Submitted to the Annals of Applied Probability

ON THE GENERATING FUNCTION OF THE PEARCEY PROCESS

BY CHRISTOPHE CHARLIER [1] AND PHILIPPE MOREILLON [1]

[1] *Department of Mathematics, KTH Royal Institute of Technology* cchalier@kth.se; phmoreil@kth.se

The Pearcey process is a universal point process in random matrix theory. In this paper, we study the generating function of the Pearcey process on any number m of intervals. We derive an integral representation for it in terms of a Hamiltonian that is related to a system of $6m+2$ coupled nonlinear equations. We also obtain asymptotics for the generating function as the size of the intervals get large, up to and including the constant term. This work generalizes some results of Dai, Xu and Zhang, which correspond to $m=1$.

1. Introduction and statement of results. In random matrix theory, the universality conjecture asserts that the microscopic behavior of the eigenvalues of large random matrices is similar for many different models. More precisely, it is expected that the local eigenvalue statistics only depend on the symmetry class of the matrix ensemble and on the nature of the point around which these statistics are considered [28, 30, 38]. The Pearcey process is one of the canonical point processes from the theory of random matrices: it models the asymptotic behavior of the eigenvalues near the points of the spectrum where the density of states admits a cusp-like singularity. This process appears in Gaussian random matrices with an external source [12, 13, 8], Hermitian random matrices with independent, not necessarily identically distributed entries [27], and in general Wishart matrices with correlated entries [32]. The Pearcey process is also universal in a sense that goes beyond random matrix theory: it appears in certain models of skew plane partitions [39] and of Brownian motions [1, 2, 8, 46, 31].

Determinantal point processes form a special class of point processes whose correlation functions can be written as determinants involving a kernel. We refer to [43] for an introduction on this subject. The Pearcey process is the determinantal point process on $\mathbb{R}$ associated with the kernel

$$K_\rho^{\mathrm{Pe}}(x,y) = \frac{\mathcal{P}(x)\mathcal{Q}''(y) - \mathcal{P}'(x)\mathcal{Q}'(y) + \mathcal{P}''(x)\mathcal{Q}(y) - \rho\mathcal{P}(x)\mathcal{Q}(y)}{x-y}, \tag{1.1}$$

where $\rho \in \mathbb{R}$,

$$\mathcal{P}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{4}t^4 - \frac{\rho}{2}t^2 + itx}dt, \qquad \mathcal{Q}(y) = \frac{1}{2\pi}\int_{\Sigma} e^{\frac{1}{4}t^4 + \frac{\rho}{2}t^2 + ity}dt,$$

and $\Sigma = (e^{\frac{\pi i}{4}}\infty, 0) \cup (0, e^{\frac{3\pi i}{4}}\infty) \cup (e^{-\frac{3\pi i}{4}}\infty, 0) \cup (0, e^{-\frac{\pi i}{4}}\infty)$. As ρ increases, the point configurations with few points near 0 are increasingly likely to occur, and when ρ tends to $+\infty$, the Pearcey process "factorizes" (in the sense of gap probabilities) into two independent Airy processes [6]. Important progress on the large gap asymptotics for any fixed $\rho \in \mathbb{R}$ have only recently been obtained in [23].

AMS 2020 subject classifications: 41A60, 60B20, 30E25.

Keywords and phrases: Pearcey point process, Generating function asymptotics, Hamiltonian, Riemann–Hilbert problems.

Acknowledgment of support: C.C. acknowledges support from the European Research Council, Grant Agreement No. 682537, the Swedish Research Council, Grant No. 2015-05430, and the Ruth and Nils-Erik Stenbäck Foundation.

This paper is inspired by the work [24] of Dai, Xu and Zhang, and is concerned with the moment generating function of the Pearcey process. Let X be a locally finite random point configuration distributed according to the Pearcey process, and let $N(x) := \#\{\xi \in X : \xi \in (-x, x)\}$ be the associated counting function. We are interested in the m-point generating function

$$F(r\vec{x}, \vec{u}) := \mathbb{E}\Big[\prod_{j=1}^{m} e^{u_j N(rx_j)}\Big], \tag{1.2}$$

where

$$r > 0, \qquad m \in \mathbb{N}_{>0}, \qquad \vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m, \qquad \vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^{+,m}_{\mathrm{ord}},$$

and $\mathbb{R}^{+,m}_{\mathrm{ord}} := \{(x_1, \ldots, x_m) : 0 < x_1 < \ldots < x_m < +\infty\}$.

Our main results are stated in Theorems 1.3 and 1.4 below and can be summarized as follows:

- Theorem 1.3 establishes an integral representation for $F(r\vec{x}, \vec{u})$ in terms of a Hamiltonian related to a system of $6m + 2$ coupled differential equations. This system of equations admits at least one solution, which we derive from the Lax pair of a Riemann-Hilbert (RH) problem. The asymptotic properties of this solution are stated in Theorem 1.1.
- Theorem 1.4 is concerned with the asymptotic properties of the generating function as the size of the intervals get large. Specifically, Theorem 1.4 gives a precise asymptotic formula, up to and including the constant term, for $F(r\vec{x}, \vec{u})$ as $r \to +\infty$.

For $m = 1$, Theorems 1.1, 1.3 and 1.4 have previously been obtained in [24]. The general case $m \geq 2$ allows to capture the correlation structure of the Pearcey process, see Corollary 1.5, and to analyze the joint fluctuations of the counting function, see Corollary 1.6. We also expect the case $m = 2$ of Theorem 1.4 to play an important role in future studies of the rigidity of the Pearcey process (we comment more on that at the end of this section).

The relevant system of $6m + 2$ coupled differential equations depends on unknown functions which are denoted by

$$p_0(r),\ q_0(r),\ p_{j,k}(r),\ q_{j,k}(r), \qquad k = 1, 2, 3, \quad j = 1, \ldots, m, \tag{1.3}$$

and is as follows:

$$\begin{cases} p_0'(r) = -\sqrt{2}\sum_{j=1}^{m} x_j p_{j,3}(r) q_{j,2}(r), \\ q_0'(r) = \sqrt{2}\sum_{j=1}^{m} x_j p_{j,2}(r) q_{j,1}(r), \\ q_{j,1}'(r) = \frac{2}{r} S_{11}(r) q_{j,1}(r) + x_j q_{j,2}(r) + \frac{2}{r} S_{31}(r) q_{j,3}(r), \\ q_{j,2}'(r) = \sqrt{2} p_0(r) x_j q_{j,1}(r) + \frac{2}{r} S_{22}(r) q_{j,2}(r) + x_j q_{j,3}(r), \\ q_{j,3}'(r) = (r x_j^2 + \frac{2}{r} S_{13}(r)) q_{j,1}(r) + \sqrt{2} q_0(r) x_j q_{j,2}(r) + \frac{2}{r} S_{33}(r) q_{j,3}(r), \\ p_{j,1}'(r) = -\frac{2}{r} S_{11}(r) p_{j,1}(r) - \sqrt{2} p_0(r) x_j p_{j,2}(r) - (r x_j^2 + \frac{2}{r} S_{13}(r)) p_{j,3}(r), \\ p_{j,2}'(r) = -x_j p_{j,1}(r) - \frac{2}{r} S_{22}(r) p_{j,2}(r) - \sqrt{2} q_0(r) x_j p_{j,3}(r), \\ p_{j,3}'(r) = -\frac{2}{r} S_{31}(r) p_{j,1}(r) - x_j p_{j,2}(r) - \frac{2}{r} S_{33}(r) p_{j,3}(r), \end{cases} \tag{1.4}$$

where $j = 1, \ldots, m$, $(\cdot)'$ denotes the derivative with respect to r, and

$$S_{kl}(r) = \sum_{j=1}^{m} p_{j,k}(r) q_{j,l}(r), \qquad k, l = 1, 2, 3.$$

Furthermore, we require the functions (1.3) to satisfy the following m relations

$$\sum_{k=1}^{3} p_{j,k}(r)q_{j,k}(r) = 0, \qquad j = 1, \ldots, m. \tag{1.5}$$

Note that the functions (1.3) also depend on $\vec{x}$, although this is not indicated in the notation. Let

$$H(r) = H(r; p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^{m})$$

be defined by

$$\begin{aligned} H(r) = {} & \sqrt{2}p_0(r)\sum_{j=1}^{m} x_j p_{j,2}(r)q_{j,1}(r) + \sqrt{2}q_0(r)\sum_{j=1}^{m} x_j p_{j,3}(r)q_{j,2}(r) \\ & + \sum_{j=1}^{m} x_j p_{j,1}(r)q_{j,2}(r) + \sum_{j=1}^{m} x_j p_{j,2}(r)q_{j,3}(r) + \sum_{j=1}^{m} r x_j^2 p_{j,3}(r)q_{j,1}(r) \\ & + \frac{1}{2r}\bigg(\Big(S_{11}(r) - S_{22}(r) + S_{33}(r)\Big)^2 \\ & - 2\sum_{k=1}^{m}\sum_{\ell=1}^{m}\Big(p_{k,1}(r)p_{\ell,3}(r) - p_{k,3}(r)p_{\ell,1}(r)\Big)\Big(q_{k,1}(r)q_{\ell,3}(r) - q_{k,3}(r)q_{\ell,1}(r)\Big)\bigg). \end{aligned} \tag{1.6}$$

It is readily checked that $q_0'(r) = \frac{\partial H}{\partial p_0}(r)$, $p_0'(r) = -\frac{\partial H}{\partial q_0}(r)$ and

$$q_{j,k}'(r) = \frac{\partial H}{\partial p_{j,k}}(r), \qquad p_{j,k}'(r) = -\frac{\partial H}{\partial q_{j,k}}(r), \qquad j = 1, \ldots, m,\ k = 1, 2, 3, \tag{1.7}$$

and therefore H is a Hamiltonian for the system (1.4)–(1.5).

THEOREM 1.1. *Let* $\rho \in \mathbb{R}$,

$$r > 0, \qquad m \in \mathbb{N}_{>0}, \qquad \vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m, \qquad \text{and} \qquad \vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^{+,m}_{\mathrm{ord}}.$$

There exists at least one solution $(p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^{m})$ *to the system of equations* (1.4) *and* (1.5) *satisfying the asymptotics* (1.8) *and* (1.11) *below. As* $r \to +\infty$, *we have*

$$p_0(r) = \frac{\sqrt{3}}{2\sqrt{2}\pi}\sum_{\ell=1}^{m} u_\ell x_\ell^{\frac{2}{3}} r^{\frac{2}{3}} + \frac{1}{\sqrt{2}}\left(\frac{\rho^3}{54} + \frac{\rho}{2}\right) + \mathcal{O}(r^{-\frac{2}{3}}), \tag{1.8a}$$

$$\begin{aligned} p_{j,1}(r) = {} & -\frac{1}{3\pi i} e^{\frac{1}{2}\theta_3(rx_j)}(rx_j)^{\frac{1}{3}}\frac{u_j}{\mathcal{A}_j} \\ & \times\left(\cos(\vartheta_j(r) - \tfrac{\pi}{3}) + \frac{\sqrt{3}}{2\pi}\sum_{\ell=1}^{m} u_\ell \frac{x_\ell^{2/3}}{x_j^{2/3}}\cos(\vartheta_j(r) + \tfrac{\pi}{3})\right)(1 + \mathcal{O}(r^{-\frac{2}{3}})), \end{aligned} \tag{1.8b}$$

$$p_{j,2}(r) = \frac{1}{3\pi i} e^{\frac{1}{2}\theta_3(rx_j)}\frac{u_j}{\mathcal{A}_j}\cos(\vartheta_j(r))(1 + \mathcal{O}(r^{-\frac{2}{3}})), \tag{1.8c}$$

$$p_{j,3}(r) = -\frac{1}{3\pi i} e^{\frac{1}{2}\theta_3(rx_j)}(rx_j)^{-\frac{1}{3}}\frac{u_j}{\mathcal{A}_j}\cos(\vartheta_j(r) + \tfrac{\pi}{3})(1 + \mathcal{O}(r^{-\frac{2}{3}})), \tag{1.8d}$$

$$q_0(r) = -\frac{\sqrt{3}}{2\sqrt{2}\pi}\sum_{\ell=1}^{m} u_\ell x_\ell^{\frac{2}{3}} r^{\frac{2}{3}} + \frac{1}{\sqrt{2}}\left(-\frac{\rho^3}{54} + \frac{\rho}{2}\right) + \mathcal{O}(r^{-\frac{2}{3}}), \tag{1.8e}$$

$$q_{j,1}(r) = 2ie^{-\frac{1}{2}\theta_3(rx_j)}(rx_j)^{-\frac{1}{3}}\mathcal{A}_j \sin(\vartheta_j(r) - \tfrac{\pi}{3})(1 + \mathcal{O}(r^{-\frac{2}{3}})), \tag{1.8f}$$

$$q_{j,2}(r) = -2ie^{-\frac{1}{2}\theta_3(rx_j)}\mathcal{A}_j \sin(\vartheta_j(r))(1 + \mathcal{O}(r^{-\frac{2}{3}})), \tag{1.8g}$$

$$\begin{aligned} q_{j,3}(r) &= 2ie^{-\frac{1}{2}\theta_3(rx_j)}(rx_j)^{\frac{1}{3}}\mathcal{A}_j \\ &\quad \times \bigg(\sin(\vartheta_j(r) + \tfrac{\pi}{3}) - \frac{\sqrt{3}}{2\pi}\sum_{\ell=1}^{m} u_\ell \frac{x_\ell^{2/3}}{x_j^{2/3}} \sin(\vartheta_j(r) - \tfrac{\pi}{3}) \bigg)(1 + \mathcal{O}(r^{-\frac{2}{3}})), \end{aligned} \tag{1.8h}$$

where $\theta_3(r) = \frac{3}{4}r^{\frac{4}{3}} + \frac{\rho}{2}r^{\frac{2}{3}}$,

$$\mathcal{A}_j = |\Gamma(1 - \tfrac{u_j}{2\pi i})| \exp\bigg(-\frac{u_j}{3} - \sum_{k=j+1}^{m} \frac{u_k}{2} - \sum_{\substack{k=1 \\ k\neq j}}^{m} \frac{u_k}{2\pi} \arctan \frac{\sqrt{3}x_k^{2/3}}{x_k^{2/3} + 2x_j^{2/3}} \bigg), \tag{1.9}$$

$$\begin{aligned} \vartheta_j(r) &= -\frac{3\sqrt{3}}{8}(rx_j)^{\frac{4}{3}} + \frac{\sqrt{3}\rho}{4}(rx_j)^{\frac{2}{3}} + \arg\Gamma(1 - \tfrac{u_j}{2\pi i}) \\ &\quad - \frac{u_j}{2\pi}\bigg(\frac{4}{3}\log(rx_j) + \log\frac{9}{2} \bigg) - \sum_{\substack{k=1 \\ k\neq j}}^{m} \frac{u_k}{2\pi} \log \frac{|x_j^{2/3} - \omega x_k^{2/3}|}{|x_j^{2/3} - x_k^{2/3}|}, \end{aligned} \tag{1.10}$$

$\omega := e^{\frac{2\pi i}{3}}$ *and* Γ *is Euler's Gamma function. Furthermore, the asymptotic formulas* (1.8) *hold uniformly for* ρ *in compact subsets of* $\mathbb{R}$, *for* $\vec{u}$ *in compact subsets of* $\mathbb{R}^m$, *and for* $\vec{x}$ *in compact subsets of* $\mathbb{R}^{+,m}_{\mathrm{ord}}$. *As* $r \to 0$, *we have*

$$p_0 = \frac{1}{\sqrt{2}}\bigg(\frac{\rho^3}{54} + \frac{\rho}{2} \bigg) + \mathcal{O}(r), \qquad q_0 = \frac{1}{\sqrt{2}}\bigg(-\frac{\rho^3}{54} + \frac{\rho}{2} \bigg) + \mathcal{O}(r), \tag{1.11a}$$

$$p_{j,1}(r) = \mathcal{O}(r), \qquad p_{j,2}(r) = \mathcal{O}(1), \qquad p_{j,3}(r) = \mathcal{O}(r), \qquad j = 1, \dots, m, \tag{1.11b}$$

$$q_{j,1}(r) = \mathcal{O}(1), \qquad q_{j,2}(r) = \mathcal{O}(r), \qquad q_{j,3}(r) = \mathcal{O}(1), \qquad j = 1, \dots, m. \tag{1.11c}$$

REMARK 1.2. Interestingly, for $m = 1$, the following stronger statement holds: there exists at least one solution $(p_0, q_0, p_{1,1}, q_{1,1}, p_{1,2}, q_{1,2}, p_{1,3}, q_{1,3})$ to the system of equations (1.4) and (1.5), which, in addition to satisfying the asymptotics (1.8) and (1.11), is also such that the functions p_0 and q_0 satisfy a system of coupled differential equations, see [24, equations (2.19)–(2.20)] (see also [12, equations (3.25)–(3.26)] for $\rho = 0$). However, it is unclear to us if this system of coupled differential equations admits a natural analogue for $m \geq 2$.

It is well-known since the works of Jimbo, Miwa, Môri and Sato [37] and of Tracy and Widom [44, 45] that the 1-point generating functions of the universal sine, Airy and Bessel point processes are naturally related to the theory of Painlevé equations. We also refer to [33, 3] for some Hamiltonian structures associated with the m-point functions of the sine and Airy processes, and to [21, 19] for some representations of the m-point functions of the Airy and Bessel processes in terms of the solution to a system of m coupled Painlevé equations. It was shown in [24, Theorem 2.2] that the 1-point function of the Pearcey process admits an elegant representation in terms of a Hamiltonian associated to a system of 8 coupled differential equations. Our first main result generalizes [24, Theorem 2.2] to arbitrary $m \in \mathbb{N}_{>0}$.

THEOREM 1.3. *Let* $\rho \in \mathbb{R}$,

$$r > 0, \qquad m \in \mathbb{N}_{>0}, \qquad \vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m, \quad \textit{and} \quad \vec{x} = (x_1, \dots, x_m) \in \mathbb{R}^{+,m}_{\mathrm{ord}}.$$

The following relation holds

$$F(r\vec{x},\vec{u}) = \exp\left(2\int_0^r H(\tau)d\tau\right), \tag{1.12}$$

with H given by (1.6), and where $(p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^m)$ is a solution to the system of equations (1.4) and (1.5) which satisfies the asymptotic formulas (1.8) and (1.11). Furthermore,

$$H(r) = \mathcal{O}(1), \qquad \textit{as } r \to 0, \tag{1.13}$$

and as $r \to +\infty$,

$$H(r) = \sum_{j=1}^m \left(\frac{\sqrt{3}}{2\pi}u_j x_j^{\frac{4}{3}} r^{\frac{1}{3}} - \frac{\rho}{2\sqrt{3}\pi}u_j x_j^{\frac{2}{3}} r^{-\frac{1}{3}} + \frac{u_j^2}{3\pi^2 r} - \frac{u_j}{3\sqrt{3}\pi r}\cos(2\vartheta_j(r))\right) + \mathcal{O}(r^{-\frac{5}{3}}), \tag{1.14}$$

where $\vartheta_j(r)$ is defined in (1.10).

Since the functions $(p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^m)$ appearing in the integral representation (1.12) are rather complicated objects, it is natural to try to approximate $F(r\vec{x},\vec{u})$ for small and large values of r with some explicit asymptotic formulas. Because $F(r\vec{x},\vec{u})$ is a Fredholm determinant on an interval of size proportional to r (see (2.1) below), the asymptotics of $F(r\vec{x},\vec{u})$ as $r \to 0$ can be easily obtained. A much more complicated question is to approximate $F(r\vec{x},\vec{u})$ for large values of r. In the case of the sine, Airy and Bessel processes, the asymptotics for the m-point generating functions are known, see [5, 15] for the sine process, [10, 17] for the Airy process, [11, 14] for the Bessel process and [20] for the transition between the Bessel and Airy processes. The asymptotics for the 1-point generating function of the Pearcey process, up to and including the notoriously difficult constant term, have been established in [24, Theorem 2.3]. We provide here the generalization of [24, Theorem 2.3] to arbitrary $m \in \mathbb{N}_{>0}$.

