Development of a parallel third order explicit multirate scheme

B. Seny *, J. Lambrechts, V. Legat and J.-F. Remacle

Institute of Mechanics, Materials and Civil Engineering
UniversitÃ© catholique de Louvain
Belgium



http://www.climate.be/SLIM



http://geuz.org/gmsh/

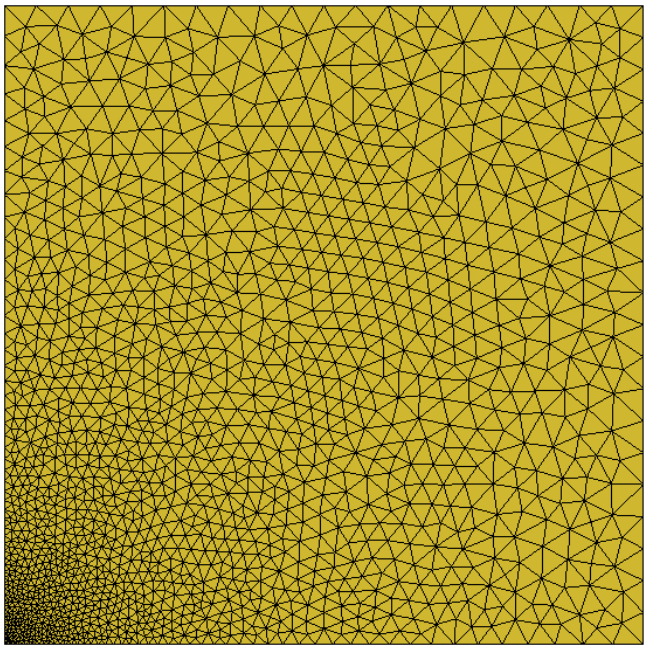
YIC 2013

02-06 September 2013

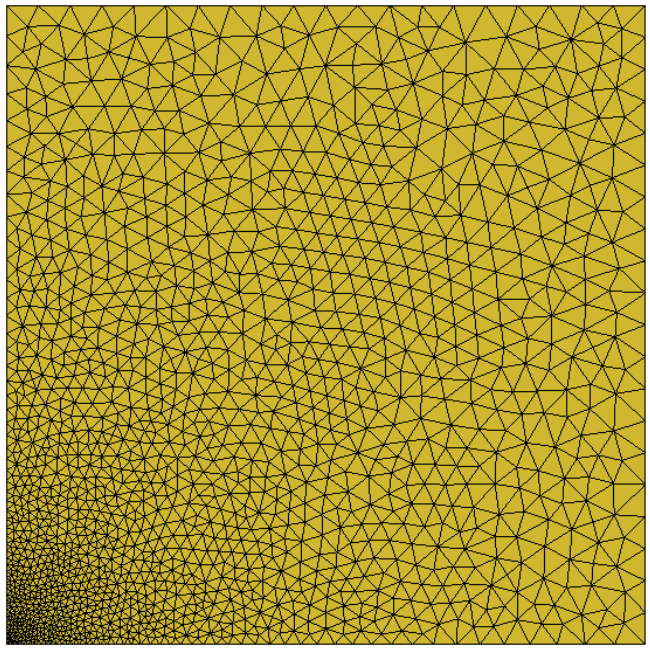
Motivation: inefficiency of explicit schemes

The Stommel Gyre testcase

Stable - $\Delta t \approx 7.22$ sec



Unstable - $\Delta t \approx 9.62$ sec



Instability due to CFL condition

$$\boxed{\Delta t < C \max([\Delta x]^n / u)}$$

BUT: $\Delta t_{min} \approx 7.22$ sec and $\Delta t_{max} \approx 109.40$ sec

Ways to tame the CFL condition

- Unconditionally stable implicit schemes
- Implicit-Explicit (IMEX) schemes
- DG schemes based on a space-time expansion [Lörcher, Gassner, Munz]
- High resolution schemes with locally varying time steps [Dawson and Kirby]
- Local time stepping methods of arbitrary high order (ADER schemes) [Dumbser et al.]
- **Explicit RK multirate time stepping methods [Constantinescu and Sandu, Schlegel et al., Seny et al.]**

Parallelization?

- Parallel multirate strategy for second order multirate schemes of Constantinescu [Seny et al.]
- **Parallel multirate strategy for third order multirate scheme of Schlegel**

Third order, four stages explicit RK method

RK43

Butcher Tableau

0				
1/2	1/2			
1/2	-1/6	2/3		
1	1/3	-1/3	1	
	1/6	1/3	1/3	1/6

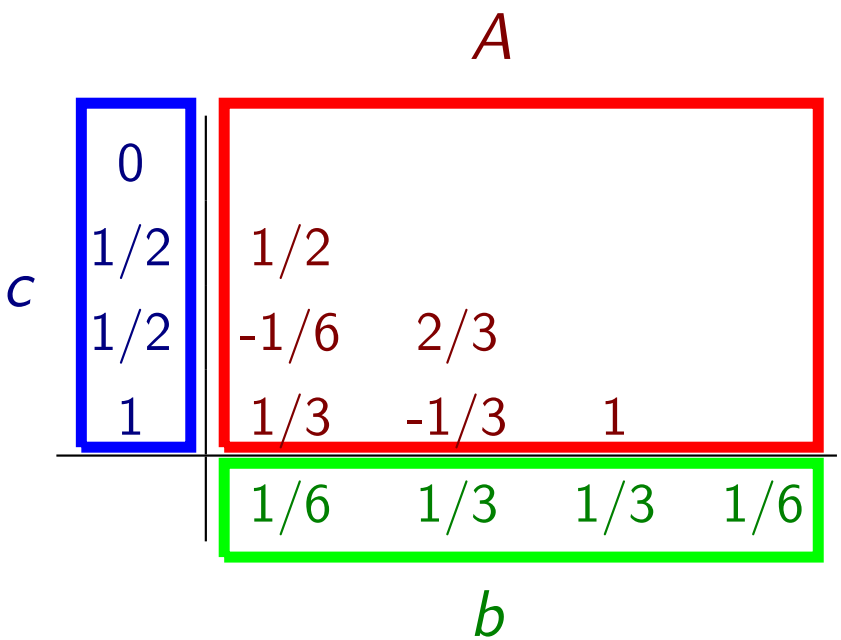
Algorithm

$$
\begin{aligned}
u^{(0)} &= u^{(n)} \\
K_0 &= f(u^{(0)}, t^n + 0\Delta t) \\
u^{(1)} &= u^{(n)} + \Delta t 1/2 K_0) \\
K_1 &= f(u^{(1)}, t^n + 1/2\Delta t) \\
u^{(2)} &= u^{(n)} + \Delta t(-1/6 K_0 + 2/3 K_1) \\
K_2 &= f(u^{(2)}, t^n + 1/2\Delta t) \\
u^{(3)} &= u^{(n)} + \Delta t(1/3 K_0 - 1/3 K_1 + 1 K_2) \\
K_3 &= f(u^{(3)}, t^n + 1\Delta t) \\
\\
u^{n+1} &= u^n + \Delta t(1/6 K_0 + 1/3 K_1 + \\
&\quad 1/3 K_2 + 1/6 K_3)
\end{aligned}
$$

Third order, four stages explicit RK method

RK43

Butcher Tableau



Algorithm

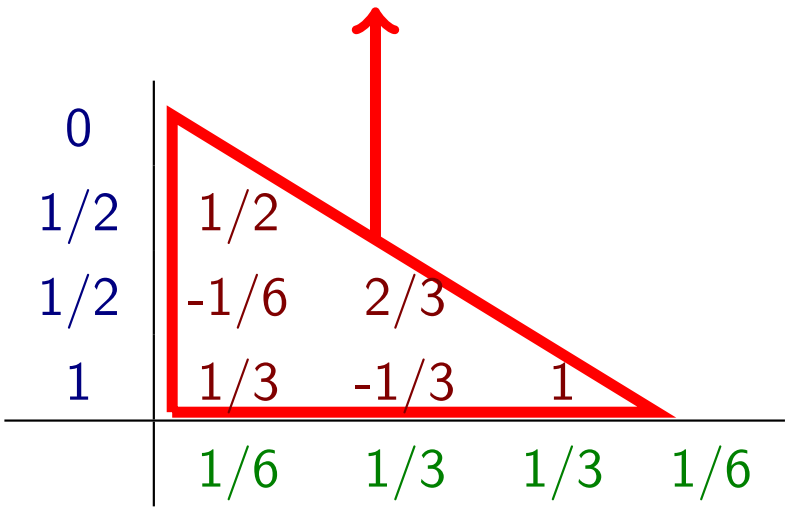
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u^{(1)} &= u^{(n)} + \Delta t 1/2 K_0) \\
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K_2 &= f(u^{(2)}, t^n + 1/2\Delta t) \\
u^{(3)} &= u^{(n)} + \Delta t(1/3 K_0 - 1/3 K_1 + 1 K_2) \\
K_3 &= f(u^{(3)}, t^n + 1\Delta t) \\
\\
u^{n+1} &= u^n + \Delta t(1/6 K_0 + 1/3 K_1 + \\
&\quad 1/3 K_2 + 1/6 K_3)
\end{aligned}
$$

Third order, four stages explicit RK method

RK43

Butcher Tableau

explicitness of the method



Algorithm

$$
\begin{aligned}
u^{(0)} &= u^{(n)} \\
K_0 &= f(u^{(0)}, t^n + 0\Delta t) \\
u^{(1)} &= u^{(n)} + \Delta t 1/2 K_0) \\
K_1 &= f(u^{(1)}, t^n + 1/2\Delta t) \\
u^{(2)} &= u^{(n)} + \Delta t(-1/6 K_0 + 2/3 K_1) \\
K_2 &= f(u^{(2)}, t^n + 1/2\Delta t) \\
u^{(3)} &= u^{(n)} + \Delta t(1/3 K_0 - 1/3 K_1 + 1 K_2) \\
K_3 &= f(u^{(3)}, t^n + 1\Delta t) \\
\\
u^{n+1} &= u^n + \Delta t(1/6 K_0 + 1/3 K_1 + \\
&\quad 1/3 K_2 + 1/6 K_3)
\end{aligned}
$$

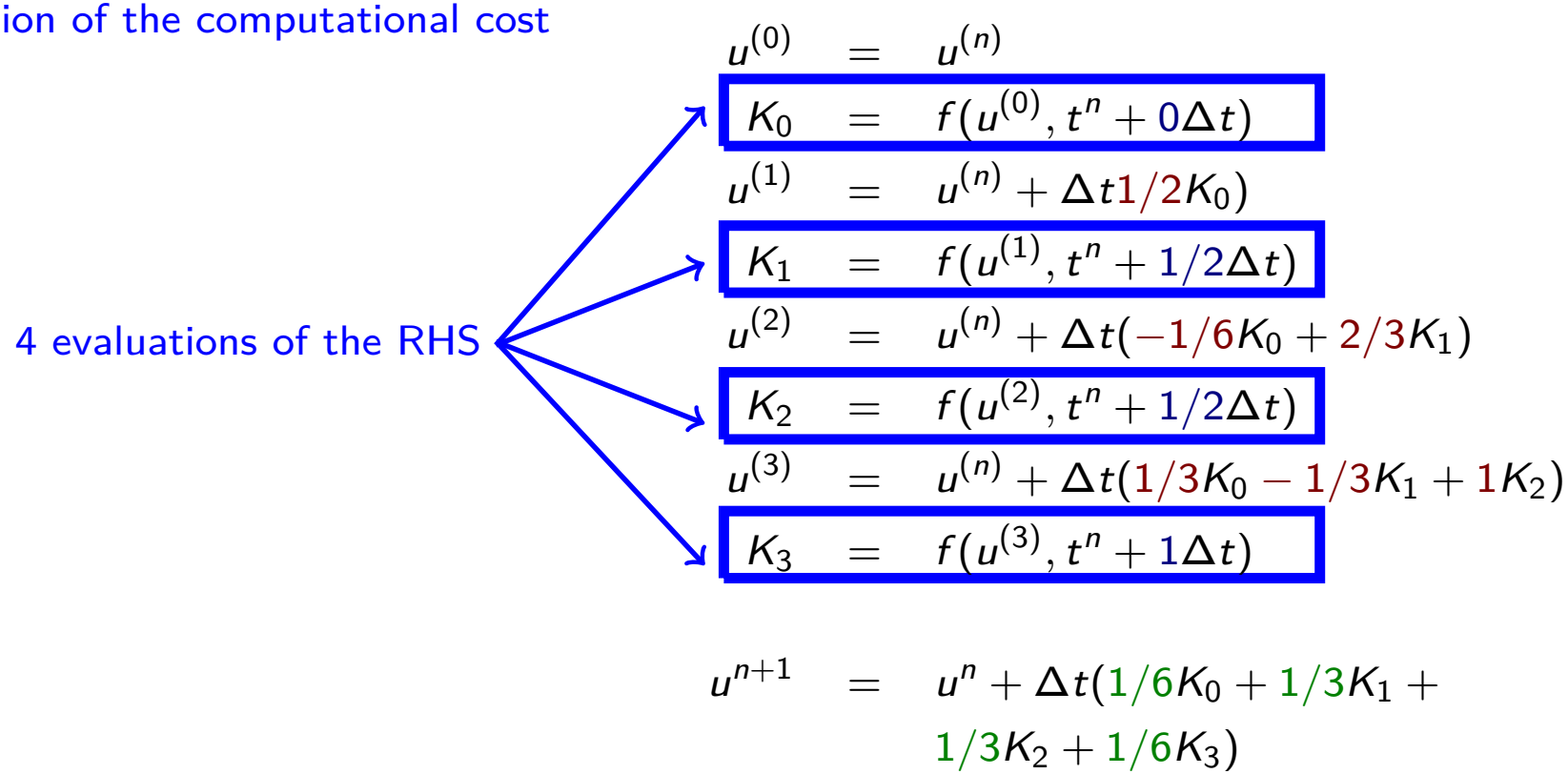
Third order, four stages explicit RK method

