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Spectra of Conformal Field Theories with Current Algebras[‡]

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Abstract: This is an elementary review of our recent work on the classification of the spectra of those two-dimensional rational conformal field theories (RCFTs) whose (maximal) chiral algebras are current algebras. We classified all possible partition functions for such theories when the defining finite-dimensional Lie algebra is simple. The concepts underlying this work are emphasized, and are illustrated using simple examples.

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Perhaps the most fundamental question one can ask about conformal field theories is “which exist?”. More precisely, fixing a chiral algebra, we would like to know the possible spectra of primary fields in any consistent theory. The possible spectra determine the universality classes of critical two-dimensional systems, and also restrict any critical string theory realizing the particular chiral algebra.

As Cardy [1] pointed out, spectra are very strongly constrained by the modular invariance of the torus partition function. For example, when the chiral algebra contains the $\mathfrak{su}(2)$ current algebra (or nontwisted affine Kac-Moody algebra) and there is a finite number of current algebra primary fields, Cappelli, Itzykson and Zuber [2] were able to classify the possible spectra using modular invariance. The analogous exercise for the next-simplest simple Lie algebra, $\mathfrak{su}(3)$, was only completed much later, by one of us [3]. Recently, however, we were able to give the complete list of possible spectra for the special case when the *maximal* chiral algebra is a current algebra based on *any* simple Lie algebra [4]. This work followed [5], where the classification was completed for the simplest series of simple Lie algebras, the A_ℓ (see [4] for further references on modular invariants and classification of spectra).

Here we present an elementary review of [4], emphasizing the concepts involved with simple illustrative examples. Let us first state the problem precisely.

Let X_ℓ be a finite-dimensional simple Lie algebra, and let $X_{\ell,k}$ denote the corresponding current algebra, at positive integer level k . The torus partition function of a conformal field theory containing a finite number of $X_{\ell,k}$ primary fields may be expressed as a sesquilinear combination of its characters:

$$Z = \sum_{\mu, \mu' \in \bar{P}_+(X_{\ell,k})} M_{\mu, \mu'} \chi_\mu^* \chi_{\mu'} \quad . \quad (1)$$

M is a non-negative integer matrix with $M_{0,0} = 1$, and χ_λ denotes the character of the integrable representation of $X_{\ell,k}$ of highest weight corresponding to dominant X_ℓ weight λ . The set of integrable highest weights of $X_{\ell,k}$ is in one-to-one correspondence with the following set of dominant weights of X_ℓ :

$$\bar{P}_+(X_{\ell,k}) = \left\{ \lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \mid \lambda_i \in \mathbb{N} \text{ and } \sum_{i=1}^{\ell} a_i^\vee \lambda_i \leq k \right\} , \quad (2)$$

where the $a_i^\vee$ are the colabels of X_ℓ . For short, we will refer to this set as the alcôve.

If we restrict to the case when the current algebra is the *maximal* chiral algebra, then M defines a permutation σ of the alcôve by $M_{\mu, \mu'} = \delta_{\mu', \sigma(\mu)}$ [6]. The characters $\chi_\lambda(\tau, z, u)$ transform linearly under the action of the modular group, with generators $T : (\tau, z, u) \mapsto (\tau + 1, z, u)$ and $S : (\tau, z, u) \mapsto (-\frac{1}{\tau}, \frac{z}{\tau}, u + \frac{z^2}{2\tau})$. The representing matrices take the form [7]

$$T_{\lambda, \lambda'} = \gamma \exp \left(\frac{2\pi i (\rho + \lambda)^2}{2n} \right) \delta_{\lambda, \lambda'} , \quad (3a)$$

$$S_{\lambda, \lambda'} = \gamma' \sum_{w \in W} (\det w) \exp \left(-\frac{2\pi i (\rho + \lambda) \cdot w(\rho + \lambda')}{n} \right) , \quad (3b)$$

where $n = k + h^\vee$, $h^\vee$ being the dual Coxeter number of X_ℓ , and $\rho = \sum_{i=1}^\ell \omega^i$ is the Weyl vector, the sum of fundamental weights ω^i . γ and γ' are constants independent of λ and λ' , and W is the Weyl group of X_ℓ . The matrices S and T are both symmetric and unitary, and satisfy $S^2 = (ST)^3 = C$. C is charge conjugation, an order 2 symmetry of the Coxeter-Dynkin diagram of X_ℓ (if non-trivial).

