\langle \phi_x | f \rangle = \sum_{n\in\mathbb{N}} \langle u_n | f \rangle \, u_n(x), \quad f \in \mathcal{D}.$$

We prove that ϕ_x is well defined and continuous on $\mathcal{D}[\mathsf{t}]$.

$$\begin{aligned}
|\langle \phi_x|f\rangle| &\le \sum_{n\in\mathbb{N}} |\langle u_n|f\rangle\, u_n(x)| = \sum_{n\in\mathbb{N}} |\alpha_n \langle u_n|f\rangle\, \alpha_n^{-1} u_n(x)| \\
&\le \left(\sum_{n\in\mathbb{N}} \alpha_n^2 |\langle u_n|f\rangle|^2\right)^{1/2} M \left(\sum_{n\in\mathbb{N}} (\alpha_n^{-1})^2\right)^{1/2} \\
&= C p_1(f),
\end{aligned}$$

with $C = M\left(\sum_{n\in\mathbb{N}} (\alpha_n^{-1})^2\right)^{1/2} > 0$. Hence $\phi_x \in \mathcal{D}^\times$.

We have

$$\int_K \langle f|\phi_x\rangle \langle \phi_x|g\rangle\, dx = \sum_{n\in\mathbb{N}} \langle f|u_n\rangle \langle u_n|g\rangle = \langle f|g\rangle .$$

Hence ϕ is a Gel'fand distribution basis.

Acting on ϕ with an operator M, as described at the end of the previous section, produces in turn continuous Riesz bases or frames.

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