RK43

Algorithm

Estimation of the computational cost

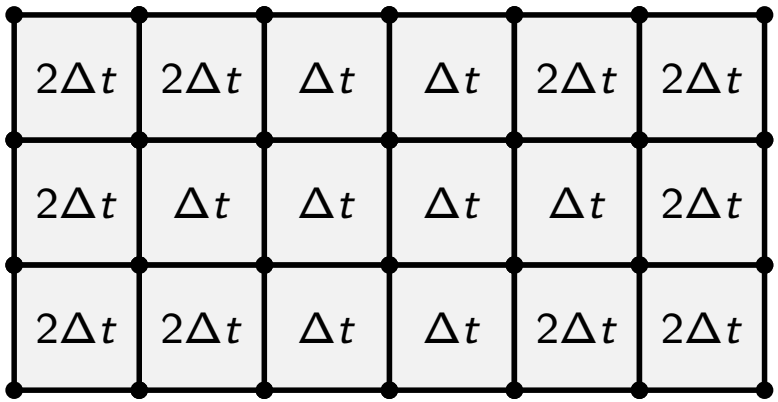
4 evaluations of the RHS



Multirate method of Schlegel

Consider locally stable regions

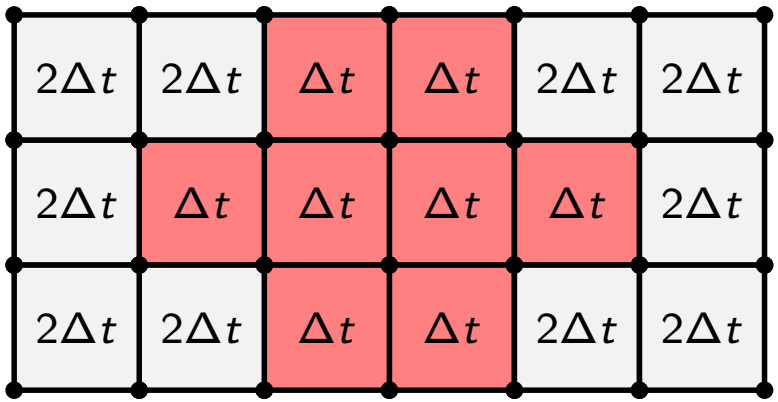
18 elements



Multirate method of Schlegel

Consider locally stable regions

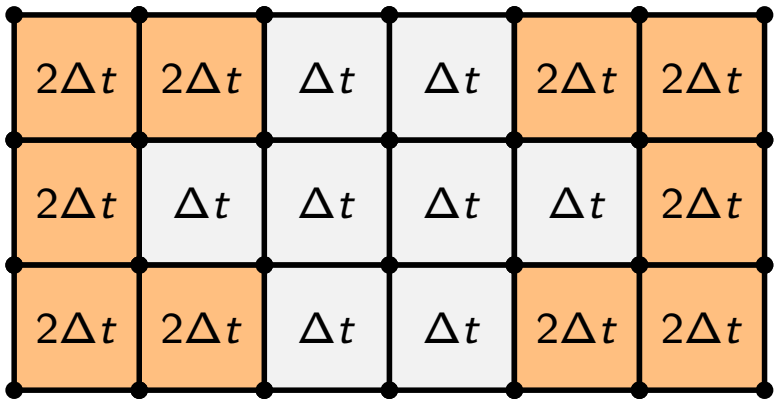
8 elements stable for Δt



Multirate method of Schlegel

Consider locally stable regions

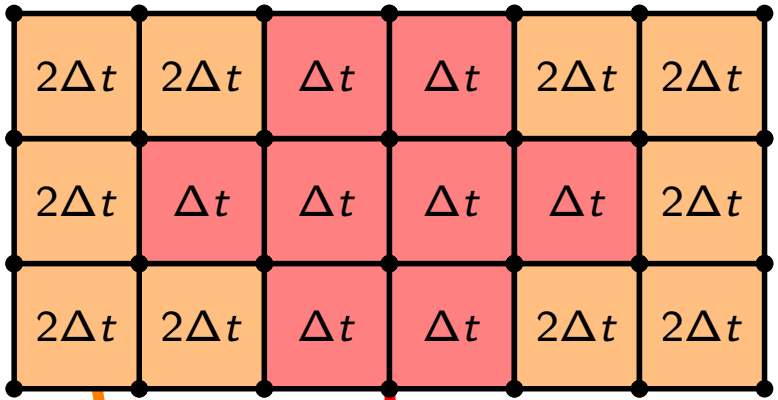
10 elements stable for $2\Delta t$



Multirate method of Schlegel

Consider locally stable regions

Naive local timestepping: SU$\approx$ 1.38



$1\times(2\Delta t)$ $2\times(\Delta t)$

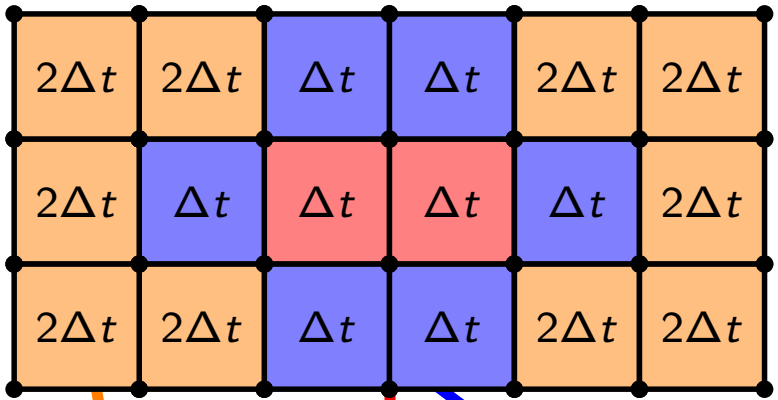
0				
1/2	1/2			
1/2	-1/6	2/3		
1	1/3	-1/3	1	
	1/6	1/3	1/3	1/6

4 (8) stages, 4 (8) evaluations of RHS

Multirate method of Schlegel

Consider locally stable regions

Insert inner buffer elements



$1\times(2\Delta t)$ $2\times(\Delta t)$ $1\times(2\Delta t)$

0				
1/2	1/2			
1/2	-1/6	2/3		
1	1/3	-1/3	1	
	1/6	1/3	1/3	1/6

4 (8) stages, 4 (8) evaluations of RHS

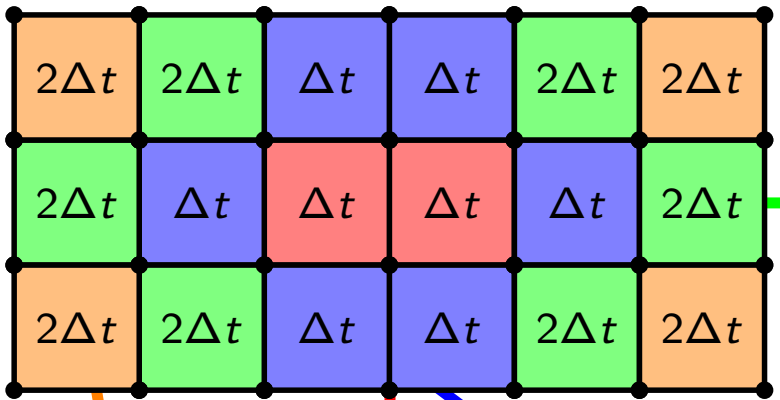
0									
1/4	1/4								
1/4	-1/12	1/3							
1/2	1/6	-1/6	1/2						
1/2	1/12	1/6	1/6	1/12					
1/2	1/12	1/6	1/6	1/12	0				
3/4	1/12	1/6	1/6	1/12	0	1/4			
3/4	1/12	1/6	1/6	1/12	0	-1/12	1/3		
1	1/12	1/6	1/6	1/12	0	1/6	-1/6	1/2	
1	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12
	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12	0

10 stages, 8 evaluations of RHS

Multirate method of Schlegel

Consider locally stable regions

Insert outer buffer elements



10 stages, 4 evaluations of RHS

$1\times(2\Delta t)$

0									
1/4	1/4								
1/4	1/4	0							
1/2	1/2	0	0						
1/2	1/2	0	0	0					
1/2	-1/6	0	0	0	2/3				
3/4	1/12	0	0	0	1/6	1/2			
3/4	1/12	0	0	0	1/6	1/2	0		
1	1/3	0	0	0	-1/3	1	0	0	
1	1/3	0	0	0	-1/3	1	0	0	0
	1/6	0	0	0	1/3	1/3	0	0	0	1/6

$1\times(2\Delta t)$ $2\times(\Delta t)$ $1\times(2\Delta t)$

0				
1/2	1/2			
1/2	-1/6	2/3		
1	1/3	-1/3	1	
	1/6	1/3	1/3	1/6

4 (8) stages, 4 (8) evaluations of RHS

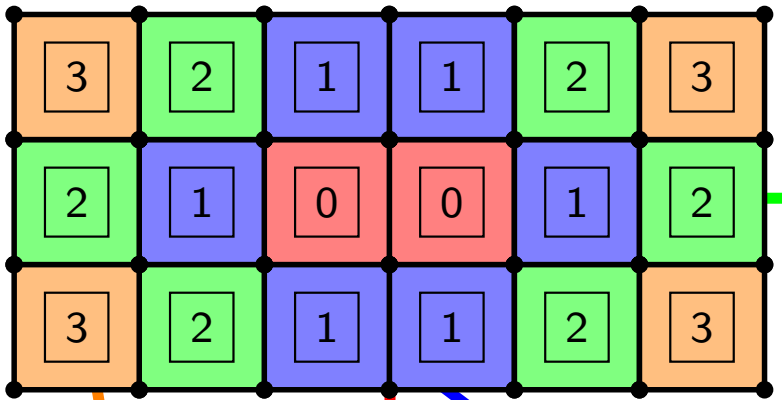
0										
1/4	1/4									
1/4	-1/12	1/3								
1/2	1/6	-1/6	1/2							
1/2	1/12	1/6	1/6	1/12						
1/2	1/12	1/6	1/6	1/12	0					
3/4	1/12	1/6	1/6	1/12	0	1/4				
3/4	1/12	1/6	1/6	1/12	0	-1/12	1/3			
1	1/12	1/6	1/6	1/12	0	1/6	-1/6	1/2		
1	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12	
	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12	0

10 stages, 8 evaluations of RHS

Multirate method of Schlegel

Consider locally stable regions

Associate tags for simplicity



$1\times(2\Delta t)$

$2\times(\Delta t)$

$1\times(2\Delta t)$

$1\times(2\Delta t)$

10 stages, 4 evaluations of RHS

0									
1/4	1/4								
1/4	1/4	0							
1/2	1/2	0	0						
1/2	1/2	0	0	0					
1/2	-1/6	0	0	0	2/3				
3/4	1/12	0	0	0	1/6	1/2			
3/4	1/12	0	0	0	1/6	1/2	0		
1	1/3	0	0	0	-1/3	1	0	0	
1	1/3	0	0	0	-1/3	1	0	0	0
	1/6	0	0	0	1/3	1/3	0	0	0	1/6

0				
1/2	1/2			
1/2	-1/6	2/3		
1	1/3	-1/3	1	
	1/6	1/3	1/3	1/6

4 (8) stages, 4 (8) evaluations of RHS

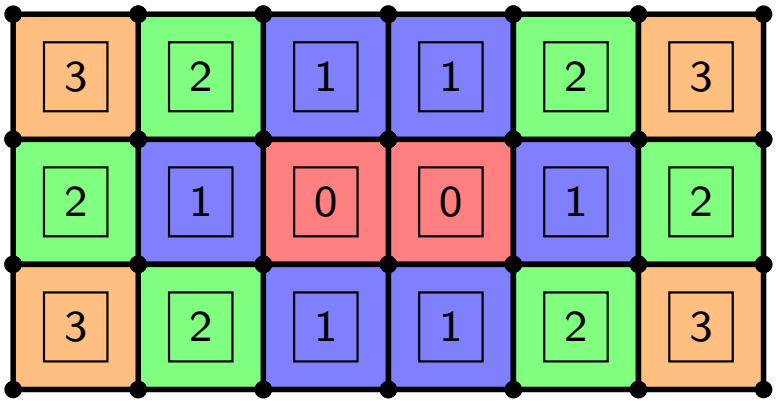
0										
1/4	1/4									
1/4	-1/12	1/3								
1/2	1/6	-1/6	1/2							
1/2	1/12	1/6	1/6	1/12						
1/2	1/12	1/6	1/6	1/12	0					
3/4	1/12	1/6	1/6	1/12	0	1/4				
3/4	1/12	1/6	1/6	1/12	0	-1/12	1/3			
1	1/12	1/6	1/6	1/12	0	1/6	-1/6	1/2		
1	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12	
	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12	0

10 stages, 8 evaluations of RHS

Multirate method of Schlegel

Consider locally stable regions

Inner/Outer buffer interface: convergence?