THEOREM 1.4. *Let*

$$\rho \in \mathbb{R}, \qquad m \in \mathbb{N}_{>0}, \qquad \vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m, \qquad \textit{and} \qquad \vec{x} = (x_1, \dots, x_m) \in \mathbb{R}_{\text{ord}}^{+,m}.$$

As $r \to +\infty$, we have

$$F(r\vec{x},\vec{u}) = \exp\Bigg(\sum_{j=1}^m u_j \mu_\rho(rx_j) + \sum_{j=1}^m \frac{u_j^2}{2}\sigma^2(rx_j) + \sum_{1\leq j<k\leq m} u_j u_k \Sigma(x_k, x_j) + \sum_{j=1}^m 2\log\left(G(1-\tfrac{u_j}{2\pi i})G(1+\tfrac{u_j}{2\pi i})\right) + \mathcal{O}(r^{-\frac{2}{3}})\Bigg), \tag{1.15}$$

where G is Barnes' G-function, and μ_ρ, σ^2 and Σ are given by

$$\mu_\rho(x) = \frac{3\sqrt{3}}{4\pi}x^{\frac{4}{3}} - \frac{\sqrt{3}\rho}{2\pi}x^{\frac{2}{3}},$$

$$\sigma^2(x) = \frac{4}{3\pi^2}\log x + \frac{1}{\pi^2}\log\frac{9}{2},$$

$$\Sigma(x_k, x_j) = \frac{1}{\pi^2}\log\frac{|x_j^{2/3} - \omega x_k^{2/3}|}{|x_j^{2/3} - x_k^{2/3}|},$$

where $\omega = e^{\frac{2\pi i}{3}}$. Furthermore, (1.15) *holds uniformly for ρ in compact subsets of $\mathbb{R}$, for $\vec{u}$ in compact subsets of $\mathbb{R}^m$, and for $\vec{x}$ in compact subsets of $\mathbb{R}^{+,m}_{\mathrm{ord}}$. The asymptotic formula* (1.15) *can also be differentiated any number of times with respect to $u_1, \ldots, u_m$ at the expense of increasing the error term in the following way. Let $\widetilde{F}(r\vec{x},\vec{u})$ be the right-hand side of* (1.15) *without the error term, and denote the error term by $\mathcal{E} = \log F(r\vec{x},\vec{u}) - \log \widetilde{F}(r\vec{x},\vec{u})$. For any $k_1, \ldots, k_m \in \mathbb{N}_{\geq 0}$, we have*

$$\partial_{u_1}^{k_1} \ldots \partial_{u_m}^{k_m} \mathcal{E} = \mathcal{O}\left(\frac{(\log r)^{k_1+\ldots+k_m}}{r^{2/3}} \right), \qquad \text{as } r \to +\infty. \tag{1.16}$$

We end this section by providing several applications of Theorem 1.4, and we also discuss its relevance in future studies of the rigidity of the Pearcey process.

Applications of Theorem 1.4. Using (1.15) with $m = 1$, (1.16),

$$\partial_u \log F(r,u)|_{u=0} = \mathbb{E}[N(r)] \quad \text{and} \quad \partial_u^2 \log F(r,u)|_{u=0} = \mathrm{Var}[N(r)],$$

it readily follows that

$$\mathbb{E}[N(r)] = \mu_\rho(r) + \mathcal{O}\left(\frac{\log r}{r^{2/3}} \right), \qquad \text{as } r \to +\infty, \tag{1.17}$$

$$\mathrm{Var}[N(r)] = \sigma^2(r) + \frac{1+\gamma_{\mathrm{E}}}{\pi^2} + \mathcal{O}\left(\frac{(\log r)^2}{r^{2/3}} \right), \qquad \text{as } r \to +\infty, \tag{1.18}$$

where γ_{E} is Euler's gamma constant. These asymptotic formulas for the expectation and variance of $N(r)$ are not new and were already obtained in [24, equations (2.30) and (2.31)]. We see from (1.18) that the large r asymptotics of $\mathrm{Var}[N(r)]$ are of the form $c_1 \log r + c_2 + o(1)$ for some explicit constants c_1 and c_2. The asymptotics for the variance of the counting functions of other classical point processes such as the sine, Airy and Bessel point processes are also of the same form [42, 15, 17, 14] (with different values for c_1 and c_2). This phenomena is expected to be universal, in the sense that it is expected to hold for many point processes in random matrix theory and other related fields, see the very general predictions [40, 41]. We emphasize that the proof of (1.17) and (1.18) only relies on Theorem 1.4 with $m = 1$. Using Theorem 1.4 with $m = 2$ allows to obtain new results on the correlation structure of the Pearcey process. More precisely, using (1.15) with $m = 2$, (1.16), and

$$\partial_u^2 \log \left(\frac{F((x_1,x_2),(u,u))}{F(x_1,u)F(x_2,u)} \right) \bigg|_{u=0} = 2\,\mathrm{Cov}\big(N(x_1), N(x_2)\big),$$

we directly obtain the following result.

COROLLARY 1.5. *Let $x_2 > x_1 > 0$ be fixed. As $r \to +\infty$,*

$$\mathrm{Cov}\big(N(rx_1), N(rx_2)\big) = \Sigma(x_k, x_j) + \mathcal{O}\left(\frac{(\log r)^2}{r^{2/3}} \right).$$

In [24, Corollary 2.4], the authors also proved that the random variable $(N(r) - \mu_\rho(r))/\sqrt{\sigma^2(r)}$ converges in distribution as $r \to +\infty$ to a normal random variable with mean 0 and variance 1. Using Theorem 1.4, we obtain the following generalization of this result.

COROLLARY 1.6. *Let $0 < x_1 < \ldots < x_m < +\infty$ be fixed and consider the random variables $N_j^{(r)}$ defined by*

$$N_j^{(r)} = \frac{N(rx_j) - \mu_\rho(rx_j)}{\sqrt{\sigma^2(rx_j)}}, \qquad j = 1, \ldots, m.$$

As $r \to +\infty$, we have

$$\big(N_1^{(r)}, N_2^{(r)}, \ldots, N_m^{(r)}\big) \quad \xrightarrow{d} \quad \mathcal{N}(\vec{0}, I_m), \tag{1.19}$$

where I_m is the $m \times m$ identity matrix, and $\mathcal{N}(\vec{0}, I_m)$ is a multivariate normal random variable of mean $\vec{0} = (0, \ldots, 0)$ and covariance matrix I_m.

PROOF. Let $a_1, \ldots, a_m \in \mathbb{R}$ be arbitrary and fixed (i.e. independent of r). It directly follows from (1.2) and (1.15) with $u_j = \frac{\sqrt{3}\pi}{2} \frac{a_j}{\sqrt{\log r}}$, $j = 1, \ldots, m$, that

$$\mathbb{E}\left[\prod_{j=1}^{m} e^{a_j N_j^{(r)}}\right] = \exp\left(\sum_{j=1}^{m} \frac{a_j^2}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\log r}}\right)\right), \qquad \text{as } r \to +\infty.$$

In other words, the moment generating function of $\big(N_1^{(r)}, N_2^{(r)}, \ldots, N_m^{(r)}\big)$ converges as $r \to +\infty$ pointwise in $\mathbb{R}^m$ to the moment generating function of $\mathcal{N}(\vec{0}, I_m)$. This implies the convergence in distribution (1.19) by standard probability theorems, see e.g. [7, Corollary of Theorem 25.10]. □

Possible future applications of Theorem 1.4. Corollary 1.6 gives information about the joint fluctuations of the counting function at m well-separated points $rx_1, \ldots, rx_m$. A more difficult question is to understand the *global rigidity* of the Pearcey process, that is, to understand the *maximum* fluctuation of the counting function. In recent years in random matrix theory, there has been a lot of progress in the study of ridigity of various point processes, see [29, 4] for important early works, [34] for the sine process, and [18] for the Airy and Bessel point processes. Of particular interest for us is the following result on the rigidity of the Pearcey process, which was proved in [16] (by combining results from [18] and [24]): for any $\epsilon > 0$, the probability that

$$\mu_\rho(x) - \left(\frac{4\sqrt{2}}{3\pi} + \epsilon\right)\log x \le N(x) \le \mu_\rho(x) + \left(\frac{4\sqrt{2}}{3\pi} + \epsilon\right)\log x \qquad \text{for all } x > r \tag{1.20}$$

tends to 1 as $r \to +\infty$. Roughly speaking, this means that with high probability and for all large x, $N(x)$ lies in a tube centered at $\mu_\rho(x)$ and of width $\big(\frac{8\sqrt{2}}{3\pi} + 2\epsilon\big)\log x$, see Figure 1 (left). Equivalently, (1.20) can be rewritten for the normalized counting function as follows

$$\lim_{r \to \infty} \mathbb{P}\left(\sup_{x > r}\left|\frac{N(x) - \mu_\rho(x)}{\log x}\right| \le \frac{4\sqrt{2}}{3\pi} + \epsilon\right) = 1, \tag{1.21}$$

see Figure 1 (right). It has also been conjectured in [16] that the upper bound (1.21) is sharp, in the sense that the following complementary lower bound is expected to hold: for any $\epsilon > 0$,

$$\lim_{r \to \infty} \mathbb{P}\left(\sup_{x > r}\left|\frac{N(x) - \mu_\rho(x)}{\log x}\right| \ge \frac{4\sqrt{2}}{3\pi} - \epsilon\right) = 1, \tag{1.22}$$

and this is also supported by Figure 1 (right). Such lower bounds are notoriously difficult to prove.

By analogy with the method developed in [22], we expect that Theorem 1.4 with $m = 2$ will be useful to establish (1.22). However, we also expect that proving (1.22) will also require other estimates that are not provided in this paper, such as the large r asymptotics for $\mathbb{E}[e^{u_1 N(rx_1) + u_2 N(rx_2)}]$ when simultaneously $|x_1 - x_2| \to 0$. This regime requires a completely different analysis than the one of Theorem 1.4, and we shall not pursue this here.

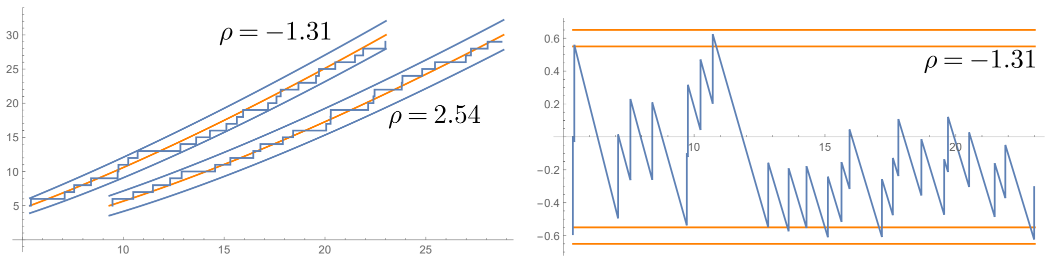

FIG 1. *Rigidity of the Pearcey process (the pictures are taken from [16]). Left: the smooth blue lines correspond to the upper and lower bounds in* (1.20) *with* $\epsilon = 0.05$*, and* $N(x)$ *is the discontinuous blue line. Right: the blue line is* $\frac{N(x)-\mu_\rho(x)}{\log x}$*, and the four orange lines correspond to the constants* $\pm\frac{4\sqrt{2}}{3\pi}+\epsilon$*,* $\pm\frac{4\sqrt{2}}{3\pi}-\epsilon$ *with* $\epsilon = 0.05$.

Outline. Relying on the fact that the Pearcey kernel K_ρ^{Pe} is known to be integrable (of size 3) in the sense of [35], we use in Section 2 the general method of [25] to express $\partial_r \log F(r\vec{x},\vec{u})$ in terms of the solution Φ to a 3×3 RH problem. In Section 3, we derive the system of equations (1.4) and establish the formula $\partial_r \log F(r\vec{x},\vec{u}) = 2H(r)$ by analyzing a natural Lax pair associated to Φ. In Section 4, we use the Deift–Zhou [26] steepest descent method to obtain the large r asymptotics of Φ. A main technical challenge here is to analyze the behavior of the global parametrix at certain points, which becomes particularly delicate for $m \geq 2$, see e.g. the asymptotic formulas of Subsection 4.3.2. The steepest descent analysis of Φ for small r is simpler and is performed in Section 5. In Section 6, we use the small and large r asymptotics of Φ together with some remarkable identities for the Hamiltonian to complete the proofs of Theorems 1.1, 1.3 and 1.4.

2. Differential identity.

The main result of this section is a differential identity which expresses $\partial_r \log F(r\vec{x},\vec{u})$ in terms of the solution Φ to a 3×3 RH problem.

It is well-known, see e.g. [43, Theorem 2], that the moment generating function (1.2) is equal to the Fredholm determinant

$$F(r\vec{x},\vec{u}) = \det\left(1-\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}}\right), \qquad \widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}} := \sum_{j=1}^{m}(1-s_j)\mathcal{K}_\rho^{\mathrm{Pe}}|_{rA_j}, \tag{2.1}$$

where $s_j := e^{u_j+\ldots+u_m} \in (0,+\infty)$, $j=1,\ldots,m$,

$$A_1 = (-x_1,x_1), \qquad A_j = (-x_j,-x_{j-1})\cup(x_{j-1},x_j), \quad j=2,\ldots,m,$$

and $\mathcal{K}_\rho^{\mathrm{Pe}}|_{rA_j}$ is the trace-class operator acting on $L^2(rA_j)$ whose kernel is K_ρ^{Pe}. We now recall a formula from [8] which expresses K_ρ^{Pe} in terms of the solution Ψ of a 3×3 RH problem.

2.1. *Background from [8].*

RH problem for Ψ.

(a) $\Psi : \mathbb{C}\setminus\{\cup_{j=0}^{5}\Sigma_j\cup\{0\}\} \to \mathbb{C}^{3\times 3}$ is analytic, where

$$\Sigma_0 = (0,+\infty), \qquad \Sigma_1 = e^{\frac{\pi i}{4}}(0,+\infty), \qquad \Sigma_2 = e^{\frac{3\pi i}{4}}(+\infty,0),$$
$$\Sigma_3 = (-\infty,0), \qquad \Sigma_4 = e^{-\frac{3\pi i}{4}}(+\infty,0), \qquad \Sigma_5 = e^{-\frac{\pi i}{4}}(0,+\infty). \tag{2.2}$$

(b) For $z \in \cup_{j=0}^{5}\Sigma_j$, we denote $\Psi_+(z)$ (resp. $\Psi_-(z)$) for the limit of $\Psi(s)$ as $s \to z$ from the left (resp. right) of $\cup_{j=0}^{5}\Sigma_j$ (here "left" and "right" refer to the orientation of $\cup_{j=0}^{5}\Sigma_j$ as indicated in (2.2)). For $z \in \Sigma_j$, we have $\Psi_+(z) = \Psi_-(z)J_j$, $j = 0, \dots, 5$, where J_0, J_1, J_2, J_3, J_4 and J_5 are respectively given by

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

(c) As $z \to \infty$, $\pm \mathrm{Im}\, z > 0$,

$$\Psi(z) = \sqrt{\frac{2\pi}{3}} e^{\frac{\rho^2}{6}} i\Psi_0 \left(I + \frac{\Psi_1}{z} + \mathcal{O}(z^{-2}) \right) \mathrm{diag}\,(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}}) L_\pm e^{\Theta(z)}, \tag{2.4}$$

where

$$\Psi_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \kappa_3(\rho) + \frac{2\rho}{3} & 0 & 1 \end{pmatrix}, \qquad \Psi_1 = \begin{pmatrix} 0 & \kappa_3(\rho) & 0 \\ \widetilde{\kappa}_6(\rho) & 0 & \kappa_3(\rho) + \frac{\rho}{3} \\ 0 & \widehat{\kappa}_6(\rho) & 0 \end{pmatrix}, \tag{2.5}$$

$$\kappa_3(\rho) = \frac{\rho^3}{54} - \frac{\rho}{6}, \qquad \kappa_6(\rho) = \frac{\rho^6}{5832} - \frac{\rho^4}{162} - \frac{\rho^2}{72} + \frac{7}{32},$$

$$\widetilde{\kappa}_6(\rho) = \kappa_6(\rho) + \frac{\rho}{3}\kappa_3(\rho) - \frac{1}{3}, \qquad \widehat{\kappa}_6(\rho) = \kappa_6(\rho) - \kappa_3(\rho)^2 + \frac{\rho^2}{9} - \frac{1}{3},$$

$$L_+ = \begin{pmatrix} -\omega & \omega^2 & 1 \\ -1 & 1 & 1 \\ -\omega^2 & \omega & 1 \end{pmatrix}, \qquad L_- = \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix},$$

$$\Theta(z) = \begin{cases} \mathrm{diag}\,(\theta_1(z), \theta_2(z), \theta_3(z)), & \mathrm{Im}\, z > 0, \\ \mathrm{diag}\,(\theta_2(z), \theta_1(z), \theta_3(z)), & \mathrm{Im}\, z < 0, \end{cases} \tag{2.6}$$

with $\theta_k(z) = \frac{3}{4}\omega^{2k} z^{\frac{4}{3}} + \frac{\rho}{2}\omega^k z^{\frac{2}{3}}$, $k = 1, 2, 3$, and $\omega = e^{\frac{2\pi i}{3}}$.

(d) $\Psi(z)$ remains bounded as $z \to 0$.