Modular invariance of (1) demands that σ commutes with the matrices S and T , i.e.

$$T_{\lambda,\lambda'} = T_{\sigma(\lambda),\sigma(\lambda')}, \quad (4a)$$

$$S_{\lambda,\lambda'} = S_{\sigma(\lambda),\sigma(\lambda')}. \quad (4b)$$

Any permutation of $X_{\ell,k}$ obeying (4a,b) is called an automorphism invariant. In [4], we classified all such automorphism invariants, for any simple Lie algebra X_ℓ .

Condition (4a), T -invariance, is easy to apply, since T is the simple, diagonal matrix of (3a). Not simple to apply directly is the condition (4b) sufficient for S -invariance, but one can find necessary (but not sufficient) conditions that are easy to impose yet still are strong restrictions. First, consider Verlinde's formula for the fusion coefficients:

$$N_{\lambda,\mu}^\nu = \sum_{\beta \in \bar{P}_+(X_{\ell,k})} \frac{S_{\lambda,\beta} S_{\mu,\beta} S_{\nu,\beta}^*}{S_{0,\beta}}. \quad (5)$$

Now, because $S_{\lambda,\mu} > 0$ for all μ only if $\lambda = 0$, we can show that $\sigma(0) = 0$ follows from (4b). Applying this and (4b) to Verlinde's formula gives

$$N_{\sigma(\lambda),\sigma(\mu)}^{\sigma(\nu)} = N_{\lambda,\mu}^\nu, \quad (6)$$

i.e. any automorphism modular invariant is a symmetry of the fusion coefficients. Since the fusion coefficients $N_{\lambda,\mu}^\nu$ are non-negative integers and are more easily calculable than the matrix S (see [8] and references therein), this last condition proved easier to impose than (4b).

The second condition necessary for (4b) that we used involved the so-called quantum dimensions, defined by

$$\mathcal{D}(\lambda) := \frac{S_{0,\lambda}}{S_{0,0}} = \prod_{\alpha > 0} \frac{\sin[\pi\alpha \cdot (\rho + \lambda)/n]}{\sin[\pi\alpha \cdot \rho/n]}. \quad (7)$$

It is easy to see that

$$\mathcal{D}(\sigma\lambda) = \mathcal{D}(\lambda). \quad (8)$$

Because the product formula (7) for $\mathcal{D}$ is considerably simpler than the Weyl-sum formula (3b) for S , the condition (8) is more manageable than the full S -invariance (4b).

