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An Algorithm for Computing the Stable Coalition Structures in Tree-Graph Communication Games

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Abstract

We construct an algorithm which provides in finite steps the stable coalition structure(s) of tree-graph communication games and an allocation of the core: the restricted marginal contribution allocation.

Key Words: stability, core, tree-graph communication games.

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1 Introduction

Aumann and Drèze (1974) introduced TU cooperative games with coalition structures and defined the core solution concept for these games. A coalition structure is stable if and only if the core of the game with that coalition structure is nonempty. Shenoy (1979) showed that the core of TU games with coalition structures may be empty.

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Myerson (1977) introduced the graph-restricted games. In this paper, we analyse the stability of particular graph-restricted games with coalition structures: tree-graph communication games in which the players are located at the nodes of an undirected tree-graph, Γ. The superadditive tree-graph communication games coincide with the Γ-component additive games defined by Potters and Reijnierse (1995). Γ-component additive games are balanced and at least the grand coalition is stable.

We prove the nonemptiness of the core of tree-graph communication games with coalition structures. More precisely, we present an algorithm which allows us to identify in finite steps the stable coalition structure(s) of these games and a core allocation: the restricted marginal contribution allocation (RMCA).

The RMCA is based on the following underlying idea: the more central is a player in a network, the more likely he emerges as a leader, and the greatest the feasible payoff he obtains.

The paper is structured as follows. Section 2 introduces tree-graph communication games. Section 3 is devoted to the algorithm. The algorithm to obtain the RMCA allows us to show the nonemptiness of the set of stable coalition structures for these games (Theorem 3.1). Section 4 illustrates our theoretical results by means of an example on the provision of an emergency service.

2 Tree-Graph Communication Games

A *cooperative n-person game in characteristic function form* is an ordered pair (N, v), where N is a finite set of n elements and $v : 2^n \longrightarrow \mathbb{R}_+$ is a function on the family of all subsets of N with $v(\emptyset) = 0$. Elements of N are called players and the real-valued function v the characteristic function of the game. Any subset S of the player set N is called a coalition and $v(S)$ the value of the coalition S in the game. Any vector $x \in \mathbb{R}^n$ is called allocation. Given $x \in \mathbb{R}^n$, $\sum_{i\in S} x_i$ will be denoted by $x(S)$.

Let $\mathcal{P}(N)$ be the set of partitions P of N. A coalition structure P on N, is a partition of N. A game with coalition structure P is a triple (N, v, P). Let $V(P) = \sum_{S\in P} v(S)$, the worth of partition P. The core of a game (N, v, P) is defined as

$$C(N, v, P) := \{x \in \mathbb{R}^n : x(S) \geq v(S) \;\forall S \subseteq N,\; x(S_k) = v(S_k) \;\forall S_k \in P\}.$$

A coalition structure P is stable whenever $C(N, v, P) \neq \emptyset$.

In this paper we assume that players are located at the nodes of an undirected graph $\langle N, E\rangle$ where N is a finite, nonempty set of nodes, and $E = \{e \subset N : |e| = 2\}$. Elements of E are called edges. Given $W \subset N$, $\langle W, E(W)\rangle$ is a subgraph of $\langle N, E\rangle$ if $E(W) = \{e \in E : e \subset W\}$. Given a node i, node j is called incident with i if $\{i, j\} \in E$. We denote by $I(i)$ the set of incident nodes with node i: $I(i) = \{j \in N : \{i, j\} \in E\}$. Given $S \subset N$, $I(S) = \bigcup_{i \in S} I(i) \setminus S$ is the set of incident nodes with S.

A path between any two nodes of N is a sequence of distinct edges in E allowing to join both nodes. A cycle is a path between a node in N and itself with at least three edges. A graph $\langle N, E\rangle$ is said to be connected if there exists a path between every pair of nodes. An undirected graph $\Gamma = \langle N, E\rangle$ is a tree, if each pair of different nodes is connected by exactly one path. In a tree a node i is a leaf if $|I(i)| \leq 1$. $L(\Gamma)$ denotes the set of leaves in the tree-graph $\Gamma = \langle N, E\rangle$.