Consider the following functions

$$\mathcal{P}_j(z) = \int_{\Gamma_j} e^{-\frac{1}{4}t^4 - \frac{\rho}{2}t^2 + itz} dt, \qquad j = 0, 1, \dots, 5, \tag{2.7}$$

where

$$\Gamma_0 = (-\infty, +\infty), \qquad \Gamma_1 = (i\infty, 0] \cup [0, \infty), \qquad \Gamma_2 = (i\infty, 0] \cup [0, -\infty),$$

$$\Gamma_3 = (-i\infty, 0] \cup [0, -\infty), \qquad \Gamma_4 = (-i\infty, 0] \cup [0, +\infty), \qquad \Gamma_5 = (-i\infty, i\infty),$$

and define

$$\widetilde{\Psi}(z) = \begin{pmatrix} \mathcal{P}_0(z) & \mathcal{P}_1(z) & \mathcal{P}_4(z) \\ \mathcal{P}_0'(z) & \mathcal{P}_1'(z) & \mathcal{P}_4'(z) \\ \mathcal{P}_0''(z) & \mathcal{P}_1''(z) & \mathcal{P}_4''(z) \end{pmatrix}, \qquad z \in \mathbb{C}. \tag{2.8}$$

It was shown in [8, Section 8.1] that the RH problem for Ψ admits a unique solution which can be explicitly written in terms of $\mathcal{P}_j$, $j = 0, \dots, 5$. For example, for $\arg z \in (\frac{\pi}{4}, \frac{3\pi}{4})$, we have $\Psi(z) = \widetilde{\Psi}(z)$. The explicit expression of Ψ in the other sectors is not needed for us so we do not write it down, but we refer the interested reader to [8, equations (8.12)–(8.17)]. The Pearcey kernel can be written as follows (see [8, equation (10.19)]):

$$K_\rho^{\mathrm{Pe}}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \widetilde{\Psi}(y)^{-1}\widetilde{\Psi}(x) \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t, \qquad x,y\in\mathbb{R}, \tag{2.9}$$

where $(\cdot)^t$ denotes the transpose operation. Let $\widetilde{K}_\rho^{\mathrm{Pe}}$ be the kernel of the operator $\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}}$ appearing in (2.1):

$$\widetilde{K}_\rho^{\mathrm{Pe}}(x,y) := \sum_{j=1}^m (1-s_j) K_\rho^{\mathrm{Pe}}(x,y)\chi_{rA_j}(y),$$

where χ_{A_j} denotes the characteristic function of A_j, i.e. $\chi_{A_j}(x) = 1$ if $x\in A_j$ and 0 otherwise. From (2.9), it is easy to see that $\widetilde{K}_\rho^{\mathrm{Pe}}$ can be written as

$$\widetilde{K}_\rho^{\mathrm{Pe}}(x,y) = \frac{\mathbf{f}(x)^t\mathbf{h}(y)}{x-y}, \tag{2.10}$$

where

$$\mathbf{f}(x) = \widetilde{\Psi}(x)\begin{pmatrix}1\\0\\0\end{pmatrix}, \ \mathbf{h}(y) = \frac{\sum_{j=1}^m (1-s_j)\chi_{rA_j}(y)}{2\pi i}\widetilde{\Psi}(y)^{-t}\begin{pmatrix}0\\1\\1\end{pmatrix} = \sum_{j=1}^m \mathfrak{s}_j\chi_{rB_j}(y)\widetilde{\Psi}(y)^{-t}\begin{pmatrix}0\\1\\1\end{pmatrix}. \tag{2.11}$$

with $\mathfrak{s}_j := \frac{s_{j+1}-s_j}{2\pi i}$ and $B_j = (-x_j, x_j)$, $j = 1,\dots,m$. Since $\vec{u}\in\mathbb{R}^m$, it follows from (1.2) that $F(r\vec{x},\vec{u})\in(0,+\infty)$. Thus, by (2.1), we have $\det(1-\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}}) > 0$ and in particular $1-\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}}$ is invertible. Using now standard identities for trace-class operators, we obtain

$$\partial_r \log F(r\vec{x},\vec{u}) = \partial_r \log\det(1-\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}}) = -\mathrm{Tr}\Big((1-\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}})^{-1}\partial_r\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}}\Big) \tag{2.12}$$

$$= -\sum_{j=1}^m x_j\Big(\lim_{v\nearrow rx_j}\mathrm{R}(v,v) + \lim_{v\searrow -rx_j}\mathrm{R}(v,v)\Big) + \sum_{j=1}^{m-1} x_j\Big(\lim_{v\searrow rx_j}\mathrm{R}(v,v) + \lim_{v\nearrow -rx_j}\mathrm{R}(v,v)\Big),$$

where R is the kernel of the resolvent operator $(1-\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}})^{-1}\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}}$. Formula (2.10) shows in particular that $\widetilde{K}_\rho^{\mathrm{Pe}}$ is integrable (of size 3) in the sense of [35]. Hence, by [25, Lemma 2.12], we have

$$\mathrm{R}(u,v) = \frac{\mathbf{F}(u)^t\mathbf{H}(v)}{u-v}, \qquad u,v\in\mathbb{R}, \tag{2.13}$$

where

$$\mathbf{F}(u) = (1-\widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}})^{-1}\mathbf{f}(u) = Y_+(u)\mathbf{f}(u), \qquad \mathbf{H}(v) = Y_+^{-t}(v)\mathbf{h}(v), \qquad u,v\in\mathbb{R}, \tag{2.14}$$

and Y is given by

$$Y(z) = I - \int_{-rx_m}^{rx_m} \frac{\mathbf{F}(w)\mathbf{h}(w)^t}{w-z}dw. \tag{2.15}$$

Furthermore, Y is the unique solution to the following RH problem.

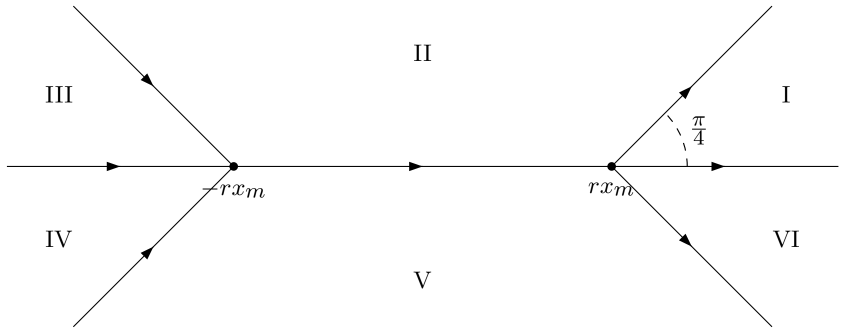

FIG 2. *Jump contours* $\Sigma_k^{(r)}$, $k = 0, 1, \ldots, 6$.

RH problem for Y.

(a) $Y : \mathbb{C} \setminus [-rx_m, rx_m] \to \mathbb{C}^{3\times 3}$ is analytic.
(b) Y satisfies the jumps

$$Y_+(x) = Y_-(x)(I - 2\pi i \mathbf{f}(x)\mathbf{h}(x)^t), \qquad x \in (-rx_m, rx_m) \setminus \cup_{j=1}^{m-1}\{-rx_j, rx_j\}. \tag{2.16}$$

(c) As $z \to \infty$, $Y(z) = I + \frac{Y_1}{z} + \mathcal{O}(z^{-2})$.
(d) As $z \to z_* \in \cup_{j=1}^{m}\{-rx_j, rx_j\}$, we have $Y(z) = \mathcal{O}(\log(z - z_*))$.

Now, we apply a transformation which changes Y into another function Φ whose jump matrices are piecewise constant. Following [24, eq (3.24)], we define $\Phi(z) = \Phi(z; r)$ as

$$\Phi(z) = \frac{\Psi_0^{-1}}{\sqrt{\frac{2\pi}{3}} e^{\frac{\rho^2}{6}} i} \begin{cases} Y(z)\Psi(z), & z \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{VI}, \\ Y(z)\widetilde{\Psi}(z), & z \in \mathrm{II}, \\ Y(z)\widetilde{\Psi}(z) \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \mathrm{V}, \end{cases} \tag{2.17}$$

where the regions I, II III, IV, V and VI are shown in Figure 2. It is easily verified from the RH problems for Y and Ψ that Φ satisfies the following RH problem.

RH problem for Φ.

(a) $\Phi : \mathbb{C} \setminus \{\cup_{j=0}^{6} \Sigma_j^{(r)} \cup \{-rx_m, rx_m\}\} \to \mathbb{C}^{3\times 3}$ is analytic, where

$$\Sigma_0^{(r)} = (rx_m, +\infty), \qquad \Sigma_1^{(r)} = rx_m + e^{\frac{\pi i}{4}}(0, +\infty), \qquad \Sigma_2^{(r)} = -rx_m + e^{\frac{3\pi i}{4}}(+\infty, 0),$$
$$\Sigma_3^{(r)} = (-\infty, -rx_m), \quad \Sigma_4^{(r)} = -rx_m + e^{-\frac{3\pi i}{4}}(+\infty, 0), \quad \Sigma_5^{(r)} = rx_m + e^{-\frac{\pi i}{4}}(0, +\infty), \tag{2.18}$$

and $\Sigma_6^{(r)} = (-rx_m, rx_m)$, see also Figure 2.

(b) For $z \in \Sigma_j^{(r)}$, we have $\Phi_+(z) = \Phi_-(z)J_j$, $j = 0, \ldots, 5$, where J_0, J_1, J_2, J_3, J_4 and J_5 are given by (2.3). For $z \in (-rx_m, rx_m) \setminus \cup_{j=1}^{m-1}\{-rx_j, rx_j\}$, we have $\Phi_+(z) = \Phi_-(z)J_6(z)$, where

$$J_6(z) = \begin{pmatrix} 1 & s_j & s_j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad z \in rA_j, \qquad j = 1, \ldots, m.$$

(c) As $z \to \infty$, $\pm\mathrm{Im}\, z > 0$,

$$\Phi(z) = \left(I + \frac{\Phi_1}{z} + \frac{\Phi_2}{z^2} + \mathcal{O}(z^{-3})\right)\mathrm{diag}\left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}}\right) L_{\pm} e^{\Theta(z)}, \tag{2.19}$$

where Φ_1,Φ_2 are independent of z and $\Phi_1 = \Psi_1 + \Psi_0^{-1} Y_1 \Psi_0$.

(d) As $z \to rx_j$, $j = 1, \ldots, m$, we have

(2.20)

$$\Phi(z) = \widehat{\Phi}_j(z) \begin{pmatrix} 1 & -\mathfrak{s}_j \log(z - rx_j) & -\mathfrak{s}_j \log(z - rx_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{cases} I, & z \in \mathrm{II}, \\ \begin{pmatrix} 1 & -s_{j+1} & -s_{j+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \mathrm{V}, \end{cases}$$

where we recall that $\mathfrak{s}_j := \frac{s_{j+1} - s_j}{2\pi i}$. The matrix $\widehat{\Phi}_j$ is analytic at rx_j and satisfies

$$\widehat{\Phi}_j(z) = \Phi_j^{(0)}(r)\Big(I + \Phi_j^{(1)}(r)(z - rx_j) + \mathcal{O}((z - rx_j)^2)\Big), \qquad z \to rx_j, \tag{2.21}$$

for some matrices $\Phi_j^{(0)}(r)$ and $\Phi_j^{(1)}(r)$.

(e) Φ satisfies the symmetry

$$\Phi(z) = -\mathrm{diag}\,(1, -1, 1)\Phi(-z)\mathcal{B}, \qquad \mathcal{B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.22}$$

PROPOSITION 2.1. *We have*

$$\partial_r \log F(r\vec{x}, \vec{u}) = -\sum_{j=1}^{m} 2\, \mathfrak{s}_j x_j \left[\left(\Phi_j^{(1)}(r)\right)_{21} + \left(\Phi_j^{(1)}(r)\right)_{31}\right]. \tag{2.23}$$

PROOF. The proof is a minor adaptation of [24, Proposition 3.5]. For $v \in \mathbb{R}$, by (2.11), (2.14) and (2.17), we have

(2.24)

$$\mathbf{F}(v) = \sqrt{\frac{2\pi}{3}} e^{\frac{\rho^2}{6}} i \Psi_0 \Phi_+(v) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{H}(v) = \sum_{j=1}^{m} \mathfrak{s}_j \chi_{rB_j}(v) \frac{\Psi_0^{-t}}{\sqrt{\frac{2\pi}{3}} e^{\frac{\rho^2}{6}} i} \Phi_+(v)^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Using (2.13), (2.17), (2.22) and (2.24), for $v \in \mathbb{R}$ we find

$$\mathrm{R}(v, v) = \sum_{j=1}^{m} \mathfrak{s}_j \chi_{rB_j}(v) \Big(\left[\Phi_+(v)^{-1}\Phi_+'(v)\right]_{21} + \left[\Phi_+(v)^{-1}\Phi_+'(v)\right]_{31} \Big) = \mathrm{R}(-v, -v),$$

which allows us to rewrite (2.12) as

$$\begin{aligned} \partial_r \log F(r\vec{x}, \vec{u}) &= -2\sum_{j=1}^{m} x_j \lim_{v \nearrow rx_j} \mathrm{R}(v, v) + 2\sum_{j=1}^{m-1} x_j \lim_{v \searrow rx_j} \mathrm{R}(v, v) \\ &= -2\sum_{j=1}^{m} x_j \mathfrak{s}_j \Big(\left[\Phi_+(x_j)^{-1}\Phi_+'(x_j)\right]_{21} + \left[\Phi_+(x_j)^{-1}\Phi_+'(x_j)\right]_{31} \Big). \end{aligned}$$

By (2.20) and (2.21), $\left[\Phi_+(x_j)^{-1}\Phi_+'(x_j)\right]_{k1} = (\Phi_j^{(1)}(r))_{k1}$, for all $j = 1, \ldots, m$ and $k = 2, 3$, which finishes the proof. □

3. Lax pair. In this section,

- we find an explicit solution to (1.4)–(1.5) in terms of Φ,
- we prove the relation $\partial_r \log F(r\vec{x},\vec{u}) = 2H(r)$,
- we derive some further identities for H which will be useful in Section 6.

These results generalize part of the content of [24, Section 4] to arbitrary $m \in \mathbb{N}_{>0}$.

PROPOSITION 3.1. *The functions* $(p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^{m})$ *defined by*

$$p_0(r) := \frac{1}{\sqrt{2}}\Big(\frac{\rho}{3} + \Phi_{1,23}\Big), \qquad q_0(r) := \frac{1}{\sqrt{2}}\Big(\frac{\rho}{3} - \Phi_{1,12}\Big), \tag{3.1}$$

$$\begin{pmatrix} p_{j,1}(r) \\ p_{j,2}(r) \\ p_{j,3}(r) \end{pmatrix} := -\mathfrak{s}_j \Phi_j^{(0)}(r)^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} q_{j,1}(r) \\ q_{j,2}(r) \\ q_{j,3}(r) \end{pmatrix} := \Phi_j^{(0)}(r) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad j = 1,\ldots,m, \tag{3.2}$$

satisfy (1.5) *and the system of coupled equations* (1.4).

REMARK 3.2. Since Φ exists by (2.14), (2.15) and (2.17), Proposition 3.1 implies that there exists at least one solution to (1.4)–(1.5).

PROOF. Following [24], we will proceed by analyzing the following Lax pair

$$L(z;r) := \partial_z \Phi(z;r) \cdot \Phi(z;r)^{-1}, \qquad U(z;r) := \partial_r \Phi(z;r) \cdot \Phi(z;r)^{-1}.$$

Since the jump matrices of Φ are independent of z and r, $L(z) = L(z;r)$ and $U(z) = U(z;r)$ are analytic in $\mathbb{C} \setminus \{-rx_m, \ldots, -rx_1, 0, rx_1, \ldots, rx_m\}$. Furthermore, using (2.22), we infer that they satisfy

$$L(z) = -\mathrm{diag}\,(1,-1,1) L(-z) \mathrm{diag}\,(1,-1,1), \quad U(z) = \mathrm{diag}\,(1,-1,1) U(-z) \mathrm{diag}\,(1,-1,1). \tag{3.3}$$

By (2.19), (3.1) and (3.3), we find

$$L(z) = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix} z + A_0(r) + \frac{L_1}{z} + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty, \tag{3.4}$$

where

$$A_0(r) = \begin{pmatrix} 0&1&0 \\ \frac{\rho}{3}&0&1 \\ 0&\frac{\rho}{3}&0 \end{pmatrix} + \left[\Phi_1, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{2}p_0(r) & 0 & 1 \\ 0 & \sqrt{2}q_0(r) & 0 \end{pmatrix}, \tag{3.5}$$

$$L_1 = \begin{pmatrix} -\frac{1}{3}&0&\frac{\rho}{3} \\ 0&0&0 \\ 0&0&\frac{1}{3} \end{pmatrix} + \left[\Phi_2, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}\right] + \left[\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}\Phi_1, \Phi_1\right] + \left[\Phi_1, \begin{pmatrix} 0&1&0 \\ \frac{\rho}{3}&0&1 \\ 0&\frac{\rho}{3}&0 \end{pmatrix}\right],$$

and where we have used the notation $[\mathcal{B}_1, \mathcal{B}_2] := \mathcal{B}_1\mathcal{B}_2 - \mathcal{B}_2\mathcal{B}_1$. Also, by (2.20)–(2.21) and (3.2), we have

$$L(z) = \frac{A_j(r)}{z - rx_j} + \mathcal{O}(1), \qquad \text{as } z \to rx_j, \quad j = 1,\ldots,m \tag{3.6}$$

with

$$A_j(r) = -\mathfrak{s}_j \Phi_j^{(0)}(r) \begin{pmatrix} 0&1&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix} \Phi_j^{(0)}(r)^{-1} = \begin{pmatrix} q_{j,1} \\ q_{j,2} \\ q_{j,3} \end{pmatrix} \begin{pmatrix} p_{j,1} & p_{j,2} & p_{j,3} \end{pmatrix}. \tag{3.7}$$

Since $\det \Phi(z)$ is constant, we have $\mathrm{Tr}L(z) = \mathrm{Tr}A_j(r) = \sum_{k=1}^{3} p_{j,k}(r)q_{j,k}(r) = 0$, which already proves (1.5). Combining (3.3), (3.4) and (3.6), we have shown that

$$L(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} + A_0(r) + \sum_{j=1}^{m} \left(\frac{A_j(r)}{z - rx_j} + \frac{A_{-j}(r)}{z + rx_j} \right), \tag{3.8}$$

where

$$A_{-j}(r) = \mathrm{diag}\,(1,-1,1)A_j(r)\mathrm{diag}\,(1,-1,1) = \begin{pmatrix} q_{j,1} \\ -q_{j,2} \\ q_{j,3} \end{pmatrix} \begin{pmatrix} p_{j,1} & -p_{j,2} & p_{j,3} \end{pmatrix}. \tag{3.9}$$

For the computation of U, we use (2.19) and (2.20)–(2.21) to obtain

$$U(z) = \mathcal{O}(z^{-1}), \qquad \text{as } z \to \infty, \qquad U(z) = -x_j \frac{A_j(r)}{z - rx_j} + \mathcal{O}(1), \qquad \text{as } z \to rx_j.$$

Using also (3.3), we conclude that

$$U(z) = \sum_{j=1}^{m} \left(-x_j \frac{A_j(r)}{z - rx_j} + x_j \frac{A_{-j}(r)}{z + rx_j} \right). \tag{3.10}$$

It remains to show that the functions $(p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^{m})$ satisfy the system of equations (1.4). For this, we note that the compatibility condition $\partial_z \partial_r \Phi(z) = \partial_r \partial_z \Phi(z)$ is equivalent to the relation

$$\partial_r L(z) - \partial_z U(z) = [U(z), L(z)]. \tag{3.11}$$

On the other hand, by (3.8) and (3.10), we have

$$\partial_r L(z) - \partial_z U(z) = A_0'(r) + \sum_{j=1}^{m} \left(\frac{A_j'(r)}{z - rx_j} + \frac{A_{-j}'(r)}{z + rx_j} \right). \tag{3.12}$$

Substituting (3.8) and (3.10) in the above two equations, and then taking $z \to \infty$, we get

$$A_0' = \sum_{j=1}^{m} x_j \left[A_{-j} - A_j, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 0 \\ -2\sum_{j=1}^{m} x_j p_{j,3} q_{j,2} & 0 & 0 \\ 0 & 2\sum_{j=1}^{m} x_j p_{j,2} q_{j,1} & 0 \end{pmatrix},$$

which yields the first two equations in (1.4). We now prove the last six equations of (1.4). A direct computation using (2.20) and (2.21) shows that

$$x_j L(z) + U(z) = \big(x_j \partial_z \Phi(z) + \partial_r \Phi(z)\big)\Phi(z)^{-1} = \partial_r \Phi_j^{(0)}(r) \cdot \Phi_j^{(0)}(r)^{-1} + o(1) \quad \text{as } z \to rx_j, \tag{3.13}$$

and using (3.8) and (3.10), we get

$$x_j L(z) + U(z) = M_j(r) - \frac{1}{r} A_j(r) + o(1) \qquad \text{as } z \to rx_j, \tag{3.14}$$

with

$$M_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ rx_j^2 & 0 & 0 \end{pmatrix} + x_j A_0 + \frac{1}{r} \sum_{\substack{k=-m \\ k \neq 0}}^{m} A_k = \begin{pmatrix} \frac{2}{r} S_{11} & x_j & \frac{2}{r} S_{31} \\ \sqrt{2} p_0 x_j & \frac{2}{r} S_{22} & x_j \\ rx_j^2 + \frac{2}{r} S_{13} & \sqrt{2} q_0 x_j & \frac{2}{r} S_{33} \end{pmatrix}. \tag{3.15}$$

In (3.15), we have omitted the r-dependence of various functions for notational convenience. Combining (3.13) and (3.14) yields

$$\partial_r \Phi_j^{(0)}(r) = \left(M_j(r) - \frac{1}{r}A_j(r)\right)\Phi_j^{(0)}(r).$$

Taking the first column of the above equation and using (1.5), (3.5) and (3.2), we get

$$\left(q'_{j,1}(r)\; q'_{j,2}(r)\; q'_{j,3}(r)\right)^t = M(r)\left(q_{j,1}(r)\; q_{j,2}(r)\; q_{j,3}(r)\right)^t, \tag{3.16}$$

and it is a direct computation to verify that (3.16) is equivalent to the third, fourth and fifth equations of (1.4). Finally, using (3.8), (3.10), (3.11) and (3.12), and letting $z \to rx_j$, we get

$$A'_j(r) = -[A_j(r), M_j(r)]. \tag{3.17}$$

Combining (3.17) with (3.5) and (3.16), we get

$$\left(p'_{j,1}(r)\; p'_{j,2}(r)\; p'_{j,3}(r)\right) = -\left(p_{j,1}(r)\; p_{j,2}(r)\; p_{j,3}(r)\right) M(r),$$

which yields the last three equations in (1.4). □

For later use, we also note that by taking $z \to \infty$ in (3.4) and then by reading the z^{-1} term of the (1,3) entry, we get

$$\sum_{j=1}^m (A_{j,13}(r) + A_{-j,13}(r)) = 2S_{31}(r) = \frac{\rho}{3} + \Phi_{1,12}(r) - \Phi_{1,23}(r) = \rho - \sqrt{2}(p_0(r) + q_0(r)), \tag{3.18}$$

where we have also used (3.8) and (3.1). In the rest of this section, we prove some identities for H which will be useful in Section 6.