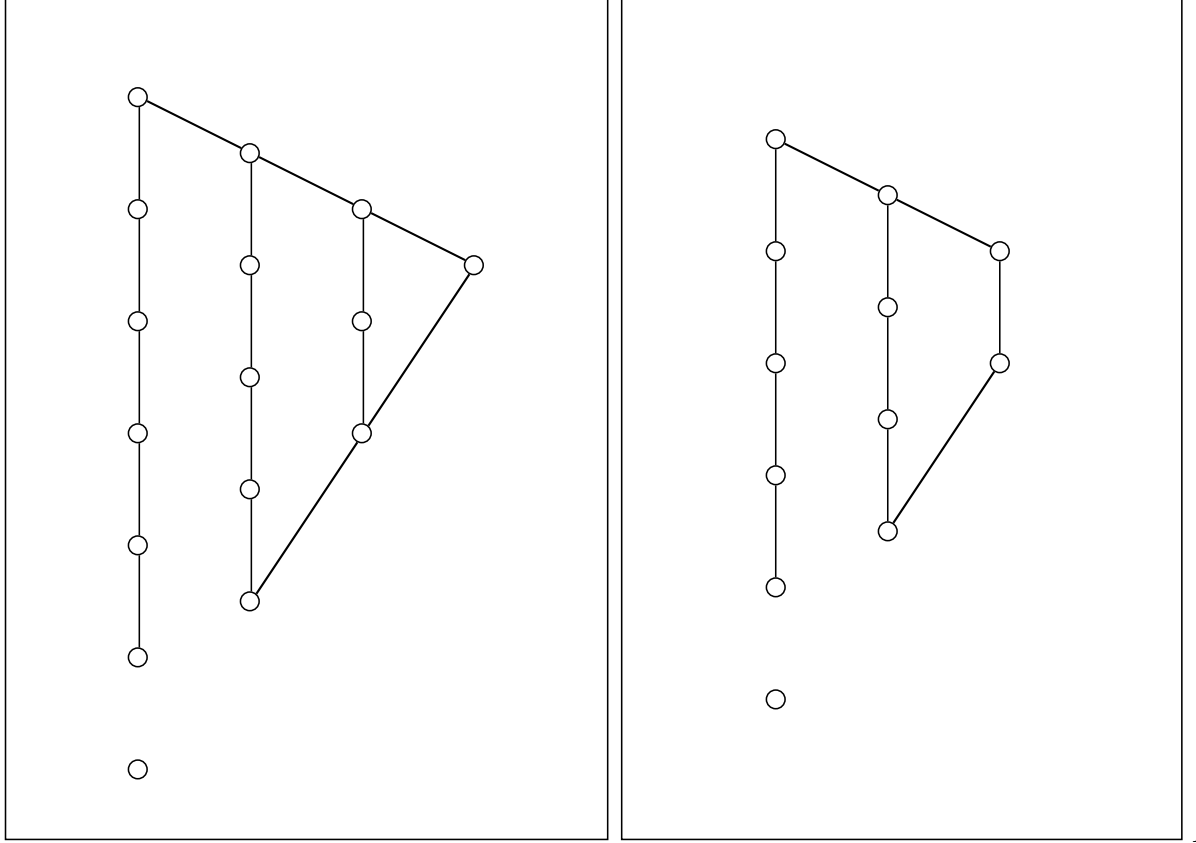


Fig-

ures 1 and 2. Circles indicate the weights of the alcôves of $G_{2,6}$ and $G_{2,5}$, in Figures 1 and 2, respectively. As explained in the text, the lines drawn rule out all but a few candidates for weights of second-lowest quantum dimension.

Now, any automorphism invariant is determined by its action on a subset of the weights of the alcôve. In particular, one can construct an automorphism invariant from its action on the fundamental weights in the alcôve [4]. Our strategy was to find all possible automorphism invariants by finding all possible actions on the fundamental weights that are consistent with (4a), (6) and (8).

First, the sets $\mathcal{Q}_2$ of weights having second-smallest quantum dimension were found. (The sets $\mathcal{Q}_1$ were previously determined by Fuchs [9], and we made use of some of his ideas.) By (8), an automorphism invariant can only permute a weight in $\mathcal{Q}_2$ to itself, or another weight in $\mathcal{Q}_2$. By checking T-invariance (4a), the list of possibilities could be immediately reduced. $\mathcal{Q}_2$ always included a fundamental weight (or some other simple weight), ω^f say, so that only a small number of possibilities for $\sigma\omega^f$ could be listed. By studying fusion rules involving ω^f and the other fundamental weights, (6) could then be used with (4a) to list all possible actions on fundamental weights, and therefore the possible full automorphism invariants.

It was important that for fixed $X_{\ell,k}$, the set of automorphism invariants forms a *group*, with composition as the group multiplication. This meant that known automorphism invariants could be “factored off”, i.e. it was often possible to simplify a potential

automorphism invariant $\tilde{\sigma}$ by multiplying by a known automorphism invariant σ . For example, suppose we know that $\tilde{\sigma}\lambda = \mu$, and a known automorphism invariant σ also obeys $\sigma\lambda = \mu$. Then, if $\tilde{\sigma}$ is an automorphism invariant, so will be $\sigma' := \sigma^{-1} \circ \tilde{\sigma}$, but with $\sigma'\lambda = \lambda$. This proved very useful, since many automorphism invariants were already known (see the references of [4]).

Henceforth we will concentrate on the simple rank-two examples of $X_\ell = G_2$ and C_2 , to illustrate our proof. The first task is to find $\mathcal{Q}_2$ for these algebras. If we consider the quantum dimension as a function of weights with real Dynkin labels λ_i , the function is quite simple, as it turns out. Along any straight line in weight space that begins and ends on weights of the alcôve, the minimum value can only occur at the ends [4]. This reduces the possible elements of $\mathcal{Q}_2$ enormously.