The distance d_{ij} between two nodes i and j on a tree $\langle N, E\rangle$ is equal to the number of edges in the path connecting i and j. The eccentricity $e(i)$ of a node i on a tree is defined as $\max_{j \in N} d_{ij}$. A node i is said central if it is the node with minimum eccentricity. The centre of a tree $\langle N, E\rangle$ is the set of all central nodes in $\langle N, E\rangle$. A well known result is that every tree has a centre containing either one node or two incident nodes (see e.g. Foulds (1992)).

Let us define tree-graph communication games. It is assumed that a coalition $S \subset N$ may communicate through the edges in $E(S)$. We say that coalition S is connected if the subgraph $\langle S, E(S)\rangle$ is a tree. We denote by $\mathcal{C}(N)$ the set of all connected coalitions. Each subset S of N may be partitioned into components in the following way: $T \subset S$ is a component in S if and only if $\langle T, E(T)\rangle$ is a tree and there is no a set T^* such that $T \subset T^* \subset S$ and $\langle T^*, E(T^*)\rangle$ is a tree. The set of all components of S is denoted by $\mathcal{K}(S)$.

A tree-graph communication game is a pair (N, v_Γ) where $v_\Gamma : 2^n \longrightarrow \mathbb{R}_+$ is such that for all $S \subset N$, $v_\Gamma(S) = \sum_{K \in \mathcal{K}(S)} v_\Gamma(K)$, and $v_\Gamma(\emptyset) = 0$.

3 Stable Coalition Structures

In this section we present our algorithm that allows us to identify both the set of stable coalition structures, $\mathcal{P}^* \subset \mathcal{P}(N)$, and the RMCA$(N, v_\Gamma) = x \in \mathbb{R}^n$ that belongs to the core of the game (N, v_Γ, P^*) for any $P^* \in \mathcal{P}^*$.

The following example shows that the core of a TU-cooperative game with coalition structure may be empty.

Example 3.1. Consider the following 4-person game: $v(i) = 0$, $\forall i \in N$; $v(12) = v(34) = 1$; $v(123) = v(124) = v(134) = v(234) = 1.8$; $v(S) = 0$ otherwise. The partition of maximum worth is $P^* = \{\{12\}, \{34\}\}$. And $C(N, v, P^*) = \emptyset$.

Shenoy (1979) showed the following two results.

1. If $C(N, v, P^*) \neq \emptyset$, then $V(P^*) \geq V(P)$ for any $P \in \mathcal{P}(N)$, and

2. if P^* is a partition satisfying $V(P^*) \geq V(P)$ for any $P \in \mathcal{P}(N)$, such that $C(N, v, P^*) = \emptyset$, then $C(N, v, P) = \emptyset$ for all partition $P \in \mathcal{P}(N)$.

Therefore, to look for the stable coalition structures of a game reverts to find the partitions of maximum worth.

The algorithm.

Input: A tree-graph communication game (N, v_Γ) with $n \geq 2$.

Output: An allocation $q \in \mathbb{R}^n$ and a collection of sets $\mathcal{S}^* = \{S : S \in \mathcal{C}(N)\}$.

I. If $n > 2$,

0. For $t = 0$. Define $\Gamma^0 = \langle N, E \rangle$ and let $L(\Gamma) = L_0$ be the set of leaves corresponding to the graph Γ^0. For all element $i \in L_0$ set $q_i = v_\Gamma(i)$. For all $i \in L_0$, let $S_i^{*0} = \{\{i\}\}$, and $\mathcal{S}^{*0} = \bigcup_{i \in L_0} S_i^{*0}$.
1. For $t \geq 1$. Let $\Gamma^t = \left\langle N \setminus \bigcup_{\tau=0}^{\tau=t-1} L_\tau, E\left(N \setminus \bigcup_{\tau=0}^{\tau=t-1} L_\tau\right) \right\rangle$ be a subgraph and $L(\Gamma^t)$ the set of leaves in the subgraph Γ^t.
   a. If $\left|L(\Gamma^t)\right| \neq 2$, or if $\left|L(\Gamma^t)\right| = 2$ and $\bigcup_{\tau=0}^{\tau=t-1} L_\tau \cup L(\Gamma^t) \neq N$, set $L_t = L\left(\Gamma^t\right)$.

