On the integrable structure of deformed sine kernel determinants

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Abstract

We study a family of Fredholm determinants associated to deformations of the sine kernel, parametrized by a weight function w. For a specific choice of w, this kernel describes bulk statistics of finite temperature free fermions. We establish a connection between these determinants and a system of integro-differential equations generalizing the fifth Painlevé equation, and we show that they allow us to solve an integrable PDE explicitly for a large class of initial data.

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1 Introduction

Context. On microscopic scales, eigenvalues of large classes of random matrices are distributed according to a limited number of universal determinantal point processes. The most classical examples of such point processes are the sine process, which arises in bulk scaling limits, and the Airy point process, which corresponds to (soft) edge scaling limits. They are determinantal point processes on the real line with correlation kernels

$$\mathrm{K}^{\sin}(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}, \qquad \mathrm{K}^{\mathrm{Ai}}(x,y) = \int_0^{+\infty} \mathrm{Ai}(x+u)\mathrm{Ai}(y+u)\mathrm{d}u,$$

where Ai is the Airy function. In recent years, particular deformations of the sine and Airy kernels, along with the associated deformed determinantal point processes, have attracted interest because of their connection with models of free fermions at finite temperature [13, 24] and with the Moshe-Neuberger-Shapiro model [22, 26, 27]. A Fredholm determinant of this finite-temperature deformation of the Airy kernel moreover characterizes the narrow wedge solution of the Kardar-Parisi-Zhang equation [1], and has been studied intensively because of this connection. More recently, another deformation of the Airy and sine kernel determinants appeared in the study of limiting behavior for edge and bulk spacing distributions of complex elliptic Ginibre matrices in weak non-hermiticity limits [6, 7].

In this work, our aim is to study a large class of Fredholm determinants of finite-temperature type deformations of the sine kernel. We will show that, just like in the Airy case, these determinants have a rich integrable structure: they are intimately connected to an integro-differential Painlevé equation, and they are the key objects for the direct and inverse scattering transform of a Zakharov-Shabat system, which allows us to solve an integrable PDE introduced by Its, Izergin, Korepin, and Slavnov [20] in a remarkably simple and explicit manner in terms of its initial data.

The sine kernel determinant. Let $\mathcal{K}^{\sin}$ be the integral operator with kernel $\mathrm{K}^{\sin}$,

$$\mathcal{K}^{\sin} f(x) = \int_{-\infty}^{+\infty} \mathrm{K}^{\sin}(x,y) f(y) \mathrm{d}y.$$

We define the sine kernel Fredholm determinant as follows,

$$F(s;\ell) := \det\left(1 - \ell \mathcal{K}^{\sin}|_{[-\frac{s}{2\pi},\frac{s}{2\pi}]}\right) := 1 + \sum_{n=1}^{\infty} \frac{(-\ell)^n}{n!} \int_{[-\frac{s}{2\pi},\frac{s}{2\pi}]^n} \det\left(\mathrm{K}^{\sin}(x_j,x_k)\right)_{j,k=1}^{n} \prod_{j=1}^{n} \mathrm{d}x_j, \quad (1.1)$$

for $s > 0, \ell \in \mathbb{C}$, where we recognize the series at the right as the standard Fredholm series. For $\ell = 1$, this quantity $F(s;1)$ is equal to the probability that a random point configuration in the sine process contains no points in the interval $[-\frac{s}{2\pi}, \frac{s}{2\pi}]$; for $0 < \ell < 1$, $F(s;\ell)$ is the probability that a random point configuration in the thinned sine process, obtained by removing each point in a configuration independently with probability $1-\ell$, contains no points in the interval $[-\frac{s}{2\pi}, \frac{s}{2\pi}]$.

Jimbo, Miwa, Mori, and Sato [21] first and Tracy and Widom [30] later proved that $F(s;\ell)$ is expressible in terms of a Painlevé transcendent: they proved

$$F(s;\ell) = \exp\left(\int_0^s \frac{\nu(x;\ell)}{x} \mathrm{d}x\right), \quad (1.2)$$

where $\nu := \nu(x;\ell)$ solves (a special case of) the σ-form of the fifth Painlevé equation, namely the second order ODE (here $'$ denotes the x-derivative)

$$(x\nu'')^2 + 4(x\nu' - \nu)(x\nu' - \nu + (\nu')^2) = 0, \quad (1.3)$$

together with the boundary condition

$$\nu(x;\ell) = -\frac{\ell}{\pi} x + O(x^2), \qquad x \to 0. \quad (1.4)$$

For later convenience, we remark that this implies in particular that

$$\partial_s \log F(s;\ell) = \frac{\nu(s;\ell)}{s}, \quad (1.5)$$

and so

$$\partial_s s \partial_s \log F(s;\ell) = \nu'(s;\ell). \quad (1.6)$$

Deformed sine kernel determinants. Observe that we can express the sine kernel as

$$\mathrm{K}^{\sin}(x,y) = \int_0^1 \cos\pi(x-y)t \mathrm{d}t = \int_{-1/2}^{1/2} \mathrm{e}^{2\pi\mathrm{i}(x-y)u} \mathrm{d}u. \quad (1.7)$$

Using this last expression, we can factorize the associated integral operator acting on $L^2(\mathbb{R})$ as follows,

$$\mathcal{K}^{\sin} = \mathcal{F}^* \chi_{(-1/2,1/2)} \mathcal{F} = \mathcal{F} \chi_{(-1/2,1/2)} \mathcal{F}^*, \quad (1.8)$$

where χ_I denotes the projection operator on the set I, and $\mathcal{F}$ and $\mathcal{F}^*$ denote the Fourier transform and its inverse acting on $L^2(\mathbb{R})$,

$$\mathcal{F} f(x) = \int_{\mathbb{R}} f(t) \mathrm{e}^{-2\pi\mathrm{i}xt} \mathrm{d}t, \text{ and } \mathcal{F}^* g(y) = \int_{\mathbb{R}} g(u) \mathrm{e}^{2\pi\mathrm{i}yu} \mathrm{d}u. \quad (1.9)$$

We can deform the sine kernel by replacing the projection operator in (1.8) by a multiplication operator,

$$\mathcal{K}_w^{\sin} = \mathcal{F}^* \mathcal{M}_w \mathcal{F}, \qquad \mathcal{M}_w f(u) = w(u) f(u), \quad (1.10)$$

for any integrable function $w:\mathbb{R}\to\mathbb{C}$, such that the associated kernel is given by

$$\mathrm{K}_w^{\sin}(x,y)=\int_{-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}(x-y)u}w(u)\mathrm{d}u. \tag{1.11}$$

We consider the Fredholm determinant

$$F_w(s):=\det\left(1-\mathcal{K}_w^{\sin}|_{[-\frac{s}{2\pi},\frac{s}{2\pi}]}\right). \tag{1.12}$$

As we will show later, $\mathcal{K}_w^{\sin}|_{[-\frac{s}{2\pi},\frac{s}{2\pi}]}$ is trace-class (see Proposition 3.1), such that the Fredholm determinant is well-defined, and we have the alternative Fredholm determinant identity

$$F_w(s):=\det\left(1-\sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right), \tag{1.13}$$

where $\mathcal{K}^{\sin}$ is the non-deformed sine kernel operator, and $\sqrt{w_s}$ denotes the multiplication operator with a square root (note that the operator $\sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}$ is independent of the choice of square root) of the function $w_s(r):=w(\frac{\pi r}{s})$ (see Proposition 3.2). The identity (1.13) has important advantages, since it allows us to characterize $F_w(s)$ in terms of a 2×2 matrix Riemann-Hilbert (RH) problem (see Section 4).

If w moreover takes values in $[0,1]$, the operator $\mathcal{K}_w^{\sin}$ is Hermitian and we will show (see also Proposition 3.1) that $0\leq\mathcal{K}_w^{\sin}<1$. It then follows from [28] that $\mathcal{K}_w^{\sin}$ defines a unique determinantal point process, which we call the (w-)deformed sine process. The Fredholm determinant (1.12) is then equal to the probability that a random configuration in this deformed sine process has no points in $[-\frac{s}{2\pi},\frac{s}{2\pi}]$.

For general integrable $w:\mathbb{R}\to\mathbb{C}$, the second Fredholm determinant identity (1.13) also has a natural point process interpretation: it is the average multiplicative statistic $\mathbb{E}\prod_{j=1}^{\infty}(1-w_s(x_j))$, where the average is with respect to the (non-deformed) sine process, and $x_1,x_2,\dots$ denote the random points in a configuration.

Remark 1.1. *The finite temperature sine kernel from [13, 22, 26] is given by*

$$K_\alpha^{\mathrm{FTsin}}(x,y)=\int_0^{+\infty}\frac{\cos\pi(x-y)u}{\alpha\mathrm{e}^{u^2}+1}\mathrm{d}u=\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}(x-y)u}}{\alpha\mathrm{e}^{4u^2}+1}\mathrm{d}u, \tag{1.14}$$

where $\alpha\in(0,1)$, and $-\frac{1}{\log\alpha}$ can be interpreted physically, in a free fermion model, as a parameter proportional to the temperature of the system. This is precisely our deformed sine kernel $\mathrm{K}_w^{\sin}$ in (1.11) for the special choice of w,

$$w(u)=\frac{1}{\alpha\mathrm{e}^{4u^2}+1}. \tag{1.15}$$

Observe that this is an even function taking values in $(0,1)$.

Remark 1.2. *Recently, [17, 18] studied other models of free fermions, different from the ones in e.g. [13]. These models are concerned with potentials with delta impurity. Already at zero temperature, certain limiting behaviors of the fermions in the bulk are again described by a deformed sine kernel of the type (1.11) but with $w(u)=h(u)\chi_I(u)$, h being a smooth function (depending on the potential) and I a compact real interval. As we will see, this choice of w does not actually belong to the class for which our results hold, and it would be interesting to see how our results generalize to such situations.*

Objectives. As already mentioned, the deformed sine kernel determinants introduced above are natural counterparts of deformed Airy kernel determinants which have been investigated intensively in recent years. Define the σ-deformed Airy kernel as

$$\mathrm{K}_\sigma^{\mathrm{Ai}}(x,y)=\int_{\mathbb{R}}\mathrm{Ai}(x+t)\mathrm{Ai}(y+t)\sigma(t)\mathrm{d}t.$$

For a suitable class of functions σ, it was shown first in [1] (see also [4, 5, 9, 23]) that

$$\partial_s^2 \log\det\left(1-\mathcal{K}_\sigma^{\mathrm{Ai}}|_{(s,+\infty)}\right) = -\int_{\mathbb{R}} \varphi_\sigma(\lambda;s)^2\sigma'(\lambda)\mathrm{d}\lambda, \tag{1.16}$$

where φ_σ solves the integro-differential Painlevé II equation

$$\partial_s^2\varphi_\sigma(\lambda;s) = \left(\lambda+s+2\int_{\mathbb{R}}\varphi_\sigma(\mu;s)^2\sigma'(\mu)\mathrm{d}\mu\right)\varphi_\sigma(\lambda;s). \tag{1.17}$$

If $\sigma=\chi_{(0,+\infty)}$, the kernel $\mathrm{K}_\sigma^{\mathrm{Ai}}$ is the classical Airy kernel, and in that case the integro-differential Painlevé II equation degenerates to the Painlevé II equation $q''(s)=sq(s)+2q(s)^3$ for $q(s)=\varphi_\sigma(0;s)$, thus recovering the classical Tracy-Widom formula [29].

Moreover, for a one-parameter family of functions $\sigma=\sigma_t$ of the form $\sigma_t(u)=\sigma(t^{-2/3}u)$, the quantity $\partial_x^2\log\det\left(1-\mathcal{K}_{\sigma_t}^{\mathrm{Ai}}|_{(-xt^{-1/3},+\infty)}\right)+\frac{x}{2t}$ solves the KdV equation, see [9, Theorem 1.3].