For example, consider the alcôve of $G_{2,6}$ drawn in Figure 1. For G_2 , $\mathcal{Q}_1 = \{0\}$, for all levels k . By considering the lines indicated, one can show that ω^1 , ω^2 , $6\omega^2$, $3\omega^1$ are the only possible candidates for elements of $\mathcal{Q}_2$. For odd levels, the situation is slightly different, as Figure 2 shows. There the case $G_{2,5}$ is drawn, and the candidates are ω^1 , ω^2 , $5\omega^2$, $2\omega^1$, $2\omega^1 + \omega^2$. Clearly then, the candidates for even k are just ω^1 , ω^2 , $k\omega^2$, $\frac{k}{2}\omega^1$, and those for odd level k are ω^1 , ω^2 , $k\omega^2$, $\frac{k-1}{2}\omega^1$, $\frac{k-1}{2}\omega^1 + \omega^2$.

Figure 3 similarly indicates the example of $C_{2,3}$. There is the complication of a diagram automorphism, and corresponding simple current [10] J in the case of C_2 . The simple current J acts as a reflection on the weights of the alcôve, as indicated. Since $\mathcal{D}(\lambda) = \mathcal{D}(J\lambda)$, we can only specify the J -orbits $[\lambda]$ of the candidates λ . Using $\mathcal{Q}_1 = [0] = \{0, k\omega^2\}$, and by considering the lines drawn in Figure 3, it is easy to see that the candidates for $C_{2,k}$ are contained in $[\omega^1]$, $[\omega^2]$ and $[k\omega^1]$.

The candidates λ for elements of $\mathcal{Q}_2$ come in two varieties: those that are independent of the level k , and those that have a single Dynkin label that grows linearly with k . The k -independent ones have quantum dimensions that tend to the ordinary (Weyl) dimension of the corresponding X_ℓ representation in the limit of large k . For large enough k then, we expect the weight λ that is the highest weight of the second-lowest-dimensional representation to be the only surviving candidate of this type. In fact, from (7) follows

$$\frac{\partial}{\partial k} \log \left(\frac{\mathcal{D}(\lambda)}{\mathcal{D}(\mu)} \right) = \frac{\pi}{n^2} \sum_{\alpha > 0} \left[(\mu + \rho) \cdot \alpha \cot \left(\pi \frac{(\mu + \rho) \cdot \alpha}{n} \right) - (\lambda + \rho) \cdot \alpha \cot \left(\pi \frac{(\lambda + \rho) \cdot \alpha}{n} \right) \right]. \quad (9)$$

For G_2 with $\lambda = \omega^1$ and $\mu = \omega^2$, the right hand side of (9) is positive for all levels $k \geq 1$. Numerically, we find that $\mathcal{D}(\omega^1) = \mathcal{D}(\omega^2)$ when $k = 3$, so (9) tells us that $\mathcal{D}(\omega^1) > \mathcal{D}(\omega^2)$ for all levels $k \geq 4$. Similarly, for C_2 with $\lambda = \omega^2$ and $\mu = \omega^1$, the right hand side is positive for all levels. Again, numerically we find that $\mathcal{D}(\omega^1) = \mathcal{D}(\omega^2)$ when $k = 3$. For C_2 then, $\mathcal{D}(\omega^2) > \mathcal{D}(\omega^1)$ for all levels $k \geq 4$. For low levels (≤ 3) the set $\mathcal{Q}_2$ can be found easily numerically, but for higher levels ($k \geq 4$), we still must compare the quantum dimensions of ω^2 for G_2 and ω^1 for C_2 to those of candidates of the second variety.