PROPOSITION 3.3. *Let H be the Hamiltonian given in (1.6) with $(p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^m)$ defined as in (3.1)–(3.2). We have*

$$\partial_r \log F(r\vec{x}, \vec{u}) = 2H(r). \tag{3.19}$$

PROOF. By (2.23), the claim (3.19) is equivalent to

$$H(r) = -\sum_{j=1}^m \mathfrak{s}_j x_j \operatorname{Tr}\left(\Phi_j^{(1)}(r)\begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}\right) = -\sum_{j=1}^m \mathfrak{s}_j x_j \left[\left(\Phi_j^{(1)}(r)\right)_{21} + \Phi_j^{(1)}(r)\Big)_{31}\right]. \tag{3.20}$$

Let $\mathcal{H}(r)$ be the right-hand side of (3.20). We must show that $H(r) = \mathcal{H}(r)$. By reading the $\mathcal{O}(1)$ term in the expansion of $\partial_z\Phi(z) = L(z)\Phi(z)$ as $z \to rx_j$ (using (3.8) and (2.20)), we obtain

$$\begin{aligned}\Phi_j^{(1)}(r) = \mathfrak{s}_j &\left[\Phi_j^{(1)}(r), \begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}\right] \\ &+ \Phi_j^{(0)}(r)^{-1}\left(\begin{pmatrix}0&0&0\\0&0&0\\rx_j&0&0\end{pmatrix} + A_0(r) + \sum_{\substack{\ell=1\\ \ell\neq j}}^m \left(\frac{A_\ell(r)}{rx_j - rx_\ell} + \frac{A_{-\ell}(r)}{rx_j + rx_\ell}\right) + \frac{A_{-j}(r)}{2rx_j}\right)\Phi_j^{(0)}(r).\end{aligned} \tag{3.21}$$

Substituting (3.21) in the right-hand side of (3.20) leads to

$$\mathcal{H}(r) = -\sum_{j=1}^{m} \mathfrak{s}_j x_j \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \Phi_j^{(0)}(r)^{-1} \Bigg[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ rx_j & 0 & 0 \end{pmatrix} + A_0(r)$$

$$+ \sum_{\substack{\ell=1 \\ \ell \neq j}}^{m} \left(\frac{A_\ell(r)}{rx_j - rx_\ell} + \frac{A_{-\ell}(r)}{rx_j + rx_\ell} \right) + \frac{A_{-j}(r)}{2rx_j} \Bigg] \Phi_j^{(0)}(r) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.22}$$

Inserting (3.2) and (3.5) in (3.22), we get

$$\mathcal{H} = \sum_{j=1}^{m} x_j \begin{pmatrix} p_{j,1} \\ p_{j,2} \\ p_{j,3} \end{pmatrix}^t \left(\begin{pmatrix} 0 & 1 & 0 \\ \sqrt{2}p_0 & 0 & 1 \\ rx_j & \sqrt{2}q_0 & 0 \end{pmatrix} + \sum_{\substack{\ell=1 \\ \ell \neq j}}^{m} \left(\frac{A_\ell}{rx_j - rx_\ell} + \frac{A_{-\ell}}{rx_j + rx_\ell} \right) + \frac{A_{-j}}{2rx_j} \right) \begin{pmatrix} q_{j,1} \\ q_{j,2} \\ q_{j,3} \end{pmatrix}.$$

The double sum in the above expression can be simplified as

$$\sum_{j=1}^{m} x_j \begin{pmatrix} p_{j,1} \\ p_{j,2} \\ p_{j,3} \end{pmatrix}^t \sum_{\substack{\ell=1 \\ \ell \neq j}}^{m} \left(\frac{A_\ell}{rx_j - rx_\ell} + \frac{A_{-\ell}}{rx_j + rx_\ell} \right) \begin{pmatrix} q_{j,1} \\ q_{j,2} \\ q_{j,3} \end{pmatrix} = \frac{1}{2} \sum_{j=1}^{m} \begin{pmatrix} p_{j,1} \\ p_{j,2} \\ p_{j,3} \end{pmatrix}^t \sum_{\substack{\ell=1 \\ \ell \neq j}}^{m} \left(A_\ell + A_{-\ell} \right) \begin{pmatrix} q_{j,1} \\ q_{j,2} \\ q_{j,3} \end{pmatrix}.$$

Using (3.7) and (3.9), it is now a direct computation to check that indeed $\mathcal{H}(r) = H(r)$, which concludes the proof. □

PROPOSITION 3.4. *Let H be the Hamiltonian given in (1.6) with $(p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^m)$ defined as in (3.1)–(3.2). We have*

$$p_0(r)q_0'(r) + \sum_{j=1}^{m}\sum_{k=1}^{3} p_{j,k}(r)q_{j,k}'(r) - H(r) \tag{3.23}$$
$$= H(r) + \frac{1}{4}\frac{d}{dr}\Big(2p_0(r)q_0(r) + \sum_{j=1}^{m} \left[p_{j,2}(r)q_{j,2}(r) + 2p_{j,3}(r)q_{j,3}(r)\right] - 3rH(r)\Big).$$

Furthermore,

$$\partial_\gamma \Big(p_0(r)q_0'(r) + \sum_{k=1}^{3}\sum_{j=1}^{m} p_{j,k}(r)q_{j,k}'(r) - H(r) \Big) = \frac{d}{dr} \Big(\sum_{k=1}^{3}\sum_{j=1}^{m} p_{j,k}(r)\partial_\gamma q_{j,k}(r) + p_0(r)\partial_\gamma q_0(r) \Big) \tag{3.24}$$

where γ is any parameter among $u_1, \ldots, u_m$.

PROOF. Formula (3.23) follows directly from (1.4) and (1.5), and formula (3.24) follows from (1.7) together with

$$\partial_\gamma H = \frac{\partial H}{\partial p_0}\partial_\gamma p_0 + \frac{\partial H}{\partial q_0}\partial_\gamma q_0 + \sum_{j=1}^{m}\sum_{k=1}^{3} \left(\frac{\partial H}{\partial p_{j,k}}\partial_\gamma p_{j,k} + \frac{\partial H}{\partial q_{j,k}}\partial_\gamma q_{j,k} \right).$$

□

4. Asymptotic analysis of $\Phi(z;r)$ as $r \to +\infty$. In this section, we perform a Deift-Zhou steepest descent analysis to obtain the large r asymptotics of Φ. The case $m = 1$ of this analysis was previously done in [24].

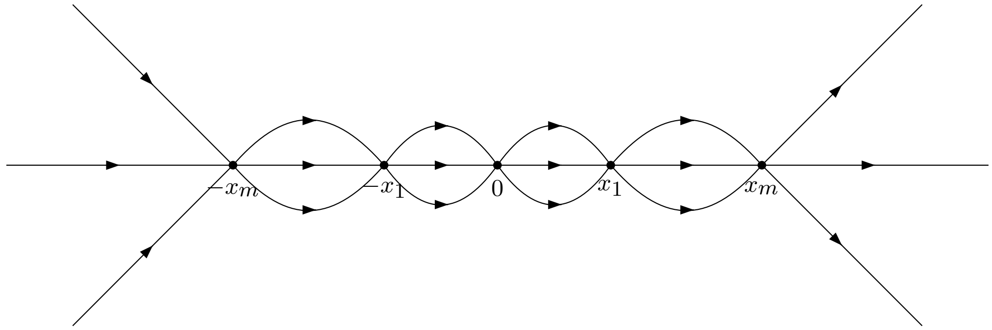

FIG 3. *Jump contours for the RH problem for S with $m = 2$.*

4.1. *First transformation: $\Phi \to T$.* Define

$$T(z) = \mathrm{diag}\left(r^{\frac{1}{3}}, 1, r^{-\frac{1}{3}}\right)\Phi(rz;r)e^{-\Theta(rz)}. \tag{4.1}$$

The jumps for T on $(-x_m, x_m)$ are given by

$$T_+(z) = T_-(z)\begin{pmatrix} e^{\theta_2(rz)-\theta_1(rz)} & s_j & s_j e^{\theta_2(rz)-\theta_3(rz)} \\ 0 & e^{\theta_1(rz)-\theta_2(rz)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad z \in (x_{j-1}, x_j),$$

$$T_+(z) = T_-(z)\begin{pmatrix} e^{\theta_{3,+}(rz)-\theta_{3,-}(rz)} & s_j e^{\theta_{2,-}(rz)-\theta_{2,+}(rz)} & s_j \\ 0 & 1 & 0 \\ 0 & 0 & e^{\theta_{3,-}(rz)-\theta_{3,+}(rz)} \end{pmatrix}, \quad z \in (-x_j, -x_{j-1}),$$

where $j = 1, \ldots, m$, $x_0 := 0$, and where we have used

$$\theta_{3,+}(z) = \theta_{2,-}(z), \quad \theta_{1,+}(z) = \theta_{3,-}(z), \quad \theta_{1,-}(z) = \theta_{2,+}(z), \qquad z < 0.$$

As $z \to \infty$, $\pm\mathrm{Im}\, z > 0$, we have

$$T(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2})\right)\mathrm{diag}\left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}}\right)L_\pm$$

where $T_1 = \frac{1}{r}\mathrm{diag}\left(r^{\frac{1}{3}}, 1, r^{-\frac{1}{3}}\right)\Phi_1 \mathrm{diag}\left(r^{-\frac{1}{3}}, 1, r^{\frac{1}{3}}\right)$.

4.2. *Second transformation: $T \to S$.* For each $j = 1, \ldots, m$, let $\gamma_{j,+}$ and $\gamma_{j,-}$ be open curves, lying in the upper and lower half planes respectively, starting at x_{j-1} and ending at x_j. We also orient the open curves $\gamma_{-j,+} := -\gamma_{j,-}$ and $\gamma_{-j,-} := -\gamma_{j,+}$ from $-x_j$ to $-x_{j-1}$. Let

$$\gamma_{m+1,+} := \Sigma_1^{(1)}, \quad \gamma_{m+1,-} := \Sigma_5^{(1)}, \quad \gamma_{-m-1,+} := \Sigma_2^{(1)}, \quad \gamma_{-m-1,-} := \Sigma_4^{(1)},$$

$s_{m+1} := 1$, $x_{m+1} := +\infty$ and for $j = 1, \ldots, m, m+1$, define

$$J_{\gamma_{j,-}}(z) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{e^{\theta_1(rz)-\theta_2(rz)}}{s_j} & 1 & -e^{\theta_1(rz)-\theta_3(rz)} \\ 0 & 0 & 1 \end{pmatrix}, \quad J_{\gamma_{j,+}}(z) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{e^{\theta_2(rz)-\theta_1(rz)}}{s_j} & 1 & e^{\theta_2(rz)-\theta_3(rz)} \\ 0 & 0 & 1 \end{pmatrix},$$

$$J_{\gamma_{-j,+}}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{e^{\theta_3(rz)-\theta_1(rz)}}{s_j} & e^{\theta_3(rz)-\theta_2(rz)} & 1 \end{pmatrix}, \quad J_{\gamma_{-j,-}}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{e^{\theta_3(rz)-\theta_2(rz)}}{s_j} & -e^{\theta_3(rz)-\theta_1(rz)} & 1 \end{pmatrix}.$$

The next transformation is defined by

$$S(z)=T(z)\begin{cases} J_{\gamma_{j,-}}(z), & \text{Im}\, z<0 \text{ and } z \text{ above } \gamma_{j,-}, & j\in\{1,\dots,m\}, \\ J_{\gamma_{j,+}}(z)^{-1}, & \text{Im}\, z>0 \text{ and } z \text{ below } \gamma_{j,+}, & j\in\{1,\dots,m\}, \\ J_{\gamma_{-j,+}}(z)^{-1}, & \text{Im}\, z>0 \text{ and } z \text{ below } \gamma_{-j,+}, & j\in\{1,\dots,m\}, \\ J_{\gamma_{-j,-}}(z), & \text{Im}\, z<0 \text{ and } z \text{ above } \gamma_{-j,-}, & j\in\{1,\dots,m\}, \\ I, & \text{otherwise}. \end{cases} \tag{4.2}$$

S satisfies the following RH problem.

RH problem for S.

(a) $S:\mathbb{C}\setminus\Sigma_S\to\mathbb{C}^{3\times 3}$ is analytic, where $\Sigma_S:=(-\infty,+\infty)\cup\bigcup_{j=1}^{m+1}\big(\gamma_{j,-}\cup\gamma_{j,+}\cup\gamma_{-j,+}\cup\gamma_{-j,-}\big)$.

(b) For $z\in\Sigma_S\setminus\cup_{j=0}^{m}\{-x_j,x_j\}$, $S_+(z)=S_-(z)J_S(z)$, where

$$J_S(z)=J_{\gamma_{j,-}}(z), \quad z\in\gamma_{j,-}, \qquad J_S(z)=J_{\gamma_{j,+}}(z), \quad z\in\gamma_{j,+},$$

$$J_S(z)=J_{\gamma_{-j,+}}(z), \quad z\in\gamma_{-j,+}, \qquad J_S(z)=J_{\gamma_{-j,-}}(z), \quad z\in\gamma_{-j,-}$$

$$J_S(z)=\begin{pmatrix} 0 & s_j & 0 \\ -s_j^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z\in(x_{j-1},x_j), \qquad J_S(z)=\begin{pmatrix} 0 & 0 & s_j \\ 0 & 1 & 0 \\ -s_j^{-1} & 0 & 0 \end{pmatrix}, \quad z\in(-x_j,-x_{j-1}),$$

where $j=1,\dots,m,m+1$.

(c) As $z\to\infty$, $\pm\text{Im}\, z>0$, we have

$$S(z)=\left(I+\frac{T_1}{z}+\mathcal{O}(z^{-2})\right)\text{diag}\left(z^{-\frac{1}{3}},1,z^{\frac{1}{3}}\right)L_\pm.$$

(d) As $z\to z_\star\in\cup_{j=1}^{m}\{-x_j,x_j\}$, we have $S(z)=\mathcal{O}(\log(z-z_\star))$.
As $z\to 0$, $S(z)=\mathcal{O}(1)$.

(e) S satisfies the symmetry $S(z)=-\text{diag}\,(1,-1,1)S(-z)\mathcal{B}$, where $\mathcal{B}$ is defined in (2.22).

4.3. *Global parametrix.* Using the definitions of $\theta_1,\theta_2,\theta_3$ given in (2.6), it is easily checked that $J_S(z)\to I$ as $r\to\infty$ for each $z\in\cup_{j=1}^{m+1}\big(\gamma_{j,-}\cup\gamma_{j,+}\cup\gamma_{-j,+}\cup\gamma_{-j,-}\big)$. The following RH problem, whose solution is denoted N and called the global parametrix, has the same jump conditions on $(-\infty,+\infty)$ than the RH problem for S, and no other jumps. We will show in Subsection 4.7 that N is a good approximation to S outside small neighborhoods of $\cup_{j=0}^{m}\{-x_j,x_j\}$. The RH problem for N is as follows.

RH problem for N.

(a) $N:\mathbb{C}\setminus(-\infty,+\infty)\to\mathbb{C}^{3\times 3}$ is analytic.

(b) N satisfies the following jump relations:

$$N_+(z)=N_-(z)\begin{pmatrix} 0 & 0 & s_j \\ 0 & 1 & 0 \\ -s_j^{-1} & 0 & 0 \end{pmatrix}, \qquad z\in(-x_j,-x_{j-1}), \quad j=1,\dots,m,m+1,$$

$$N_+(z)=N_-(z)\begin{pmatrix} 0 & s_j & 0 \\ -s_j^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad z\in(x_{j-1},x_j), \qquad j=1,\dots,m,m+1.$$

(c) As $z \to \infty$, $\pm\mathrm{Im}\, z > 0$, we have

$$N(z) = \left(I + \frac{1}{z}N_1 + \mathcal{O}(z^{-2})\right)\mathrm{diag}\,(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}})L_{\pm}, \tag{4.3}$$

for a certain matrix N_1.

(d) As $z \to z_\star \in \cup_{j=1}^m \{-x_j, x_j\}$, $N(z) = \mathcal{O}(1)$.
As $z \to 0$, $N(z) = \mathcal{O}(1)\mathrm{diag}\,(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}})\mathcal{O}(1)$.

(e) N satisfies the symmetry $N(z) = -\mathrm{diag}\,(1,-1,1)N(-z)\mathcal{B}$.

For $m = 1$, the above RH problem was solved explicitly in [24, Section 5.3]. Let us define

$$\beta_j := \frac{1}{2\pi i}u_j = \frac{1}{2\pi i}\log\frac{s_j}{s_{j+1}}, \qquad j = 1, \ldots, m. \tag{4.4}$$

Inspired by [24], we consider three functions d_1, d_2, d_3 defined by

$$d_1(z) = \begin{cases} \lambda(z^{\frac{1}{3}}), & \mathrm{Im}\, z > 0, \\ \lambda(\omega^{-1}z^{\frac{1}{3}}), & \mathrm{Im}\, z < 0, \end{cases} \quad d_2(z) = \begin{cases} \lambda(\omega^{-1}z^{\frac{1}{3}}), & \mathrm{Im}\, z > 0, \\ \lambda(z^{\frac{1}{3}}), & \mathrm{Im}\, z < 0, \end{cases} \quad d_3(z) = \lambda(\omega z^{\frac{1}{3}}),$$

where

$$\lambda(z) = \prod_{j=1}^m \left(\frac{z^2 - \omega x_j^{2/3}}{z^2 - x_j^{2/3}}\right)^{\beta_j}, \qquad z \in \mathbb{C} \setminus \left((-x_m^{\frac{1}{3}}, x_m^{\frac{1}{3}}) \cup \omega^{-1}(-x_m^{\frac{1}{3}}, x_m^{\frac{1}{3}})\right). \tag{4.5}$$

The branch structure for λ is such that $\lambda(z) = 1 + \mathcal{O}(z^{-1})$ as $z \to \infty$, and

$$\lambda_+(z) = \lambda_-(z)\begin{cases} s_j, & z \in (-x_j^{\frac{1}{3}}, -x_{j-1}^{\frac{1}{3}}) \cup e^{-\frac{2\pi i}{3}}(x_{j-1}^{\frac{1}{3}}, x_j^{\frac{1}{3}}), \quad j \in \{1, \ldots, m\}, \\ s_j^{-1}, & z \in e^{\frac{\pi i}{3}}(x_{j-1}^{\frac{1}{3}}, x_j^{\frac{1}{3}}) \cup (x_{j-1}^{\frac{1}{3}}, x_j^{\frac{1}{3}}), \quad j \in \{1, \ldots, m\}, \end{cases} \tag{4.6}$$

where the boundary values λ_+ and λ_- are taken with respect to the orientation of the contour as stated in (4.5). Using (4.5)–(4.6), it can be verified that

$$N(z) := C_N \mathrm{diag}\,(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}})L_\pm \mathrm{diag}\,(d_1(z), d_2(z), d_3(z)), \qquad \pm\mathrm{Im}\, z > 0, \tag{4.7}$$

is the unique solution to the RH problem for N, where

$$C_N := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i\sqrt{3}\sum_{j=1}^m \beta_j x_j^{2/3} & 0 & 1 \end{pmatrix}. \tag{4.8}$$

It the next subsections we compute more detailed asymptotic expansions than those stated in conditions (c) and (d) of the RH problem for N.