Our objective in this work is to describe similar connections between the deformed sine kernel determinants, an integro-differential Painlevé equation, and an integrable PDE. To avoid technical complications, we will restrict ourselves to Schwartz functions $w:\mathbb{R}\to\mathbb{C}$, even though we believe most of the results hold under weaker smoothness and decay assumptions on w. First, we will reveal a simple connection between $F_w(s)$ and the Zakharov-Shabat system, which we can re-write here as a system of integro-differential equations. For w even, it reduces to a single integro-differential equation. We will also explain how we can see this integro-differential equation as a generalization of the fifth Painlevé equation. Secondly, we will consider a one-parameter family of even functions w, namely we will take $w(u)$ of the form $W(u^2-y)$, $y\in\mathbb{R}$, for some function W. Then, the Fredholm determinants $F_{W(.^2-y)}(s)$ are connected to an integrable PDE introduced in [20]. As we will see, they drive the direct and inverse scattering transform which allows to solve this integrable PDE explicitly in terms of its initial data. It is striking that the map transforming initial data into scattering data and vice versa consists only of applying simple integral transforms.

2 Statement of results

Connection with the Zakharov-Shabat system. Our first result connects the determinant $F_w(s)$ to a system of integro-differential equations, arising from the Zakharov-Shabat system, in general, and to an integro-differential generalization of the Painlevé V equation, in the case where w is symmetric. In particular, we are going to prove the following characterization.

Theorem 2.1. *Let $w:\mathbb{R}\to\mathbb{C}$ be a Schwartz function.*

(i) For every $s>0$ such that $F_w(s)\neq 0$, we have the identity

$$\partial_s s\partial_s \log F_w(s) = \frac{1}{\pi}\int_{\mathbb{R}}\lambda w'(\lambda)\phi(\lambda;s)\psi(\lambda;s)\mathrm{d}\lambda, \tag{2.1}$$

where ϕ,ψ solve the system of equations

$$\partial_s\phi(\lambda;s) = \mathrm{i}\lambda\phi(\lambda,s) - \frac{1}{2\pi\mathrm{i}s}\int_{\mathbb{R}}\phi^2(\mu;s)w'(\mu)\mathrm{d}\mu\ \psi(\lambda;s), \tag{2.2}$$

$$\partial_s\psi(\lambda;s) = \frac{1}{2\pi\mathrm{i}s}\int_{\mathbb{R}}\psi^2(\mu;s)w'(\mu)\mathrm{d}\mu\ \phi(\lambda;s) - \mathrm{i}\lambda\psi(\lambda;s). \tag{2.3}$$

with $\lambda\to\pm\infty$ asymptotics

$$\phi(\lambda;s)\sim \mathrm{e}^{\mathrm{i}s\lambda}, \qquad \psi(\lambda;s)\sim\mathrm{e}^{-\mathrm{i}s\lambda}. \tag{2.4}$$

Moreover, if $w:\mathbb{R}\to[0,1]$ takes values in $[0,1]$, $F_w(s)\neq 0$ for all $s>0$.

(ii) If w is even, $w(\lambda) = w(-\lambda)$ for all $u \in \mathbb{R}$, we have

$$\partial_s s \partial_s \log F_w(s) = \frac{1}{\pi} \int_{\mathbb{R}} \lambda w'(\lambda)\phi(\lambda; s)\phi(-\lambda; s)\mathrm{d}\lambda, \tag{2.5}$$

where ϕ solves the integro-differential equation

$$\partial_s \phi(\lambda; s) = \mathrm{i}\lambda\phi(\lambda; s) - \frac{1}{2\pi \mathrm{i} s} \int_{\mathbb{R}} \phi^2(\mu; s)w'(\mu)\mathrm{d}\mu \; \phi(-\lambda; s), \tag{2.6}$$

with $\lambda \to \pm\infty$ asymptotics $\phi(\lambda; s) \sim \mathrm{e}^{\mathrm{i}s\lambda}$.

Remark 2.2. *The system (2.2)–(2.3) is nothing else than the Zakharov-Shabat system [31]*

$$\partial_s \phi(\lambda; s) = \mathrm{i}\lambda\phi(\lambda; s) - \mathrm{i}\beta(s)\psi(\lambda; s),$$
$$\partial_s \psi(\lambda; s) = \mathrm{i}\gamma(s)\phi(\lambda; s) - \mathrm{i}\lambda\psi(\lambda; s),$$

along with the integral identities

$$\beta(s) = \frac{-1}{2\pi s} \int_{\mathbb{R}} \phi^2(\mu; s)w'(\mu)\mathrm{d}\mu, \qquad \gamma(s) = \frac{-1}{2\pi s} \int_{\mathbb{R}} \psi^2(\mu; s)w'(\mu)\mathrm{d}\mu,$$

for the potentials β and γ. Such integral expressions are, in the literature of integrable differential equations, often referred to as trace formulas, see e.g. [15].

Back to Painlevé V. Since w is assumed to be smooth, the choice $w(\lambda) = \chi_{(-\frac{1}{2},\frac{1}{2})}(\lambda)$ is not admissible. However, we can approximate $\chi_{(-\frac{1}{2},\frac{1}{2})}(\lambda)$ by a sequence of smooth functions w, and then in the limit, it is natural to interpret $w'(\lambda) = \delta_{-\frac{1}{2}}(\lambda) - \delta_{\frac{1}{2}}(\lambda)$ as a sum of δ-functions. By doing this, the results of Theorem 2.1 (ii) degenerate to the Painlevé V expression (1.2), as we explain next.

In this degenerate case, the integro-differential equation (2.6) reduces to the system

$$\partial_s \phi\left(\frac{1}{2}; s\right) = \frac{\mathrm{i}}{2}\phi\left(\frac{1}{2}; s\right) - \frac{1}{2\pi \mathrm{i} s}\left(\phi^2\left(-\frac{1}{2}; s\right) - \phi^2\left(\frac{1}{2}; s\right)\right)\phi\left(-\frac{1}{2}; s\right) \tag{2.7}$$

and

$$\partial_s \phi\left(-\frac{1}{2}; s\right) = -\frac{\mathrm{i}}{2}\phi\left(-\frac{1}{2}; s\right) - \frac{1}{2\pi \mathrm{i} s}\left(\phi^2\left(-\frac{1}{2}; s\right) - \phi^2\left(\frac{1}{2}; s\right)\right)\phi\left(\frac{1}{2}; s\right). \tag{2.8}$$

Hence, defining $v(x), u(x)$ by

$$v(x) = \frac{1}{2\pi \mathrm{i}}\phi\left(\frac{1}{2}; \frac{x}{2\mathrm{i}}\right)\phi\left(-\frac{1}{2}; \frac{x}{2\mathrm{i}}\right), \qquad u(x) = \frac{\phi^2\left(\frac{1}{2}; \frac{x}{2\mathrm{i}}\right)}{\phi^2\left(-\frac{1}{2}; \frac{x}{2\mathrm{i}}\right)}, \tag{2.9}$$

we see after a straightforward computation that v, u solve the coupled system of differential equations

$$xv' = v^2(u - \frac{1}{u}), \quad xu' = xu - 2v(u-1)^2. \tag{2.10}$$

This system is well-known to be related to the Painlevé V equation, see [16] or [10, Equations (4.106-7) with $\alpha = 0 = \beta$]: it implies that $u(x)$ solves the Painlevé V equation (again, see [16] or [10, Equation (1.22) with $A = B = 0, C = 1, D = -1/2$]

$$u'' = \frac{u}{x} - \frac{u'}{x} - \frac{u(u+1)}{2(u-1)} + (u')^2\frac{3u-1}{2u(u-1)}, \tag{2.11}$$

and that

$$\sigma(x) = \int_0^x \sigma'(\xi)\mathrm{d}\xi \quad \text{with} \quad \sigma'(x) := -\frac{1}{2\pi \mathrm{i}}\phi\left(\frac{1}{2}; \frac{x}{2\mathrm{i}}\right)\phi\left(-\frac{1}{2}; \frac{x}{2\mathrm{i}}\right),$$

solves the σ-form of the Painlevé V equation

$$(x\sigma'')^2 = (\sigma - x\sigma' + 2(\sigma')^2)^2 - 4(\sigma')^4. \tag{2.12}$$

Defining $\nu(s) = \sigma(2\mathrm{i}s)$, such that

$$\nu'(s) = -\frac{1}{\pi}\phi\left(\frac{1}{2};s\right)\phi\left(-\frac{1}{2};s\right),$$

it is now straightforward to check that $\nu(s)$ solves (1.3) and the above expression in the right hand side is indeed the degenerate case of (2.5) as it should be and it corresponds to equation (1.6) ($\ell = 1$).

Remark 2.3. *Another integro-differential generalization of the Painlevé V equation related to $F_w(s)$ appeared in [7]. For smooth, even and exponentially fast decaying to zero at $\pm\infty$ functions w, the authors there established an expression for the Fredholm determinant $F_w(s)$ in terms of a solution of an integro-differential equation of Painlevé type [7, Theorem 6.7]. More precisely, they gave a formula of the form*

$$\frac{d^2}{ds^2}\log F_w(s) = -\left(\int_0^\infty r(z;s)w'(z)dz\right)^2, \tag{2.13}$$

where $r(z;s)$ solves an integro–differential equation that for $w = \chi_{(-1/2,1/2)}$ degenerates to an ODE for $r(s) = r(1/2;s)$ (see e.g. [2, Equation (3.6.36)]) related to the (special case of the) Painlevé V sigma form (1.3) for $\nu(s)$, by the transformation

$$\frac{d}{ds}\left(\frac{\nu(s)}{s}\right) = -r^2(s).$$

For the specific choice of w given by

$$w(z) = \Phi(\alpha(z+1)) - \Phi(\alpha(z-1)),\ \text{with}\ \Phi(z) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^z \mathrm{e}^{-y^2}dy, \quad \alpha > 0, \tag{2.14}$$

(where $\alpha = s/\sigma > 0$ and σ is a parameter) the authors of [7] also proved that the Fredholm determinant $F_w(s)$ describes the limiting behavior in the weak non-hermiticity limit of the bulk spacing distribution of the real parts of eigenvalues in the complex elliptic Ginibre ensemble.

Second deformation and integrable PDE. In order to relate the Fredholm determinants $F_w(s)$ to an integrable PDE, we now introduce an additional deformation, by deforming the function $w = w_y$ depending on a parameter $y \in \mathbb{R}$. A convenient deformation will turn out to be as follows. Let $W : \mathbb{R} \to \mathbb{C}$ be smooth and decaying fast at $+\infty$, such that the even function $w(\lambda) = W(\lambda^2 - y)$ is a Schwartz function. We consider the Fredholm determinant

$$Q_W(y,s) := \det\left(1 - \mathcal{K}_w^{\sin}|_{[-\frac{s}{2\pi},\frac{s}{2\pi}]}\right) = F_w(s). \tag{2.15}$$

By a simple change of variables in the integrals of the Fredholm series, we see that alternative expressions for Q_W and F_w are

$$Q_W(y,s) = \det\left(1 - \mathcal{K}_{w_{y,s}}^{\sin}|_{[-1/2,1/2]}\right), \qquad F_w(s) = \det\left(1 - \mathcal{K}_{w_{0,s}}^{\sin}|_{[-1/2,1/2]}\right) \tag{2.16}$$

where

$$w_{y,s}(\zeta) = W\left(\frac{\pi^2\zeta^2}{s^2} - y\right). \tag{2.17}$$

Denoting $w_{y,s}$ also for the corresponding multiplication operator, we have, analogous to (1.13) and by Proposition 3.2 below, the alternative representation

$$Q_W(y,s) := \det\left(1 - \sqrt{w_{y,s}}\mathcal{K}^{\sin}\sqrt{w_{y,s}}\right). \tag{2.18}$$

Remark 2.4. *We see from Remark 1.1 that for the special choice*

$$W(r) = \frac{1}{\mathrm{e}^{4r}+1}, \tag{2.19}$$

we can identify $Q_W(y,s)$ with the Fredholm determinant of the finite temperature sine kernel:

$$\det\left(1 - \mathrm{K}_\alpha^{\mathrm{FTsin}}|_{[-\frac{s}{2\pi},\frac{s}{2\pi}]}\right) = Q_W\left(y = -\frac{\log\alpha}{4}, s\right).$$

For $y \in \mathbb{R}, s > 0$ such that $Q_W(y,s) \neq 0$, define $\sigma_W(y,s)$ and $q_W(y,s)$ by

$$\sigma_W(y,s) := \log Q_W(y,s), \qquad q_W(y,s)^2 = -\partial_s^2\sigma_W(y,s). \tag{2.20}$$

Then, q_W and σ_W are solutions to PDEs which appeared more than thirty years ago in the ground-breaking work [20] by Its, Izergin, Korepin, and Slavnov, but which do not seem to have been investigated much in recent years.