a.1. $\forall i \in L_t$ let $\mathcal{S}_i^t = \left\{ S \in \mathcal{C}(N) : S \subset \bigcup_{\tau=0}^{\tau=t-1} L_\tau \cup \{i\}, i \in S \right\}$

a.2. $\forall S \in \mathcal{S}_i^t$ define $w(S) = v_\Gamma(S) - \sum_{\substack{j \in S \\ j \neq i}} q_j$

a.3. Define $\mathcal{S}_i^{*t} = \left\{ S^* \in \mathcal{S}_i^t : w(S^*) \geq w(S), \forall S \in \mathcal{S}_i^t \right\}$

a.4. $\forall i \in L_t$ set $q_i = w(S^*)$ where $S^* \in \mathcal{S}_i^{*t}$ and $\mathcal{S}^{*t} = \bigcup_{i \in L_t} \mathcal{S}_i^{*t}$.

b. If $\left| L(\Gamma^t) \right| = 2$ and $\bigcup_{\tau=0}^{\tau=t-1} L_\tau \cup L(\Gamma^t) = N$, select arbitrarily an element i of $L(\Gamma^t)$, set $L_t = \{i\}$, and go back to *a.1.* After, proceed with the remaining element from 1.

II. If $n = 2$, set $t = 0$ and then go to b.

The process finishes at stage $t = T$ where $N \backslash \bigcup_{\tau=0}^{\tau=T} L_\tau = \emptyset$.

Define the RMCA(N, v_Γ) as the vector $x \in \mathbb{R}^n$ such that $x_i = q_i$ for $i \in N$, and $\mathcal{S}^* = \bigcup_{t=0}^{t=T} \mathcal{S}^{*t}$. Note that $\forall S \in \mathcal{S}^*$, $v_\Gamma(S) = x(S)$. The coalitions in $\mathcal{S}^*$ allow us to identify the set of stable coalition structures for tree-graph communication games.

Theorem 3.1. *For any tree-graph communication game (N, v_Γ), there exists a set of partitions $\mathcal{P}^* \subset \mathcal{P}(N)$ such that $C(N, v_\Gamma, P^*) \neq \emptyset$ for any $P^* \in \mathcal{P}^*$.*

Proof. We will show that RMCA(N, v_Γ) belongs to the core of the game (N, v_Γ, P^*), $\forall P^* \in \mathcal{P}^*$. The theorem is proved in three steps. In step 1 we construct a family of sets $\mathcal{F}(x)$ associated to the RMCA x. In step 2 we show that this family forms a partition $P^*$of N. Finally, in step 3 it is established that $x \in C(N, v_\Gamma, P^*)$.

Step 1. The construction of the set $\mathcal{F}(x)$ has the following stages.

Stage 0: Let $i_T \in N$ be the element whose payoff is determined in the last stage of the algorithm above described. Select a coalition $S(i_T) \in \mathcal{S}^*$, with $i_T \in S(i_T)$. Let $\mathcal{F}_0 = \{S(i_T)\}$ and $I_0 = \{i_T\}$. Define $\mathcal{A}_0^* = \{S \in \mathcal{S}^* \colon S(i_T) \cap S = \emptyset\}$.

Stage t : For $t \geq 1$, define $\mathcal{F}^t = \bigcup_{\tau=0}^{\tau=t-1} \mathcal{F}_\tau$, $B_t = \bigcup_{S \in \mathcal{F}^t} S$ and $I_t = I(B_t)$. For each $k \in I_t$ select a set $S(k) \in \mathcal{A}_{t-1}^*$, with $k \in S(k)$, and define $\mathcal{F}_t = \{S(k)\}_{k \in I_t}$. Define $\mathcal{A}_t^* = \{S \in \mathcal{A}_{t-1}^* \colon S(k) \cap S = \emptyset$ for each $k \in I_t\}$.