0									
1/4	1/4								
1/4	1/4	0							
1/2	1/2	0	0						
1/2	1/2	0	0	0					
1/2	-1/6	0	0	0	2/3				
3/4	1/12	0	0	0	1/6	1/2			
3/4	1/12	0	0	0	1/6	1/2	0		
1	1/3	0	0	0	-1/3	1	0	0	
1	1/3	0	0	0	-1/3	1	0	0	0
	1/6	0	0	0	1/3	1/3	0	0	0	1/6

This method is third order accurate!

$$\sum_{i=1}^{s+1} \tilde{c}_i \sum_{j=1}^{s} (A_{ij} + A_{i-1,j}) c_j = 1/3$$

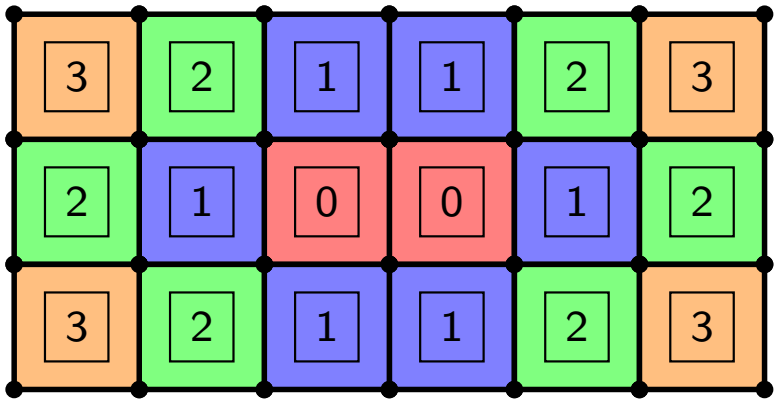
"*It is an open question whether multirate schemes constructed from methods satisfying the additional third order conditions are generally of third order.*"
M. Schlegel

0									
1/4	1/4								
1/4	-1/12	1/3							
1/2	1/6	-1/6	1/2						
1/2	1/12	1/6	1/6	1/12					
1/2	1/12	1/6	1/6	1/12	0				
3/4	1/12	1/6	1/6	1/12	0	1/4			
3/4	1/12	1/6	1/6	1/12	0	-1/12	1/3		
1	1/12	1/6	1/6	1/12	0	1/6	-1/6	1/2	
1	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12
	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12	0

Multirate method of Schlegel

Consider locally stable regions

Inner/Outer buffer interface: conservation?



0									
1/4	1/4								
1/4	1/4	0							
1/2	1/2	0	0						
1/2	1/2	0	0	0					
1/2	-1/6	0	0	0	2/3				
3/4	1/12	0	0	0	1/6	1/2			
3/4	1/12	0	0	0	1/6	1/2	0		
1	1/3	0	0	0	-1/3	1	0	0	
1	1/3	0	0	0	-1/3	1	0	0	0
	1/6	0	0	0	1/3	1/3	0	0	0	1/6

This method is not conservative!

$b^{Inner} \neq b^{Outer}$

But:

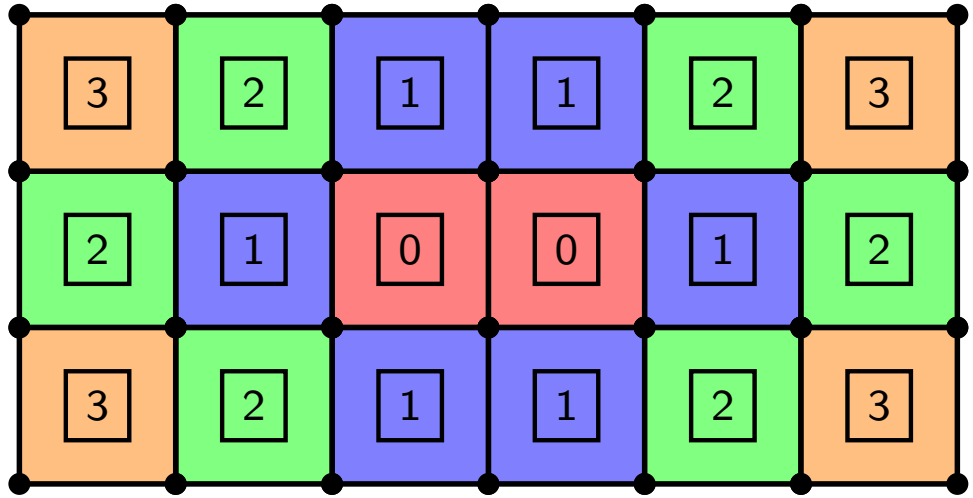
Choose the most accurate flux (inner buffer) for interface terms.

$\Rightarrow$ Consistency?

0									
1/4	1/4								
1/4	-1/12	1/3							
1/2	1/6	-1/6	1/2						
1/2	1/12	1/6	1/6	1/12					
1/2	1/12	1/6	1/6	1/12	0				
3/4	1/12	1/6	1/6	1/12	0	1/4			
3/4	1/12	1/6	1/6	1/12	0	-1/12	1/3		
1	1/12	1/6	1/6	1/12	0	1/6	-1/6	1/2	
1	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12
	1/12	1/6	1/6	1/12	0	1/12	1/6	1/6	1/12	0

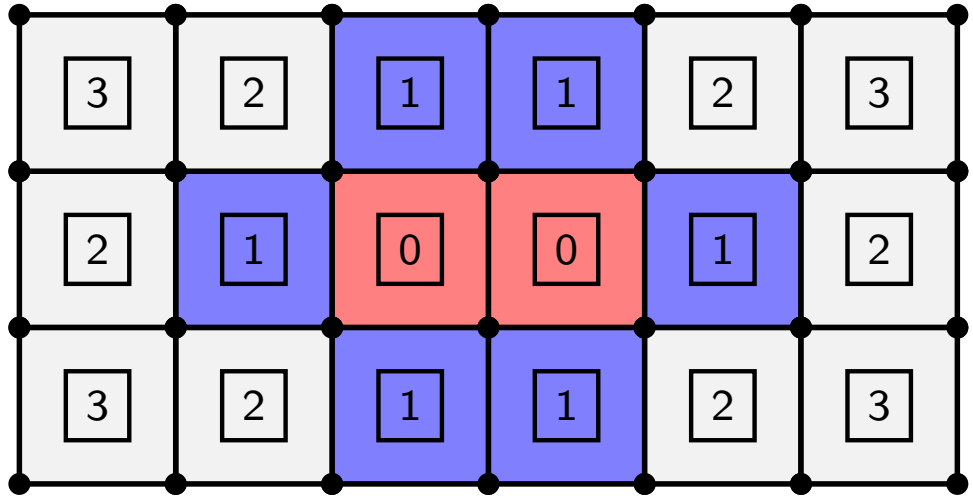
Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



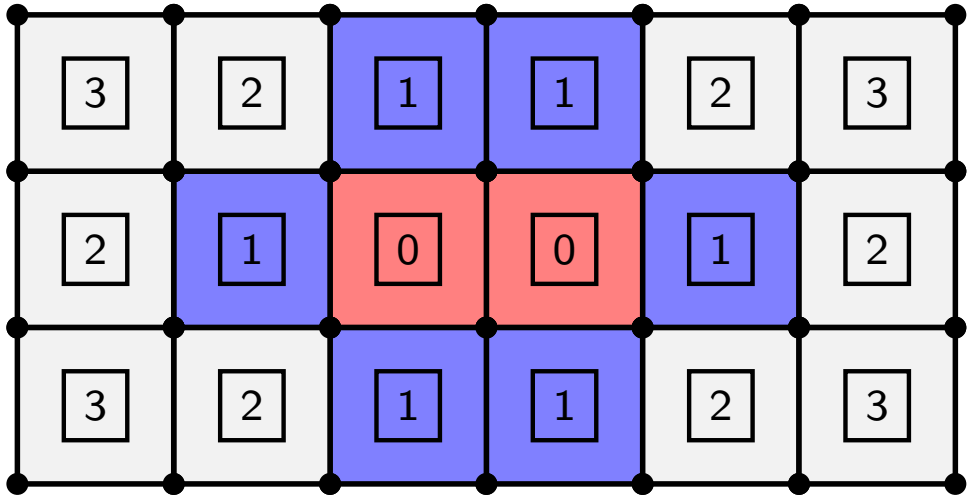
Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



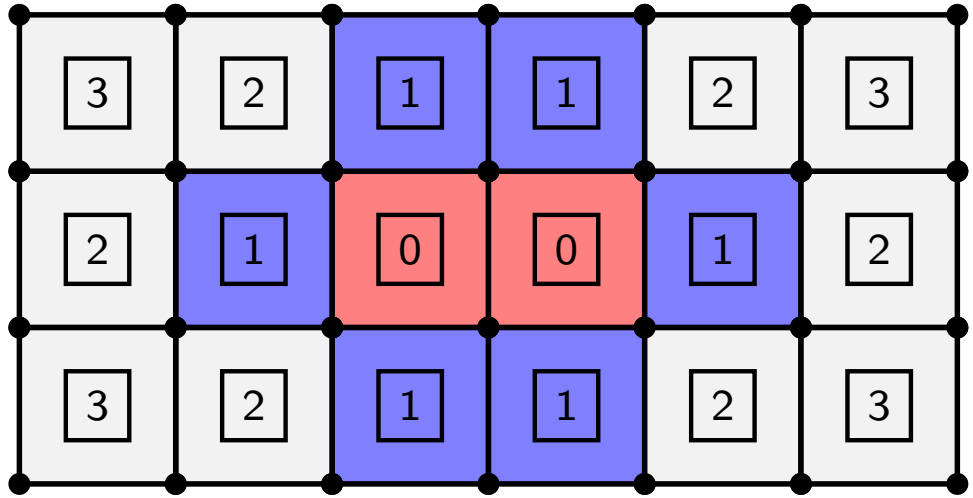
Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



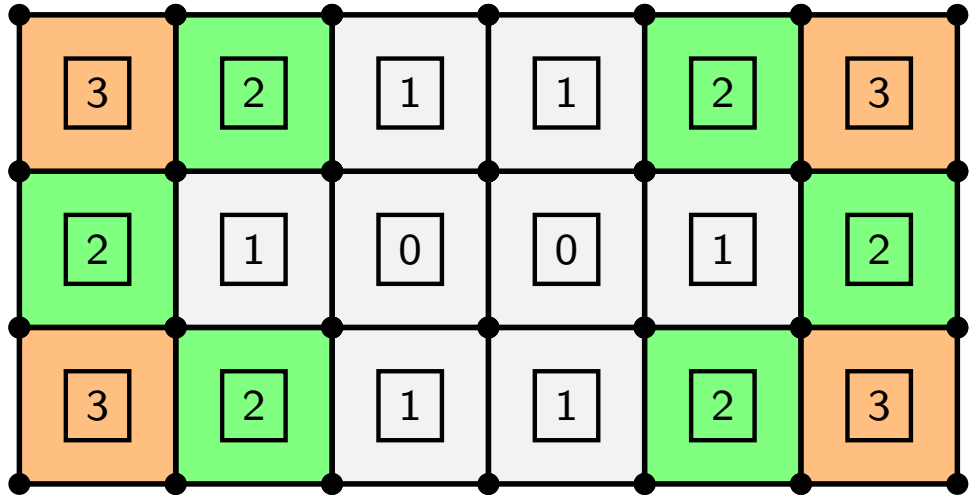
Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



Sequential algorithm for RK43S

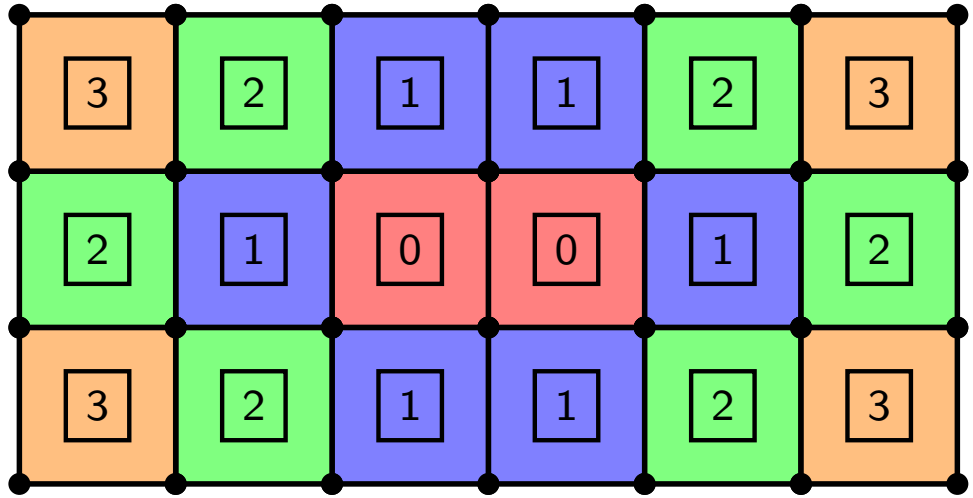
	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



0 $\quad U^{n+\frac{1}{2}} = U^n + \frac{\Delta t}{2}(\frac{1}{6}K_0 + \frac{1}{3}K_1 + \frac{1}{3}K_2 + \frac{1}{6}K_3)$