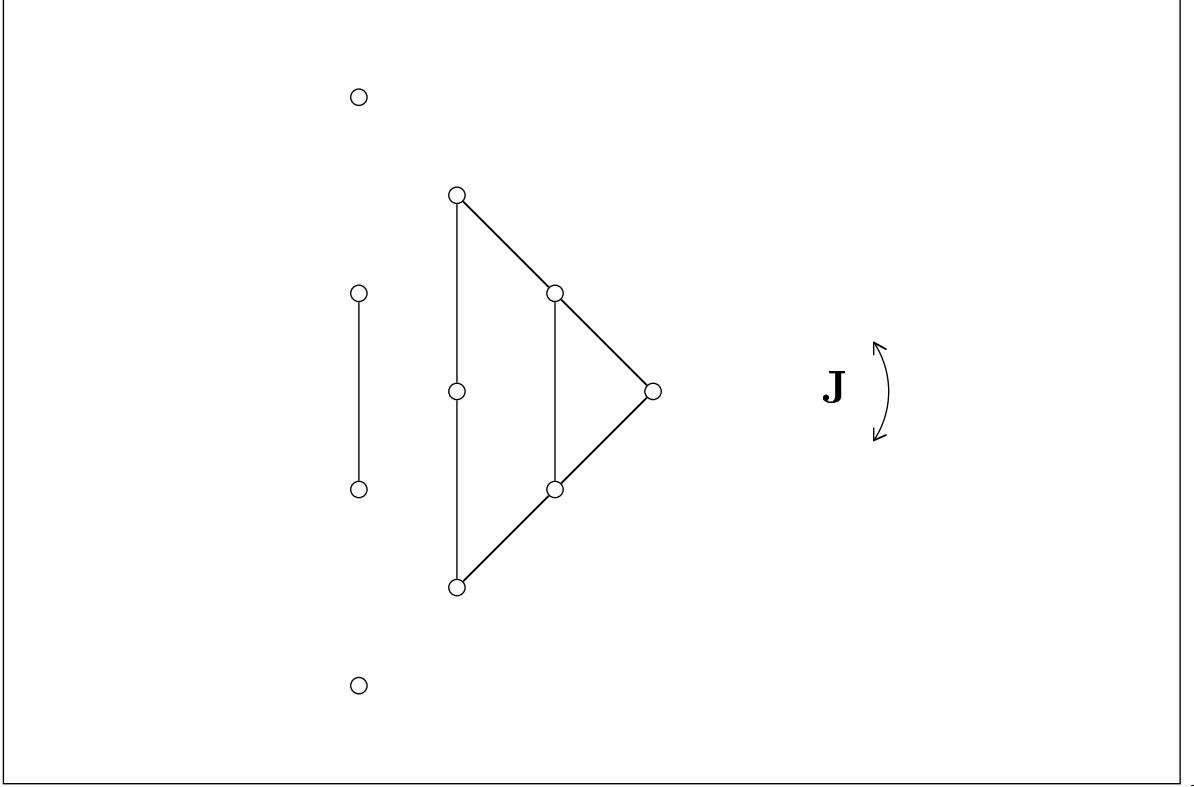


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ure 3. Candidates for $C_{2,3}$ weights of second-lowest quantum dimension. The simple current J reflects the weights as shown.

Consider C_2 first. We will use the well-known level-rank duality of quantum dimensions. Suppose λ is a weight in the alcôve of $C_{\ell,k}$, and λ' is the weight in the alcôve of $C_{k,\ell}$ whose Young tableau is the transpose of that of λ . Then $\mathcal{D}(\lambda) = \mathcal{D}'(\lambda')$, where $\mathcal{D}$ indicates a $C_{\ell,k}$ quantum dimension, and $\mathcal{D}'$ one for $C_{k,\ell}$. For the algebra $C_{k,2}$ dual to $C_{2,k}$, one can show that $\mathcal{D}'(\omega'^k) > \mathcal{D}'(\omega'^1)$ for all $k \geq 4$, in a manner similar to the argument of the previous paragraph. Therefore, rank-level duality tells us that

$$\mathcal{D}(k\omega^1) = \mathcal{D}'(\omega'^k) > \mathcal{D}'(\omega'^1) = \mathcal{D}(\omega^1). \quad (10)$$

So, for all levels $k \geq 4$, we find $\mathcal{Q}_2 = [\omega^1]$. The lower levels can be treated numerically, with results: $k=1$, $\mathcal{Q}_2 = [\omega^1]$; $k=2$, $\mathcal{Q}_2 = [\omega^2] \cup [2\omega^1]$; $k=3$, $\mathcal{Q}_2 = [\omega^1] \cup [\omega^2] \cup [3\omega^1]$.

The situation is more complicated for G_2 ; for example, there is no level-rank duality available. However, the quantum dimension of a k -dependent candidate λ^k is a monotonically increasing function of k for large enough level. Now, the quantum dimension of the surviving k -independent candidate ω^f (ω^2 for G_2) is also an increasing function of k , but one that converges to its Weyl dimension $\dim(\omega^f)$. Therefore, if for a certain level $k = k_1$ we have $\mathcal{D}(\lambda^{k_1}) \geq \dim(\omega^f)$, then $\mathcal{D}(\lambda^k) > \mathcal{D}(\omega^f)$ is guaranteed for all higher levels $k > k_1$.