4.3.1. *Asymptotics of $N(z)$ as $z \to \infty$.* As $z \to \infty$, $\pm\mathrm{Im}\, z > 0$, we have

$$N(z) = \left(I + \frac{1}{z}N_1 + \mathcal{O}(z^{-2})\right)\mathrm{diag}\,(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}})L_\pm \tag{4.9}$$

where N_1 is of the form

$$N_1 = \begin{pmatrix} 0 & i\sqrt{3}\sum_{j=1}^m \beta_j x_j^{2/3} & 0 \\ \star & 0 & i\sqrt{3}\sum_{j=1}^m \beta_j x_j^{2/3} \\ 0 & \star & 0 \end{pmatrix}. \tag{4.10}$$

The entries $(N_1)_{21}$ and $(N_1)_{32}$ can also be computed explicitly, but their expressions are longer and not important for us.

4.3.2. *Asymptotics of $N(z)$ as $z \to x_j$, $j=1,\ldots,m$.* As $z \to x_j$, $\mathrm{Im}\, z > 0$, $j=1,\ldots,m$,

$$\begin{aligned}
d_1(z) &= d_{1,x_j}^{(0)}(z-x_j)^{-\beta_j}\big(1+d_{1,x_j}^{(1)}(z-x_j)+\mathcal{O}((z-x_j)^2)\big),\\
d_{1,x_j}^{(0)} &= \left(\frac{3\sqrt{3}x_j}{2}\right)^{\beta_j} e^{-\frac{\pi i\beta_j}{6}} \prod_{\substack{k=1\\k\neq j}}^{m} \frac{(x_j^{2/3}-\omega x_k^{2/3})^{\beta_k}}{(x_j^{2/3}-x_k^{2/3})_+^{\beta_k}},\\
d_{1,x_j}^{(1)} &= \frac{(\omega-5)\beta_j}{6(\omega-1)x_j} + \sum_{\substack{k=1\\k\neq j}}^{m} \frac{2(\omega-1)x_k^{2/3}\beta_k}{3x_j^{1/3}(x_j^{2/3}-x_k^{2/3})(x_j^{2/3}-\omega x_k^{2/3})},\\
d_2(z) &= d_{2,x_j}^{(0)}(z-x_j)^{\beta_j}\big(1+d_{2,x_j}^{(1)}(z-x_j)+\mathcal{O}((z-x_j)^2)\big),\\
d_{2,x_j}^{(0)} &= \left(\frac{2}{3\sqrt{3}x_j}\right)^{\beta_j} e^{-\frac{\pi i\beta_j}{6}} \prod_{\substack{k=1\\k\neq j}}^{m} \frac{(x_j^{2/3}-x_k^{2/3})_+^{\beta_k}}{(x_j^{2/3}-\omega^2 x_k^{2/3})^{\beta_k}},\\
d_{2,x_j}^{(1)} &= \frac{(1-5\omega)\beta_j}{6(\omega-1)x_j} + \sum_{\substack{k=1\\k\neq j}}^{m} \frac{2(1-\omega^2)x_k^{2/3}\beta_k}{3x_j^{1/3}(x_j^{2/3}-x_k^{2/3})(x_j^{2/3}-\omega^2 x_k^{2/3})},\\
d_3(z) &= d_{3,x_j}^{(0)}\big(1+d_{3,x_j}^{(1)}(z-x_j)+\mathcal{O}((z-x_j)^2)\big),\\
d_{3,x_j}^{(0)} &= e^{\frac{\pi i\beta_j}{3}} \prod_{\substack{k=1\\k\neq j}}^{m} \frac{(x_j^{2/3}-\omega^2 x_k^{2/3})^{\beta_k}}{(x_j^{2/3}-\omega x_k^{2/3})^{\beta_k}},\\
d_{3,x_j}^{(1)} &= \frac{2(\omega+1)\beta_j}{3(\omega-1)x_j} + \sum_{\substack{k=1\\k\neq j}}^{m} \frac{2(\omega^2-\omega)x_k^{2/3}\beta_k}{3x_j^{1/3}(x_j^{2/3}-\omega x_k^{2/3})(x_j^{2/3}-\omega^2 x_k^{2/3})}.
\end{aligned}$$

In the above asymptotic expansions, all branches are the principal ones: for example, for the product appearing in $d_{1,x_j}^{(0)}$, we have

$$\left|\prod_{\substack{k=1\\k\neq j}}^{m} \frac{(x_j^{2/3}-\omega x_k^{2/3})^{\beta_k}}{(x_j^{2/3}-x_k^{2/3})_+^{\beta_k}}\right| = \exp\left(-\sum_{\substack{k=1\\k\neq j}}^{m} i\beta_k \arctan\frac{\sqrt{3}x_k^{2/3}}{x_k^{2/3}+2x_j^{2/3}} - \sum_{k=j+1}^{m}\pi i\beta_k\right),$$

$$\arg\prod_{\substack{k=1\\k\neq j}}^{m} \frac{(x_j^{2/3}-\omega x_k^{2/3})^{\beta_k}}{(x_j^{2/3}-x_k^{2/3})_+^{\beta_k}} = -\sum_{\substack{k=1\\k\neq j}}^{m} i\beta_k \log\frac{|x_j^{2/3}-\omega x_k^{2/3}|}{|x_j^{2/3}-x_k^{2/3}|} \mod 2\pi.$$

Hence, as $z \to x_j$, $\mathrm{Im}\, z > 0$, $j=1,\ldots,m$,

$$N(z) = \big(N_{x_j}^{(0)} + (z-x_j)N_{x_j}^{(1)} + \mathcal{O}\big((z-x_j)^2\big)\big)(z-x_j)^{-\beta_j\sigma_{3,1}}, \tag{4.11}$$

where $\sigma_{3,1} = \mathrm{diag}\,(1,-1,0)$ and

$$\begin{aligned}
N_{x_j}^{(0)} &= C_N \mathrm{diag}\,(x_j^{-1/3},1,x_j^{1/3}) L_+ \mathrm{diag}\,(d_{1,x_j}^{(0)}, d_{2,x_j}^{(0)}, d_{3,x_j}^{(0)}),\\
N_{x_j}^{(1)} &= C_N\Big(\mathrm{diag}\,(x_j^{-1/3},1,x_j^{1/3}) L_+ \mathrm{diag}\,(d_{1,x_j}^{(0)}d_{1,x_j}^{(1)}, d_{2,x_j}^{(0)}d_{2,x_j}^{(1)}, d_{3,x_j}^{(0)}d_{3,x_j}^{(1)})\\
&\qquad + \mathrm{diag}\,(-\tfrac{1}{3}x_j^{-4/3}, 0, \tfrac{1}{3}x_j^{-2/3}) L_+ \mathrm{diag}\,(d_{1,x_j}^{(0)}, d_{2,x_j}^{(0)}, d_{3,x_j}^{(0)})\Big).
\end{aligned} \tag{4.12}$$

4.3.3. *Asymptotics of $N(z)$ as $z \to 0$.* As $z \to 0$, $\mathrm{Im}\, z > 0$,

$$d_\ell(z) = d_{\ell,0}^{(0)}(1 + d_{\ell,0}^{(1)} z^{\frac{2}{3}} + \mathcal{O}(z^{\frac{4}{3}})), \qquad \ell = 1,2,3,$$

$$d_{1,0}^{(0)} = e^{-\pi i \sum_{k=1}^m \beta_k}(-\omega)^{\sum_{k=1}^m \beta_k} = s_1^{-\frac{2}{3}}, \qquad d_{2,0}^{(0)} = d_{3,0}^{(0)} = \omega^{\sum_{k=1}^m \beta_k} = s_1^{\frac{1}{3}},$$

$$d_{1,0}^{(1)} = (1-\omega^2)\sum_{k=1}^m \beta_k x_k^{-\frac{2}{3}}, \quad d_{2,0}^{(1)} = (\omega - 1)\sum_{k=1}^m \beta_k x_k^{-\frac{2}{3}}, \quad d_{1,0}^{(1)} = (\omega^2 - \omega)\sum_{k=1}^m \beta_k x_k^{-\frac{2}{3}},$$

where the branches are the principal ones. As $z \to 0$, $\mathrm{Im}\, z > 0$,

(4.13)

$$N(z) = C_N \mathrm{diag}\,(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}}) L_+ \mathrm{diag}\,(d_{1,0}^{(0)}, d_{2,0}^{(0)}, d_{3,0}^{(0)})\Big(I + z^{\frac{2}{3}}\mathrm{diag}\,(d_{1,0}^{(1)}, d_{2,0}^{(1)}, d_{3,0}^{(1)}) + \mathcal{O}(z^{\frac{4}{3}})\Big).$$

Local parametrices. For each $p \in \{-x_m, \dots, -x_1, 0, x_1, \dots, x_m\}$, we let $\mathcal{D}_p$ be a small open disk centered at p. The local parametrix $P^{(p)}$ is defined inside $\mathcal{D}_p$, has the same jumps as S in $\mathcal{D}_p$, and satisfies $S(z)P^{(p)}(z)^{-1} = \mathcal{O}(1)$ as $z \to p$. Furthermore, we require $P^{(p)}$ to satisfy the following matching condition with $P^{(\infty)}$ on $\partial\mathcal{D}_p$:

$$P^{(p)}(z) = (I + o(1))P^{(\infty)}(z), \qquad \text{as } r \to +\infty, \tag{4.14}$$

uniformly for $z \in \partial\mathcal{D}_p$.

4.4. *Local parametrix near x_j, $j = 1, \dots, m$.* Since the (1,2) and (2,1) entries of $J_S(z)$ have each a discontinuity at $z = x_j$, we can follow [36] and build $P^{(x_j)}$ using the model RH problem Φ_{HG} which is presented in Appendix A. We also refer to [24, Section 5.5] for more details about this construction. The local parametrix $P^{(x_j)}$ is of the form

$$P^{(x_j)}(z) = E_{x_j}(z)\begin{pmatrix} \Phi_{\mathrm{HG},11}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & \Phi_{\mathrm{HG},12}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & 0 \\ \Phi_{\mathrm{HG},21}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & \Phi_{\mathrm{HG},22}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.15}$$
$$\times (s_j s_{j+1})^{-\frac{\sigma_{3,1}}{4}} e^{\pm\frac{1}{2}(\theta_2(rz)-\theta_1(rz))\sigma_{3,1}} A_{x_j}(z),$$

$$A_{x_j}(z) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{\theta_2(rz)-\theta_3(rz)} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \{z : \mathrm{Im}\, z > 0\} \cap \mathcal{D}_{x_j} \setminus (\Omega_{j,+} \cup \Omega_{j+1,+}), \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{\theta_1(rz)-\theta_3(rz)} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \{z : \mathrm{Im}\, z < 0\} \cap \mathcal{D}_{x_j} \setminus (\Omega_{j,-} \cup \Omega_{j+1,-}), \\ I, & \text{otherwise}, \end{cases}$$

where $\sigma_{3,1} = \mathrm{diag}\,(1,-1,0)$, $\pm$ stands for $\pm\mathrm{Im}\, z > 0$, E_{x_j} is analytic in $\mathcal{D}_{x_j}$ and given by

$$E_{x_j}(z) = N(z)(s_j s_{j+1})^{\frac{\sigma_{3,1}}{4}} \begin{cases} \sqrt{\frac{s_{j+1}}{s_j}}^{\sigma_{3,1}}, & \mathrm{Im}\, z > 0 \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \mathrm{Im}\, z < 0 \end{cases} e^{-\frac{1}{2}(\theta_2(rx_j)-\theta_1(rx_j))\sigma_{3,1}}(r^{\frac{4}{3}} f_{x_j}(z))^{\beta_j \sigma_{3,1}},$$

and f_{x_j} is given by

$$f_{x_j}(z) = r^{-\frac{4}{3}}\left[(\theta_2(rz) - \theta_1(rz)) - (\theta_2(rx_j) - \theta_1(rx_j))\right] = \frac{i\sqrt{3}}{4}\Big(3(z^{\frac{4}{3}} - x_j^{\frac{4}{3}}) - \frac{2\rho}{r^{\frac{2}{3}}}(z^{\frac{2}{3}} - x_j^{\frac{2}{3}})\Big).$$

It is easily checked that $A_{x_j}(z)$ is exponentially small as $r \to +\infty$ uniformly for $z \in \mathcal{D}_{x_j}$, and that

(4.16)

$$f_{x_j}(x_j) = 0,\; f'_{x_j}(x_j) = i\left(\sqrt{3}x_j^{1/3} - \frac{\rho}{\sqrt{3}x_j^{1/3}r^{2/3}}\right),\; f''_{x_j}(x_j) = i\left(\frac{1}{\sqrt{3}x_j^{2/3}} + \frac{\rho}{3\sqrt{3}x_j^{4/3}r^{2/3}}\right).$$

Using (4.11), we obtain

$$E_{x_j}(x_j) = N_{x_j}^{(0)}\sqrt{s_{j+1}}^{\sigma_{3,1}}e^{-\frac{1}{2}(\theta_2(rx_j)-\theta_1(rx_j))\sigma_{3,1}}(r^{\frac{4}{3}}|f'(x_j)|)^{\beta_j\sigma_{3,1}}, \tag{4.17}$$

$$E_{x_j}(x_j)^{-1}E'_{x_j}(x_j) = \begin{pmatrix} \frac{f''_{x_j}(x_j)}{2f'_{x_j}(x_j)}\beta_j + d_{1,x_j}^{(1)} & \frac{i\mathfrak{c}_j^{-2}}{3\sqrt{3}x_j}\frac{d_{2,x_j}^{(0)}}{d_{1,x_j}^{(0)}} & -\frac{i\mathfrak{c}_j^{-1}}{3\sqrt{3}x_j}\frac{d_{3,x_j}^{(0)}}{d_{1,x_j}^{(0)}} \\ -\frac{i\mathfrak{c}_j^{2}}{3\sqrt{3}x_j}\frac{d_{1,x_j}^{(0)}}{d_{2,x_j}^{(0)}} & -\frac{f''_{x_j}(x_j)}{2f'_{x_j}(x_j)}\beta_j + d_{2,x_j}^{(1)} & -\frac{i\mathfrak{c}_j}{3\sqrt{3}x_j}\frac{d_{3,x_j}^{(0)}}{d_{2,x_j}^{(0)}} \\ \frac{i\mathfrak{c}_j}{3\sqrt{3}x_j}\frac{d_{1,x_j}^{(0)}}{d_{3,x_j}^{(0)}} & \frac{i\mathfrak{c}_j^{-1}}{3\sqrt{3}x_j}\frac{d_{2,x_j}^{(0)}}{d_{3,x_j}^{(0)}} & d_{3,x_j}^{(1)} \end{pmatrix}, \tag{4.18}$$

$$\mathfrak{c}_j = \sqrt{s_{j+1}}e^{-\frac{1}{2}(\theta_2(rx_j)-\theta_1(rx_j))}\left(r^{\frac{4}{3}}|f'_{x_j}(x_j)|\right)^{\beta_j}. \tag{4.19}$$

Using (A.2), we obtain

(4.20)

$$P^{(x_j)}(z)N(z)^{-1} = I + \frac{1}{r^{\frac{4}{3}}f_{x_j}(z)}E_{x_j}(z)\begin{pmatrix} \Phi_{\mathrm{HG},1}(\beta_j)_{11} & \Phi_{\mathrm{HG},1}(\beta_j)_{12} & 0 \\ \Phi_{\mathrm{HG},1}(\beta_j)_{21} & \Phi_{\mathrm{HG},1}(\beta_j)_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}E_{x_j}(z)^{-1} + \mathcal{O}(r^{-\frac{8}{3}}),$$

as $r \to +\infty$ uniformly for $z \in \partial\mathcal{D}_{x_j}$.

4.5. *Local parametrix near* $-x_j$, $j = 1, \ldots, m$. $P^{(-x_j)}$ can be constructed in terms of Φ_{HG} in a similar way as $P^{(x_j)}$. Alternatively, we can use the symmetry stated in condition (e) of the RH problem for S. This observation saves us some effort and allows us to see immediately that

$$P^{(-x_j)}(z) = -\mathrm{diag}\,(1,-1,1)P^{(x_j)}(z)\mathcal{B}, \qquad z \in \mathcal{D}_{-x_j}. \tag{4.21}$$

4.6. *Local parametrix near* 0. For the local parametrix $P^{(0)}$, we need to use the model RH problem for Ψ from [8] (and which is recalled in Subsection 2.1 for the convenience of the reader). Define

$$P^{(0)}(z) = E_0(z)\Psi(rz)e^{-\Theta(rz)}\mathrm{diag}\,(s_1^{-\frac{2}{3}}, s_1^{\frac{1}{3}}, s_1^{\frac{1}{3}}), \tag{4.22}$$

where E_0 is analytic inside $\mathcal{D}_0$ and given by

$$E_0(z) = -\sqrt{\frac{3}{2\pi}}e^{-\frac{\rho^2}{6}}iN(z)\mathrm{diag}\,(s_1^{\frac{2}{3}}, s_1^{-\frac{1}{3}}, s_1^{-\frac{1}{3}})L_{\pm}^{-1}\mathrm{diag}\,((rz)^{\frac{1}{3}}, 1, (rz)^{-\frac{1}{3}})\Psi_0^{-1}, \quad \pm\mathrm{Im}\, z > 0,$$

and

$$\Psi_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \kappa_3 + \frac{2\rho}{3} & 0 & 1 \end{pmatrix}, \qquad \kappa_3 = \frac{\rho^3}{54} - \frac{\rho}{6}.$$

In a similar way as in [24, Proposition 5.13], we verify that $P^{(0)}$ has the same jumps as S inside $\mathcal{D}_0$. It is also clear from condition (d) of the RH problem for Ψ that $P^{(0)}(z)$ remains bounded as $z \to 0$. Finally, using (4.22) and (2.4), we obtain

(4.23)
$$P^{(0)}(z)N(z)^{-1} = I + \frac{1}{r^{\frac{2}{3}}z}\widehat{E}_0(z)\widehat{\Psi}_1\widehat{E}_0(z)^{-1} + \mathcal{O}(r^{-\frac{4}{3}}), \qquad \widehat{\Psi}_1 = \begin{pmatrix} 0 & \kappa_3 & 0 \\ 0 & 0 & \kappa_3 + \frac{\rho}{3} \\ 0 & 0 & 0 \end{pmatrix},$$

as $r \to +\infty$ uniformly for $z \in \partial\mathcal{D}_0$, where

$$\widehat{E}_0(z) := N(z)\mathrm{diag}\,(s_1^{\frac{2}{3}}, s_1^{-\frac{1}{3}}, s_1^{-\frac{1}{3}})L_{\pm}^{-1}\mathrm{diag}\,(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}}).$$

For future reference, using (4.13) we note that $\widehat{E}_0(0) = C_N$.