Theorem 2.5. *Let $W : \mathbb{R} \to \mathbb{C}$ be such that $W(.^2 - y)$ is a Schwartz function for every $y \in \mathbb{R}$. Then, for any $s > 0$, $y \in \mathbb{R}$ such that $Q_W(y,s) \neq 0$, $q = q_W$ solves the PDE*

$$\partial_s\left(\frac{\partial_s\partial_y q}{2q}\right) = \partial_y(q^2) - 1, \tag{2.21}$$

and $\sigma = \sigma_W$ solves the PDE

$$(\partial_s^2\partial_y\sigma)^2 = 4\partial_s^2\sigma\left(-2s\partial_s\partial_y\sigma + 2\partial_y\sigma - (\partial_s\partial_y\sigma)^2\right). \tag{2.22}$$

In particular, if W takes values in $[0,1]$, $q_W(y,s)$ and $\sigma_W(y,s)$ are defined for all $s > 0$, $y \in \mathbb{R}$, and solve (2.21) and (2.22).

Remark 2.6. *Note that the choice of square root for $q_W(y,s)$ in (2.20) is unimportant for the above result, as long as the relevant derivatives exist locally. Indeed, the equation for q is invariant under the map $q \mapsto -q$. Similarly, the choice of logarithm for $\sigma_W(y,s)$ is unimportant as long as the relevant derivatives exist locally, because σ in (2.22) appears only under a y- or s-derivative. For our asymptotic results below, it is however important to fix our choice of logarithm in (2.20): we therefore define $\sigma_W(y,s)$ as the smooth in y and s logarithm of $Q_W(y,s)$ which is such that $\lim_{s\to 0}\sigma_W(y,s) = 0$.*

Remark 2.7. *For W given by (2.19), the PDEs for $q_W(y,s)$ and $\sigma_W(y,s)$ were already established in [20, Equations (5.14–15)]. This determinant was investigated in [20] to describe the zero temperature equal time one-point correlation function in the impenetrable one dimensional Bose gas [20, Equation (1.10)]. Even if the general techniques developed in [20] form a cornerstone of the modern RH technology which we use here, the connection between $Q_W(y,s)$ and the PDE (2.22) was derived in [20] using methods that are quite different from ours. We believe therefore that it is interesting and enlightening to revisit this connection using the RH methods available nowadays, as we do.*

Asymptotics and scattering. We constructed a class of solutions $\sigma_W(y,s)$ to the PDE (2.22) parametrized by a function W. It is then natural to ask to what extent these are general solutions of this PDE, and whether we can identify the function W in terms of properties of the solution σ_W. To that end, we consider the asymptotic behavior of $\sigma_W(y,s)$ as $s \to 0$. Afterwards, we will see that these asymptotics allow to find a simple relation between the initial data as $s \to 0$ and the function W.

Theorem 2.8. *Let $W : \mathbb{R} \to \mathbb{C}$ be such that $W(.^2 - y)$ is a Schwartz function for every $y \in \mathbb{R}$. Then, $\sigma_W(y,s)$ has the following asymptotics:*

$$\sigma_W(y,s) \sim -\frac{2s}{\pi}\int_0^{+\infty} W(\lambda^2 - y)\mathrm{d}\lambda, \tag{2.23}$$

as $s \to 0$, and also as $y \to -\infty$ with either $s > 0$ fixed or $s \to 0$.

Finally, we can employ the above result to construct an explicit solution method for the PDE (2.22), as stated next.

Theorem 2.9. *Let $f : \mathbb{R} \to \mathbb{C}$ be C^∞ and decaying fast at $-\infty$, such that $f(y - .^2)$ is a Schwartz function for all $y \in \mathbb{R}$. Define*

$$W(r) = -2\int_0^\infty f'(-u^2 - r)\mathrm{d}u. \tag{2.24}$$

Then, for every $s > 0$, $y \in \mathbb{R}$ for which $\sigma_W(y,s) \neq 0$, the function $\sigma_W(y,s)$ solves the PDE (2.22) with initial data

$$\lim_{s\to 0} \frac{1}{s}\sigma_W(y,s) = f(y). \tag{2.25}$$

Remark 2.10. *The expression for W in terms of f and the inverse given in (2.23) can alternatively be rewritten in terms of the direct and inverse Abel integral transform, after an explicit change of variable. For the sake of simpler and more explicit formulas, we choose however not to do this.*

Remark 2.11. *It is striking that the map between initial data and the function W, which we can interpret as scattering data, is so explicit and described by a simple integral transform. For other integrable equations whose solutions are constructed via scattering theory and characterized by RH problems, like the Korteweg-de Vries equation, this map is far more involved, and simplifies to an integral transform only in small dispersion limits, see e.g. [19, 25].*

Remark 2.12. *We believe that a more general class of solutions to (2.22) can be constructed by considering Jànossy densities of the sine process instead of average multiplicative statistics. This is suggested by similar results for deformed Airy kernel determinants and the KdV equation [12], and by the more general considerations in [11].*

Remark 2.13. *Many integrable PDEs admit solutions involving Fredholm determinant expressions of the form $\det(1-K_{x,t})$, where $K_{x,t}$ depends on the variables x, t and on scattering data. It is natural to ask whether our method to find an explicit relation between scattering data and solution data could apply to other integrable PDEs. The general principle to apply the method would consist of (1) finding a region in the (x,t)-plane where $K_{x,t}$ has small trace, such that $\log\det(1 - K_{x,t}) \sim -\mathrm{Tr}\, K_{x,t}$, (2) computing the trace of $K_{x,t}$ explicitly, and (3) inverting the expression for the trace to obtain an explicit expression for the scattering data in terms of the trace.*

Methodology and outline. In Section 3, we gather important properties of the Fredholm determinants $F_w(s)$ and $Q_W(y,s)$, which allow in particular to obtain their $s \to 0$ asymptotics, as well as that of their derivatives with respect to s and y. Next, in Section 4, we characterize the determinants $F_w(s)$ and $Q_W(y,s)$ in terms of a 2×2 matrix-valued RH problem. We also study $s \to 0$ asymptotics of the solution to the RH problem. In Section 5, we use the RH characterization to study the s-dependence of $F_w(s)$ in detail, which allows us to relate $F_w(s)$ to a Zakharov-Shabat system, and to prove Theorem 2.1. In Section 6, we similarly study the y-dependence of $Q_W(y,s)$. In particular, we derive a Lax pair associated to the RH problem and use it to prove Theorem 2.5. We emphasize that the Lax pair is of an unusual type, since it does not only contain derivatives in the deformation parameters y and s, but also in the spectral variable λ (i.e., the complex variable of the RH problem). The results in this section will yield the proof of Theorem 2.5, and by combining it with the small s asymptotics from Section 3, we also prove Theorem 2.8. In Section 7, we finally complete the proof of Theorem 2.9.

3 Preliminaries on the Fredholm determinants $F_w(s)$ and $Q_W(y,s)$

In this section, we will prove that the Fredholm determinants $F_w(s)$ and $Q_W(y,s)$ as well as their derivatives with respect to s and y are well-defined under our assumptions on w and W, and we will establish their $s \to 0$ asymptotics.

3.1 Trace-class operators

We start by collecting useful properties of the integral operator $\chi_I \mathcal{K}_w^{\sin} \chi_I$ acting on $L^2(\mathbb{R})$ with kernel $\chi_I(x) \mathrm{K}_w^{\sin}(x,y) \chi_I(y)$,

$$\left(\chi_I \mathcal{K}_w^{\sin} \chi_I\right) f(x) = \chi_I(x) \int_I \mathrm{K}_w^{\sin}(x,y) f(y) dy, \tag{3.1}$$

where I is a compact real interval, χ_I is the indicator function of I, and $\mathrm{K}_w^{\sin}$ is defined in (1.11).

Proposition 3.1. *Let I be a compact real interval.*

(i) For any integrable $w : \mathbb{R} \to \mathbb{C}$, the integral operator $\chi_I \mathcal{K}_w^{\sin} \chi_I : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a trace-class operator.

(ii) For any integrable $w : \mathbb{R} \to [0,1]$, the integral operator $\chi_I \mathcal{K}_w^{\sin} \chi_I : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is Hermitian, and $0 \leq \chi_I \mathcal{K}_w^{\sin} \chi_I < 1$; in particular $1 - \chi_I \mathcal{K}_w^{\sin} \chi_I$ is invertible on $L^2(\mathbb{R})$.

Proof. Let $w : \mathbb{R} \to \mathbb{C}$ be integrable. For the first part, we use the factorization (1.10) to conclude that

$$\chi_I \mathcal{K}_w^{\sin} \chi_I = (\chi_I \mathcal{F}^* \mathcal{M}_{\sqrt{w}})(\mathcal{M}_{\sqrt{w}} \mathcal{F} \chi_I),$$

for any choice of square root of w. But it is not hard to see that $\chi_I \mathcal{F}^* \mathcal{M}_{\sqrt{w}}$ and $\mathcal{M}_{\sqrt{w}} \mathcal{F} \chi_I$ are Hilbert-Schmidt operators, hence $\chi_I \mathcal{K}_w^{\sin} \chi_I$ is trace-class on $L^2(\mathbb{R})$.

Next, suppose that w takes values in $[0,1]$. Since $0 \leq \mathcal{M}_w \leq 1$, it follows from the factorization (1.10) and the fact that the Fourier transform $\mathcal{F}$ is unitary that $0 \leq \chi_I \mathcal{K}_w^{\sin} \chi_I \leq 1$. Finally, to prove that $\chi_I \mathcal{K}_w^{\sin} \chi_I < 1$ and that $1 - \chi_I \mathcal{K}_w^{\sin} \chi_I$ is invertible, it is enough to prove that 1 is not an eigenvalue of $\chi_I \mathcal{K}_w^{\sin} \chi_I$. By contradiction, suppose that there exists a non-zero function $f \in L^2(\mathbb{R})$ such that $\chi_I \mathcal{K}_w^{\sin} \chi_I f = f$. Then, f would be supported inside I, and we would have

$$\langle f, \chi_I \mathcal{K}_w^{\sin} \chi_I f \rangle = \langle f, f \rangle. \tag{3.2}$$

But then from the one side, we would have by direct computation

$$\langle f, \chi_I \mathcal{K}_w^{\sin} \chi_I f \rangle = \int_{\mathbb{R}} \left| \mathcal{F} f(u) \right|^2 w(u) \mathrm{d}u \tag{3.3}$$

while on the other side by Plancherel's theorem

$$\langle f, f \rangle = \int_{\mathbb{R}} \left| \mathcal{F} f(u) \right|^2 \mathrm{d}u. \tag{3.4}$$

Consequently,

$$\int_{\mathbb{R}} \left| \mathcal{F} f(u) \right|^2 (1 - w(u)) \mathrm{d}u = 0, \tag{3.5}$$

and $\mathcal{F} f(u) = 0$ for a.e. $u \in \mathbb{R}$ for which $w(u) \neq 1$. Since f is compactly supported, it follows from Paley-Wiener's theorem that $\mathcal{F} f$ extends to an entire function, and this implies by the identity theorem that $\mathcal{F} f = 0$, and thus $f = 0$, which is a contradiction. □

As a consequence, we have that the Fredholm determinant $F_w(s)$ is well-defined for every $s > 0$, provided that $w : \mathbb{R} \to \mathbb{C}$ is integrable. By replacing $w(\lambda)$ with $W(\lambda^2 - y)$, we see that $Q_W(y,s)$ is well-defined for any $y \in \mathbb{R}$, $s > 0$, provided that $\lambda \mapsto W(\lambda^2 - y)$ is integrable for any $y \in \mathbb{R}$. Moreover, if w takes values in $[0,1]$, then $F_w(s)$ is non-zero for every $s > 0$; if W takes values in $[0,1]$, then $Q_W(y,s)$ is non-zero for every $s > 0$, $y \in \mathbb{R}$.

