# Certifying unstability of Switched Systems using Sum of Squares Programming

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## 1 Introduction

The joint spectral radius (JSR) of a set of matrices characterizes the maximal asymptotic growth rate of an infinite product of matrices of the set. This quantity appears in a number of applications including the stability of switched and hybrid systems. A popular method used for the stability analysis of these systems searches for a Lyapunov function with convex optimization tools. We investigate dual formulations for this approach and leverage these dual programs for developing new analysis tools for the JSR.

## 2 Joint Spectral Radius and Lyapunov functions

A switched linear system is characterized by a finite set of matrices $\mathscr{A} \triangleq \{A_1, A_2, \ldots, A_m\} \subset \mathbb{R}^{n\times n}$ and the iteration

$$x_k = A_{\sigma_k} x_{k-1}, \quad \sigma_k \in [m]. \tag{1}$$

The maximal asymptotic growth rate of this iteration is given by the *joint spectral radius* (JSR). The JSR $\rho(\mathscr{A})$ of a finite set of matrices $\mathscr{A}$ is defined as

$$\rho(\mathscr{A}) = \lim_{k\to\infty} \max_{\sigma \in [m]^k} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}.$$

This definition is independent of the norm used.

If there exists a homogeneous (strictly) *positive* function $f$, often referred to as *Lyapunov function*, such that

$$f(A_\sigma x) \le \overline{\gamma} f(x) \quad \text{for } \sigma = 1, \ldots, m \tag{2}$$

then $\rho(\mathscr{A}) \le \overline{\gamma}$.

If we restrict $f$ to be an homogeneous polynomial of degree $2d$ and restrict the postivity conditions on $f$ and on the constraint (2) to Sum of Squares condition then the search for such function $f$ can be formulated as a Sum of Squares Program and can be solved using Semidefinite Programming.

Let $\rho_{\text{SOS-}2d}(\mathscr{A})$ be the minimal value of $\overline{\gamma}$ such that the Sum of Squares Program is feasible. We have

$$\min\left\{\binom{n+d-1}{d}, m\right\}^{-\frac{1}{2d}} \rho_{\text{SOS-}2d}(\mathscr{A}) \le \rho(\mathscr{A}) \le \rho_{\text{SOS-}2d}(\mathscr{A}). \tag{3}$$

## 3 Contributions

We show that if there exists measures $\mu_\sigma$ for $\sigma = 1, \ldots, m$ such that[1]

$$\sum_{\sigma=1}^{m} A_\sigma \# \mu_\sigma \ge \underline{\gamma} \sum_{\sigma=1}^{m} \mu_\sigma \tag{4}$$

then $\rho(\mathscr{A}) \ge \underline{\gamma}$.

If we represent $\mu_\sigma$ by its moments of degree $2d$ and we use the moment relaxation on the existence of a measure with these moments and on the constraint (4) then the search for such measures $\mu_\sigma$ is the dual of the Sum of Squares Program of Section 2.

Given measures feasible with $\underline{\gamma}$, we provide an algorithm that extracts a trajectory $\sigma_1, \sigma_2, \ldots$ of asymptotic growth rate

$$\lim_{k\to\infty} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k} \ge m^{-\frac{1}{2d}} \underline{\gamma}. \tag{5}$$

Moreover, if the measures are atomic then with another algorithm, we can extract a periodic trajectory with asymtotic growth rate $\underline{\gamma}$.

We can generalize these result when the allowed switching sequences of the system (1) are constrained by an automaton. The guarantees (3) and (5) holds with $m$ replaced by $\rho(A)$, the spectral radius of the adjacency matrix of the automaton [2, 1].

[1] $A\#\mu$ is the *pushforward measure* defined to be the measure such that $\langle f, A\#\mu\rangle = \langle f \circ A, \mu\rangle$ for any function $f$.

## References

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