Stage T* : This construction ends at stage T* when $B_{T^*} = N$. Define $\mathcal{F}(x) = \mathcal{F}^{T^*}$.

Step 2. We show that the family $\mathcal{F}(x)$ is a partition of N. By construction of $\mathcal{F}(x)$, we have that $\bigcup_{S\in\mathcal{F}(x)} S = N$. To prove that the elements of the set $\mathcal{F}(x)$ are disjoint reverts to show that $\forall S(k), S(k') \in \mathcal{F}(x)$ it is satisfied that (a) $\forall t$ and $\forall k \in I_t$, $S(k) \cap S(k') = \emptyset$ for $k' \in \bigcup_{\tau=0}^{\tau=t-1} I_\tau$, and (b) $\forall t$ and $\forall k, k' \in I_t$, $S(k) \cap S(k') = \emptyset$.

(a) By construction, the elements in I_t are a subset of the set of leaves of the tree $\langle B_t \cup I_t, E(B_t \cup I_t)\rangle$, and taking into account the way of proceeding of the algorithm we have that $\forall k \in I_t$, $S(k) \cap B_t = \emptyset$, which in turns implies $S(k) \cap S(k')$ $=\emptyset$ for $k' \in \bigcup_{\tau=0}^{\tau=t-1} I_\tau$.

(b) Notice that $k, k' \in I_t$ are connected by means of elements of B_t. If $S(k) \cap S(k') \neq \emptyset$ both elements would be connected by a different element which does not belong to B_t. This implies the presence of a cycle which is not possible.

Step 3. We show that $x \in C(N, v_\Gamma, P^*)$. Set $P^* = \mathcal{F}(x)$. By construction we have that $\forall S \in \mathcal{F}(x)$ is verified that $x(S) = v_\Gamma(S)$. It remains to prove that $x(S) \geq v_\Gamma(S)$, $\forall S \subset N$. In a tree-graph communication game it is sufficient to prove these inequalities for the sets $S \in \mathcal{C}(N)$. Let $S \in \mathcal{C}(N)$ and consider $\bar{k} \in S$ the last element whose payoff is determined within S according to the algorithm. Then, we have that: $x(S) = x_{\bar{k}} + \sum\limits_{i\in S, i\neq\bar{k}} x_i \geq v_\Gamma(S) - \sum\limits_{i\in S, i\neq\bar{k}} x_i + \sum\limits_{i\in S, i\neq\bar{k}} x_i = v_\Gamma(S)$.

Finally, since $\forall i \in N$ we have that $\left|S_i^{*t}\right| \geq 1$, $t = 0, ..., T$, we obtain the set $\mathcal{P}^*$ of partitions of $\mathcal{P}(N)$ such that $C(N, v_\Gamma, P^*) \neq \emptyset$, $\forall P^* \in \mathcal{P}^*$. Note that, according to the algorithm used to calculate the RMCA, the set $\mathcal{P}^*$ contains all partitions of maximum worth. □

Remark 1. The maximum number of stages needed to calculate the RMCA and the stable coalition structure is $\frac{n+1}{2}$ if n is odd, or $\frac{n}{2} + 1$ if n is even.

Remark 2. Our proof of Theorem 3.1 is constructive in the sense that it determines the set of partitions of maximum worth for which the core of the game is nonempty. An alternative proof of this theorem is the following. Consider the game (N, v_Γ, P^*) where P^* is a partition of maximum worth. The superadditive cover $(N, \widehat{v}_\Gamma)$ of the game (N, v_Γ) is such that, for all $S \subseteq N$, $\widehat{v}_\Gamma(S) = \max_{P \in \mathcal{P}(S)} \{\sum_{T \in P} v_\Gamma(T)\}$ where $\mathcal{P}(S)$ is the set of partitions of S. For this game we have that $C(N, \widehat{v}_\Gamma) = C(N, v_\Gamma, P^*)$ (see Aumann and Drèze (1974)). Moreover, the zero normalized game of $\widehat{v}_\Gamma$, $\widehat{v}_\Gamma^0$ is a Γ-component additive game and therefore, it has a nonempty core (see Potters and Reijnierse (1995)). Hence, $C(N, \widehat{v}_\Gamma) = C(N, v_\Gamma, P^*) \neq \emptyset$. This alternative proof, as well as the proof of the next proposition, is due to an anonymous referee.