3.2 Expressions in terms of integrable operators

Next, we prove that $F_w(s)$ and $Q_W(y,s)$ can be expressed as Fredholm determinants involving only the non-deformed sine kernel operator and a multiplication operator, more precisely we prove (1.13) and (2.18). Similar identities are well-known for other operators, see e.g. [1, 23], and the proof below contains no conceptual novelties, but is included for the convenience of the reader.

Proposition 3.2. *(i) Let $w:\mathbb{R}\to\mathbb{C}$ be integrable. Then,*

$$F_w(s) = \det\left(1-\sqrt{w_{0,s}}\mathcal{K}^{\sin}\sqrt{w_{0,s}}\right), \tag{3.6}$$

where $w_{0,s}(u) = w\left(\frac{\pi u}{s}\right)$.

(ii) Let $y\in\mathbb{R}$ and $W:\mathbb{R}\to\mathbb{C}$ be such that $W(.^2-y)$ is integrable. Then,

$$Q_W(y,s) = \det\left(1-\sqrt{w_{y,s}}\mathcal{K}^{\sin}\sqrt{w_{y,s}}\right), \tag{3.7}$$

where $w_{y,s}$ is given by (2.17).

Proof. It sufficient to prove (3.6), since (3.7) then follows upon setting $w(u)=W(u^2-y)$. By (2.16) and (1.10), we have

$$F_w(s) = \det\left(1-\chi_{(-1/2,1/2)}\mathcal{F}^*\sqrt{w_{0,s}}.\sqrt{w_{0,s}}\mathcal{F}\chi_{(-1/2,1/2)}\right). \tag{3.8}$$

Now we verify that $\chi_{(-1/2,1/2)}\mathcal{F}^*\sqrt{w_{0,s}}$ and $\sqrt{w_{0,s}}\mathcal{F}\chi_{(-1/2,1/2)}$ are Hilbert-Schmidt operators, and use the property $\det(1-AB)=\det(1-BA)$ for Hilbert-Schmidt operators A,B, such that

$$F_w(s) = \det\left(1-\sqrt{w_{0,s}}\mathcal{F}\chi_{(-1/2,1/2)}\mathcal{F}^*\sqrt{w_{0,s}}\right) = \det\left(1-\sqrt{w_{0,s}}\mathcal{K}^{\sin}\sqrt{w_{0,s}}\right).$$

For the last equality we used (1.8). □

The advantage of the representations (3.6)–(3.7) is that they involve operators with integrable kernels. Indeed, we can write

$$\sqrt{w_{y,s}(u)}\mathrm{K}^{\sin}(u,v)\sqrt{w_{y,s}(v)} = \frac{\vec{f}(u)^{\top}\vec{g}(v)}{u-v}, \tag{3.9}$$

where

$$\vec{f}(u) = \frac{\sqrt{w_{y,s}(u)}}{2\pi\mathrm{i}}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\pi u}\\ \mathrm{e}^{-\mathrm{i}\pi u}\end{pmatrix}, \quad \vec{g}(v) = \sqrt{w_{y,s}(v)}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\pi v}\\ -\mathrm{e}^{\mathrm{i}\pi v}\end{pmatrix}. \tag{3.10}$$

This integrable representation of the kernel will enable us later to characterize $F_w(s)$ in terms of a 2×2 matrix RH problem, and to derive underlying differential equations.

3.3 Small s asymptotics

The following result describes the small s behavior of the Fredholm determinants $Q_W(y,s)$ and their logarithmic derivatives with respect to s and y. Below, we write as before $\sigma_W(y,s)=\log Q_W(y,s)$.

Proposition 3.3. *Let $y\in\mathbb{R}$ and let $W:\mathbb{R}\to\mathbb{C}$ be such that $W(.^2-y)$ is a Schwartz function for every $y\in\mathbb{R}$. Then,*

$$\lim_{s\to 0}\frac{1}{s}\sigma_W(y,s) = -\frac{2}{\pi}\int_0^\infty W(u^2-y)\mathrm{d}u, \tag{3.11}$$

and

$$\lim_{s\to 0} s\partial_s\sigma_W(y,s)=0, \qquad \lim_{s\to 0}\partial_y\sigma_W(y,s)=0. \tag{3.12}$$

Proof. We first recall that

$$Q_W(y,s) = \det(1-\mathcal{L}_{y,s}), \qquad \mathcal{L}_{y,s} := \chi_{(-1/2,1/2)}\mathcal{F}^*\sqrt{w_{y,s}}\cdot\sqrt{w_{y,s}}\mathcal{F}\chi_{(-1/2,1/2)}, \tag{3.13}$$

and then prove that the trace-norm of $\mathcal{L}_{y,s}$ is $O(s)$ as $s \to 0$. Both factors in the above factorization are Hilbert-Schmidt operators. Hence, the trace-norm is bounded by the product of Hilbert-Schmidt norms:

$$\|\mathcal{L}_{y,s}\|_1 \leq \left\|\chi_{(-1/2,1/2)}\mathcal{F}^*\sqrt{w_{y,s}}\right\|_2 \, \left\|\sqrt{w_{y,s}}\mathcal{F}\chi_{(-1/2,1/2)}\right\|_2,$$

and it is straightforward to estimate the Hilbert-Schmidt norms on the right: for the first one, we have

$$\begin{aligned}\left\|\chi_{(-1/2,1/2)}\mathcal{F}^*\sqrt{w_{y,s}}\right\|_2^2 &\leq \int_{-\infty}^{+\infty}\int_{-1/2}^{1/2}|w_{y,s}(y)|\mathrm{d}x\mathrm{d}y = \|w_{y,s}\|_1 \\ &= 2\int_0^\infty W\left(\frac{\pi^2 r^2}{s^2}-y\right)\mathrm{d}r = \frac{2s}{\pi}\int_0^\infty W\left(\lambda^2-y\right)\mathrm{d}\lambda.\end{aligned}$$

Similar estimates hold for the second factor, such that the operator norm of $\mathcal{L}_{y,s}$ is $O(s)$ as $s \to 0$. Hence, as $s \to 0$, we have

$$Q_W(y,s) = 1 - \operatorname{Tr}\mathcal{L}_{y,s} + O(s^2), \qquad s \to 0.$$

Taking the logarithm and computing the trace explicitly as

$$\operatorname{Tr}\mathcal{L}_{y,s} = \frac{2s}{\pi}\int_0^\infty W(\lambda^2-y)\mathrm{d}\lambda,$$

we obtain (3.11).

Moreover, the operator $1-\mathcal{L}_{y,s}$ is invertible for small s and by a Neumann expansion, we have the operator norm bound

$$\|\left(1-\mathcal{L}_{y,s}\right)^{-1}\| \leq 2, \tag{3.14}$$

for s sufficiently small.

Then by Jacobi's identity and similar norm estimates as before, we obtain for the s-derivative

$$\begin{aligned}|\partial_s \log Q_W(y,s)| &= \left|\operatorname{Tr}\left[\left(\partial_s\mathcal{L}_{y,s}\right)\left(1-\mathcal{L}_{y,s}\right)^{-1}\right]\right| \\ &\leq 2\ \left\|\partial_s w_{y,s}\right\|_1 \\ &= 8\int_0^\infty \left|W'\left(\frac{\pi^2r^2}{s^2}-y\right)\frac{\pi^2r^2}{s^3}\right|\mathrm{d}r \\ &= \frac{8}{\pi}\int_0^\infty |W'(\lambda^2-y)|\lambda^2\mathrm{d}\lambda.\end{aligned}$$

For the y-derivative, we similarly obtain

$$|\partial_y \log Q_W(y,s)| \leq 2\ \left\|\partial_y w_{y,s}\right\|_1 = \frac{4s}{\pi}\int_0^\infty |W'(\lambda^2-y)|\mathrm{d}\lambda.$$

The results in (3.12) now follow directly since $\sigma_W = \log Q_W$. ☐

4 RH characterization of $F_w(s)$ and $Q_W(y,s)$

Our objective here is to characterize $F_w(s)$ and $Q_W(y,s)$ in terms of the unique solution of a 2×2 matrix RH problem. To that end, we closely follow the method developed in [8, 9] for the study of deformed Airy kernel determinants, relying on the more general theory of [3, 14, 20].

4.1 Characterization by a RH problem

We will express $F_w(s)$ and $Q_W(y,s)$ in terms of the solution $U(\lambda) = U(\lambda; s)$ of the following RH problem, which depends on a parameter $s > 0$ and on a function $w : \mathbb{R} \to \mathbb{C}$, which we will assume to be Schwartz.

RH problem for U.

(1) $U : \mathbb{C} \setminus \mathbb{R} \to GL(2, \mathbb{C})$ is analytic.

(2) $U(\lambda; s)$ has continuous boundary values $U_\pm(\lambda; s)$ when λ approaches $\mathbb{R}$ from either above $(+)$ or below $(-)$ and they satisfy the jump condition

$$U_+(\lambda; s) = U_-(\lambda; s) \underbrace{\begin{pmatrix} 1 & 1 - w(\lambda) \\ 0 & 1 \end{pmatrix}}_{=J_U(\lambda)}, \quad \lambda \in \mathbb{R}. \tag{4.1}$$

(3) There exists a matrix $U_1 = U_1(s)$ such that we have as $\lambda \to \infty$,

$$U(\lambda; s) = \left(I + \frac{U_1}{\lambda} + O(\lambda^{-2}) \right) e^{is\lambda\sigma_3} \begin{cases} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, & \operatorname{Im} \lambda > 0, \\ \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, & \operatorname{Im} \lambda < 0, \end{cases} \tag{4.2}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The RH problem and its solution depend on the choice of w. When this dependence will become important, we will write $U^{(w)}$ instead of U for the solution of the RH problem.

The next subsection will be devoted to the proof of the following result, which essentially follows from the theory of integrable kernel operators developed by Its, Izergin, Korepin, and Slavnov [20].

Proposition 4.1. *(i) Let $w : \mathbb{R} \to \mathbb{C}$ be a Schwartz function. The RH problem for $U^{(w)}$ is uniquely solvable if and only if $F_w(s) \neq 0$, and in that case we have the identity*

$$\partial_s \log F_w(s) = \frac{1}{2\pi i s} \int_{\mathbb{R}} \left[U_+^{(w)}(\lambda; s)^{-1} \frac{d}{d\lambda} U_+^{(w)}(\lambda; s) \right]_{2,1} w'(\lambda) \lambda d\lambda. \tag{4.3}$$

(ii) Let $y \in \mathbb{R}$ and $W : \mathbb{R} \to \mathbb{C}$ be such that $W(.^2 - y)$ is a Schwartz function. The RH problem for $U^{(w)}$ is uniquely solvable if and only if $Q_W(y,s) \neq 0$, and in that case we have the identity

$$\partial_s \log Q_W(y,s) = \frac{1}{2\pi i s} \int_{\mathbb{R}} \left[U_+^{(w)}(\lambda; s)^{-1} \frac{d}{d\lambda} U_+^{(w)}(\lambda; s) \right]_{2,1} w'(\lambda) \lambda d\lambda. \tag{4.4}$$

Observe that part (ii) of the above result is a simple consequence of part (i), obtained by setting $w(\lambda) = W(\lambda^2 - y)$. The next subsection will be devoted to the proof of part (i).

4.2 Proof of Proposition 4.1

We will first relate $F_w(s)$ to another RH problem. Indeed, the RH problem for U is obtained by transformations of the standard RH problem obtained using the Its-Izergin-Korepin-Slavnov method from [20], associated to the integrable kernel written in equations (3.9)–(3.10), which is described below. As before, we let $s > 0$ and $w : \mathbb{R} \to \mathbb{C}$ be a Schwartz function. Let $\vec{f}, \vec{g}$ be as in (3.10) with $y = 0$.

Consider the following RH problem for $Y(\zeta) = Y(\zeta; s)$.

RH problem for Y.

(1) $Y : \mathbb{C} \setminus \mathbb{R} \to GL(2,\mathbb{C})$ is analytic.

(2) $Y(\zeta; s)$ has continuous boundary values $Y_\pm(\zeta; s)$ when approaching $\mathbb{R}$ from above or below, they are related by

$$Y_+(\zeta; s) = Y_-(\zeta; s) \underbrace{\left(I - 2\pi \mathrm{i} \vec{f}(\zeta)\vec{g}(\zeta)^\top\right)}_{=J_Y(\zeta;s)}, \quad \zeta \in \mathbb{R}, \tag{4.5}$$

for $\vec{f}, \vec{g}$ given in (3.10) with $y = 0$.