4.7. *Final transformation.* Define

$$R(z) = \begin{cases} S(z)N(z)^{-1}, & z \in \mathbb{C} \setminus \big(\cup_{j=1}^m (\mathcal{D}_{x_j} \cup \mathcal{D}_{-x_j}) \cup \mathcal{D}_0 \big), \\ S(z)P^{(p)}(z)^{-1}, & z \in \mathcal{D}_p,\ p \in \{-x_m, \ldots, -x_1, 0, x_1, \ldots, x_m\}. \end{cases} \tag{4.24}$$

Using the analysis of Subsections 4.4–4.6, we conclude that R is analytic in $\cup_{j=1}^m(\mathcal{D}_{x_j} \cup \mathcal{D}_{-x_j}) \cup \mathcal{D}_0$, and therefore R is analytic in $\mathbb{C} \setminus \Sigma_R$, where

$$\Sigma_R := \cup_{j=1}^m(\partial\mathcal{D}_{x_j} \cup \partial\mathcal{D}_{-x_j}) \cup \partial\mathcal{D}_0 \cup \Sigma_S \setminus \big(\cup_{j=1}^m (\mathcal{D}_{x_j} \cup \mathcal{D}_{-x_j}) \cup \mathcal{D}_0 \cup \mathbb{R} \big).$$

For convenience, we orient the boundaries of the $2m+1$ disks in the *clockwise* direction, and for $z \in \Sigma_R$, we define $J_R(z) := R_-^{-1}(z)R_+(z)$. Using the definitions (2.6) of $\theta_1, \theta_2, \theta_3$, condition (b) of the RH problem for S and (4.24), we verify that

$$J_R(z) = I + \mathcal{O}(e^{-cr}), \qquad \text{as } r \to +\infty \text{ uniformly for } z \in \Sigma_R \setminus \big(\cup_{j=1}^m (\mathcal{D}_{x_j} \cup \mathcal{D}_{-x_j}) \cup \mathcal{D}_0 \big)$$

for a certain $c > 0$. Also, by (4.20), (4.21) and (4.23), we have

$$J_R(z) = I + \mathcal{O}(r^{-\frac{4}{3}}), \qquad \text{as } r \to +\infty, \text{ uniformly for } z \in \cup_{j=1}^m(\partial\mathcal{D}_{x_j} \cup \partial\mathcal{D}_{-x_j}),$$
$$J_R(z) = I + \frac{J^{(1)}(z)}{r^{2/3}} + \mathcal{O}(r^{-\frac{4}{3}}), \quad \text{as } r \to +\infty, \text{ uniformly for } z \in \partial\mathcal{D}_0,$$

where $J^{(1)}(z) = \frac{1}{z}\widehat{E}_0(z)\widehat{\Psi}_1\widehat{E}_0(z)^{-1}$. Hence, R satisfies a small norm RH problem [26], and we have

$$R(z) = I + R^{(1)}(z)r^{-\frac{2}{3}} + \mathcal{O}(r^{-\frac{4}{3}}), \qquad \text{as } r \to +\infty, \tag{4.25}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, with

$$R^{(1)}(z) = \frac{1}{2\pi i}\oint_{\partial\mathcal{D}_0} \frac{J^{(1)}(x)dx}{x - z} = \begin{cases} \frac{C_N\widehat{\Psi}_1 C_N^{-1}}{z}, & z \notin \mathcal{D}_0, \\ \frac{C_N\widehat{\Psi}_1 C_N^{-1}}{z} - J^{(1)}(z), & z \in \mathcal{D}_0, \end{cases} \tag{4.26}$$

and where we recall that $\partial\mathcal{D}_0$ is oriented in the clockwise direction. (the method of [26] also implies that R exists for all sufficiently large r; note however that here we already know from (2.15) and from $\det(1 - \widetilde{\mathcal{K}}_\rho^{\mathrm{Pe}}) > 0$ that Y exists for *all* $r > 0$, which implies by the transformations $Y \to \Phi \to T \to S \to R$ that R also exists for *all* $r > 0$). Finally, the same analysis as in [20, Section 3.5] shows that for any $k_1, \ldots, k_m \in \mathbb{N}_{\geq 0}$ with $k_1 + \ldots + k_m \geq 1$, we have

(4.27)
$$\partial_{u_1}^{k_1} \ldots \partial_{u_m}^{k_m} R(z) = \partial_{u_1}^{k_1} \ldots \partial_{u_m}^{k_m} R^{(1)}(z)r^{-\frac{2}{3}} + \mathcal{O}((\log r)^{k_1+\ldots+k_m} r^{-\frac{4}{3}}), \qquad \text{as } r \to +\infty,$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$.

5. Asymptotic analysis of Φ as $r \to 0$. The analysis of Φ as $r \to 0$ is much simpler than the large r analysis of Section 4. Here we generalize [24, Section 6] to arbitrary $m \in \mathbb{N}_{>0}$. Let $\delta > 0$ be fixed. Define

(5.1)
$$\widetilde{N}(z) := -\sqrt{\tfrac{3}{2\pi}} e^{-\frac{\rho^2}{6}} i \Psi_0^{-1} \Psi(z) \times \begin{cases} J_1, & \text{for } \arg z < \frac{\pi}{4} \text{ and } \arg(z - r x_m) > \frac{\pi}{4}, \\ J_2, & \text{for } \arg z > \frac{3\pi}{4} \text{ and } \arg(z + r x_m) < \frac{3\pi}{4}, \\ J_4^{-1}, & \text{for } \arg z < -\frac{3\pi}{4} \text{ and } \arg(z + r x_m) > -\frac{3\pi}{4}, \\ J_5^{-1}, & \text{for } \arg z > -\frac{\pi}{4} \text{ and } \arg(z - r x_m) < -\frac{\pi}{4}, \end{cases}$$

and for $|z| < \delta$, define

(5.2)
$$\widetilde{P}^{(0)}(z) := -\sqrt{\tfrac{3}{2\pi}} e^{-\frac{\rho^2}{6}} i \Psi_0^{-1} \widetilde{\Psi}(z)$$
$$\left(I + \sum_{j=1}^{m} \frac{s_j - s_{j+1}}{2\pi i} \Big(\log(z - r x_j) - \log(z + r x_j) \Big) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{cases} J_1^{-1}, & z \in \mathrm{I}, \\ I, & z \in \mathrm{II}, \\ J_2^{-1}, & z \in \mathrm{III}, \\ J_2^{-1} J_3^{-1}, & z \in \mathrm{IV}, \\ J_1^{-1} J_0^{-1} J_5^{-1}, & z \in \mathrm{V}, \\ J_1^{-1} J_0^{-1}, & z \in \mathrm{VI}, \end{cases}$$

where the principal branches are chosen for the log, and we recall that $s_{m+1} := 1$, the regions I, II, III, IV, V and VI are shown in Figure 2, $\widetilde{\Psi}$ is defined in (2.8), and the matrices J_j, $j = 0, \ldots, 5$ are defined in (2.3). Let

$$\widetilde{R}(z) := \begin{cases} \Phi(z) \widetilde{N}(z)^{-1}, & |z| > \delta, \\ \Phi(z) \widetilde{P}^{(0)}(z)^{-1}, & |z| < \delta. \end{cases} \tag{5.3}$$

The definitions (5.1) and (5.2) also ensure that $\widetilde{R}$ has no jumps on $\cup_{j=0}^{5} \Sigma_j^{(r)}$. Since

$$\widetilde{P}_-^{(0)}(z)^{-1} \widetilde{P}_+^{(0)}(z) = J_5 J_0 J_1 \begin{pmatrix} 1 & s_j - 1 & s_j - 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s_j & s_j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Phi_-(z)^{-1} \Phi_+(z) \tag{5.4}$$

holds for all $z \in (-r x_j, -r x_{j-1}) \cup (r x_{j-1}, r x_j)$, it follows that $\widetilde{R}(z)$ is also analytic in $(-r x_m, r x_m) \setminus \cup_{j=1}^{m-1} \{-r x_j, r x_j\}$. Furthermore, from a direct inspection of (5.2) and (5.3), we see that the singularities of $\widetilde{R}(z)$ at $z = -r x_m, \ldots, -r x_1, 0, r x_1, \ldots, r x_m$ are removable. Hence, $\widetilde{R}(z)$ is analytic for $z \in \mathbb{C} \setminus \{z : |z| = \delta\}$. Let us orient the circle $|z| = \delta$ in the clockwise direction, and define $J_{\widetilde{R}} := \widetilde{R}_-^{-1} \widetilde{R}_+$. By (5.3), $J_{\widetilde{R}}(z) = \widetilde{P}^{(0)}(z) \widetilde{N}(z)^{-1}$, and by (2.4), (2.19) and (5.3), $\widetilde{R}(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$. We also check using (5.1) and (5.2) that $J_{\widetilde{R}}(z) = I + \mathcal{O}(r)$ as $r \to 0$ uniformly for $|z| = \delta$. Thus

$$\widetilde{R}(z) = I + \mathcal{O}(r), \qquad \text{as } r \to 0 \text{ uniformly for } z \in \mathbb{C} \setminus \{z : |z| = \delta\}. \tag{5.5}$$

6. Proof of the main results.

6.1. *Proof of Theorem 1.1.* We already proved in Section 3 that the functions $(p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^{m})$ defined by (3.1)–(3.2) exist and satisfy (1.4)–(1.5). In this subsection we complete the proof of Theorem 1.1 by obtaining the asymptotic formulas (1.8) and (1.11).

Asymptotics of p_0, q_0, $p_{j,k}$ *and* $q_{j,k}$ *as* $r \to +\infty$. We first compute the asymptotics of p_0 and q_0 using (3.1). Using (4.25) and (4.26), we see that $R(z) = I + \frac{R_1}{z} + \mathcal{O}(z^{-2})$ as $z \to \infty$, where

$$R_1 = C_N \widehat{\Psi}_1 C_N^{-1} r^{-\frac{2}{3}} + \mathcal{O}(r^{-\frac{4}{3}}), \qquad \text{as } r \to +\infty. \tag{6.1}$$

Inverting the transformations $\Phi \mapsto T \mapsto S \mapsto R$ in the region outside the disks using (4.1), (4.2) and (4.24), we get

$$\Phi(rz) = \mathrm{diag}\,(r^{-\frac{1}{3}}, 1, r^{\frac{1}{3}}) R(z) N(z) e^{\Theta(rz)}, \qquad z \in \mathbb{C} \setminus \left(\cup_{j=1}^m \left(\mathcal{D}_{x_j} \cup \mathcal{D}_{-x_j}\right) \cup \mathcal{D}_0 \right).$$

From this expression, (2.19) and (4.9), we deduce that

$$\Phi_1 = r\, \mathrm{diag}\,(r^{-\frac{1}{3}}, 1, r^{\frac{1}{3}})(R_1 + N_1)\mathrm{diag}\,(r^{\frac{1}{3}}, 1, r^{-\frac{1}{3}}),$$

which implies by (4.10) and (6.1) that

$$\Phi_{1,12} = i\sqrt{3} r^{\frac{2}{3}} \sum_{j=1}^m \beta_j x_j^{\frac{2}{3}} + \frac{\rho^3}{54} - \frac{\rho}{6} + \mathcal{O}(r^{-\frac{2}{3}}),\ \Phi_{1,23} = i\sqrt{3} r^{\frac{2}{3}} \sum_{j=1}^m \beta_j x_j^{\frac{2}{3}} + \frac{\rho^3}{54} + \frac{\rho}{6} + \mathcal{O}(r^{-\frac{2}{3}}),$$

as $r \to +\infty$. Substituting these asymptotics in (3.1) gives the large r asymptotics of p_0 and q_0 stated in (1.8a) and (1.8e).

We now compute the large r asymptotics of $p_{j,k}$, $j = 1, \ldots, m$, $k = 1, 2, 3$ using the definitions (3.2). It follows from (2.20) and (2.21) that

$$\Phi_j^{(0)}(r) = \lim_{\substack{z \to x_j \\ z \in \mathrm{II}}} \Phi(rz) \begin{pmatrix} 1 & \mathfrak{s}_j \log(rz - rx_j) & \mathfrak{s}_j \log(rz - rx_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.2}$$

For $z \in \mathcal{D}_{x_j}$ and z outside the lenses, by (4.1), (4.2) and (4.24) we have

$$\Phi(rz) = \mathrm{diag}\,(r^{-\frac{1}{3}}, 1, r^{\frac{1}{3}}) R(z) P^{(x_j)}(z) e^{\Theta(rz)}. \tag{6.3}$$

Hence, using also (4.15), we get

$$\Phi_j^{(0)}(r) = \mathrm{diag}\,(r^{-\frac{1}{3}}, 1, r^{\frac{1}{3}}) R(x_j) E_{x_j}(x_j) \lim_{\substack{z \to x_j \\ z \in \mathrm{II}}} \Bigg[\begin{pmatrix} \Phi_{\mathrm{HG},11}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & \Phi_{\mathrm{HG},12}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & 0 \\ \Phi_{\mathrm{HG},21}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & \Phi_{\mathrm{HG},22}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & 0 \\ 0 & 0 & 1 \end{pmatrix} (s_j s_{j+1})^{-\frac{\sigma_{3,1}}{4}} \widetilde{\Theta}(z) \begin{pmatrix} 1 & \mathfrak{s}_j \log(rz - rx_j) & \mathfrak{s}_j \log(rz - rx_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Bigg] \tag{6.4}$$

where

$$\widetilde{\Theta}(z) = e^{\frac{1}{2}(\theta_2(rz) - \theta_1(rz))\sigma_{3,1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{\theta_2(rz) - \theta_3(rz)} \\ 0 & 0 & 1 \end{pmatrix} e^{\Theta(rz)} = e^{-\frac{\theta_3(rz)}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & e^{\frac{3}{2}\theta_3(rz)} \end{pmatrix}. \tag{6.5}$$

Using (4.4), we note that

$$\frac{\sin(\pi\beta_j)}{\pi} = \frac{1}{\Gamma(\beta_j)\Gamma(1-\beta_j)} = -\frac{\mathfrak{s}_j}{\sqrt{s_j s_{j+1}}}.$$

Therefore, using (A.4), we can rewrite (6.4) as

$$\Phi_j^{(0)}(r) = e^{-\frac{\theta_3(rx_j)}{2}} \operatorname{diag}(r^{-\frac{1}{3}}, 1, r^{\frac{1}{3}}) R(x_j) E_{x_j}(x_j) \tag{6.6}$$
$$\times \operatorname{diag}(\Upsilon_j^{(0)}, 1) \begin{pmatrix} 1 & \mathfrak{s}_j(\log r - \log(r^{\frac{4}{3}}|f'_{x_j}(x_j)|i)) & \mathfrak{s}_j(\log r - \log(r^{\frac{4}{3}}|f'_{x_j}(x_j)|i)) \\ 0 & 1 & 1 \\ 0 & 0 & e^{\frac{3}{2}\theta_3(rx_j)} \end{pmatrix},$$

where

$$\Upsilon_j^{(0)} = \begin{pmatrix} \frac{\Gamma(1-\beta_j)}{(s_j s_{j+1})^{1/4}} & \frac{(s_j s_{j+1})^{1/4}}{\Gamma(\beta_j)} \left(\frac{\Gamma'(1-\beta_j)}{\Gamma(1-\beta_j)} + 2\gamma_{\mathrm{E}} - i\pi \right) \\ \frac{\Gamma(1+\beta_j)}{(s_j s_{j+1})^{1/4}} & \frac{-(s_j s_{j+1})^{1/4}}{\Gamma(-\beta_j)} \left(\frac{\Gamma'(-\beta_j)}{\Gamma(-\beta_j)} + 2\gamma_{\mathrm{E}} - i\pi \right) \end{pmatrix}, \tag{6.7}$$

and γ_{E} is Euler's gamma constant. Combining (6.6) with (3.2), we get

$$\begin{pmatrix} q_{j,1} & q_{j,2} & q_{j,3} \end{pmatrix}^t = e^{-\frac{\theta_3(rx_j)}{2}} \operatorname{diag}(r^{-\frac{1}{3}}, 1, r^{\frac{1}{3}}) R(x_j) E_{x_j}(x_j) \begin{pmatrix} \Upsilon_{j,11}^{(0)} & \Upsilon_{j,21}^{(0)} & 0 \end{pmatrix}^t,$$

$$\begin{pmatrix} p_{j,1} & p_{j,2} & p_{j,3} \end{pmatrix}^t = -e^{\frac{\theta_3(rx_j)}{2}} \mathfrak{s}_j \operatorname{diag}(r^{\frac{1}{3}}, 1, r^{-\frac{1}{3}}) R(x_j)^{-t} E_{x_j}(x_j)^{-t} \operatorname{diag}((\Upsilon_j^{(0)})^{-t}, 1) \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^t.$$

We then obtain (1.8b), (1.8c), (1.8d), (1.8f), (1.8g) and (1.8h) after a long computation. We omit the details.

Asymptotics of p_0, q_0, $p_{j,k}$ *and* $q_{j,k}$ *as* $r \to 0$. Using (5.3) and (5.5), we have, for $|z| > \delta$,

$$\Phi(z) = \left(\frac{\sqrt{2\pi}}{\sqrt{3}} e^{\frac{\rho^2}{6}} i\right)^{-1} (I + \mathcal{O}(r)) \Psi_0^{-1} \Psi(z), \qquad \text{as } r \to 0.$$

Using also (2.4) and (3.1), we obtain (1.11a).

We now compute the asymptotics of $p_{j,k}$ and $q_{j,k}$, $j = 1, \dots, m$, $k = 1, 2, 3$ as $r \to 0$. Recall that $p_{j,k}$ and $q_{j,k}$ are defined in (3.2). Using (5.2) and (5.3), for $z \in \mathrm{II} \cap \{z : |z| < \delta r^{-1}\}$ we get

$$\Phi(rz) = \widetilde{R}(rz) \frac{\Psi_0^{-1}}{\frac{\sqrt{2\pi}}{\sqrt{3}} e^{\frac{\rho^2}{6}} i} \widetilde{\Psi}(rz) \left(I - \sum_{j=1}^m \mathfrak{s}_j \Big(\log(rz - rx_j) - \log(rz + rx_j) \Big) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

It then follows from (6.2) that

$$\Phi_j^{(0)}(r) = \widetilde{R}(rx_j) \frac{\Psi_0^{-1}}{\frac{\sqrt{2\pi}}{\sqrt{3}} e^{\frac{\rho^2}{6}} i} \widetilde{\Psi}(rx_j) \left(I + \sum_{j=1}^m \mathfrak{s}_j \log(2rx_j) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

and then by (2.8) and (5.5), we get

$$\Phi_j^{(0)}(r) = \frac{\Psi_0^{-1}}{\frac{\sqrt{2\pi}}{\sqrt{3}} e^{\frac{\rho^2}{6}} i} (\widetilde{\Psi}(0) + \mathcal{O}(r)) \left(I + \sum_{j=1}^m \mathfrak{s}_j \log(2rx_j) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

On the other hand, by [24, equation (7.22)], we have

$$\widetilde{\Psi}(0) \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t = \begin{pmatrix} \mathcal{P}_0(0) & 0 & \mathcal{P}_0''(0) \end{pmatrix}^t, \qquad \widetilde{\Psi}(0)^{-t} \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^t = \begin{pmatrix} 0 & \frac{1}{\mathcal{P}_1'(0)} & 0 \end{pmatrix}^t,$$

where $\mathcal{P}_1'(0) \neq 0$. The asymptotics of (1.11b) and (1.11c) now directly follows from (3.2).