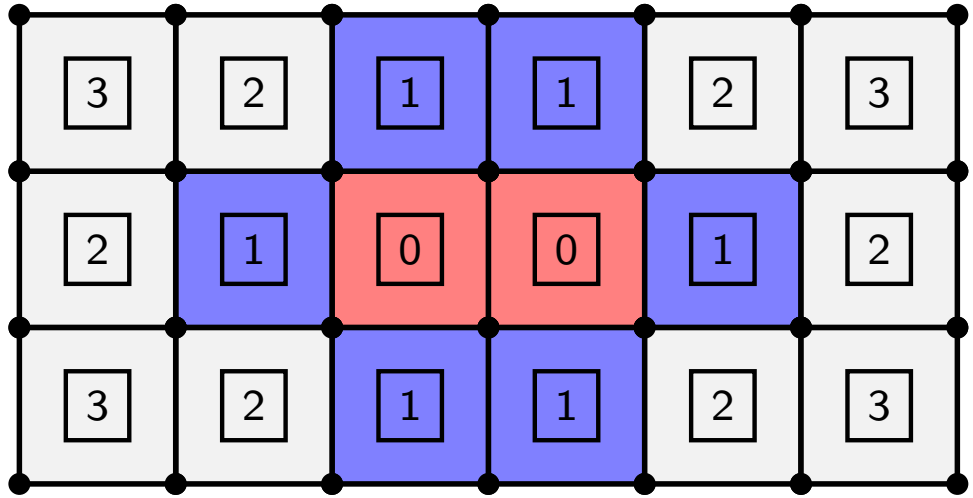
Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



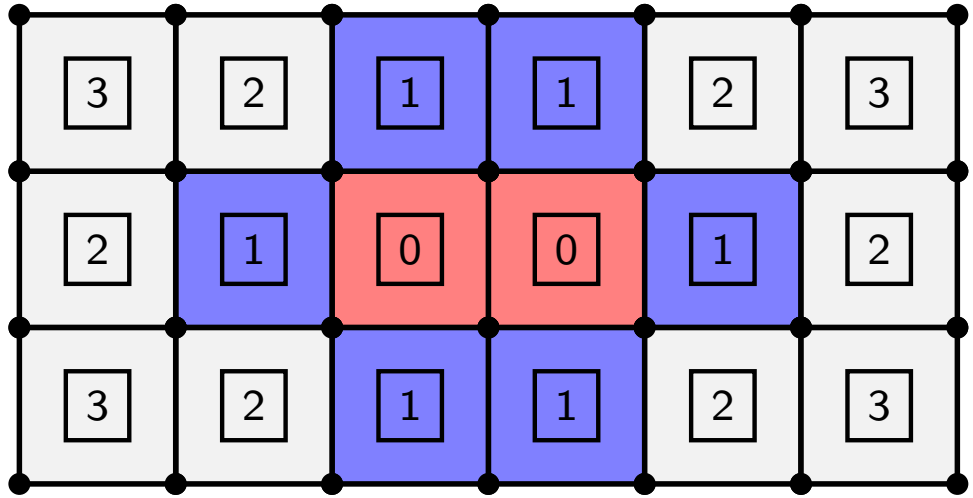
Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



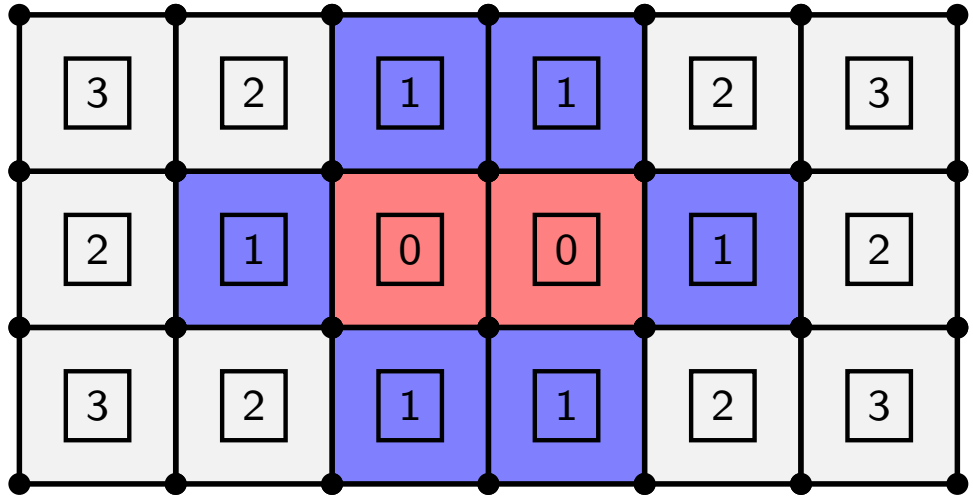
Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



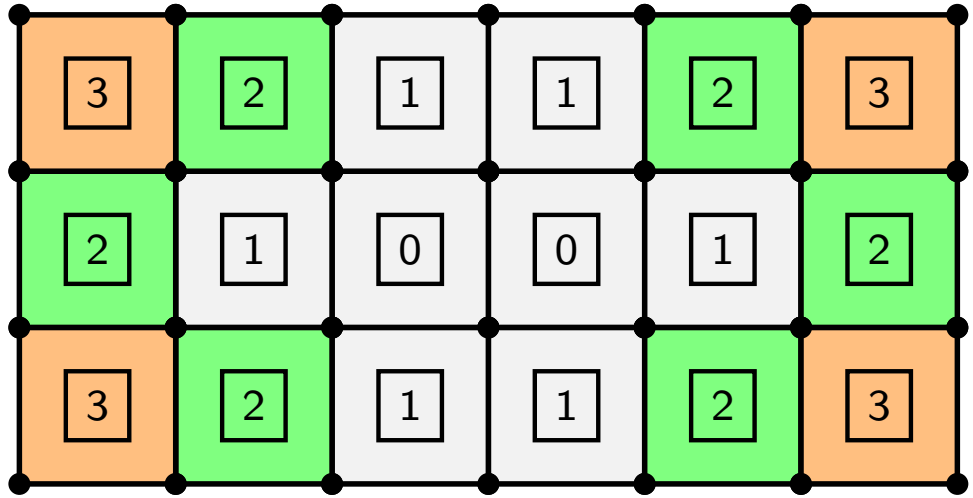
Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

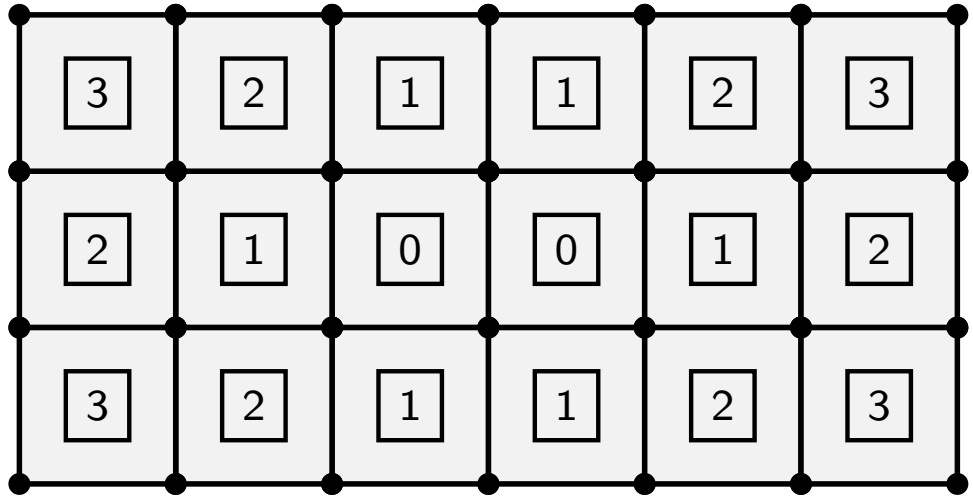


0 $$U^{n+1} = U^{n+\frac{1}{2}} + \frac{\Delta t}{2}(\frac{1}{6}K_0 + \frac{1}{3}K_1 + \frac{1}{3}K_2 + \frac{1}{6}K_3)$$

1 $$U^{n+1} = U^n + \Delta t(\frac{1}{12}K_0 + \frac{1}{6}K_1 + \frac{1}{6}K_2 + \frac{1}{12}K_3 + \frac{1}{12}K_5 + \frac{1}{6}K_6 + \frac{1}{6}K_7 + \frac{1}{12}K_8)$$

Sequential algorithm for RK43S

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		



2 $U^{n+1} = U^n + \Delta t(\frac{1}{6}K_0 + \frac{1}{3}K_4 + \frac{1}{3}K_5 + \frac{1}{6}K_9)$

3 $U^{n+1} = U^n + \Delta t(\frac{1}{6}K_0 + \frac{1}{3}K_1 + \frac{1}{3}K_2 + \frac{1}{6}K_3)$

Advection-Diffusion testcase (1)

Physical, numerical and multirate setup

Equation

- $\frac{\partial c}{\partial t} + v_y \frac{\partial c}{\partial y} = \nu(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2})$