(3) As $\zeta \to \infty$, we have

$$Y(\zeta; s) = I + O(\zeta^{-1}). \tag{4.6}$$

From the general theory developed in [14, 20], it follows that the solution of this RH problem exists and is unique if and only if $F_w(s) \neq 0$, and it is then related to the operator $\mathcal{R}$ defined by

$$\begin{aligned}\mathcal{R} &= \left(1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right)^{-1} - 1 && (4.7)\\ &= \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\left(1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right)^{-1} && (4.8)\\ &= \left(1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right)^{-1}\sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}, && (4.9)\end{aligned}$$

where we denote for brevity $w_s(u) = w_{0,s}(u) = w\left(\frac{\pi u}{s}\right)$. In particular, $\mathcal{R}$ has a kernel

$$R(u, v) = \frac{\vec{F}(u)^\top \vec{G}(v)}{u - v}, \tag{4.10}$$

where $\vec{F}$, $\vec{G}$ are characterized by the RH solution Y,

$$\vec{F}(u) = Y_+(u)\vec{f}(u), \qquad \vec{G}(v) = Y_+^{-\top}(v)\vec{g}(v), \tag{4.11}$$

or equivalently

$$\vec{F}(u) = \left((1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s})^{-1}\vec{f}\,\right)(u),\ \vec{G}(v) = \left((1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s})^{-1}\vec{g}\,\right)(v). \tag{4.12}$$

As a byproduct, we have

$$R(u, v) = \frac{\vec{f}(u)^\top Y_+^\top(u) Y_+^{-\top}(v)\vec{g}(v)}{u - v}. \tag{4.13}$$

To compute $R(\zeta, \zeta)$ on the diagonal, we need to be able to differentiate $Y_+(\zeta)$ with respect to ζ. Since w is smooth and fast decaying at $\pm\infty$, the jump matrix is smooth and close to I at $\pm\infty$. It then follows from the general theory of RH problems, see e.g. [14], that the boundary values $Y_\pm$ are indeed differentiable, such that

$$R(\zeta, \zeta) = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\vec{f}^{\,\top}(\zeta)Y_+^\top(\zeta)\right)Y_+^{-\top}(\zeta)\vec{g}(\zeta). \tag{4.14}$$

and we can also verify from this expression that $\frac{R(\zeta,\zeta)}{w_s(\zeta)}$ is integrable on the real line.

With a first transformation of the RH problem for Y, we are able to simplify the jump matrix. Notice that the jump matrix $J_Y(\zeta; s)$ is explicitly given by

$$J_Y(\zeta; s) = \begin{pmatrix} 1 - w_s(\zeta) & w_s(\zeta)\mathrm{e}^{2\pi \mathrm{i}\zeta} \\ -w_s(\zeta)\mathrm{e}^{-2\pi \mathrm{i}\zeta} & 1 + w_s(\zeta)\end{pmatrix}. \tag{4.15}$$

In order to transform this jump matrix to an upper-triangular jump matrix, we define

$$\Psi(\zeta;s) := \begin{cases} Y(\zeta;s)\begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\zeta} & \mathrm{e}^{\pi\mathrm{i}\zeta} \\ \mathrm{e}^{-\pi\mathrm{i}\zeta} & 0 \end{pmatrix}, & \operatorname{Im}\zeta > 0, \\ \\ Y(\zeta;s)\begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\zeta} & 0 \\ \mathrm{e}^{-\pi\mathrm{i}\zeta} & -\mathrm{e}^{-\pi\mathrm{i}\zeta} \end{pmatrix}, & \operatorname{Im}\zeta < 0. \end{cases} \tag{4.16}$$

We then obtain the following RH problem for $\Psi(\zeta) = \Psi(\zeta;s)$.

RH problem for Ψ.

(1) $\Psi : \mathbb{C} \setminus \mathbb{R} \to GL(2,\mathbb{C})$ is analytic.

(2) $\Psi(\zeta;s)$ has continuous boundary values $\Psi_\pm(\zeta;s)$ when approaching $\mathbb{R}$ from either left or right, and

$$\Psi_+(\zeta;s) = \Psi_-(\zeta;s)\underbrace{\begin{pmatrix} 1 & 1-w_s(\zeta) \\ 0 & 1 \end{pmatrix}}_{=J_\Psi(\zeta;s)}, \quad \zeta \in \mathbb{R}. \tag{4.17}$$

(3) As $\zeta \to \infty$, we have

$$\Psi(\zeta;s) = \left(I + O(\zeta^{-1})\right)\mathrm{e}^{\pi\mathrm{i}\zeta\sigma_3}\begin{cases} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, & \operatorname{Im}\zeta > 0, \\ \\ \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, & \operatorname{Im}\zeta < 0. \end{cases} \tag{4.18}$$

Indeed, we have

$$\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\pi\zeta} & 0 \\ \mathrm{e}^{-\mathrm{i}\pi\zeta} & -\mathrm{e}^{\mathrm{i}\pi\zeta} \end{pmatrix} J_Y(\zeta;s)\begin{pmatrix} \mathrm{e}^{\mathrm{i}\pi\zeta} & \mathrm{e}^{\mathrm{i}\pi\zeta} \\ \mathrm{e}^{-\mathrm{i}\pi\zeta} & 0 \end{pmatrix} = J_\Psi(\zeta;s), \tag{4.19}$$

implying that $\Psi_+(\zeta;s) = \Psi_-(\zeta;s)J_\Psi(\zeta;s)$. Also notice that

$$\mathrm{e}^{\pi\mathrm{i}\zeta\sigma_3}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\zeta} & \mathrm{e}^{\pi\mathrm{i}\zeta} \\ \mathrm{e}^{-\pi\mathrm{i}\zeta} & 0 \end{pmatrix}, \quad \mathrm{e}^{\pi\mathrm{i}\zeta\sigma_3}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\zeta} & 0 \\ \mathrm{e}^{-\pi\mathrm{i}\zeta} & -\mathrm{e}^{-\pi\mathrm{i}\zeta} \end{pmatrix}, \tag{4.20}$$

thus the asymptotic condition for $\Psi(\zeta;s)$ holds.

Finally, we define

$$U(\lambda;s) := \Psi(\zeta = \frac{s}{\pi}\lambda;s), \tag{4.21}$$

and now it is straightforward to check that $U(\lambda;s)$ solves the RH problem for U given at the beginning of this section.

We can finally prove Proposition 4.1.

Proof. We use the representation for $F_w(s)$ given in (3.6), together with Jacobi's identity and the formula for the kernel $R(\zeta,\zeta)$ given in (4.14) to compute the s-logarithmic derivative of $F_w(s)$:

$$\begin{aligned}
\partial_s \log F_w(s) &= \partial_s \log\det\left(1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right) \\
&= -\operatorname{Tr}\left[\left(1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right)^{-1}\partial_s(\sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s})\right] \\
&= -\int_{\mathbb{R}} \frac{\partial_s w_s(\zeta)}{w_s(\zeta)} R(\zeta,\zeta)\mathrm{d}\zeta \\
&= -\int_{\mathbb{R}} \frac{\partial_s w_s(\zeta)}{w_s(\zeta)}\frac{\mathrm{d}}{\mathrm{d}\zeta}\vec{f}^{\top}(\zeta)\vec{g}(\zeta)\mathrm{d}\zeta - \int_{\mathbb{R}} \frac{\partial_s w_s(\zeta)}{w_s(\zeta)}\vec{f}^{\top}(\zeta)\frac{\mathrm{d}}{\mathrm{d}\zeta}Y_+^{\top}(\zeta)Y_+^{-\top}(\zeta)\vec{g}(\zeta)\mathrm{d}\zeta.
\end{aligned}$$

Note that the integral in the third line converges because $R(\zeta,\zeta)/w_s(\zeta)$ is integrable. The first term gives

$$-\int_{\mathbb{R}} \frac{\partial_s w_s(\zeta)}{w_s(\zeta)} \frac{\mathrm{d}}{\mathrm{d}\zeta} \vec{f}^{\top}(\zeta)\vec{g}(\zeta)\mathrm{d}\zeta = -\int_{\mathbb{R}} \partial_s w_s(\zeta)\mathrm{d}\zeta. \tag{4.22}$$

To compute the second term, we use the transformation

$$U_+(\lambda) = Y_+(\lambda s/\pi) \underbrace{\begin{pmatrix} \mathrm{e}^{\mathrm{i}\lambda s} & \mathrm{e}^{\mathrm{i}\lambda s} \\ \mathrm{e}^{-\mathrm{i}\lambda s} & 0 \end{pmatrix}}_{=\Phi(\lambda s/\pi)} \tag{4.23}$$

to replace Y inside the integral. We first take the transpose and then change the variable $\zeta = s\lambda/\pi$, to obtain

$$\begin{aligned}
&-\int_{\mathbb{R}} \frac{\partial_s w_s(\zeta)}{w_s(\zeta)} \vec{f}^{\top}(\zeta) \frac{\mathrm{d}}{\mathrm{d}v} Y_+^{\top}(\zeta) Y_+^{-\top}(\zeta)\vec{g}(\zeta)\mathrm{d}\zeta \\
&= -\int_{\mathbb{R}} \frac{(\partial_s w_s)(s\lambda/\pi)}{w_s(s\lambda/\pi)} \vec{g}^{\top}(s\lambda/\pi) Y_+^{-1}(s\lambda/\pi) \frac{\mathrm{d}}{\mathrm{d}\lambda} Y_+^{\top}(s\lambda/\pi)\vec{f}(s\lambda/\pi)\mathrm{d}\lambda \\
&= -\int_{\mathbb{R}} \frac{(\partial_s w_s)(s\lambda/\pi)}{w_s(s\lambda/\pi)} \vec{g}^{\top}(s\lambda/\pi)\Phi(s\lambda/\pi) \frac{\mathrm{d}}{\mathrm{d}\lambda} \Phi^{-1}(s\lambda/\pi)\vec{f}(s\lambda/\pi)\mathrm{d}\lambda \\
&\quad - \int_{\mathbb{R}} \frac{(\partial_s w_s)(s\lambda/\pi)}{w_s(s\lambda/\pi)} \vec{g}^{\top}(s\lambda/\pi)\Phi(s\lambda/\pi)U_+^{-1}(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} U_+(\lambda)\Phi(s\lambda/\pi)^{-1}\vec{f}(s\lambda/\pi)\mathrm{d}\lambda.
\end{aligned}$$

The first term here gives

$$-\int_{\mathbb{R}} \frac{(\partial_s w_s)(s\lambda/\pi)}{w_s(s\lambda/\pi)} \vec{g}^{\top}(s\lambda/\pi)\Phi(s\lambda/\pi) \frac{\mathrm{d}}{\mathrm{d}\lambda} \Phi^{-1}(s\lambda/\pi)\vec{f}(s\lambda/\pi)\mathrm{d}\lambda = \int_{\mathbb{R}} (\partial_s w_s)(\zeta)\mathrm{d}\zeta, \tag{4.24}$$

thus it cancels out with the term obtained in (4.22). What is left, is given by (see (3.10))

$$\begin{aligned}
&-\int_{\mathbb{R}} \frac{(\partial_s w_s)(s\lambda/\pi)}{w_s(s\lambda/\pi)} \vec{g}^{\top}(s\lambda/\pi)\Phi(s\lambda/\pi)U_+^{-1}(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} U_+(\lambda)\Phi(s\lambda/\pi)^{-1}\vec{f}(s\lambda/\pi)\mathrm{d}\lambda \\
&= -\int_{\mathbb{R}} \frac{(\partial_s w_s)(s\lambda/\pi)}{2\pi\mathrm{i}} (0,1)U_+^{-1}(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} U_+(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\lambda \\
&= -\int_{\mathbb{R}} \frac{(\partial_s w_s)(s\lambda/\pi)}{2\pi\mathrm{i}} \left[U_+^{-1}(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} U_+(\lambda) \right]_{2,1} \mathrm{d}\lambda.
\end{aligned}$$