6.2. *Proof of Theorem 1.3*. The asymptotics of $H(r)$ as $r \to 0$ given by (1.13) are directly obtained from (1.6) and (1.11a)–(1.11c). Since $F(0\vec{x}, \vec{u}) = 1$ by (1.2), the integral representation (1.12) follows by integrating (3.19) from 0 to an arbitrary $r > 0$. To compute the asymptotics of $H(r)$ as $r \to +\infty$, we follow the method of [24] and rely on (3.20). Using (2.20) and (2.21), we obtain

$$\Phi_j^{(1)}(r) = \frac{1}{r}\Phi_j^{(0)}(r)^{-1} \lim_{\substack{z\to x_j \\ z\in \mathrm{II}}} \left[\Phi(rz) \begin{pmatrix} 1 & \mathfrak{s}_j \log(rz - rx_j) & \mathfrak{s}_j \log(rz - rx_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]', \tag{6.8}$$

where $'$ denotes the derivative with respect to z. We see from (6.6) that

$$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \Phi_j^{(0)}(r)^{-1} = e^{\frac{1}{2}\theta_3(rx_j)} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \mathrm{diag}\,((\Upsilon_j^{(0)})^{-1}, 1) E_{x_j}(x_j)^{-1} R(x_j)^{-1} \mathrm{diag}\,(r^{\frac{1}{3}}, 1, r^{-\frac{1}{3}}).$$

Also, by (6.3) and (4.15), we have

$$\lim_{\substack{z\to x_j \\ z\in \mathrm{II}}} \left[\Phi(rz) \begin{pmatrix} 1 & \mathfrak{s}_j \log(rz - rx_j) & \mathfrak{s}_j \log(rz - rx_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathrm{diag}\,(r^{-\frac{1}{3}}, 1, r^{\frac{1}{3}})$$

$$\times \lim_{\substack{z\to x_j \\ z\in \mathrm{II}}} \left[R(z) E_{x_j}(z) \begin{pmatrix} \Phi_{\mathrm{HG},11}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & \Phi_{\mathrm{HG},12}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & 0 \\ \Phi_{\mathrm{HG},21}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & \Phi_{\mathrm{HG},22}(r^{\frac{4}{3}} f_{x_j}(z); \beta_j) & 0 \\ 0 & 0 & 1 \end{pmatrix} (s_j s_{j+1})^{-\frac{\sigma_{3,1}}{4}} \widetilde{\Theta}(z) \right]' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where $\widetilde{\Theta}$ is defined in (6.5). A direct computation using (A.4) shows that this last limit is given by

$$\begin{aligned} &e^{-\frac{\theta_3(rx_j)}{2}} \mathrm{diag}\,(r^{-\frac{1}{3}}, 1, r^{\frac{1}{3}}) \big[R'(x_j) E_{x_j}(x_j) \mathrm{diag}\,(\Upsilon_j^{(0)}, 1) + R(x_j) E'_{x_j}(x_j) \mathrm{diag}\,(\Upsilon_j^{(0)}, 1) \\ &+ r^{\frac{4}{3}} f'_{x_j}(x_j) R(x_j) E_{x_j}(x_j) \mathrm{diag}\,(\Upsilon_j^{(0)} \Upsilon_j^{(1)}, 0) - \tfrac{r\theta_3'(rx_j)}{2} R(x_j) E_{x_j}(x_j) \mathrm{diag}\,(\Upsilon_j^{(0)}, 1) \big] \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t \end{aligned} \tag{6.9}$$

where $\Upsilon_{j,21}^{(1)} = \frac{\beta_j \pi}{\sqrt{s_j s_{j+1}} \sin(\pi\beta_j)}$. Combining the above equations with (3.20), we then find

$$\begin{aligned} H(r) = -\frac{1}{r} \sum_{j=1}^{m} \mathfrak{s}_j x_j \Big[&\mathrm{diag}\,((\Upsilon_j^{(0)})^{-1}, 1) E_{x_j}(x_j)^{-1} R(x_j)^{-1} R'(x_j) E_{x_j}(x_j) \mathrm{diag}\,(\Upsilon_j^{(0)}, 1) \\ &+ \mathrm{diag}\,((\Upsilon_j^{(0)})^{-1}, 1) E_{x_j}(x_j)^{-1} E'_{x_j}(x_j) \mathrm{diag}\,(\Upsilon_j^{(0)}, 1) + r^{\frac{4}{3}} f'_{x_j}(x_j) \mathrm{diag}\,(\Upsilon_j^{(1)}, 0) \Big]_{21}. \end{aligned} \tag{6.10}$$

The first part in the sum in (6.10) decays as $r \to +\infty$ by (4.25); more precisely

$$\mathrm{diag}\,((\Upsilon_j^{(0)})^{-1}, 1) E_{x_j}(x_j)^{-1} R(x_j)^{-1} R'(x_j) E_{x_j}(x_j) \mathrm{diag}\,(\Upsilon_j^{(0)}, 1) = \mathcal{O}(r^{-\frac{2}{3}}), \quad \text{as } r \to +\infty.$$

For the second part in (6.10), we use (6.7) and get

$$\begin{aligned} &\big[\mathrm{diag}\,((\Upsilon_j^{(0)})^{-1}, 1) E_{x_j}(x_j)^{-1} E'_{x_j}(x_j) \mathrm{diag}\,(\Upsilon_j^{(0)}, 1)\big]_{21} \\ &= \tfrac{\Gamma(1-\beta_j)^2}{\sqrt{s_j s_{j+1}}} \big[E_{x_j}(x_j)^{-1} E'_{x_j}(x_j)\big]_{21} - \tfrac{\Gamma(1+\beta_j)^2}{\sqrt{s_j s_{j+1}}} \big[E_{x_j}(x_j)^{-1} E'_{x_j}(x_j)\big]_{12} \\ &+ \tfrac{\Gamma(1-\beta_j)\Gamma(1+\beta_j)}{\sqrt{s_j s_{j+1}}} \Big(\big[E_{x_j}(x_j)^{-1} E'_{x_j}(x_j)\big]_{22} - \big[E_{x_j}(x_j)^{-1} E'_{x_j}(x_j)\big]_{11} \Big). \end{aligned}$$

This expression can be further simplified using (4.18) and (4.19). After a rather long computation, as $r \to +\infty$ we obtain

$$\frac{\Gamma(1-\beta_j)\Gamma(1+\beta_j)}{\sqrt{s_j s_{j+1}}}\Big(\big[E_{x_j}(x_j)^{-1}E_{x_j}'(x_j)\big]_{22} - \big[E_{x_j}(x_j)^{-1}E_{x_j}'(x_j)\big]_{11}\Big)$$
$$= \frac{1}{\mathfrak{s}_j}\left(\frac{\beta_j^2}{3x_j} + 2i\beta_j \mathrm{Im}\,(d_{1,x_j}^{(1)})\right) + \mathcal{O}(r^{-\frac{2}{3}}),$$
$$\frac{\Gamma(1-\beta_j)^2}{\sqrt{s_j s_{j+1}}}\big[E_{x_j}(x_j)^{-1}E_{x_j}'(x_j)\big]_{21} - \frac{\Gamma(1+\beta_j)^2}{\sqrt{s_j s_{j+1}}}\big[E_{x_j}(x_j)^{-1}E_{x_j}'(x_j)\big]_{12}$$
$$= \frac{2}{3\sqrt{3}x_j}\frac{i\beta_j}{\mathfrak{s}_j}\cos(2\vartheta_j(r)) + \mathcal{O}(r^{-\frac{2}{3}}),$$

where we recall that $\vartheta_j(r)$ is defined in (1.10). For the third and last term in (6.10), we use (A.5) and (4.16) to write

$$\big[r^{\frac{4}{3}} f_{x_j}'(x_j)\mathrm{diag}\,(\Upsilon_j^{(1)},0)\big]_{21} = r^{\frac{4}{3}} f_{x_j}'(x_j)\Upsilon_{j,21}^{(1)} = \frac{i\beta_j}{\mathfrak{s}_j}\left(-\sqrt{3}x_j^{\frac{1}{3}}r^{\frac{4}{3}} + \frac{\rho}{\sqrt{3}x_j^{1/3}}r^{\frac{2}{3}}\right).$$

Substituting the above formulas in (6.10), and noting the remarkable simplification

$$\sum_{j=1}^m 2i\beta_j x_j \mathrm{Im}\,(d_{1,x_j}^{(1)}) = \sum_{j=1}^m 2i\beta_j x_j \mathrm{Im}\left(\frac{(\omega-5)\beta_j}{6(\omega-1)x_j}\right)$$
$$+ \sum_{1\le j\neq k\le m}\mathrm{Re}\left(\frac{4(\omega-1)x_k^{2/3}\beta_k x_j^{2/3}\beta_j}{3(x_j^{2/3}-x_k^{2/3})(x_j^{2/3}-\omega x_k^{2/3})}\right) = \sum_{j=1}^m 2i\beta_j x_j \mathrm{Im}\left(\frac{(\omega-5)\beta_j}{6(\omega-1)x_j}\right) = \sum_{j=1}^m \beta_j^2,$$

we obtain (1.14). This finishes the proof of Theorem 1.3.

6.3. *Proof of Theorem 1.4.* Integrating (3.23) from 0 to an arbitrary $r>0$, we get

$$\int_0^r H(\tau)d\tau = \int_0^r \Big(p_0(\tau)q_0'(\tau) + \sum_{j=1}^m\sum_{k=1}^3 p_{j,k}(\tau)q_{j,k}'(\tau) - H(\tau)\Big)d\tau$$
$$- \frac{1}{4}\Big[2p_0(\tau)q_0(\tau) + \sum_{j=1}^m\Big(p_{j,2}(\tau)q_{j,2}(\tau) + 2p_{j,3}(\tau)q_{j,3}(\tau)\Big) - 3\tau H(\tau)\Big]_{\tau=0}^r. \tag{6.11}$$

For convenience, we write $\vec{0} = (0,\ldots,0)\in\mathbb{R}^m$,

$$\vec{\beta} := (\beta_1,\ldots,\beta_m), \qquad \vec{\beta}_\ell := (\beta_1,\ldots,\beta_\ell,0,\ldots,0), \qquad \vec{\beta}_\ell' := (\beta_1,\ldots,\beta_{\ell-1},\beta_\ell',0,\ldots,0).$$

We also write explicitly the dependence of $p_{j,k}$, $q_{j,k}$, p_0, q_0 and H in $\beta_1,\ldots,\beta_m$ using the notation $p_{j,k}(r;\vec{\beta})$, $q_{j,k}(r;\vec{\beta})$, $p_0(r;\vec{\beta})$, $q_0(r;\vec{\beta})$ and $H(r;\vec{\beta})$. By (4.4), the parameter γ in (3.24) can also be chosen to be any parameter among $\beta_1,\ldots,\beta_m$. Let $\ell\in\{1,\ldots,m\}$. Using (3.24) with $\vec{\beta}=\vec{\beta}_{\ell-1}$ and $\gamma=\beta_\ell$, and integrating in r from 0 to r and integrating in β_ℓ from 0 to β_ℓ, we get

$$\int_0^r \Big(p_0(\tau;\vec{\beta})q_0'(\tau;\vec{\beta}_\ell) + \sum_{j=1}^m\sum_{k=1}^3 p_{j,k}(\tau;\vec{\beta}_\ell)q_{j,k}'(\tau;\vec{\beta}_\ell) - H(\tau;\vec{\beta}_\ell)\Big)d\tau$$
$$- \int_0^r \Big(p_0(\tau;\vec{\beta}_{\ell-1})q_0'(\tau;\vec{\beta}_{\ell-1}) + \sum_{j=1}^m\sum_{k=1}^3 p_{j,k}(\tau;\vec{\beta}_{\ell-1})q_{j,k}'(\tau;\vec{\beta}_{\ell-1}) - H(\tau;\vec{\beta}_{\ell-1})\Big)d\tau$$

(6.12)
$$= \int_0^{\beta_\ell} \left(\sum_{k=1}^{3} \sum_{j=1}^{m} p_{j,k}(r;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_{j,k}(r;\vec{\beta}'_\ell) + p_0(r;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_0(r;\vec{\beta}'_\ell) \right) d\beta'_\ell,$$

where we have used (1.11a)–(1.11c) to conclude that

$$\int_0^{\beta_\ell} \left(\sum_{k=1}^{3} \sum_{j=1}^{m} p_{j,k}(0;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_{j,k}(0;\vec{\beta}'_\ell) + p_0(0;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_0(0;\vec{\beta}'_\ell) \right) d\beta'_\ell = 0.$$

Summing (6.12) over $\ell = 1, \ldots, m$, we then obtain

$$\int_0^{r} \left(p_0(\tau;\vec{\beta}) q_0'(\tau;\vec{\beta}) + \sum_{j=1}^{m} \sum_{k=1}^{3} p_{j,k}(\tau;\vec{\beta}) q'_{j,k}(\tau;\vec{\beta}) - H(\tau;\vec{\beta}) \right) d\tau$$
$$- \int_0^{r} \left(p_0(\tau;\vec{0}) q_0'(\tau;\vec{0}) + \sum_{j=1}^{m} \sum_{k=1}^{3} p_{j,k}(\tau;\vec{0}) q'_{j,k}(\tau;\vec{0}) - H(\tau;\vec{0}) \right) d\tau$$
$$= \sum_{\ell=1}^{m} \int_0^{\beta_\ell} \left(\sum_{k=1}^{3} \sum_{j=1}^{m} p_{j,k}(r;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_{j,k}(r;\vec{\beta}'_\ell) + p_0(r;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_0(r;\vec{\beta}'_\ell) \right) d\beta'_\ell. \tag{6.13}$$

If $\beta_\ell = 0$ (or equivalently, if $s_\ell = s_{\ell+1}$), it follows from (3.2) that $p_{\ell,1}(r) = p_{\ell,2}(r) = p_{\ell,3}(r) = 0$ for all $r > 0$. Also, by (3.20), we have $H(r;\vec{0}) = 0$, and by (2.11) and (2.16), we have $Y|_{\vec{\beta}=\vec{0}} \equiv I$. Hence, by (3.1) and the relations $\Phi_1 = \Psi_1 + \Psi_0^{-1} Y_1 \Psi_0$ and (2.5), we have $p_0(r;\vec{0}) = \frac{1}{\sqrt{2}}(\frac{\rho^3}{54} + \frac{\rho}{2})$ and $q_0(r;\vec{0}) = \frac{1}{\sqrt{2}}(-\frac{\rho^3}{54} + \frac{\rho}{2})$. Thus,

$$\int_0^{r} \left(p_0(\tau;\vec{0}) q_0'(\tau;\vec{0}) + \sum_{j=1}^{m} \sum_{k=1}^{3} p_{j,k}(\tau;\vec{0}) q'_{j,k}(\tau;\vec{0}) - H(\tau;\vec{0}) \right) d\tau = 0$$

and (6.13) can be simplified as

$$\int_0^{r} \left(p_0(\tau;\vec{\beta}) q_0'(\tau;\vec{\beta}) + \sum_{j=1}^{m} \sum_{k=1}^{3} p_{j,k}(\tau;\vec{\beta}) q'_{j,k}(\tau;\vec{\beta}) - H(\tau;\vec{\beta}) \right) d\tau \tag{6.14}$$
$$= \sum_{\ell=1}^{m} \int_0^{\beta_\ell} \left(\sum_{k=1}^{3} \sum_{j=1}^{\ell} p_{j,k}(r;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_{j,k}(r;\vec{\beta}'_\ell) + p_0(r;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_0(r;\vec{\beta}'_\ell) \right) d\beta'_\ell.$$

Substituting (6.14) in (6.11), we obtain

(6.15)
$$\int_0^{r} H(\tau;\vec{\beta}) d\tau = \sum_{\ell=1}^{m} \int_0^{\beta_\ell} \left(\sum_{k=1}^{3} \sum_{j=1}^{\ell} p_{j,k}(r;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_{j,k}(r;\vec{\beta}'_\ell) + p_0(r;\vec{\beta}'_\ell)\partial_{\beta'_\ell} q_0(r;\vec{\beta}'_\ell) \right) d\beta'_\ell$$
$$- \frac{1}{4}\left(2p_0(r;\vec{\beta}) q_0(r;\vec{\beta}) - \sum_{j=1}^{m} \left(2p_{j,1}(r;\vec{\beta}) q_{j,1}(r;\vec{\beta}) + p_{j,2}(r;\vec{\beta}) q_{j,2}(r;\vec{\beta}) \right) - 3rH(r;\vec{\beta}) \right)$$
$$+ \frac{1}{2} p_0(0;\vec{\beta}) q_0(0;\vec{\beta}),$$

where we have also used (1.5). Using (1.8b)–(1.8d), (1.8f)–(1.8h) and (4.27), we get

$$p_{j,1}(r;\vec{\beta}_\ell)\partial_{\beta_\ell}q_{j,1}(r;\vec{\beta}_\ell) = p_{j,1}(r;\vec{\beta}_\ell)q_{j,1}(r;\vec{\beta}_\ell)\partial_{\beta_\ell}\log q_{j,1}(r;\vec{\beta}_\ell)$$

$$= -\frac{2i\beta_j}{3}\bigg(\sin\left(2\vartheta_j(r;\vec{\beta}_\ell)-\tfrac{2\pi}{3}\right)+\left(\sin(2\vartheta_j(r;\vec{\beta}_\ell))-\tfrac{\sqrt{3}}{2}\right)\sum_{n=1}^{\ell}\sqrt{3}i\beta_n\frac{x_n^{2/3}}{x_j^{2/3}}\bigg)$$

(6.16)

$$\times\left(\partial_{\beta_\ell}\log\mathcal{A}_j(\vec{\beta}_\ell)+\cot(\vartheta_j(r;\vec{\beta}_\ell)-\tfrac{\pi}{3})\partial_{\beta_\ell}\vartheta_j\right)\left(1+\mathcal{O}\Big(\frac{\log r}{r^{2/3}}\Big)\right), \qquad \text{as } r\to+\infty,$$

where we have explicitly written the dependence of ϑ_j and $\mathcal{A}_j$ in r and $\vec{\beta}_\ell$. For the quantities $p_{j,2}(r;\vec{\beta}_\ell)\partial_{\beta_\ell}q_{j,2}(r;\vec{\beta}_\ell)$ and $p_{j,3}(r;\vec{\beta}_\ell)\partial_{\beta_\ell}q_{j,3}(r;\vec{\beta}_\ell)$, we obtain after another computation (using again (1.8b)–(1.8d), (1.8f)–(1.8h) and (4.27)) that

(6.17)

$$p_{j,2}(r;\vec{\beta}_\ell)\partial_{\beta_\ell}q_{j,2}(r;\vec{\beta}_\ell) = -\frac{2i\beta_j}{3}\sin\left(2\vartheta_j\right)\left(\partial_{\beta_\ell}\log\mathcal{A}_j(\vec{\beta}_\ell)+\cot(\vartheta_j)\partial_{\beta_\ell}\vartheta_j\right)\left(1+\mathcal{O}\Big(\frac{\log r}{r^{2/3}}\Big)\right),$$

(6.18)

$$p_{j,3}(r;\vec{\beta}_\ell)\partial_{\beta_\ell}q_{j,3}(r;\vec{\beta}_\ell) = \bigg[\tfrac{-2i\beta_j}{3}\sin\left(2\vartheta_j+\tfrac{2\pi}{3}\right)\left(\partial_{\beta_\ell}\log\mathcal{A}_j(\vec{\beta}_\ell)+\cot(\vartheta_j+\tfrac{\pi}{3})\partial_{\beta_\ell}\vartheta_j\right)$$

$$+\tfrac{2i\beta_j}{3}\left(\sin(2\vartheta_j)-\tfrac{\sqrt{3}}{2}\right)\bigg(\Big\{\partial_{\beta_\ell}\log\mathcal{A}_j(\vec{\beta}_\ell)+\cot(\vartheta_j-\tfrac{\pi}{3})\partial_{\beta_\ell}\vartheta_j\Big\}\sum_{n=1}^{\ell}\sqrt{3}i\beta_n\tfrac{x_n^{2/3}}{x_j^{2/3}}+\sqrt{3}i\tfrac{x_\ell^{2/3}}{x_j^{2/3}}\bigg)\bigg]$$

$$\times\left(1+\mathcal{O}\Big(\frac{\log r}{r^{2/3}}\Big)\right)$$

as $r\to+\infty$, where $\vartheta_j=\vartheta_j(r;\vec{\beta}_\ell)$. Furthermore, using (1.9) and (4.4), we find

$$\partial_{\beta_\ell}\log\mathcal{A}_j(\vec{\beta}_\ell) = \begin{cases} -\frac{2\pi i}{3}+\partial_{\beta_\ell}\log|\Gamma(1-\beta_\ell)|, & \text{if } j=\ell, \\ \log\frac{|x_j^{2/3}-\omega x_\ell^{2/3}|}{|x_j^{2/3}-x_\ell^{2/3}|}, & \text{if } j<\ell. \end{cases}$$

Combining the asymptotics (6.16)–(6.18), we get

$$\sum_{k=1}^{3}\sum_{j=1}^{\ell}p_{j,k}(r;\vec{\beta}_\ell)\partial_{\beta_\ell}q_{j,k}(r;\vec{\beta}_\ell) = \sum_{j=1}^{\ell}\beta_j\left(\frac{x_\ell^{2/3}}{x_j^{2/3}}-\frac{2}{\sqrt{3}}\sin(2\vartheta_j)\frac{x_\ell^{2/3}}{x_j^{2/3}}-2i\partial_{\beta_\ell}\vartheta_j\right)+\mathcal{O}\left(\frac{\log r}{r^{2/3}}\right)$$

as $r\to+\infty$, where again $\vartheta_j=\vartheta_j(r;\vec{\beta}_\ell)$. Using (3.18) and the fact that $p_{j,1}(r)=p_{j,2}(r)=p_{j,3}(r)=0$ if $\beta_j=0$, we note that

$$p_0(r;\vec{\beta}_\ell)\partial_{\beta_\ell}q_0(r;\vec{\beta}_\ell) = -\sqrt{2}\sum_{j=1}^{\ell}p_{j,3}(r;\vec{\beta}_\ell)q_{j,1}(r;\vec{\beta}_\ell)\partial_{\beta_\ell}q_0(r;\vec{\beta}_\ell)-\left(q_0(r;\vec{\beta}_\ell)-\frac{\rho}{\sqrt{2}}\right)\partial_{\beta_\ell}q_0(r;\vec{\beta}_\ell).$$