Parameters

- $v_y = 0.1$ ms^{-1}
- $\nu = 15 \cdot 10^{-6}$ m^{2}s^{-1}

Boundaries

- Top and bottom: periodic
- Left and right: walls

Mesh

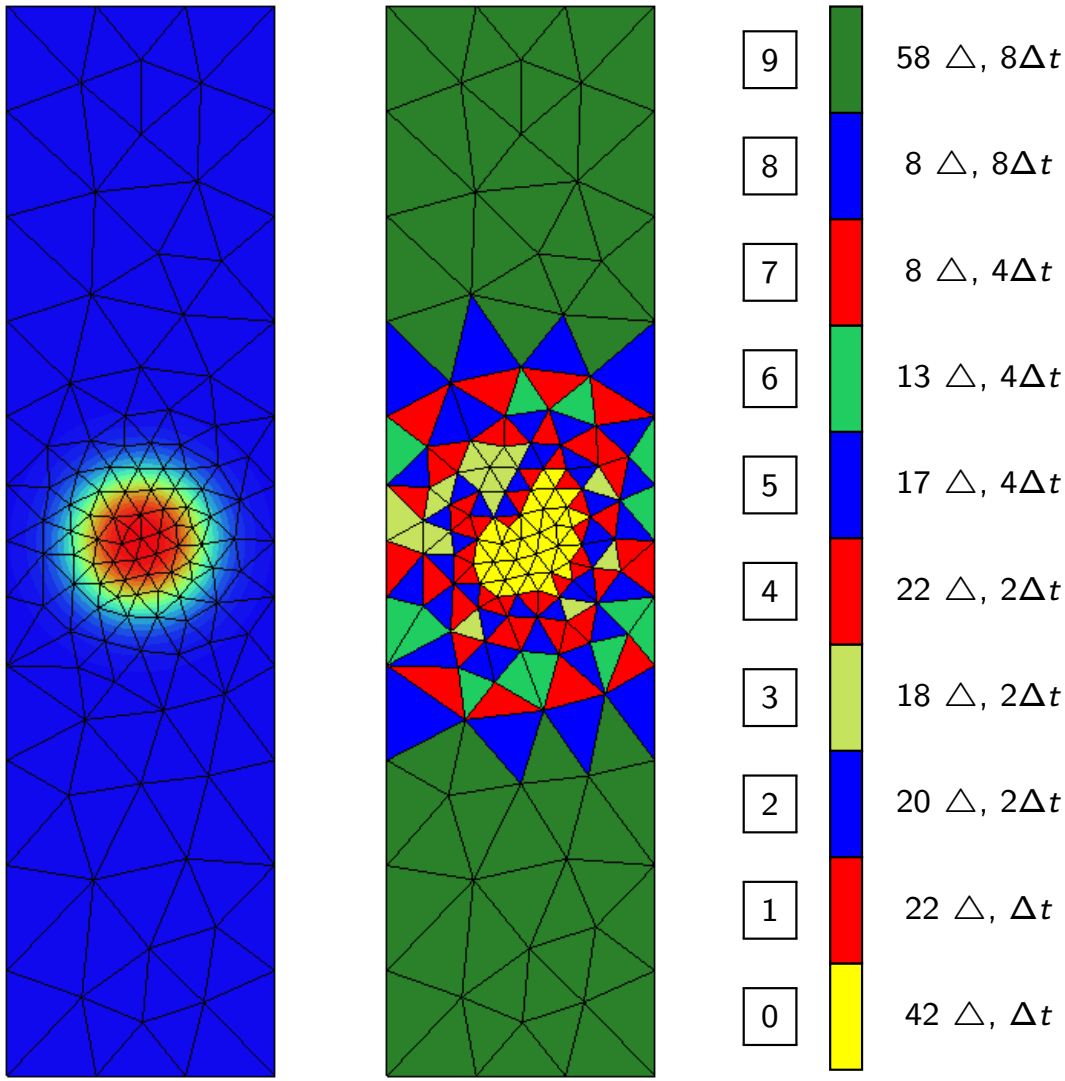
- 228 P_3^{DG} triangles

CFL

- $\Delta t_{\text{max}} / \Delta t_{\text{min}} \approx 17.31$

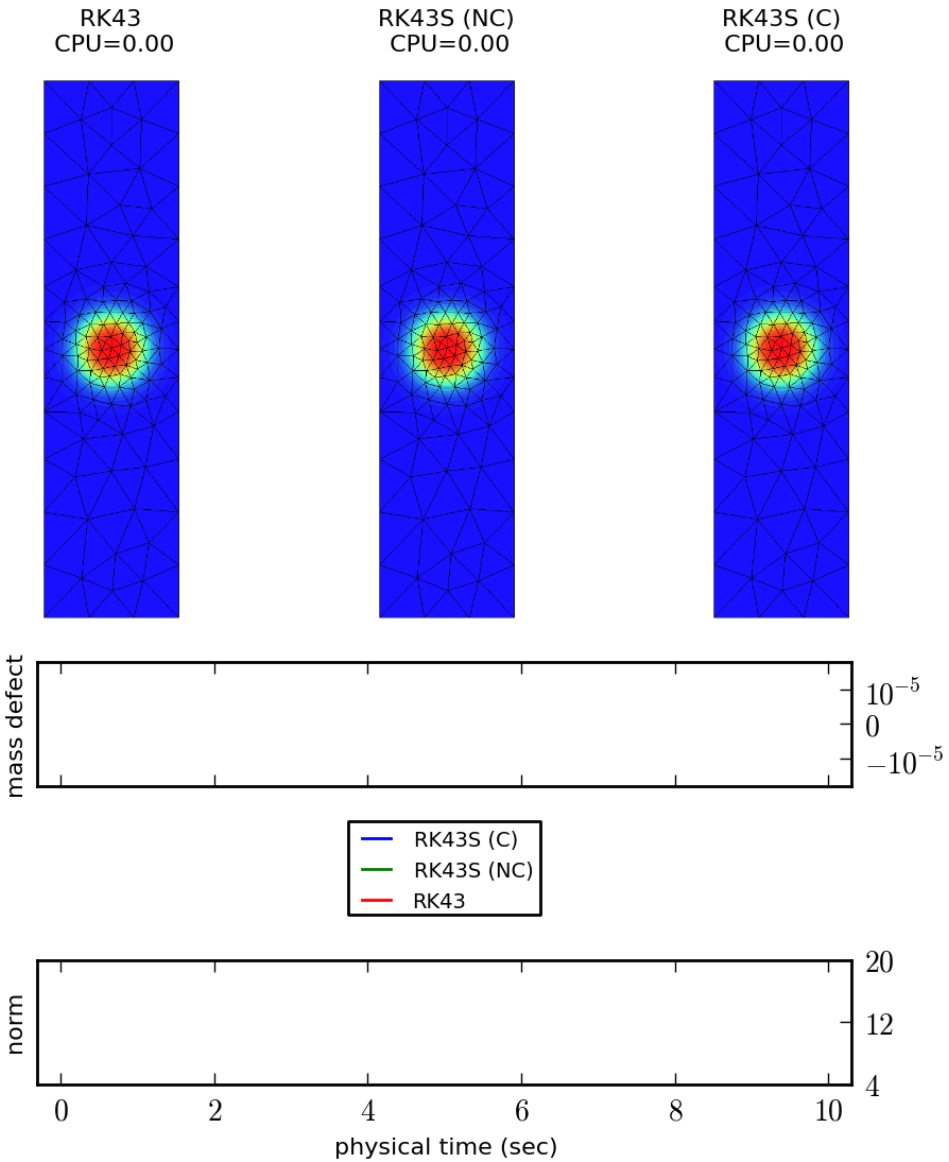
Multirate

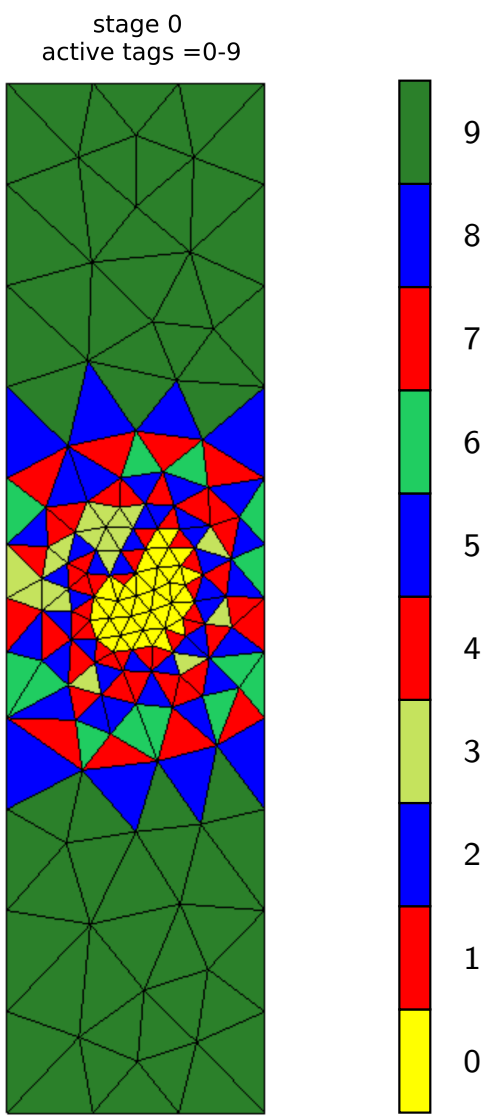
- RK43S $\rightarrow$ Th. SU $= 2.04$



Advection-Diffusion testcase (2)

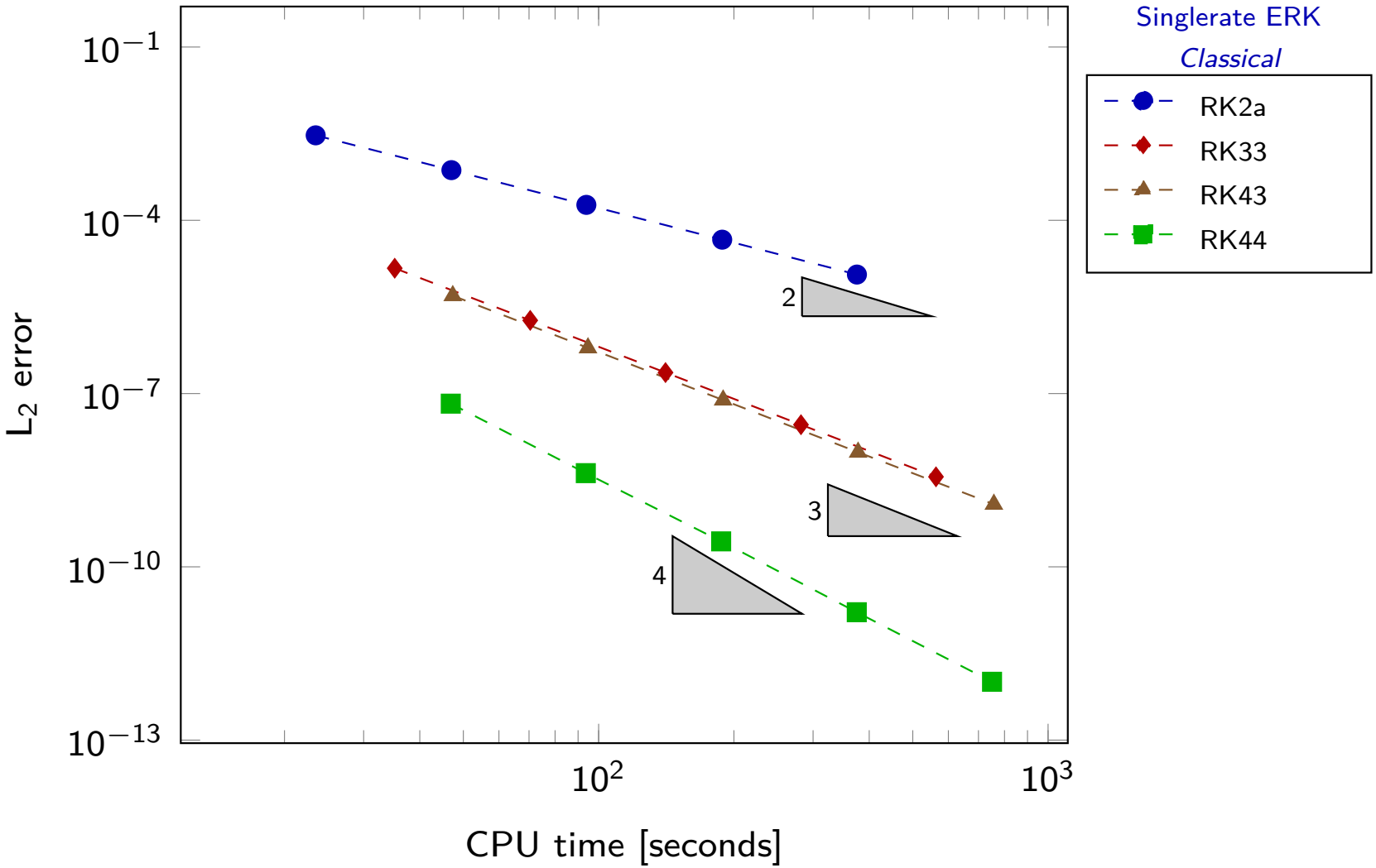
Algorithm, experimental speedup, conservation, accuracy.

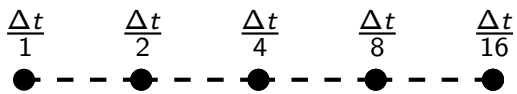


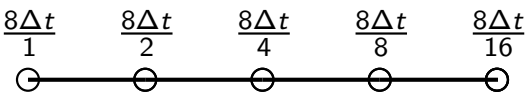


Advection-Diffusion testcase (3)

Temporal convergence (after 10 sec)

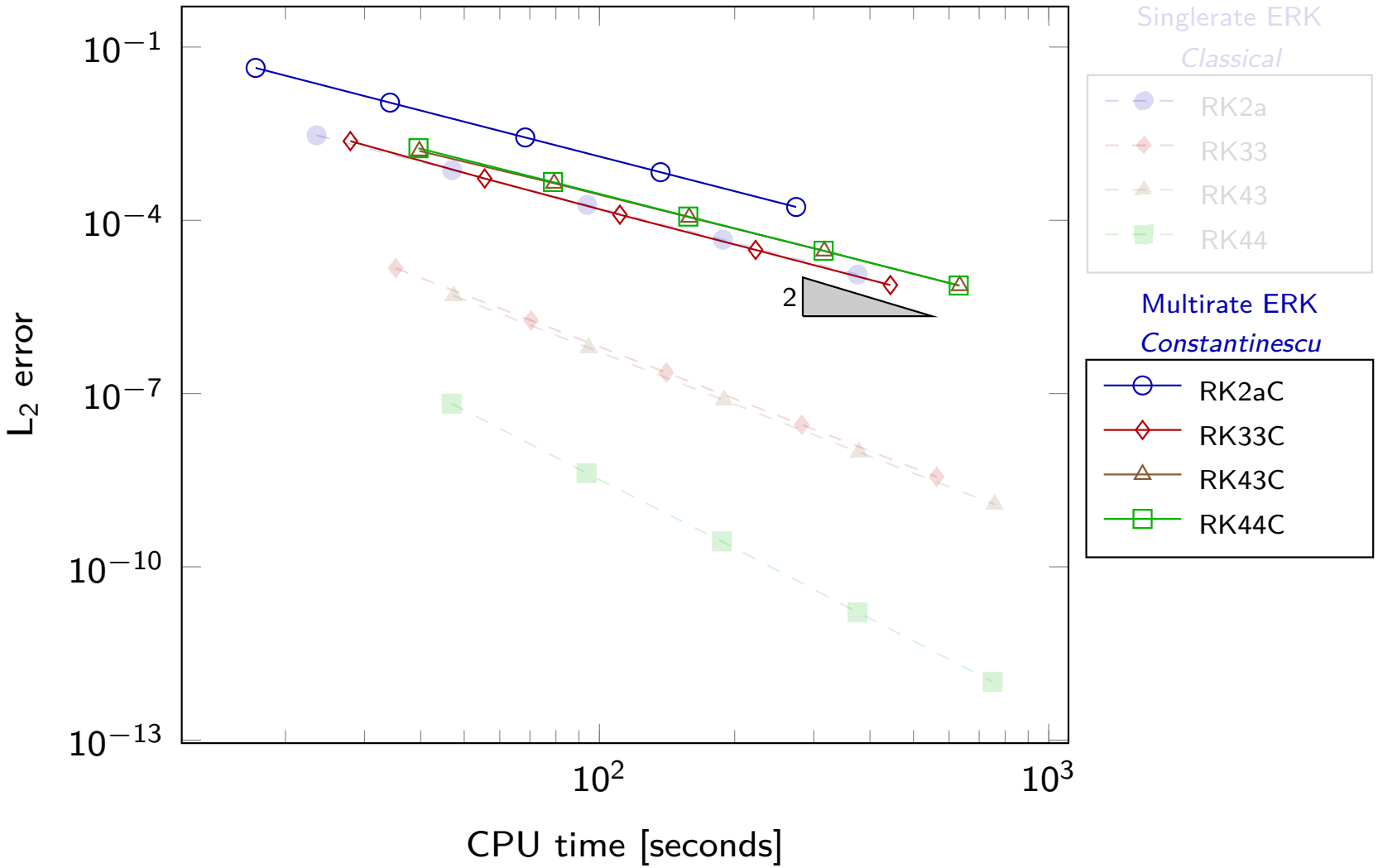


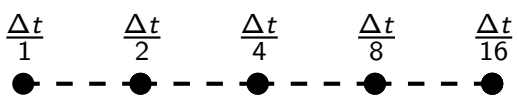


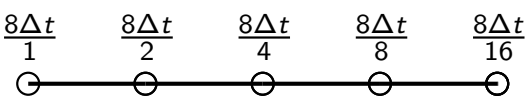


Advection-Diffusion testcase (3)

Temporal convergence (after 10 sec)