Now we compute

$$\partial_s w_s(\zeta)|_{\zeta=s\lambda/\pi} = -\frac{\lambda}{s} w'(\lambda), \tag{4.25}$$

and substitute this into the last integral, to arrive at (4.3). □

4.3 Analytical properties of the RH problem

Later on, we will need to differentiate the RH solution $U(\lambda) = U^{(w)}(\lambda; s)$ with respect to the parameter $s > 0$, and also with respect to $y \in \mathbb{R}$ if $w(\lambda) = W(\lambda^2 - y)$. In addition, we will need to differentiate the asymptotic expansion (4.2) with respect to y, s, and λ. To justify that we can differentiate the solution and the asymptotic expansion, we rely on the well understood functional analysis behind RH problems, developed in [14]. For this, it is important that the jump matrices are sufficiently smooth and decay sufficiently fast at $\pm\infty$, and this is the origin of our requirement that w is a Schwartz function. Without entering into too much technical detail, let us outline how this general theory of RH problems applies to our RH problem. Recall the RH problems for U and

Y, and define

$$T(\lambda) = T(\lambda; y, s) = Y^{(w)}(s\lambda/\pi; s) = U^{(w)}(\lambda; s) \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, & \operatorname{Im}\lambda > 0, \\ \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, & \operatorname{Im}\lambda < 0, \end{cases} \mathrm{e}^{-\mathrm{i}s\lambda\sigma_3}, \tag{4.26}$$

with $w(u) = W(u^2 - y)$. Then T is analytic in $\mathbb{C} \setminus \mathbb{R}$, $T(\lambda) \to I$ as $\lambda \to \infty$, and

$$T_+(\lambda) = T_-(\lambda) \underbrace{\begin{pmatrix} 1 - w(\lambda) & w(\lambda)\mathrm{e}^{2\mathrm{i}s\lambda} \\ -w(\lambda)\mathrm{e}^{-2\mathrm{i}s\lambda} & 1 + w(\lambda) \end{pmatrix}}_{= J_T(\lambda)}.$$

These RH properties can be reformulated as an integral equation, namely we have

$$T(\lambda) = I + \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{+\infty} T_-(\xi)(J_T(\xi) - I)\frac{\mathrm{d}\xi}{\xi - \lambda},$$

and the boundary value T_- satisfies the integral equation

$$(T_- - I) = C_-\left[(T_- - I)(J_T - I)\right] + C_-[J_T - I],$$

where C_- is the Cauchy transform defined as

$$C_-[f](\lambda) = \frac{1}{2\pi\mathrm{i}} \lim_{\epsilon \to 0_+} \int_{-\infty}^{+\infty} f(\xi)\frac{\mathrm{d}\xi}{\xi - \lambda + \mathrm{i}\epsilon}.$$

Alternatively, writing $C_{J_T}[f] = C_-[f(J_T - I)]$, we have

$$T_- - I = (1 - C_{J_T})^{-1}\left[C_-[J_T - I]\right], \tag{4.27}$$

and the invertibility of the operator $1 - C_{J_T}$ is equivalent to the solvability of the RH problem (see e.g. [14, Section 2]. The above equation can be differentiated with respect to s to obtain

$$\partial_s T_- = (1 - C_{J_T})^{-1}\left[C_-[(T_- - I)\partial_s J_T] + C_-\partial_s J_T\right], \tag{4.28}$$

provided that J_T is smooth and J_T as well as its s-derivative are sufficiently fast decaying as $\lambda \to \pm\infty$. Taking the derivative with respect to y works similarly. Expanding the last equation as $\lambda \to \infty$, one shows that the expansion

$$T(\lambda) = I + \frac{U_1}{\lambda} + O(\lambda^{-2}), \quad U_1 = -\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} T_-(\xi)(J_T - I)(\xi)\mathrm{d}\xi,$$

continues to hold after differentiating it with respect to s or y. In particular we have

$$\partial_s T(\lambda) = \frac{\partial_s U_1}{\lambda} + O(\lambda^{-2}), \qquad \partial_s U_1 = -\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \partial_s \left(T_-(\xi)(J_T - I)(\xi)\right)\mathrm{d}\xi,$$

and similarly for the y-derivative.

For the λ-derivative, the argument is different: here we can see that the RH conditions for T imply RH conditions for T', which yield

$$T'(\lambda) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{+\infty} \left(T'_-(\xi)(J_T(\xi) - I) + T_-(\xi)J'_T(\xi)\right)\frac{\mathrm{d}\xi}{\xi - \lambda},$$

and

$$T'_- = (1 - C_{J_T})^{-1}\left[C_-[T_- J'_T]\right], \tag{4.29}$$

and expanding this as $\lambda \to \infty$, we find that

$$T'(\lambda) = -\frac{U_1}{\lambda^2} + O(\lambda^{-3}).$$

We can similarly take higher order partial derivatives of T or U, and accordingly differentiate expansion (4.2) multiple times, provided that J_T is smooth and decaying sufficiently fast at $\pm\infty$, which is the case since w is a Schwartz function. In what follows, we will sometimes tacitly rely on arguments like the above to justify taking derivatives and derivatives of asymptotic expansions.

We conclude this Section with some results on the small s behavior of the large λ asymptotic coefficient U_1. They will be used in the proof of Theorem 2.5 in Section 6.

Proposition 4.2. *Let $W : \mathbb{R} \to \mathbb{C}$ be such that $W(.^2 - y)$ is a Schwartz function for every $y \in \mathbb{R}$. Then, for any $y \in \mathbb{R}$, we have*

$$\lim_{s\to 0} sU_1(y,s) = 0, \qquad \lim_{s\to 0} \partial_y U_1(y,s) = 0. \tag{4.30}$$

Proof. Let T be as in (4.26). For $s = 0$, we can construct an explicit solution to the RH conditions for T and we call it T_0. We construct T_0 analytic off the real line such that $T_0(\lambda) \to I$ as $\lambda \to \infty$, and

$$T_{0,+}(\lambda) = T_{0,-}(\lambda) \underbrace{\begin{pmatrix} 1-w(\lambda) & w(\lambda) \\ -w(\lambda) & 1+w(\lambda) \end{pmatrix}}_{=J_{T_0}(\lambda)},$$

as follows:

$$T_0(\lambda) = I + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} w(\xi) \frac{\mathrm{d}\xi}{\xi - \lambda} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Next we set

$$R(\lambda) = T(\lambda) T_0(\lambda)^{-1}.$$

Then R solves a small norm RH problem as $s \to 0$, in the sense that R is analytic in $\mathbb{C} \setminus \mathbb{R}$, that $R(\lambda) \to I$ as $\lambda \to \infty$, and that

$$R_+(\lambda) = R_-(\lambda) J_R(\lambda), \qquad \lambda \in \mathbb{R},$$

where $J_R(\lambda) = T_{0,-}(\lambda) J_T(\lambda) J_{T_0}(\lambda)^{-1} T_{0,-}^{-1}(\lambda)$, such that $J_R - I$ is $O(s)$ as $s \to 0$, in L^2-, L^1-, and L^∞- norms. Indeed, for any of these norms combined with a sub-multiplicative norm on 2×2 matrices, since $T_{0,-}$ and $T_{0,-}^{-1}$ are uniformly bounded, there exists a constant $C > 0$ such that

$$\|J_R - I\| \le C \|J_T J_{T_0}^{-1} - I\|, \tag{4.31}$$

and

$$J_T(\lambda) J_{T_0}^{-1}(\lambda) - I = \begin{pmatrix} w(\lambda)^2 (\mathrm{e}^{2\mathrm{i}\lambda s} - 1) & w(\lambda)(1 - w(\lambda))(\mathrm{e}^{2\mathrm{i}\lambda s} - 1) \\ w(\lambda)(1 + w(\lambda))(1 - \mathrm{e}^{-2\mathrm{i}\lambda s}) & w(\lambda)^2 (\mathrm{e}^{-2\mathrm{i}\lambda s} - 1) \end{pmatrix}, \tag{4.32}$$

with $w(\lambda) = W(\lambda^2 - y)$, whose entries are, for any $y \in \mathbb{R}$, $O(s\lambda w(\lambda))$ as $s \to 0$, uniformly for $\lambda \in \mathbb{R}$. Similarly as outlined before, it then follows that $R_- - I$ satisfies the integral equation

$$(R_- - I) = C_- \left[(R_- - I)(J_R - I)\right] + C_-[J_R - I],$$

hence

$$(R_- - I) = (1 - C_{J_R})^{-1} \left[C_-[J_R - I]\right], \quad \text{where} \quad C_{J_R}[h] = C_-[h(J_R - I)].$$

Since C_{J_R} has small operator norm, the right hand side can now be written as a Neumann series. This integral equation can also be differentiated with respect to y, in order to obtain

$$\partial_y R_- = (1 - C_{J_R})^{-1} \left[C_-[(R_- - I)\partial_y J_R] + C_- \partial_y J_R\right],$$

provided that J_R is smooth and decays sufficiently fast at $\pm\infty$.

As $\lambda \to \infty$, we obtain

$$R(\lambda) = I + \frac{R_1}{\lambda} + O(\lambda^{-2}),$$

and

$$\partial_y R(\lambda) = \frac{\partial_y R_1}{\lambda} + O(\lambda^{-2}), \qquad \partial_y R_1 = -\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \partial_y \left(R_-(\xi)(J_R - I)(\xi) \right) \mathrm{d}\xi.$$

Using (4.26) and the fact that $T = RT_0$, we obtain

$$U^{(w)}(\lambda; s) = R(\lambda; y, s) T_0(\lambda) \mathrm{e}^{\mathrm{i}s\lambda\sigma_3} \begin{cases} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, & \operatorname{Im}\lambda > 0, \\ \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, & \operatorname{Im}\lambda < 0. \end{cases} \tag{4.33}$$

As $s \to 0$, it is straightforward to check that R_1 and $\partial_y R_1$ are $O(s)$, since $J_R - I$ is $O(s)$ in L^∞-, L^1-, and L^2-norms. Expanding (4.33) as $\lambda \to \infty$, we obtain that $U_1(y, s) = O(1)$ as $s \to \infty$. Differentiating with respect to y, we obtain that $\partial_y U_1(y, s) = \partial_y R_1(y, s) = O(s)$ as $s \to 0$. $\square$

5 Differential equations in s

Next, we derive a differential equation for the matrix-valued function $U(\lambda; s)$ with respect to the parameter s. This will allow us to prove Theorem 2.1.

5.1 Lax equation

Proposition 5.1. *Let $w : \mathbb{R} \to \mathbb{C}$ be a Schwartz function. Let $s > 0$ be such that $F_w(s) \neq 0$. Then, the unique solution $U(\lambda; s)$ to the RH problem for U solves the following linear differential equation,*

$$\partial_s U(\lambda; s) = M(\lambda; s) U(\lambda; s), \qquad M(\lambda; s) = \mathrm{i} \begin{pmatrix} \lambda & -\beta(s) \\ \gamma(s) & -\lambda \end{pmatrix}, \tag{5.1}$$

where

$$\beta(s) := 2 \left[U_1(s)\right]_{1,2}, \quad \gamma(s) := 2 \left[U_1(s)\right]_{2,1}, \tag{5.2}$$

and $U_1(s)$ is defined by (4.2). Moreover, if w is even,

$$\beta(s) = -\gamma(s), \qquad \partial_s \alpha = \mathrm{i}\gamma^2, \quad \text{with} \quad \alpha(s) := 2 \left[U_1(s)\right]_{1,1}. \tag{5.3}$$

Proof. Since $J_U(\lambda)$ does not depend on s, we can deduce that

$$M(\lambda; s) := \left(\partial_s U(\lambda; s)\right) U(\lambda; s)^{-1} \tag{5.4}$$

is analytic for all $\lambda \in \mathbb{C}$, since it has no jump on the real line and smooth boundary values. Recall the asymptotic behavior of U as $\lambda \to \infty$, given in (4.2), and write

$$U_1(s) = \frac{1}{2} \begin{pmatrix} \alpha(s) & \beta(s) \\ \gamma(s) & -\alpha(s) \end{pmatrix}. \tag{5.5}$$

Here we note that $U_1(s)$ has indeed zero trace, since the determinant of U is identically equal to -1, which can be checked from the RH conditions for U.