Proposition 3.1. *The RMCA allocates to the central node of the tree (or to one of the two central nodes) the maximum payoff compatible with the game (N, v_Γ, P^*) having a nonempty core.*

Proof. Let $x = RMCA(N, v_\Gamma) \in C(N, v_\Gamma, P^*)$ and i_T be the central player in the tree Γ. Consider the game (N, v'_Γ) defined as $v'_\Gamma(S) = v'_\Gamma(S \cup \{i_T\}) = v_\Gamma(S)\ \forall S \subseteq N \backslash \{i_T\}$. This game is also a tree-graph communication game.

Let $y = RMCA(N, v'_\Gamma) \in C(N, v'_\Gamma, P'_*)$ where $P'_* \in \mathcal{P}(N)$ is a partition of maximum worth associated to v'_Γ, such that, w.l.o.g, $P'_* = \{\{i_T\} \cup S_1, S_2, ... S_m\}$, $S_k \cap S_j = \emptyset$ for all $k \neq j$ and $\bigcup_{k=1}^m S_k = N \backslash \{i_T\}$.

Notice that: a) $y_{i_T} = 0$ as i_T is a null player in the game v'_Γ; and b) $y_i = x_i\ \forall i \in N \backslash \{i_T\}$ as the algorithm determines the payoff of players in $N \backslash \{i_T\}$ regardless the worth of coalitions containing player i_T.

By construction of the allocation y we have that

$$\begin{array}{l} v_\Gamma(S_1) = v'_\Gamma(\{i_T\} \cup S_1) = \sum_{i \in S_1} y_i + y_{i_T} = \sum_{i \in S_1} y_i \text{ and} \\ v_\Gamma(S_k) = v'_\Gamma(S_k) = \sum_{i \in S_k} y_i,\ k = 2, ..., m \end{array} \tag{3.1}$$

Now, suppose that we take an allocation $x' \in C(N, v_\Gamma, P^*)$ such that $x'_{i_T} > x_{i_T}$. Hence we have that

$$\sum_{k=1}^m v_\Gamma(S_k) \leq \sum_{i \in N \backslash \{i_T\}} x'_i < \sum_{i \in N \backslash \{i_T\}} x_i = \sum_{i \in N \backslash \{i_T\}} y_i = (\text{by } (3.1)) = \sum_{k=1}^m v_\Gamma(S_k),$$

where the first inequality holds as we suppose x' to be a core element.

Therefore we get that $\sum_{k=1}^m v_\Gamma(S_k) < \sum_{k=1}^m v_\Gamma(S_k)$ which is a contradiction. So, we conclude that x' is not a core element. □

4 An Example : the Provision of an Emergency Service

A set of municipalities located on the nodes of a tree-graph decide to share a common emergency service (e.g., fire-station or ambulance service) that requires the construction of one or more plants of provision (in the case in which some subgroup of municipalities prefers to construct and share its own emergency service). Each provision plant charges municipalities that use the service with a price that maximizes its profits. The profits obtained by each plant have to be divided among the users. In sharing the benefits, municipalities take into account the quality of the service which depends negatively on the distance between the municipalities and the location of the plant.

It is assumed that each subgroup of municipalities S decides to locate the plant in the point of the graph so that the aggregate distance is minimized, i.e., $y^*(S) = \arg\min_{y\in\mathcal{L}(\Gamma)} \sum_{i\in S} d(i,y)$ where $\mathcal{L}(\Gamma)$ denotes the set of all possible locations of the emergency service in the tree Γ. In this case, $d(S,y^*(S)) = \sum_{i\in S} d(i,y^*(S))$ represents the measure of distance between S and the plant. We introduce a parameter $k \geq 0$ that denotes the effect of the aggregate distance on the quality of service provided. The benefits obtained by the provision plant attending the set of municipalities S will be denoted by $\Pi(S)$.