It is not difficult to show directly from (7) that all k -dependent candidates for G_2 have quantum dimensions that are increasing functions of k for $k \geq 2$. Numerically, we found that they all also exceeded $\dim(\omega^f) = \dim(\omega^2) = 7$ for $k \geq 6$. Therefore, the k -dependent

candidates can be eliminated, so that $\mathcal{Q}_2 = \{\omega^2\}$ for all levels $k \geq 6$. Treating the low levels numerically, we finally find $\mathcal{Q}_2 = \{\omega^2\}$ for $k = 1$ and all $k \geq 5$, $\mathcal{Q}_2 = \{\omega^1\}$ for $k = 2$, $\mathcal{Q}_2 = \{\omega^1, \omega^2, 3\omega^2\}$ for $k = 3$, and $\mathcal{Q}_2 = \{\omega^2, 2\omega^1\}$ for $k = 4$.

The weights of second-lowest quantum dimension for both C_2 and G_2 have now been listed. Now we must select a fundamental field, or some other simple field, $\omega^f \in \mathcal{Q}_2$, and restrict the possibilities $\sigma\omega^f \in \mathcal{Q}_2$ using T -invariance, eqn. (4a). To convey the method, it will be sufficient to restrict to particular levels, and consider just $G_{2,4}$ and $C_{2,3}$.

Level $k = 4$ is the most interesting for G_2 . Choosing $\omega^f = \omega^2$, we see that $\sigma\omega^f = \omega^f$ and $\sigma\omega^f = 2\omega^1$ are the only possibilities, since $\mathcal{Q}_2 = \{\omega^2, 2\omega^1\}$. From the Kac-Peterson formula for the modular matrix T , eqn. (3a), T -invariance means that if $\sigma\lambda = \mu$, we must have $(\lambda + \rho)^2 \equiv (\mu + \rho)^2 \pmod{2(k + h^\vee)}$. With $\lambda = \omega^2$ and $\mu = 2\omega^1$ this yields $8\frac{2}{3} \equiv 24\frac{2}{3} \pmod{(2k + 8)}$. This means $\sigma\omega^f = 2\omega^1$ is possible *only* for level $k = 4$.

So, for $G_{2,4}$ there remain two possibilities: either $\sigma\omega^2 = \omega^2$, or $\sigma\omega^2 = 2\omega^1$. The first possibility implies that $\sigma = id$. To see this, we use the symmetry of the fusion coefficients (6) on the following fusion rule:

$$\omega^2 \otimes \omega^2 = (0)_1^{4\frac{2}{3}} \oplus (\omega^1)_2^{12\frac{2}{3}} \oplus (\omega^2)_1^{8\frac{2}{3}} \oplus (2\omega^2)_2^{14}. \quad (11)$$

We write the fusion rule for all levels as a single tensor product decomposition for G_2 , with subscripts indicating the *threshold level* of the fusions. For example, (11) indicates that the representation of highest weight $2\omega^2$ is contained in the fusion product $\omega^2 \times \omega^2$ for all levels greater than or equal to 2. For convenience, the values $(\lambda + \rho)^2$ for highest weight λ are also indicated, as superscripts. By (6), if $\sigma\omega^2 = \omega^2$, the right hand side of (11) must be invariant under the action of σ . The only nontrivial possibility is that σ interchanges $(\omega^1)_2^{12\frac{2}{3}}$ and $(2\omega^2)_2^{14}$. But T -invariance demands that the superscripts be equivalent $\pmod{2(k + h^\vee) = 16}$. Therefore, $\sigma\omega^1 = \omega^1$ as well as $\sigma\omega^2 = \omega^2$. Since σ fixes all the fundamental weights of $G_{2,4}$, we conclude $\sigma = id$.