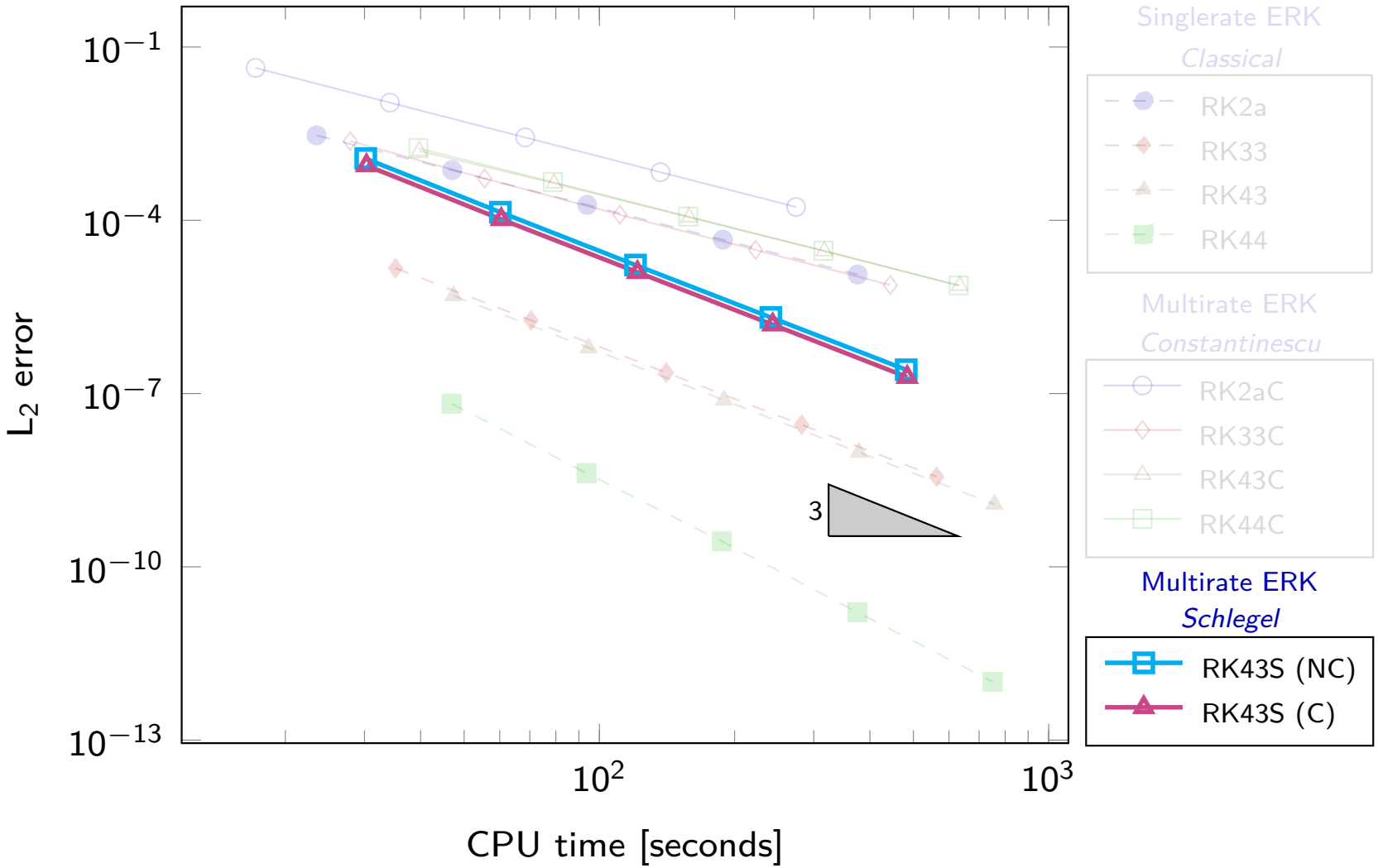


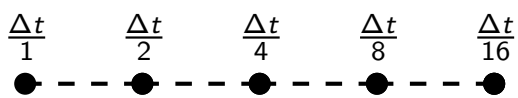


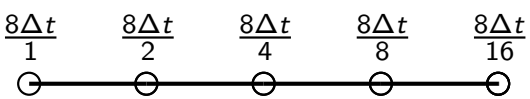


Advection-Diffusion testcase (3)

Temporal convergence (after 10 sec)

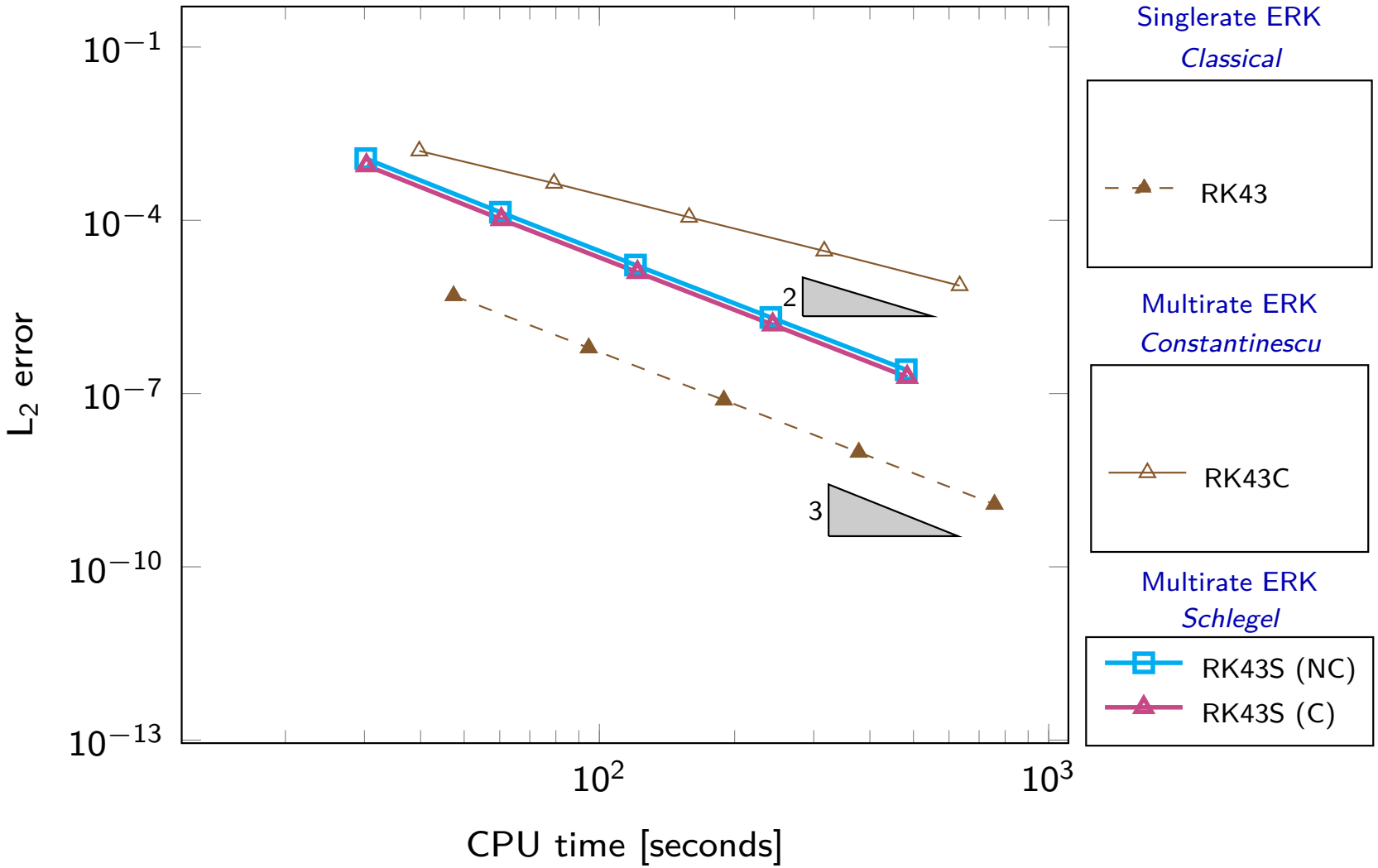


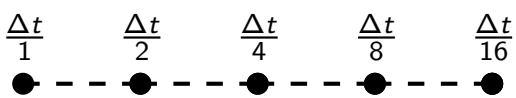


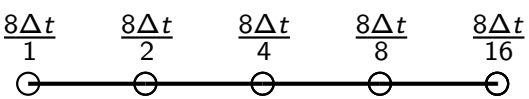


Advection-Diffusion testcase (3)

Temporal convergence (after 10 sec)





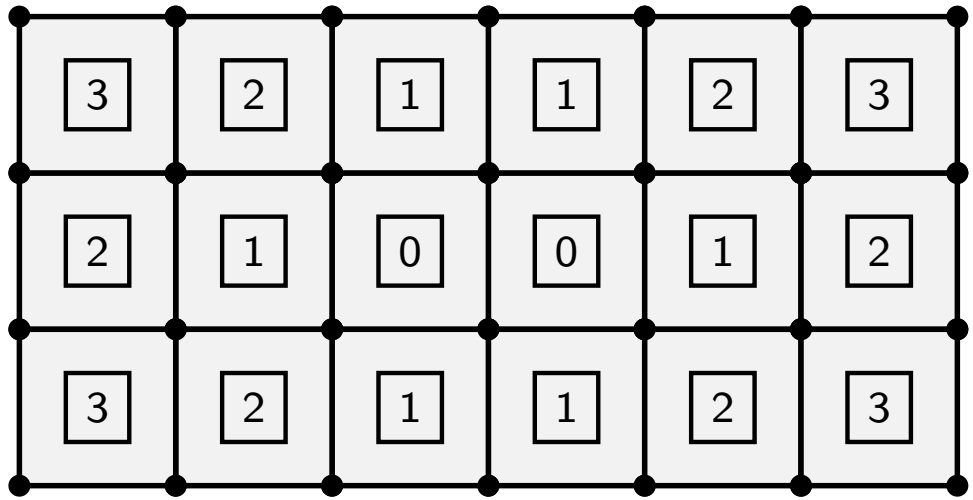


Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Total number of RHS evaluations = 104



The computational load varies at each stage of the algorithm and involves subsets of elements.

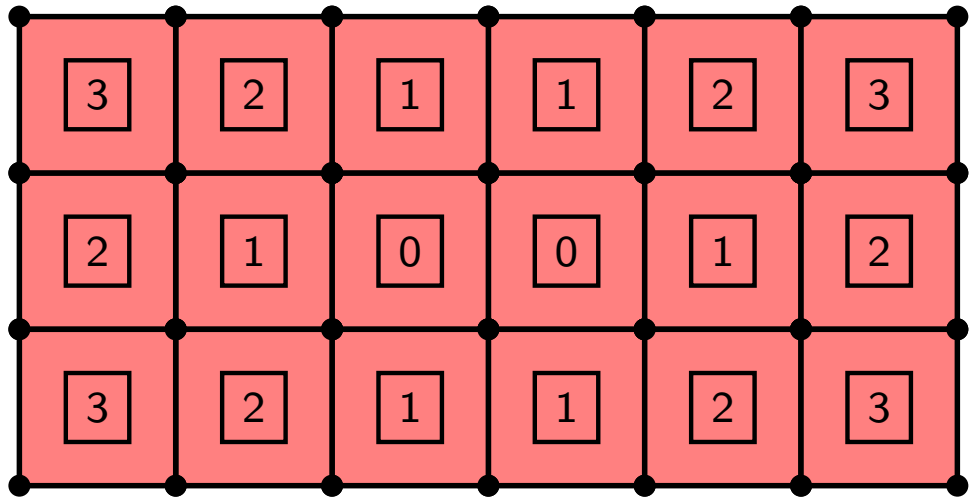
$\Rightarrow$ 3 different subsets

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Total number of RHS evaluations = 104



Subset 1: 36 RHS evaluations

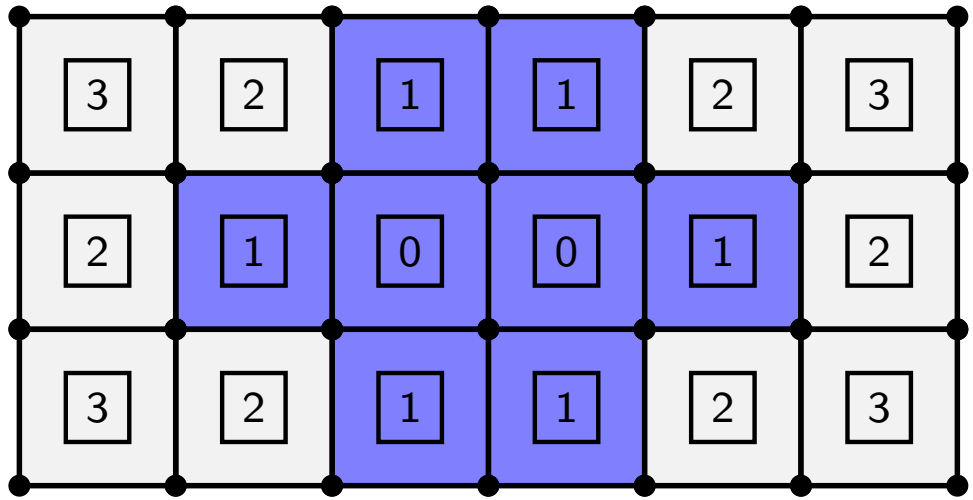
34.6 % of total computational load

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Total number of RHS evaluations = 104