The profits derived by a set of municipalities S is identified with the characteristic function of the following tree-graph communication game:

$$v_\Gamma(S) = \begin{cases} \Pi(S)\cdot [d(S,y^*(S))]^{-k} & \text{for all } S\in\mathcal{C}(N),\ |S|\geq 2 \\ \sum_{K\in\mathcal{K}(S)} v_\Gamma(K) & \text{for all } S\notin\mathcal{C}(N) \end{cases} \tag{4.1}$$

Moreover, we assume $v_\Gamma(\{i\}) = \Pi(\{i\})$ for all $i\in N$.

The next numerical example highlights how our algorithm works.

Example 4.1. Consider the tree-graph communication game (N, v_Γ) associated to the following tree $\langle N,E\rangle = \langle\{1,2,3,4,5,6\},\{\{1,2\},\{1,3\},\{3,4\},\{4,5\},\{4,6\}\}\rangle$, with the distances between municipalities given by: $d(1,2) = 5$, $d(1,3) = 10$, $d(3,4) = 10$, $d(4,5) = 8$, $d(4,6) = 8$, and $\Pi(S) = s^2$ for all $S\subseteq N$.

From (4.1) the characteristic function of this example is:

$$\begin{array}{lll} v_\Gamma(i) = 1,\ \forall i \in N; & v_\Gamma(12) = 4\,[5]^{-k}; & v_\Gamma(13) = 4\,[10]^{-k}; \\ v_\Gamma(34) = 4\,[10]^{-k}; & v_\Gamma(45) = 4\,[8]^{-k}; & v_\Gamma(46) = 4\,[8]^{-k}; \\ v_\Gamma(123) = 9\,[15]^{-k}; & v_\Gamma(134) = 9\,[20]^{-k}; & v_\Gamma(345) = 9\,[18]^{-k}; \\ v_\Gamma(346) = 9\,[18]^{-k}; & v_\Gamma(456) = 9\,[16]^{-k}; & v_\Gamma(1234) = 16\,[35]^{-k}; \\ v_\Gamma(1345) = 16\,[38]^{-k}; & v_\Gamma(1346) = 16\,[38]^{-k}; & v_\Gamma(3456) = 16\,[26]^{-k}; \\ v_\Gamma(12345) = 25\,[53]^{-k}; & v_\Gamma(12346) = 25\,[53]^{-k}; & v_\Gamma(13456) = 25\,[46]^{-k}; \\ v_\Gamma(N) = 36\,[71]^{-k}; & & \end{array}$$

and $v_\Gamma(S) = \sum_{K\in\mathcal{K}(S)} v_\Gamma(K), \forall S \notin \mathcal{C}(N)$.

- For $k = 0.4$, the efficient structure of provision is $P^* = \{N\}$ with the emergency service located at node 4 and with $RMCA(N, v_\Gamma, P^*) = (1.1, 1, 1.44, 1, 1, 1)$.

- For $k = 0.42$, the efficient structure of provision is $P^* = \{12, 3456\}$ with two emergency services located at node 4 and at node 1 or 2, and with $RMCA(N, v_\Gamma, P^*) = (1.03, 1, 1.07, 1, 1, 1)$.

To show how proceeds our algorithm, we focus on the case $k = 0.4$. The characteristic function of the game with $k = 0.4$ is:

$$\begin{array}{llll} v_\Gamma(i) = 1,\ \forall i \in N; & v_\Gamma(12) = 2.10; & v_\Gamma(13) = 1.59; & v_\Gamma(34) = 1.59; \\ v_\Gamma(45) = 1.74; & v_\Gamma(46) = 1.74; & v_\Gamma(123) = 3.05; & v_\Gamma(134) = 2.71; \\ v_\Gamma(345) = 2.83; & v_\Gamma(346) = 2.83; & v_\Gamma(456) = 2.97; & v_\Gamma(1234) = 3.86; \\ v_\Gamma(1345) = 3.73; & v_\Gamma(1346) = 3.73; & v_\Gamma(3456) = 4.35; & v_\Gamma(12345) = 5.11; \\ v_\Gamma(12346) = 5.11; & v_\Gamma(13456) = 5.40; & v_\Gamma(N) = 6.54; & \end{array}$$

and $v_\Gamma(S) = \sum_{K\in K(S)} v_\Gamma(K),\ \forall S \notin \mathcal{C}(N)$

The algorithm.