Substituting these asymptotics in the definition of M, and using Liouville's theorem, we conclude that $M(\lambda; s)$ is a polynomial of degree 1 in λ, more precisely we find

$$M(\lambda; s) = \mathrm{i} \begin{pmatrix} \lambda & -\beta(s) \\ \gamma(s) & -\lambda \end{pmatrix}. \tag{5.6}$$

If w is even, we have in addition the symmetry relation

$$\sigma_1 U(-\lambda;s)\sigma_3 = U(\lambda;s). \tag{5.7}$$

Indeed, we verify that the left hand side also satisfies the RH problem for U, hence the equality follows by uniqueness of the solution. So

$$M(\lambda;s) = \sigma_1 \partial_s U(-\lambda;s) U(-\lambda;s)^{-1}\sigma_1 = \mathrm{i}\begin{pmatrix} \lambda & \gamma(s) \\ -\beta(s) & -\lambda \end{pmatrix} \tag{5.8}$$

from which we conclude that $\beta(s) = -\gamma(s)$. Moreover, expanding the identity $\det U \equiv -1$ as $\lambda \to \infty$ yields the equation

$$4\widetilde{\alpha} - \alpha^2 = -\gamma^2, \tag{5.9}$$

where $\widetilde{\alpha}$ is twice the $(1,1)$-entry of the $O(\lambda^{-2})$-term in (4.2). Expanding $(\partial_s U)U^{-1}$ as $\lambda \to \infty$ (in particular, the $(1,1)$-entry of the $1/\lambda$-term) implies

$$2\mathrm{i}\partial_s \alpha = -\gamma^2 + 4\widetilde{\alpha} - \alpha^2.$$

Combining the last two identities, we get the second equality in (5.3). $\square$

Corollary 5.2. *Let $w : \mathbb{R} \to \mathbb{C}$ be a Schwartz function, and let $s > 0$ be such that $F_w(s) \neq 0$. Then, $\phi(\lambda;s) := U_{1,1}(\lambda;s)$ and $\psi(\lambda;s) := U_{2,1}(\lambda;s)$ satisfy the system of equations*

$$\partial_s \phi = \mathrm{i}\lambda\phi - \mathrm{i}\beta(s)\psi, \tag{5.10}$$

$$\partial_s \psi = \mathrm{i}\gamma(s)\phi - \mathrm{i}\lambda\psi. \tag{5.11}$$

Moreover, if w is even,

$$\psi(\lambda;s) = \phi(-\lambda;s), \qquad \partial_s\phi(\lambda;s) = \mathrm{i}\lambda\phi(\lambda;s) + \mathrm{i}\gamma(s)\phi(-\lambda;s). \tag{5.12}$$

We now look for an expression of $\beta(s), \gamma(s)$ in terms of ϕ, ψ.

Proposition 5.3. *Let $w : \mathbb{R} \to \mathbb{C}$ be a Schwartz function, and let $s > 0$ be such that $F_w(s) \neq 0$. We have the relations*

$$\int_{\mathbb{R}} \phi(\lambda;s)\psi(\lambda;s)w'(\lambda)\mathrm{d}\lambda = 0, \tag{5.13}$$

$$\beta(s) = -\frac{1}{2\pi s}\int_{\mathbb{R}} \phi^2(\lambda;s)w'(\lambda)\mathrm{d}\lambda, \tag{5.14}$$

$$\gamma(s) = -\frac{1}{2\pi s}\int_{\mathbb{R}} \psi^2(\lambda;s)w'(\lambda)\mathrm{d}\lambda. \tag{5.15}$$

For w even, they reduce to

$$\gamma(s) = \frac{1}{2\pi s}\int_{\mathbb{R}} \phi^2(\lambda;s)w'(\lambda)\mathrm{d}\lambda, \tag{5.16}$$

and

$$\int_{\mathbb{R}} \phi(\lambda;s)\phi(-\lambda;s)w'(\lambda)\mathrm{d}\lambda = 0. \tag{5.17}$$

Proof. We start by computing

$$\Delta\left(\left(\partial_\lambda U\right)U^{-1}\right)(\lambda;s) := \left(\left(\partial_\lambda U\right)U^{-1}\right)_+(\lambda;s) - \left(\left(\partial_\lambda U\right)U^{-1}\right)_-(\lambda;s). \tag{5.18}$$

Using the jump condition for U and the fact that $\det U = -1$, we compute

$$\left(\left(\partial_\lambda U\right)U^{-1}\right)_+(\lambda;s) = \left(\left(\partial_\lambda U\right)U^{-1}\right)_-(\lambda;s) + U_-(\lambda;s)\begin{pmatrix} 0 & -w'(\lambda) \\ 0 & 0 \end{pmatrix} U_-^{-1}(\lambda;s) \tag{5.19}$$

$$= \left(\left(\partial_\lambda U\right)U^{-1}\right)_- + \begin{pmatrix} -\phi(\lambda)\psi(\lambda;s)w'(\lambda) & \phi^2(\lambda;s)w'(\lambda) \\ -\psi^2(\lambda;s)w'(\lambda) & \phi(\lambda;s)\psi(\lambda;s)w'(\lambda) \end{pmatrix}. \tag{5.20}$$

Integrating this relation entry by entry along the real line, we get

$$\int_{\mathbb{R}} \Delta \left((\partial_\lambda U) U^{-1} \right)_{11} (\lambda; s) \mathrm{d}\lambda = -\int_{\mathbb{R}} \phi(\lambda; s)\psi(\lambda; s) w'(\lambda) \mathrm{d}\lambda, \tag{5.21}$$

$$\int_{\mathbb{R}} \Delta \left((\partial_\lambda U) U^{-1} \right)_{12} (\lambda; s) \mathrm{d}\lambda = \int_{\mathbb{R}} \phi^2(\lambda; s) w'(\lambda) \mathrm{d}\lambda, \tag{5.22}$$

$$\int_{\mathbb{R}} \Delta \left((\partial_\lambda U) U^{-1} \right)_{21} (\lambda; s) \mathrm{d}\lambda = -\int_{\mathbb{R}} \psi^2(\lambda; s) w'(\lambda) \mathrm{d}\lambda, \tag{5.23}$$

$$\int_{\mathbb{R}} \Delta \left((\partial_\lambda U) U^{-1} \right)_{22} (\lambda; s) \mathrm{d}\lambda = \int_{\mathbb{R}} \phi(\lambda; s)\psi(\lambda; s) w'(\lambda) \mathrm{d}\lambda. \tag{5.24}$$

Now the integrals in the left hand side can be computed by contour deformation and residue computation:

$$\int_{\mathbb{R}} \Delta \left((\partial_\lambda U) U^{-1} \right)_{ij} (\lambda; s) \mathrm{d}\lambda = 2\pi \mathrm{i} \operatorname{res}_{\lambda=\infty} \left[(\partial_\lambda U) U^{-1} \right]_{ij}, \tag{5.25}$$

where $\operatorname{res}_{\lambda=\infty}$ denotes the formal residue at infinity (note that the singularity at ∞ of $\left[(\partial_\lambda U) U^{-1} \right]_{ij}$ is not necessarily isolated, but that the function does admit an asymptotic Laurent series expansion at ∞).

Using the asymptotic condition for U (4.2) we get

$$\operatorname{res}_{\lambda=\infty} \left((\partial_\lambda U) U^{-1} \right) = -\mathrm{i}s \left[U_1(s), \sigma_3 \right] = -\mathrm{i}s \begin{pmatrix} 0 & -\beta(s) \\ \gamma(s) & 0 \end{pmatrix}. \tag{5.26}$$

Thus we obtain formulae (5.13), (5.14), (5.15).

Finally, if w is even then as already noticed we have $\beta(s) = -\gamma(s)$ and $\psi(\lambda; s) = \phi(-\lambda; s)$. Thus in this case the same computation yields to

$$\int_{\mathbb{R}} \phi(\lambda; s)\phi(-\lambda; s) w'(\lambda) \mathrm{d}\lambda = 0 \tag{5.27}$$

$$\gamma(s) = \frac{1}{2\pi s} \int_{\mathbb{R}} \phi^2(\lambda; s) w'(\lambda) \mathrm{d}\lambda. \tag{5.28}$$

□

Substituting the integral identities from Proposition 5.3 into the differential equations from Corollary 5.2, we recognize the integro-differential equations for ϕ, ψ appearing in Theorem 2.1. The $\lambda \to \infty$ asymptotics for ϕ, ψ present in Theorem 2.1 follow directly from the RH conditions for U.

5.2 Second logarithmic derivative

In order to complete the proof of Theorem 2.1, it remains to prove the expression for the second logarithmic s-derivative of F_w in terms of ϕ and ψ (notice that the closed expression we have is actually for the derivative of s times the logarithmic derivative of F_w, slightly different from the actual second logarithmic derivative as in the Airy case, comparing to equation (1.16)).

Proposition 5.4. *Let $w : \mathbb{R} \to \mathbb{C}$ be a Schwartz function, and let $s > 0$ be such that $F_w(s) \neq 0$. Then we have the identity*

$$\partial_s s \partial_s \log F_w(s) = \frac{1}{\pi} \int_{\mathbb{R}} \phi(\lambda; s)\psi(\lambda; s) w'(\lambda) \lambda \mathrm{d}\lambda, \tag{5.29}$$

which, in case w is even, reduces to

$$\partial_s s \partial_s \log F_w(s) = \frac{1}{\pi} \int_{\mathbb{R}} \phi(\lambda; s)\phi(-\lambda; s) w'(\lambda) \lambda \mathrm{d}\lambda. \tag{5.30}$$

Proof. We first observe that using the notation in Corollary 5.2, we can express

$$\left[U_+^{-1}(\lambda;s)\partial_\lambda U_+(\lambda;s)\right]_{2,1} = \psi(\lambda;s)\partial_\lambda\phi(\lambda;s) - \phi(\lambda,s)\partial_\lambda\psi(\lambda;s). \tag{5.31}$$

Thus by Proposition 4.1, we get

$$\partial_s \log F_w(s) = \frac{1}{2\pi \mathrm{i} s}\int_{\mathbb{R}} \lambda w'(\lambda)(\psi(\lambda;s)\partial_\lambda\phi(\lambda;s) - \phi(\lambda;s)\partial_\lambda\psi(\lambda;s))\mathrm{d}\lambda. \tag{5.32}$$

Taking another s-derivative we get

$$\partial_s s\partial_s \log F_w(s) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} \lambda w'(\lambda)\partial_s(\psi(\lambda;s)\partial_\lambda\phi(\lambda;s) - \phi(\lambda;s)\partial_\lambda\psi(\lambda;s))\mathrm{d}\lambda. \tag{5.33}$$

The integrand can be further simplified by using the differential equations in s for ϕ, ψ (5.10), (5.11). In particular we obtain

$$\partial_s(\psi(\lambda;s)\partial_\lambda\phi(\lambda;s) - \phi(\lambda;s)\partial_\lambda\psi(\lambda;s)) = 2\mathrm{i}\phi(\lambda;s)\psi(\lambda;s). \tag{5.34}$$

Thus we can conclude

$$\partial_s s\partial_s \log F_w(s) = \frac{1}{\pi}\int_{\mathbb{R}} \lambda w'(\lambda)\phi(\lambda;s)\psi(\lambda;s)\mathrm{d}\lambda. \tag{5.35}$$

In the case where w is even, then $\psi(\lambda;s) = \phi(-\lambda;s)$ and equation (5.30) is obtained as well. □

The proof of Theorem 2.1 is now completed.

6 The corresponding PDE

In the previous section, we considered the dependence on the parameter s of $F_w(s)$ and $Q_W(y,s)$, and of the associated RH solution U. In this section we will study the y-dependence of $Q_W(y,s)$.