Subset 1: 36 RHS evaluations

34.6 % of total computational load

Subset 2: 48 RHS evaluations

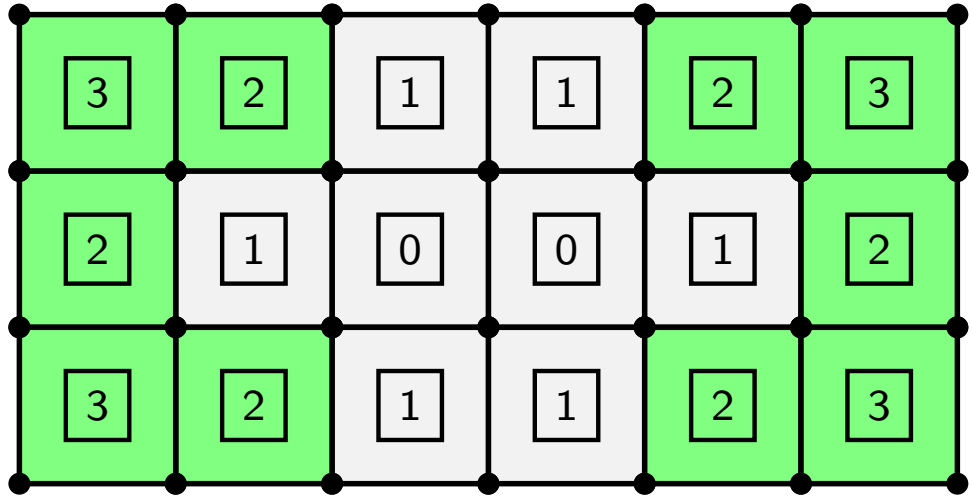
46.2 % of total computational load

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Total number of RHS evaluations = 104



Subset 1: 36 RHS evaluations

34.6 % of total computational load

Subset 2: 48 RHS evaluations

46.2 % of total computational load

Subset 3: 20 RHS evaluations

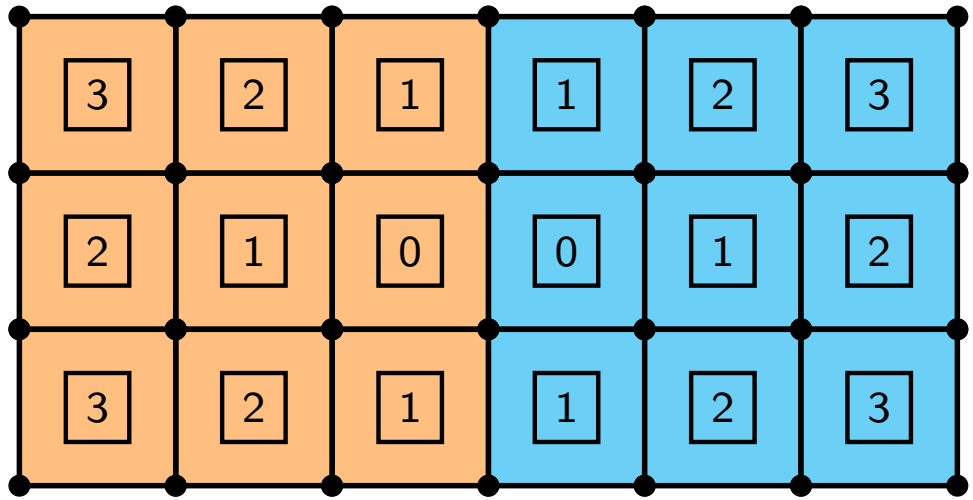
19.2 % of total computational load

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Equal sharing of the workload between 2 partitions



P1: 50%

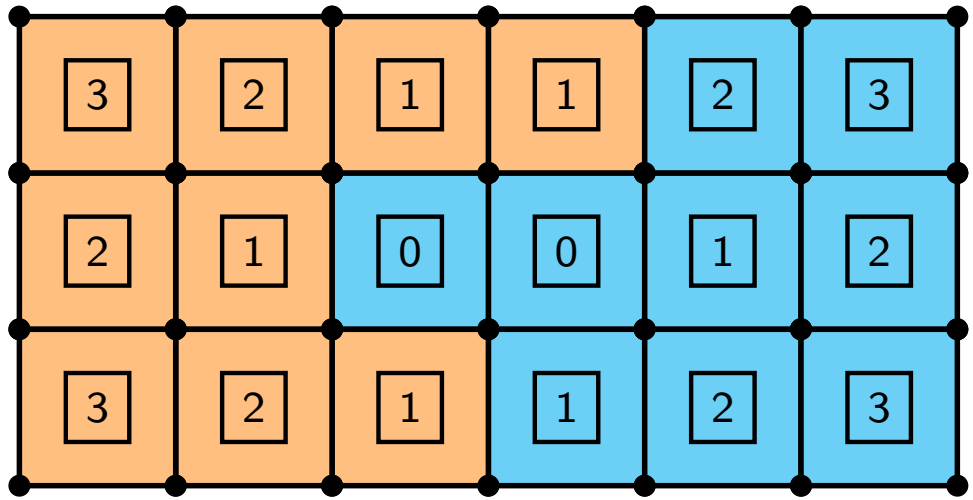
P2: 50 %

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Equal sharing of the workload between 2 partitions



P1: 50%

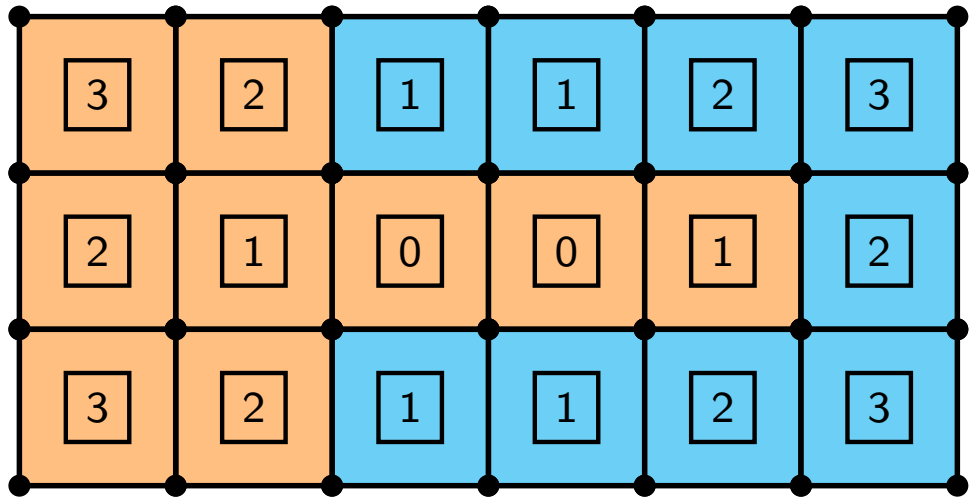
P2: 50 %

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Equal sharing of the workload between 2 partitions



P1: 50%

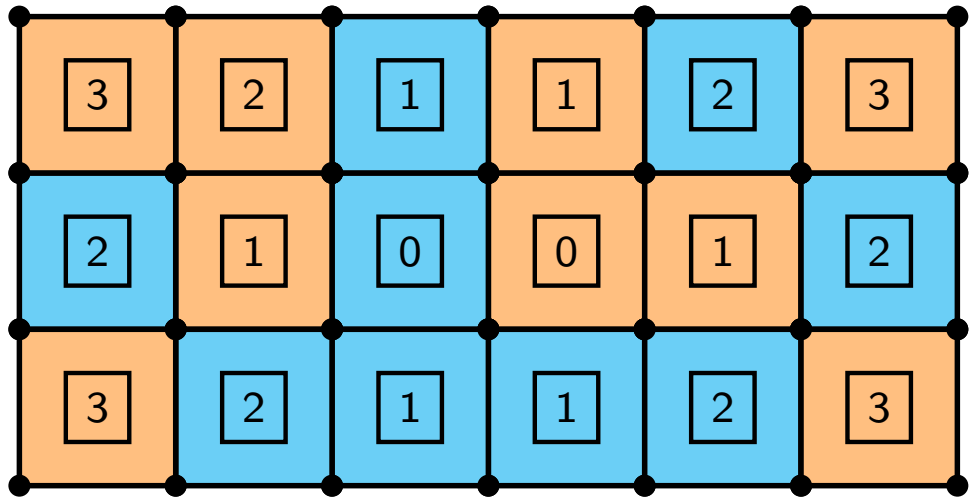
P2: 50 %

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Equal sharing of the workload between 2 partitions



P1: 50%

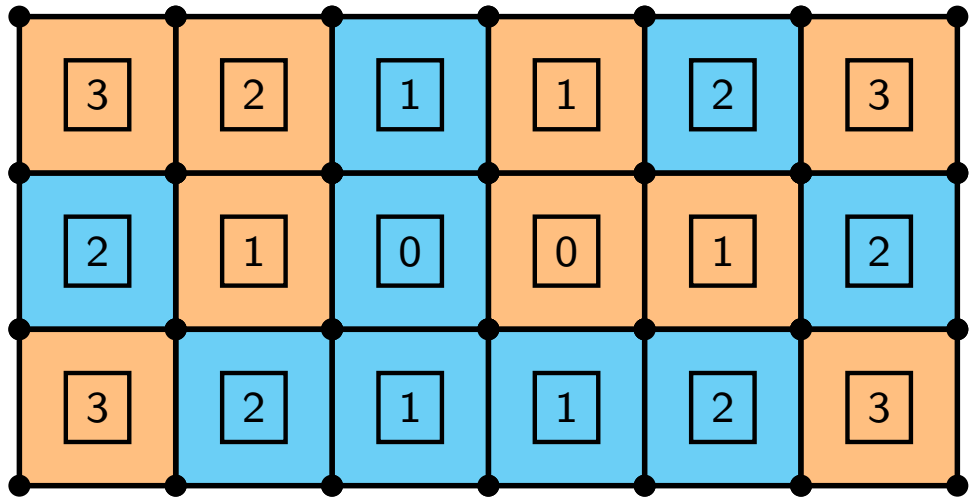
P2: 50 %

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Equal sharing of the workload between 2 partitions



P1: 50%

P2: 50 %

What about communication?

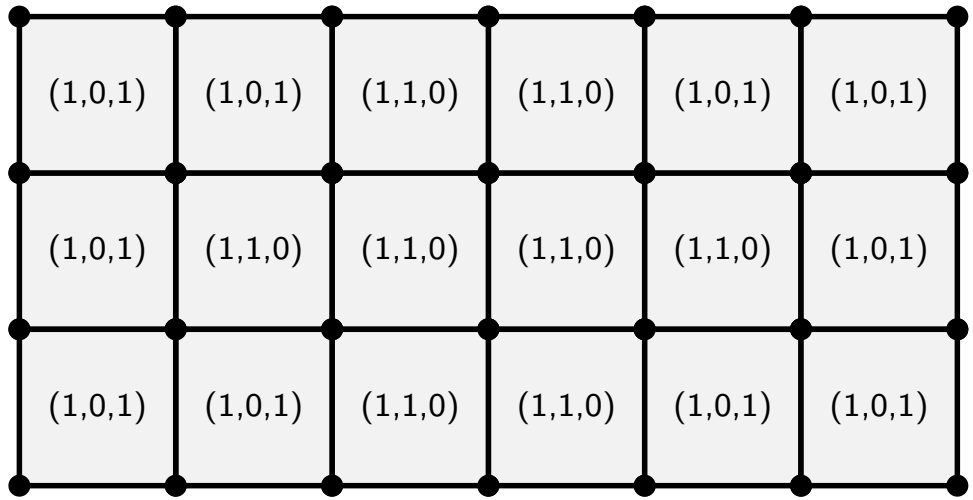
⇒assign weights to the interfaces

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Associate weighted graph for MeTiS partitioner



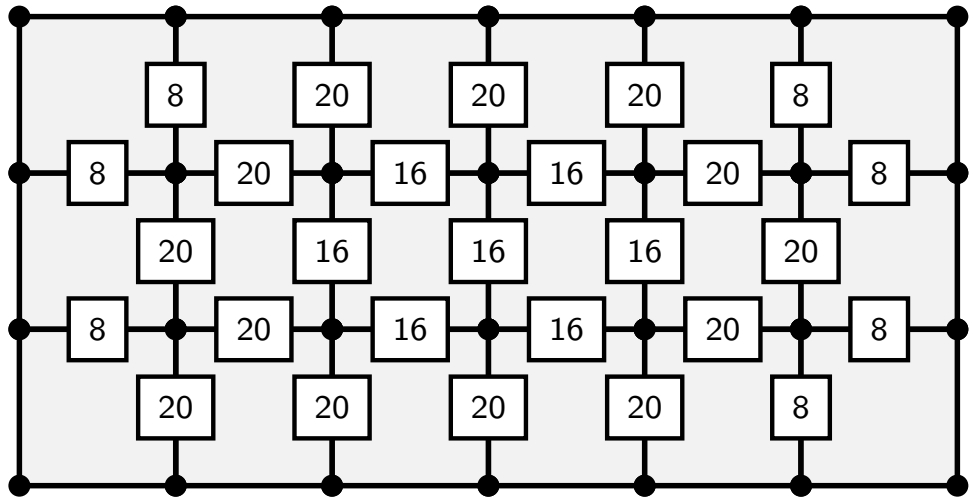
(Subset 1?, Subset 2?, Subset 3?)