0. $L_0 = \{2, 5, 6\}$, then $q_2 = v_\Gamma(2) = 1$, $q_5 = v_\Gamma(5) = 1$, $q_6 = v_\Gamma(6) = 1$, with $\mathcal{S}^{*0} = \{\{2\}, \{5\}, \{6\}\}$.

1. $\Gamma^1 = \langle N \setminus L_0, E(N \setminus L_0)\rangle$; $L_1 = \{1, 4\}$ with $|L_1| = 2$ and $L_0 \cup L_1 \neq N$. Here $\mathcal{S}_1^1 = \{\{1\}, \{1, 2\}\}$ with $w(\{1\}) = v_\Gamma(1) = 1$ and $w(\{1, 2\}) = v_\Gamma(12) - q_2 = 1.1$, $\mathcal{S}_1^{*1} = \{\{1, 2\}\}$ and $q_1 = w(\{1, 2\}) = 1.1$.

Moreover, $\mathcal{S}_4^1 = \{\{4\}, \{4,5\}, \{4,6\}, \{4,5,6\}\}$ with $w(\{4\}) = v_\Gamma(4) = 1$, $w(\{4,5\}) = v_\Gamma(45) - q_5 = 0.74$, $w(\{4,6\}) = v_\Gamma(46) - q_6 = 0.74$ and $w(\{4,5,6\}) = v_\Gamma(456) - q_5 - q_6 = 0.97$, $\mathcal{S}_4^{*1} = \{\{4\}\}$ and $q_4 = w(\{4\}) = 1$. Now $\mathcal{S}^{*1} = \{\{12\}, \{4\}\}$

2. $\Gamma^2 = \langle N \setminus \{L_0 \cup L_1\}, E(N \setminus \{L_0 \cup L_1\})\rangle$; $L_2 = \{3\}$ with $|L_2| = 1$.

$$\mathcal{S}_3^2 = \left\{ \begin{array}{c} \{3\}, \{1,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{3,4,5\}, \{3,4,6\}, \\ \{1,2,3,4\}, \{1,3,4,5\}, \{1,3,4,6\}, \{3,4,5,6\}, \{1,2,3,4,5\}, \\ \{1,2,3,4,6\}, \{1,3,4,5,6\}, \{N\} \end{array} \right\}$$

and $S_3^{*2} = \{\{N\}\}$. Thus, $q_3 = w(\{N\}) = v_\Gamma(N) - q_1 - q_2 - q_4 - q_5 - q_6 = 1.44$ with $\mathcal{S}^{*2} = \{\{N\}\}$. Hence, $\mathcal{S}^* = \{\{2\}, \{5\}, \{6\}, \{12\}, \{4\}, \{N\}\}$, $P^* = \{N\}$, with $RMCA(N, v_\Gamma, P^*) = (1.1, 1, 1.44, 1, 1, 1) \in C(N, v_\Gamma, P^*)$ and $V(P^*) \geq V(P)$, $\forall P \in \mathcal{P}(N)$.

References

Aumann R.J. and J.H. Drèze, (1974). Cooperative games with coalition structures, *International Journal of Game Theory,* **3**, 217-237.

Foulds, L.R. (1992). Graph theory applications, Springer-Verlag, New York.

Myerson, R. (1977). Graphs and cooperation in games, *Mathematics of Operations Research,* **2**, 225-229.

Potters J. and H. Reijnierse, (1995). Γ-component additive games, *International Journal of Game Theory,* **24**, 49-56.

Shenoy, P.P. (1979). On coalition formation: a game-theoretical approach, *International Journal of Game Theory,* **8**, 133-164.