Integrating this identity in β_ℓ from 0 to an arbitrary $\beta_\ell\in i\mathbb{R}$, we get

$$\int_0^{\beta_\ell}p_0(r;\vec{\beta}_\ell')\partial_{\beta_\ell'}q_0(r;\vec{\beta}_\ell')d\beta_\ell' = -\sqrt{2}\sum_{j=1}^{\ell}\int_0^{\beta_\ell}p_{j,3}(r;\vec{\beta}_\ell')q_{j,1}(r;\vec{\beta}_\ell')\partial_{\beta_\ell'}q_0(r;\vec{\beta}_\ell')d\beta_\ell'$$

$$-\left[\frac{1}{2}q_0(r;\vec{\beta}_\ell')^2-\frac{\rho}{\sqrt{2}}q_0(r;\vec{\beta}_\ell')\right]_{\beta_\ell'=0}^{\beta_\ell}. \tag{6.19}$$

Using (1.8d), (1.8e), (1.8f) and (4.27), as $r \to +\infty$ we get

$$-\sqrt{2}\sum_{j=1}^{\ell} p_{j,3}(r;\vec{\beta}_\ell)q_{j,1}(r;\vec{\beta}_\ell)\partial_{\beta_\ell} q_0(r;\vec{\beta}_\ell) = \sum_{j=1}^{\ell} \frac{x_\ell^{2/3}}{x_j^{2/3}}\beta_j\left(\frac{2}{\sqrt{3}}\sin(2\vartheta_j(r;\vec{\beta}_\ell))-1\right)+\mathcal{O}\left(\frac{\log r}{r^{2/3}}\right). \tag{6.20}$$

Hence, substituting (6.16)–(6.20) in (6.15), we obtain

$$\begin{aligned}\int_0^r H(\tau;\vec{\beta})d\tau &= -2\sum_{\ell=1}^{m}\int_0^{\beta_\ell}\left\{\sum_{j=1}^{\ell-1} i\beta_j\partial_{\beta_\ell'}\vartheta_j(r;\vec{\beta}_\ell')+i\beta_\ell'\partial_{\beta_\ell'}\vartheta_\ell(r;\vec{\beta}_\ell')\right\}d\beta_\ell' \\ &-\left(\frac{1}{2}q_0(r;\vec{\beta})^2-\frac{\rho}{\sqrt{2}}q_0(r;\vec{\beta})\right)+\left(\frac{1}{2}q_0(r;\vec{0})^2-\frac{\rho}{\sqrt{2}}q_0(r;\vec{0})\right)+\frac{1}{2}p_0(0;\vec{\beta})q_0(0;\vec{\beta}) \\ &-\frac{1}{4}\left(2p_0(r;\vec{\beta})q_0(r;\vec{\beta})-\sum_{j=1}^{m}\left(2p_{j,1}(r;\vec{\beta})q_{j,1}(r;\vec{\beta})+p_{j,2}(r;\vec{\beta})q_{j,2}(r;\vec{\beta})\right)-3rH(r;\vec{\beta})\right) \\ &+\mathcal{O}\left(\frac{\log r}{r^{2/3}}\right), \qquad \text{as } r\to+\infty.\end{aligned} \tag{6.21}$$

Since $p_0(r;\vec{0}) = \frac{1}{\sqrt{2}}(\frac{\rho^3}{54}+\frac{\rho}{2})$, $q_0(r;\vec{0}) = \frac{1}{\sqrt{2}}(-\frac{\rho^3}{54}+\frac{\rho}{2}) = q_0(0;\vec{\beta})$, we see that

$$\left(\frac{1}{2}q_0(r;\vec{0})^2-\frac{\rho}{\sqrt{2}}q_0(r;\vec{0})\right)+\frac{1}{2}p_0(0;\vec{\beta})q_0(0;\vec{\beta}) = -\frac{\rho}{2\sqrt{2}}q_0(0;\vec{\beta}) = \frac{\rho^4}{216}-\frac{\rho^2}{8}. \tag{6.22}$$

Also, using (1.8), (1.14) and (3.18), we obtain

$$\begin{aligned}&-\left(\frac{1}{2}q_0(r)^2-\frac{\rho}{\sqrt{2}}q_0(r)\right)-\frac{1}{4}\left(2p_0(r)q_0(r)-\sum_{j=1}^{m}\left(2p_{j,1}(r)q_{j,1}(r)+p_{j,2}(r)q_{j,2}(r)\right)-3rH(r)\right) \\ &= \frac{q_0(r)}{\sqrt{2}}\left(\frac{\rho}{2}+\sum_{j=1}^{m}p_{j,3}(r)q_{j,1}(r)\right)+\frac{3}{4}rH(r)+\frac{1}{4}\sum_{j=1}^{m}\left(2p_{j,1}(r)q_{j,1}(r)+p_{j,2}(r)q_{j,2}(r)\right) \\ &= \sum_{j=1}^{m}\left(\frac{3\sqrt{3}}{4}i\beta_j(rx_j)^{\frac{4}{3}}-\frac{\sqrt{3}\rho}{2}i\beta_j(rx_j)^{\frac{2}{3}}-\beta_j^2\right)-\frac{\rho^4}{216}+\frac{\rho^2}{8}+\mathcal{O}(r^{-\frac{2}{3}}), \qquad \text{as } r\to+\infty.\end{aligned} \tag{6.23}$$

It follows from [24, equation (7.51)] that

$$-2\sum_{j=1}^{m}\int_0^{\beta_\ell} i\beta_\ell'\partial_{\beta_\ell'}\vartheta_\ell(r;\vec{\beta}_\ell')d\beta_\ell' = \sum_{j=1}^{m}\log\left(G(1+\beta_j)G(1-\beta_j)\right) + \sum_{j=1}^{m}\beta_j^2\left(1-\frac{4}{3}\log(rx_j)-\log\left(\frac{9}{2}\right)\right). \tag{6.24}$$

For $m \geq 2$, we also need the relation

$$-2\sum_{\ell=1}^{m}\int_0^{\beta_\ell}\sum_{j=1}^{\ell-1} i\beta_j\partial_{\beta_\ell'}\vartheta_j(r;\vec{\beta}_\ell')d\beta_\ell' = -2\sum_{\ell=1}^{m}\sum_{j=1}^{\ell-1}\beta_j\beta_\ell\log\frac{|x_j^{2/3}-\omega x_\ell^{2/3}|}{|x_j^{2/3}-x_\ell^{2/3}|}, \tag{6.25}$$

which can be proved directly from (1.10) and a direct computation. The asymptotic formula (1.15) (without the error term) now follows after substituting (6.22) and (6.23) in (6.21) and performing a rather long calculation which uses (6.24) and (6.25). The fact the the error term in (1.15) is $\mathcal{O}(r^{-\frac{2}{3}})$ and not $\mathcal{O}(r^{-\frac{2}{3}}\log r)$ follows directly from (1.12) and (1.14), and (1.16) follows from (4.27). This finishes the proof of Theorem 1.4.

APPENDIX A: CONFLUENT HYPERGEOMETRIC MODEL RH PROBLEM

In this appendix we recall a well-known model RH problem, whose solution depends on a parameter $\beta\in i\mathbb{R}$ and is denoted $\Phi_{\mathrm{HG}}(\cdot)=\Phi_{\mathrm{HG}}(\cdot;\beta)$.

(a) $\Phi_{\mathrm{HG}}:\mathbb{C}\setminus\Sigma_{\mathrm{HG}}\to\mathbb{C}^{2\times 2}$ is analytic, where $\Sigma_{\mathrm{HG}}=e^{\frac{\pi i}{4}}(-\infty,\infty)\cup e^{\frac{\pi i}{2}}(-\infty,\infty)\cup e^{\frac{3\pi i}{4}}(-\infty,\infty)$.

(b) Φ_{HG} satisfies the jump relations

$$\Phi_{\mathrm{HG},+}(z)=\Phi_{\mathrm{HG},-}(z)\widetilde{J}_k,\qquad z\in\Gamma_k^{\mathrm{HG}},\quad k=1,\dots,6,\tag{A.1}$$

where

$$\Gamma_1^{\mathrm{HG}}=(0,i\infty),\qquad \Gamma_2^{\mathrm{HG}}=(0,e^{\frac{3\pi i}{4}}\infty),\qquad \Gamma_3^{\mathrm{HG}}=(e^{-\frac{3\pi i}{4}}\infty,0),$$
$$\Gamma_4^{\mathrm{HG}}=(-i\infty,0),\qquad \Gamma_5^{\mathrm{HG}}=(e^{-\frac{\pi i}{4}}\infty,0),\qquad \Gamma_6^{\mathrm{HG}}=(0,e^{\frac{\pi i}{4}}\infty),$$

and

$$\widetilde{J}_1=\begin{pmatrix}0 & e^{-i\pi\beta}\\ -e^{i\pi\beta} & 0\end{pmatrix},\ \widetilde{J}_4=\begin{pmatrix}0 & e^{i\pi\beta}\\ -e^{-i\pi\beta} & 0\end{pmatrix},\ \widetilde{J}_2=\widetilde{J}_6=\begin{pmatrix}1 & 0\\ e^{i\pi\beta} & 1\end{pmatrix},\ \widetilde{J}_3=\widetilde{J}_5=\begin{pmatrix}1 & 0\\ e^{-i\pi\beta} & 1\end{pmatrix}.$$

(c) As $z\to\infty$, $z\notin\Sigma_{\mathrm{HG}}$, we have

$$\Phi_{\mathrm{HG}}(z)=\left(I+\frac{\Phi_{\mathrm{HG},1}(\beta)}{z}+\mathcal{O}(z^{-2})\right)z^{-\beta\sigma_3}e^{-\frac{z}{2}\sigma_3}\begin{cases}e^{i\pi\beta\sigma_3}, & \frac{\pi}{2}<\arg z<\frac{3\pi}{2},\\ \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, & -\frac{\pi}{2}<\arg z<\frac{\pi}{2},\end{cases}\tag{A.2}$$

where $z^{\beta}=|z|^{\beta}e^{i\beta\arg z}$ with $\arg z\in(-\frac{\pi}{2},\frac{3\pi}{2})$ and

$$\Phi_{\mathrm{HG},1}(\beta)=\beta^2\begin{pmatrix}-1 & \tau(\beta)\\ -\tau(-\beta) & 1\end{pmatrix},\qquad \tau(\beta)=\frac{-\Gamma(-\beta)}{\Gamma(\beta+1)}.\tag{A.3}$$

(d) $\Phi_{\mathrm{HG}}(z)=\mathcal{O}(\log z)$ as $z\to 0$.

This model RH problem can be solved explicitly using confluent hypergeometric functions [36]. By a computation similar to [15, equation (A.9)] and [24, equation (A.10)], we have

$$\Phi_{\mathrm{HG}}(z)=\Upsilon^{(0)}(I+\Upsilon^{(1)}z+\mathcal{O}(z^2))\begin{pmatrix}1 & \frac{\sin(\pi\beta)}{\pi}\log z\\ 0 & 1\end{pmatrix},\qquad \text{as } z\to 0,\ \arg z\in(\tfrac{3\pi}{4},\tfrac{5\pi}{4}),\tag{A.4}$$

where

$$\Upsilon^{(0)}=\begin{pmatrix}\Gamma(1-\beta) & \frac{1}{\Gamma(\beta)}\left(\frac{\Gamma'(1-\beta)}{\Gamma(1-\beta)}+2\gamma_{\mathrm{E}}-i\pi\right)\\ \Gamma(1+\beta) & \frac{-1}{\Gamma(-\beta)}\left(\frac{\Gamma'(-\beta)}{\Gamma(-\beta)}+2\gamma_{\mathrm{E}}-i\pi\right)\end{pmatrix},\qquad \Upsilon_{21}^{(1)}=\frac{\beta\pi}{\sin(\pi\beta)},\tag{A.5}$$

γ_{E} is Euler's gamma constant and

$$\log z=\log|z|+i\arg z,\qquad \arg z\in\left(-\tfrac{\pi}{2},\tfrac{3\pi}{2}\right).$$

REFERENCES

[1] M. Adler, N. Orantin and P. van Moerbeke, Universality for the Pearcey process, *Phys. D* **239** (2010), 924–941.

[2] M. Adler and P. van Moerbeke, PDEs for the Gaussian ensemble with external source and the Pearcey distribution, *Comm. Pure Appl. Math.* **60** (2007), 1261–1292.

[3] G.W. Anderson, A. Guionnet and O. Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics **118**, Cambridge University Press, Cambridge, 2010.

[4] L.-P. Arguin, D. Belius, and P. Bourgade, Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.* **349** (2017), 703–751.

[5] E. Basor and H. Widom, Toeplitz and Wiener-Hopf determinants with piecewise continuous symbols, *J. Funct. Anal.* **50** (1983), 387–413.

[6] M. Bertola and M. Cafasso, The transition between the gap probabilities from the Pearcey to the Airy process–a Riemann–Hilbert approach, *Int. Math. Res. Not.* **2012** (2012), 1519–1568.

[7] P. Billingsley, Probability and measure. Anniversary edition. Wiley Series in Probability and Statistics, *John Wiley and Sons, Inc., Hoboken, NJ* (2012).

[8] P.M. Bleher and A.B.J. Kuijlaars, Large n limit of Gaussian random matrices with external source, part III: double scaling limit, *Comm. Math. Phys.* **270** (2007), 481–517.

[9] A. Borodin, *Determinantal point processes*, The Oxford handbook of random matrix theory, edited by G. Akemann, J. Baik, and P. Di Francesco, 231–249, Oxford Univ. Press, Oxford, 2011.

[10] T. Bothner and R. Buckingham, Large deformations of the Tracy-Widom distribution I. Non-oscillatory asymptotics, *Comm. Math. Phys.*, **359** (2018), 223–263.

[11] T. Bothner, A. Its and A. Prokhorov, On the analysis of incomplete spectra in random matrix theory through an extension of the Jimbo-Miwa-Ueno differential, *Adv. Math.* **345** (2019), 483–551.

[12] E. Brézin and S. Hikami, Level spacing of random matrices in an external source, *Phys. Rev. E.* **58** (1998), 7176–7185.

[13] E. Brézin and S. Hikami, Universal singularity at the closure of a gap in a random matrix theory, *Phys. Rev. E.* **57** (1998), 4140–4149.

[14] C. Charlier, Exponential moments and piecewise thinning for the Bessel point process, *Int. Math. Res. Not. IMRN*, doi:rnaa054.

[15] C. Charlier, Large gap asymptotics for the generating function of the sine point process, *Proc. Lond. Math. Soc.*, doi:plms.12393.

[16] C. Charlier, Upper bounds for the maximum deviation of the Pearcey process, *Random Matrices Theory Appl.* **10** (2021), no. 4, 8 pp.

[17] C. Charlier and T. Claeys, Large gap asymptotics for Airy kernel determinants with discontinuities, *Comm. Math. Phys.* **375** (2020), 1299–1339.

[18] C. Charlier and T. Claeys, Global rigidity and exponential moments for soft and hard edge point processes, *Prob. Math. Phys.* **2** (2021), 143–197, DOI: 10.2140.

[19] C. Charlier and A. Doeraene, The generating function for the Bessel point process and a system of coupled Painlevé V equations, *Random Matrices Theory Appl.* **8** (2019), 31 pp.

[20] C. Charlier and J. Lenells, The hard-to-soft edge transition: exponential moments, central limit theorems and rigidity, arXiv:2104.11494.

[21] T. Claeys and A. Doeraene, The generating function for the Airy point process and a system of coupled Painlevé II equations, *Stud. Appl. Math.* **140** (2018), no. 4, 403–437.

[22] T. Claeys, B. Fahs, G. Lambert, and C. Webb, How much can the eigenvalues of a random Hermitian matrix fluctuate?, *Duke Math. J.* **170** (2021), DOI: 10.1215.

[23] D. Dai, S.-X. Xu and L. Zhang, Asymptotics of Fredholm determinant associated with the Pearcey kernel, *Comm. Math. Phys.* **382** (2021), no. 3, 1769–1809.

[24] D. Dai, S.-X. Xu and L. Zhang, On the deformed Pearcey determinant, *Adv. Math.* **400** (2022), Paper No. 108291, 64 pp.

[25] P. Deift, A. Its and X. Zhou, A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, *Ann. of Math.* **146** (1997), 149–235.

[26] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. Math.* **137** (1993), 295–368.
[27] L. Erdős, T. Krüger and D. Schröder, Cusp universality for random matrices I: local Law and the complex Hermitian case, *Comm. Math. Phys.* **378** (2020), 1203–1278.
[28] L. Erdős and H.-T. Yau, Universality of local spectral statistics of random matrices, *Bull. Amer. Math. Soc.* **49** (2012), 377–414.
[29] L. Erdős, H.-T. Yau, and J. Yin, Rigidity of eigenvalues of generalized Wigner matrices, *Adv. Math.* **229** (2012), no. 3, 1435–1515.
[30] P.J. Forrester, *Log-gases and Random Matrices*, London Mathematical Society Mono-graphs Series, **34**, Princeton University Press, Princeton, NJ, 2010.
[31] D. Geudens and L. Zhang, Transitions between critical kernels: from the tacnode kernel and critical kernel in the two-matrix model to the Pearcey kernel, *Int. Math. Res. Not. IMRN* **2015** (2015), 5733–5782.
[32] W. Hachem, A. Hardy and J. Najim, Large complex correlated Wishart matrices: the Pearcey kernel and expansion at the hard edge, *Electron. J. Probab.* **21** (2016), 36 pp.
[33] J. Harnad, C.A. Tracy and H. Widom, Hamiltonian structure of equations appearing in random matrices (Cambridge, 1992), *NATO Adv. Sci. Inst. Ser. B Phys.* **315** (1993), Plenum, New York, 231–245.
[34] D. Holcomb and E. Paquette, The maximum deviation of the Sine-β counting process, *Electron. Commun. Probab.* **23** (2018), paper no. 58, 13 pp.
[35] A. Its, A.G. Izergin, V.E. Korepin and N.A. Slavnov, Differential equations for quantum correlation functions, *Internat. J. Modern Phys. B*, **4**, (1990), no. 5, 1003–1037.
[36] A. Its and I. Krasovsky, Hankel determinant and orthogonal polynomials for the Gaussian weight with a jump, *Contemporary Mathematics* **458** (2008), 215–248.
[37] M. Jimbo, T. Miwa, Y. Môri and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, *Phys. D* **1** (1980), 80–158.
[38] A.B.J. Kuijlaars, Universality, Chapter 6 in *The Oxford Handbook on Random Matrix Theory*, (G. Akemann, J. Baik, and P. Di Francesco, eds.), Oxford University Press, 2011, pp. 103–134.
[39] A. Okounkov and N. Reshetikhin, Random skew plane partitions and the Pearcey process, *Comm. Math. Phys.* **269** (2007), 571–609.
[40] N.R. Smith, P. Le Doussal, S.N. Majumdar and G. Schehr, Counting statistics for non-interacting fermions in a d-dimensional potential, *Phys. Rev. E* **103** (2021), L030105.
[41] N.R. Smith, P. Le Doussal, S.N. Majumdar and G. Schehr, Full counting statistics for interacting trapped fermions, *SciPost Phys.* **11** (2021), no. 6, No. 110, 56 pp.
[42] A. Soshnikov, Gaussian fluctuation for the number of particles in Airy, Bessel, sine, and other determinantal random point fields, *J. Statist. Phys.* **100** (2000), 491–522.
[43] A. Soshnikov, Determinantal random point fields, *Russian Math. Surveys* **55** (2000), no. 5, 923–975.
[44] C.A. Tracy and H. Widom, Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** (1994), no. 1, 151–174.
[45] C.A. Tracy and H. Widom, Level spacing distributions and the Bessel kernel. *Comm. Math. Phys.* **161** (1994), no. 2, 289–309.
[46] C.A. Tracy and H. Widom, The Pearcey process, *Comm. Math. Phys.* **263** (2006), 381–400.