The other possibility, $\sigma\omega^2 = 2\omega^1$ is realized in an automorphism invariant of the Galois type first found by Verstegen. We denote this invariant σ_{g2} . As illustrated in Figure 4, it interchanges the weights ω^2 and $2\omega^1$, and the pair $\omega^1, 4\omega^2$, and fixes all others. It is simple to show that σ_{g2} is the *only* invariant sending ω^2 to $2\omega^1$, using the group property of automorphism invariants. For $\sigma_{g2}^{-1} \circ \sigma$ fixes ω^2 , and we showed in the previous paragraph that any such invariant must be the identity.

Therefore, the full set of invariants for $G_{2,4}$ is just $\{id, \sigma_{g2}\}$. For all other levels $k \geq 1$, we found that the identity is the unique automorphism invariant for G_2 .

For $C_{2,3}$ we found above that $\mathcal{Q}_2 = [\omega^1] \cup [\omega^2] \cup [3\omega^1]$. More explicitly, $\mathcal{Q}_2 = \{(\omega^1)^5, (\omega^1 + 2\omega^2)^{17}, (\omega^2)^{13/2}, (2\omega^2)^{25/2}, (3\omega^1)^{13}\}$, where the superscripts again indicate the “norm squared” $(\lambda + \rho)^2$. Since $2(k + h^\vee) = 12$ for $C_{2,3}$, T -invariance tells us that the only possibilities are $\sigma\omega^1 = \omega^1$, or $\sigma\omega^1 = \omega^1 + 2\omega^2$. The former possibility is consistent only with $\sigma = id$, as can be seen from the following fusion rule:

$$\omega^1 \times \omega^1 = (0)^{5/2} + (\omega^2)^{13/2} + (2\omega^1)^{17/2}. \quad (12)$$

If $\sigma\omega^1 = \omega^1$, then the right hand side of (12) is fixed by σ . But comparison of the
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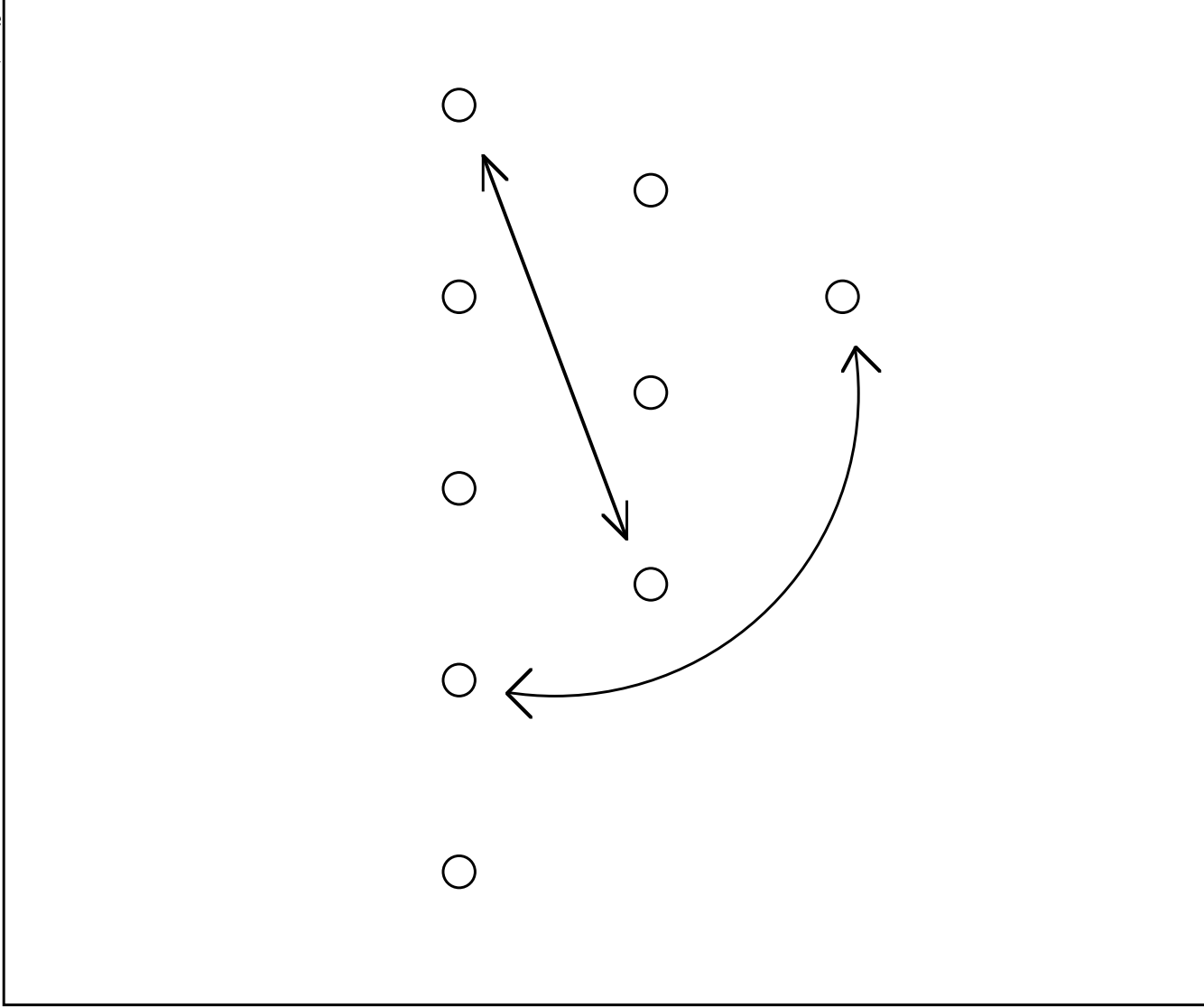