6.1 Lax pair

We now consider the RH problem for U corresponding to $w(\lambda) = W(\lambda^2 - y)$, with $W : \mathbb{R} \to \mathbb{C}$ such that w is a Schwartz function for every $y \in \mathbb{R}$. For the ease of notations, we write

$$U(\lambda;y,s) := U^{(w)}(\lambda;s),$$

and accordingly for $U_1(y,s)$. Since the jump condition for U does not depend on s, the differential equation (5.1) for $U(\lambda;s)$ in s derived in the previous section continues to hold for $U(\lambda;y,s)$. Instead of writing $U_1 = U_1(y,s)$ as in (5.5), it is now more convenient to write

$$U_1(y,s) = -\mathrm{i}\begin{pmatrix} p(y,s) & -q(y,s) \\ q(y,s) & -p(y,s). \end{pmatrix} \tag{6.1}$$

In other words, we set $p = \mathrm{i}\alpha/2$, $q = \mathrm{i}\gamma = -\mathrm{i}\beta/2$, which we can do because w is even, such that $\beta = -\gamma$ by (5.3). The differential equation (5.1) then reads

$$\partial_s U(\lambda;y,s) = \begin{pmatrix} \mathrm{i}\lambda & -\mathrm{i}q(y,s) \\ \mathrm{i}q(y,s) & -\mathrm{i}\lambda \end{pmatrix} U(\lambda;y,s), \tag{6.2}$$

and the second identity in (5.3) becomes

$$\partial_s p(y,s) = q(y,s)^2. \tag{6.3}$$

Observe that the jump condition for $U(\lambda;y,s)$ does depend on y, and because of this, we cannot directly derive a Lax equation in y, in order to obtain the PDE (2.21) as compatibility relation of a Lax pair. Instead, inspired by [20, Section 5], we introduce the differential operator

$$D_{\lambda,y} = \partial_\lambda + 2\lambda\partial_y, \tag{6.4}$$

which allows us to obtain a second linear differential equation for $U(\lambda;y,s)$.

Proposition 6.1. *Let $y \in \mathbb{R}$ and let $W : \mathbb{R} \to \mathbb{C}$ be such that $W(.^2 - y)$ is a Schwartz function. Then, for any $s > 0$ such that $Q_W(y, s) \neq 0$, we have*

$$D_{\lambda,y} U(\lambda; y, s) = L(\lambda; y, s) U(\lambda; y, s), \quad L(\lambda; y, s) = \begin{pmatrix} \mathrm{i}s - \mathrm{i}\partial_y p(y,s) & \mathrm{i}\partial_y q(y,s) \\ -\mathrm{i}\partial_y q(y,s) & -\mathrm{i}s + \mathrm{i}\partial_y p(y,s) \end{pmatrix}. \tag{6.5}$$

Proof. Recall that U is differentiable with respect to y, and that (4.2) can be differentiated with respect to y.

Notice that, since $w(\lambda) = W(\lambda^2 - y)$, we have

$$D_{\lambda,y}(w(\lambda)) = 2\lambda W'(\lambda^2 - y) - 2\lambda W'(\lambda^2 - y) = 0. \tag{6.6}$$

Thus, defining

$$L(\lambda; y, s) = D_{\lambda,y} U(\lambda; y, s) U(\lambda; y, s)^{-1}, \tag{6.7}$$

we can conclude, from the first two properties of the RH problem for U and the fact that L has continuous boundary values, that $L(\lambda; y, s)$ is an entire function of λ. Moreover, from the asymptotic expansion of $U(\lambda; y, s)$ as $\lambda \to \infty$ together with Liouville's theorem, we conclude that $L(\lambda; y, s)$ is independent of λ and that it has precisely the form given in (6.5). □

The compatibility condition between equations (5.1) and (6.5), obtained by imposing that $\partial_s D_{\lambda,y} U = D_{\lambda,y} \partial_s U$, reads

$$\partial_s L - D_{\lambda,y} M + [L, M] = 0, \tag{6.8}$$

and is equivalent to a coupled system of PDEs for $p(y,s)$ and $q(y,s)$, namely

$$\partial_y \partial_s p(y,s) = 2q(y,s)\partial_y q(y,s), \tag{6.9}$$

$$\partial_y \partial_s q(y,s) + 2q(y,s)(s - \partial_y p(y,s)) = 0. \tag{6.10}$$

The first identity reveals nothing new, since it is the y-derivative of (6.3). Dividing the second identity by $2q(y,s)$, taking another s-derivative, and using again (6.3), we obtain a PDE for $q(y,s)$ only, which is precisely (2.21).

6.2 Local differential identity

The second logarithmic s-derivative of $Q_W(y,s)$ can also be expressed in terms of the solution $q(y,s)$ of the PDE (2.21).

Proposition 6.2. *Let $W : \mathbb{R} \to \mathbb{C}$ be such that $W(.^2 - y)$ is a Schwartz function for every $y \in \mathbb{R}$. Let $s > 0$, $y \in \mathbb{R}$ be such that $Q_W(y,s) \neq 0$, and let $q(y,s)$ be as in (6.1). Then, we have*

$$\partial_s^2 \log Q_W(y,s) = -q(y,s)^2, \tag{6.11}$$

and q solves the PDE (2.21).

Proof. We adapt formula (5.30) to the case of $Q_W(y,s) = F_w(s)$, so that

$$\partial_s s \partial_s \log Q_W(y,s) = \frac{1}{\pi} \int_{\mathbb{R}} \lambda w'(\lambda) \phi(\lambda; y, s) \phi(-\lambda; y, s) \mathrm{d}\lambda. \tag{6.12}$$

We can now repeat the same procedure used in the proof of Proposition 5.3 to see that

$$\Delta\left(\lambda \partial_\lambda U U^{-1}\right) = \begin{pmatrix} -\lambda\phi(\lambda; y,s)\phi(-\lambda; y,s) w'(\lambda) & \lambda\phi^2(\lambda; y,s) w'(\lambda) \\ -\lambda\phi^2(-\lambda; y,s) w'(\lambda) & \lambda\phi(\lambda; y,s)\phi(-\lambda; y,s) w'(\lambda) \end{pmatrix}. \tag{6.13}$$

We consider the $(1,1)$ entry and we take the integral over $\mathbb{R}$:

$$-\int_{\mathbb{R}} \lambda\phi(\lambda; y,s)\phi(-\lambda; y,s) w'(\lambda) \mathrm{d}\lambda = \int_{\mathbb{R}} \left[\Delta\left(\lambda\partial_\lambda U U^{-1}\right)\right]_{1,1} \mathrm{d}\lambda. \tag{6.14}$$

As we did in the proof of Proposition 5.3, we can now compute the integral in the right hand side by residue computation: expressing (5.9) in terms of $p(y,s)$ and $q(y,s)$ and using (6.3), we obtain

$$\begin{aligned} -\frac{1}{\pi}\int_{\mathbb{R}} \left[\Delta\left(\lambda\partial_\lambda U U^{-1}\right)\right]_{1,1} \mathrm{d}\lambda = -2\mathrm{ires}_{\lambda=\infty}\left[\left(\lambda\partial_\lambda U U^{-1}\right)\right]_{1,1} &= -p(y,s) - sq(y,s)^2 \\ &= -p(y,s) - s\partial_s p(y,s). \end{aligned} \tag{6.15}$$

We conclude that

$$\partial_s s \partial_s \log Q_W(y,s) = -\partial_s(sp(y,s)). \tag{6.16}$$

We can now integrate this equation in s, between 0 and s. In Proposition 3.3 and Proposition 4.2, we showed respectively that $s\partial_s \log Q_W(y,s)$ and $sp(y,s)$ tend to zero as $s \to 0$. Hence the integration constant can be set to zero, and we obtain

$$\partial_s \log Q_W(y,s) = -p(y,s). \tag{6.17}$$

By differentiating in s and replacing again equation (5.3), we conclude that

$$\partial_s^2 \log Q_W(y,s) = -q^2(y,s), \tag{6.18}$$

and obtain (6.11). Since we already showed that q solves (2.21), the result is proved. □

6.3 Sigma-form of the PDE

We are now going to derive the PDE written in (2.22) for $\sigma_W(y,s) = \log Q_W(y,s)$. We call it a *sigma-form* referring to the fact that in the classical case (when $w = \chi_{(-1/2,1/2)}$), the quantity $s\partial_s \log F(s;1)$ evolves according to the Painlevé V sigma-form, recall equations (1.3) and (1.5).

Proposition 6.3. *Let $y \in \mathbb{R}$ and let $W : \mathbb{R} \to \mathbb{C}$ be such that $W(.^2 - y)$ is a Schwartz function. Let $\sigma(y,s) = \sigma_W(y,s) = \log Q_W(y,s)$. Then, for any $s > 0$ such that $Q_W(y,s) \neq 0$, σ solves the PDE* (2.22).

Proof. Recall from Proposition 6.2 and (6.17) that

$$q^2(y,s) = -\partial_s^2 \sigma(y,s) = \partial_s p(y,s). \tag{6.19}$$

We can thus re-write (6.10) as

$$\partial_s \partial_y q = -2sq + 2q\partial_y p. \tag{6.20}$$

Multiplying with $\partial_y q$, we find

$$\frac{1}{2}\partial_s\left((\partial_y q)^2\right) = -s\partial_y(q^2) + \partial_y p\ \partial_y(q^2),$$

or

$$\frac{1}{2}\partial_s\left((\partial_y q)^2\right) = s\partial_s^2\partial_y\sigma + \partial_s\partial_y\sigma\ \partial_s^2\partial_y\sigma.$$

Integrating in s, we get

$$\frac{1}{2}(\partial_y q)^2 = s\partial_s\partial_y\sigma - \partial_y\sigma + \frac{1}{2}\left(\partial_s\partial_y\sigma\right)^2 + c,$$

and the integration constant c is equal to 0. Indeed it follows from Propositions 3.3 and 4.2 that

$$\lim_{s\to 0}\partial_y p(y,s) = \lim_{s\to 0}\partial_y q(y,s) = \lim_{s\to 0}\partial_y\sigma(y,s) = 0,$$

hence by taking the limit $s \to 0$ on both sides we obtain $c = 0$. We also have

$$\frac{(\partial_y q)^2}{2} = \frac{\left(\partial_y(q^2)\right)^2}{8q^2} = -\frac{\left(\partial_s^2\partial_y\sigma\right)^2}{8\partial_s^2\sigma}.$$

Substituting this, we obtain

$$\frac{\left(\partial_s^2\partial_y\sigma\right)^2}{4\partial_s^2\sigma} = -2s\partial_s\partial_y\sigma + 2\partial_y\sigma - \left(\partial_s\partial_y\sigma\right)^2,$$

which we recognize as (2.22). □

Propositions 6.2 and 6.3 together yield Theorem 2.5. Combining this with (3.11), we also obtain Theorem 2.8.

7 Proof of Theorem 2.9

We are now ready to prove Theorem 2.9. Let f satisfy the assumptions from Theorem 2.9, and define

$$W(r) = -2\int_0^\infty f'(-u^2 - r)\mathrm{d}u. \tag{7.1}$$

We then have

$$W'(r) = 2\int_0^\infty f''(-u^2 - r)\mathrm{d}u. \tag{7.2}$$

Since f is C^∞ and f as well as its derivatives decay fast at $-\infty$, it is easy to see that W is infinitely many times differentiable, and that $\lim_{r\to+\infty} r^k W^{(\ell)}(r) = 0$ for every positive integers k, ℓ: indeed, we have

$$|W^{(\ell)}(r)| \leq 2\int_0^\infty |f^{(\ell+1)}(-u^2 - r)|\mathrm{d}u = \int_r^\infty |f^{(\ell+1)}(-t)|\frac{\mathrm{d}t}{\sqrt{t-r}},$$

and by the fast decay of $f^{(\ell+1)}$ at $-\infty$, the right hand side decays fast as $r \to +\infty$.

As a consequence, $w(u) = W(u^2 - y)$ is a Schwartz function, and it is an admissible choice to apply Theorem 2.8. Hence, we have

$$\lim_{s\to 0}\frac{1}{s}\sigma_W(y,s) = -\frac{2}{\pi}\int_0^{+\infty} W(\lambda^2 - y)\mathrm{d}\lambda.$$

Now we substitute (7.1), and obtain

$$\lim_{s\to 0}\frac{1}{s}\sigma_W(y,s) = \frac{4}{\pi}\int_0^\infty\left(\int_0^{+\infty} f'(-u^2 - \lambda^2 + y)\mathrm{d}u\right)\mathrm{d}\lambda.$$

Changing to polar coordinates, we get

$$\lim_{s\to 0}\frac{1}{s}\sigma_W(y,s) = 2\int_0^\infty f'(-\rho^2 + y)\rho\mathrm{d}\rho = \int_0^\infty f'(-v + y)\mathrm{d}v = f(y),$$

since $f(-\infty) = 0$.

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Statements and Declarations

The authors have no competing interests to declare that are relevant to the content of this article.

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