Figure 4. Indicated is σ_{g2} , the $G_{2,4}$ automorphism modular invariant of Galois type. Its action on the weights of the alcôve is shown using arrows. The weights ω^2 and $2\omega^1$ are interchanged, as are ω^1 and $4\omega^2$; all others are fixed.

The other possibility, $\sigma\omega^1 = \omega^1 + 2\omega^2$, is realized in an invariant of the simple current type, illustrated in Figure 5. But if we denote this as σ_J , then $\sigma_J^{-1} \circ \sigma$ fixes ω^1 . Therefore it also fixes ω^2 by the argument above, and so must be the identity. We conclude that the only invariants for $C_{2,3}$ are the identity, and one simple current invariant, pictured in Fig. 5.

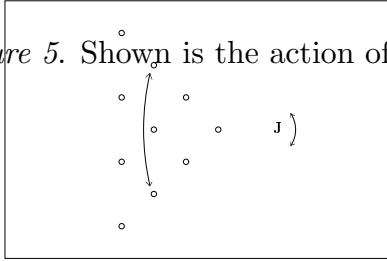
The classification of all automorphism modular invariants of current algebras $X_{\ell,k}$ for all simple X_ℓ proceeds essentially in the same manner, but with more complications. To conclude, we'll comment on our findings [4].

No new surprises were found; all modular invariants listed were previously known.

The existence of infinite series of modular invariants of the orthogonal algebras (B_ℓ and C_ℓ) at level 2 was only recently uncovered [11], however. In [4] we gave the first explicit, direct formulas for these invariants. Our formulas make it clear that they are of a new type (dubbed generalized Galois invariants) that also include invariants in the more general context of rational conformal field theory.

Our catalogue of automorphism invariants contains invariants constructible by the orbifold procedure, the so-called simple current invariants, and those constructible using the Galois properties of modular matrices [11]. Along with the new generalized Galois invariants, there is also the very exceptional $E_{8,4}$ invariant first discovered by Fuchs and Versteegen. It is neither a simple current invariant nor a Galois (nor generalized Galois) invariant. A natural question then is “does there exist a method of constructing modular invariants that includes the $E_{8,4}$ in a more general class?”.

Figure 5. Shown is the action of the simple current invariant σ_J on the alcove of $C_{2,3}$.



Of course, one such method of constructing invariants is that implied by our classification proof. But this procedure is not very direct, although it *can* in principle be applied to rational conformal field theories other than those realizing current algebras.

Modular invariants corresponding to conformal field theories realizing *extensions* of chiral algebras, the extension invariants, are not amenable to our method. For example, the group property of automorphism invariants does not carry over to extension invariants. But perhaps some variation will work. In a sense, however, it is the automorphism invariants that are the most natural. As soon as the chiral algebra is extended, a better description of the theory would be as an automorphism invariant of the extended algebra [6]. On the other hand, the famous A-D-E pattern of $A_{1,k}$ invariants [2] only appears when both automorphism and extension invariants are included. No pattern of a similar nature is (as yet) transparent in our classification.

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