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Associate weighted graph for MeTiS partitioner



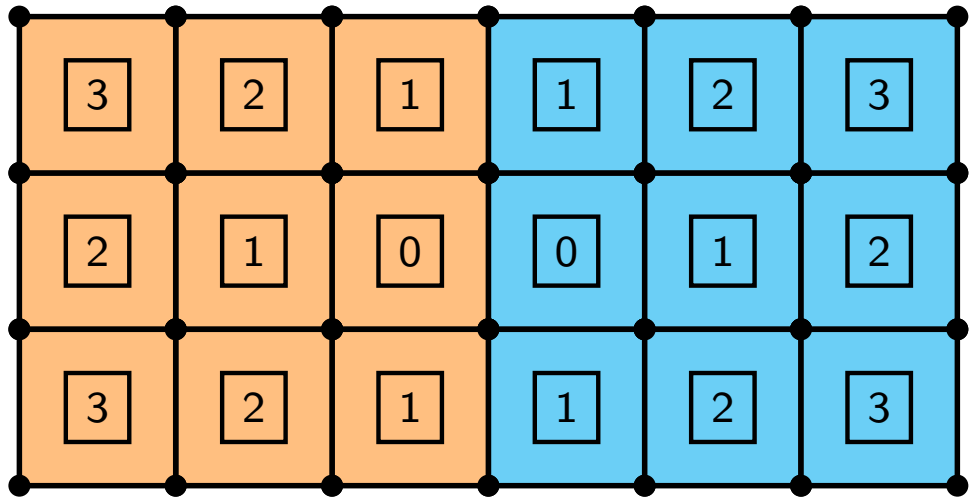
communications between two elements

Parallelization and partitioning

Load balancing and communication

	3	2	1	0
1	K_0	K_0	K_0	K_0
2			K_1	K_1
3			K_2	K_2
4			K_3	K_3
5	K_1	K_4		
6	K_2	K_5	K_5	K_0
7			K_6	K_1
8			K_7	K_2
9			K_8	K_3
10	K_3	K_9		

Optimal solution



P1: 50%

P2: 50 %

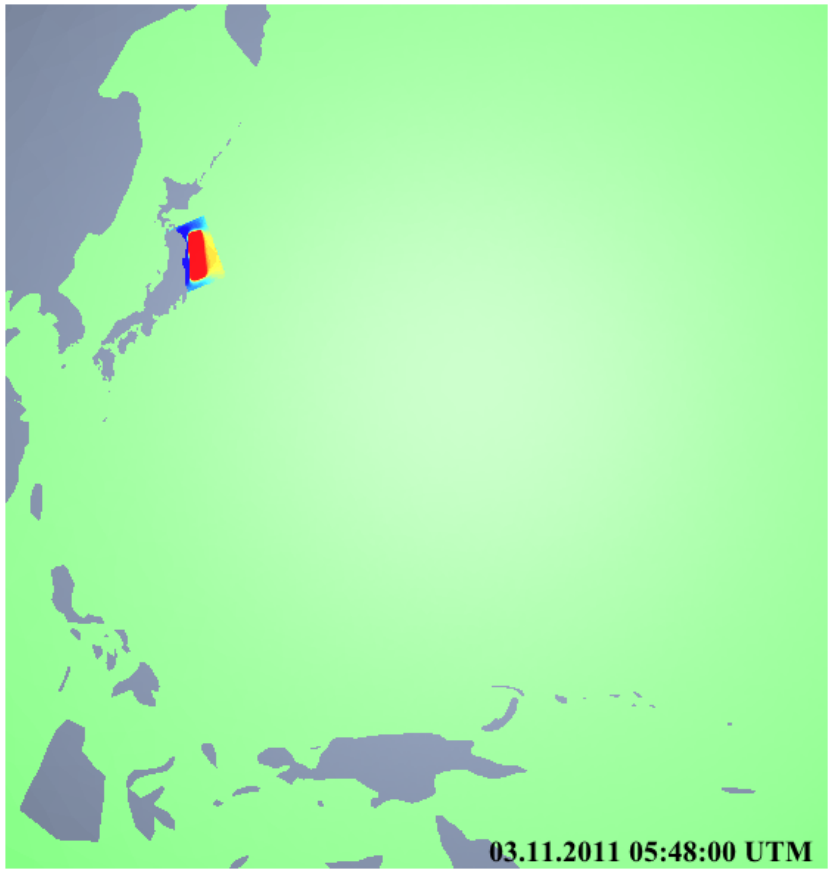
Communication volume = 56

Use non-blocking communications!

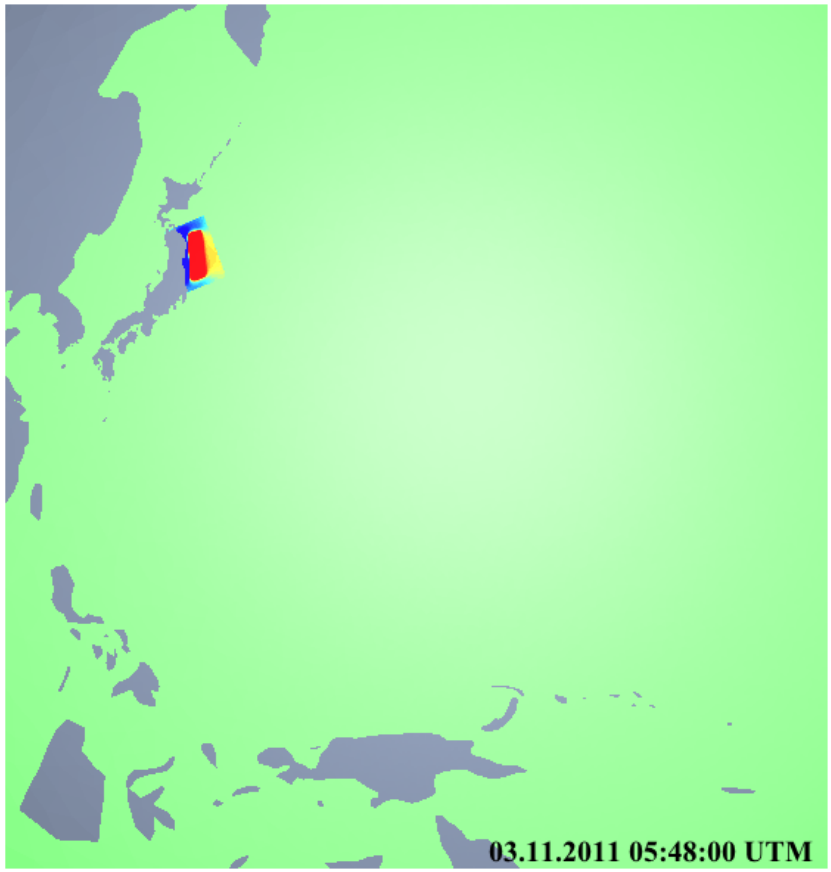
Illustration: Japan's tsunami

Simulation

Singlerate (RK43)



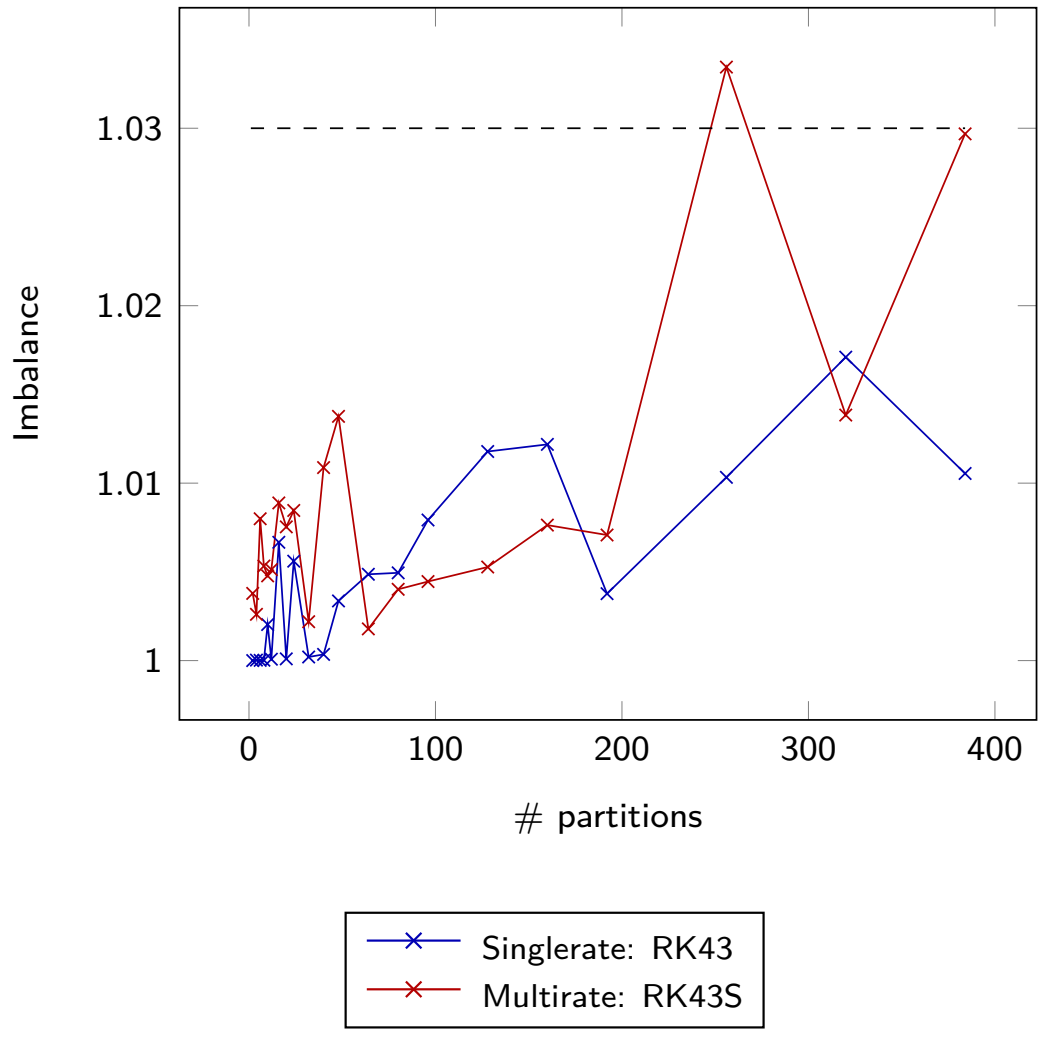
Multirate (RK43S)



Tsunami: parallel performance (1)

Quality of mesh partitionings

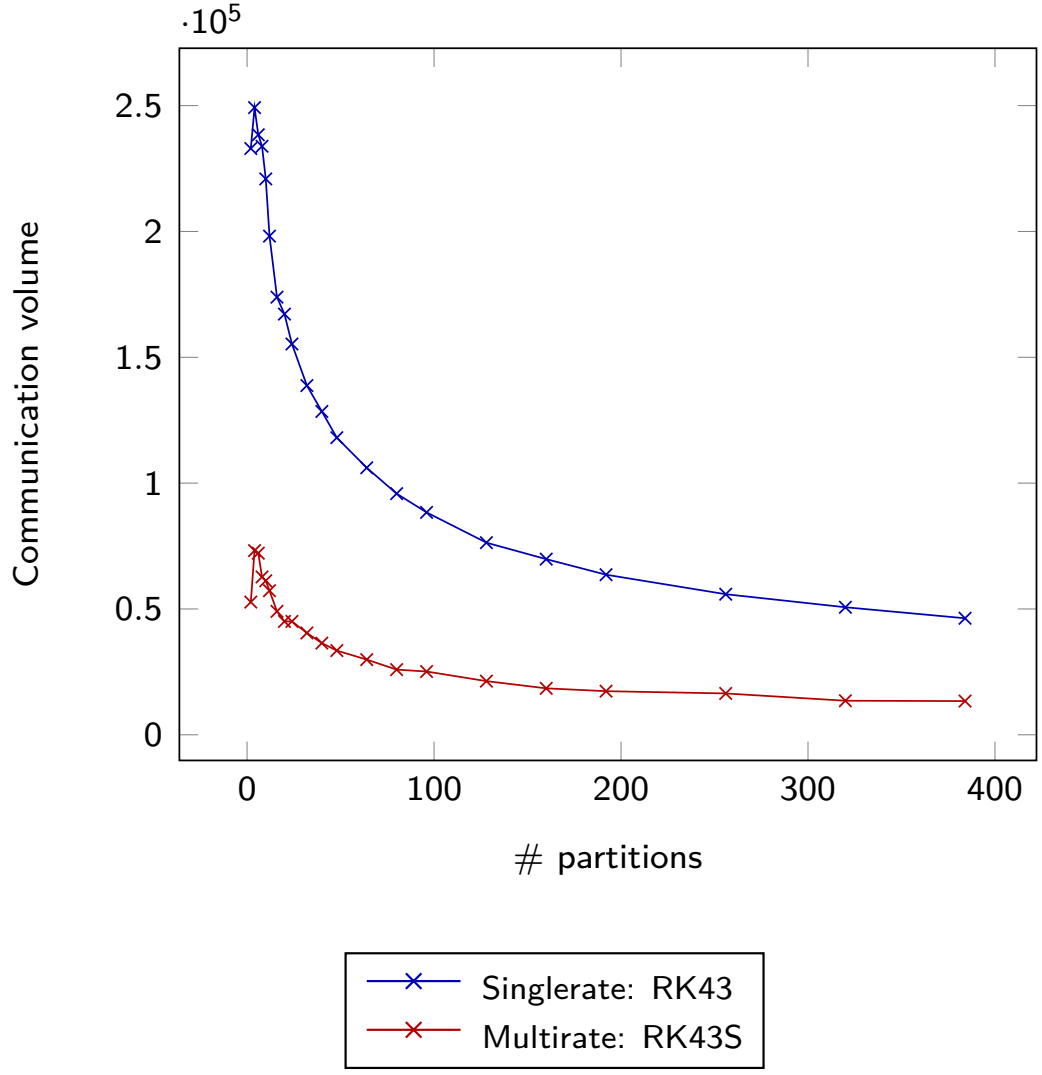
- 1757467 P_3^{DG} triangles
- 7 levels of refinement
- multirate speedup ≈ 10.91



Tsunami: parallel performance (1)

Quality of mesh partitionings

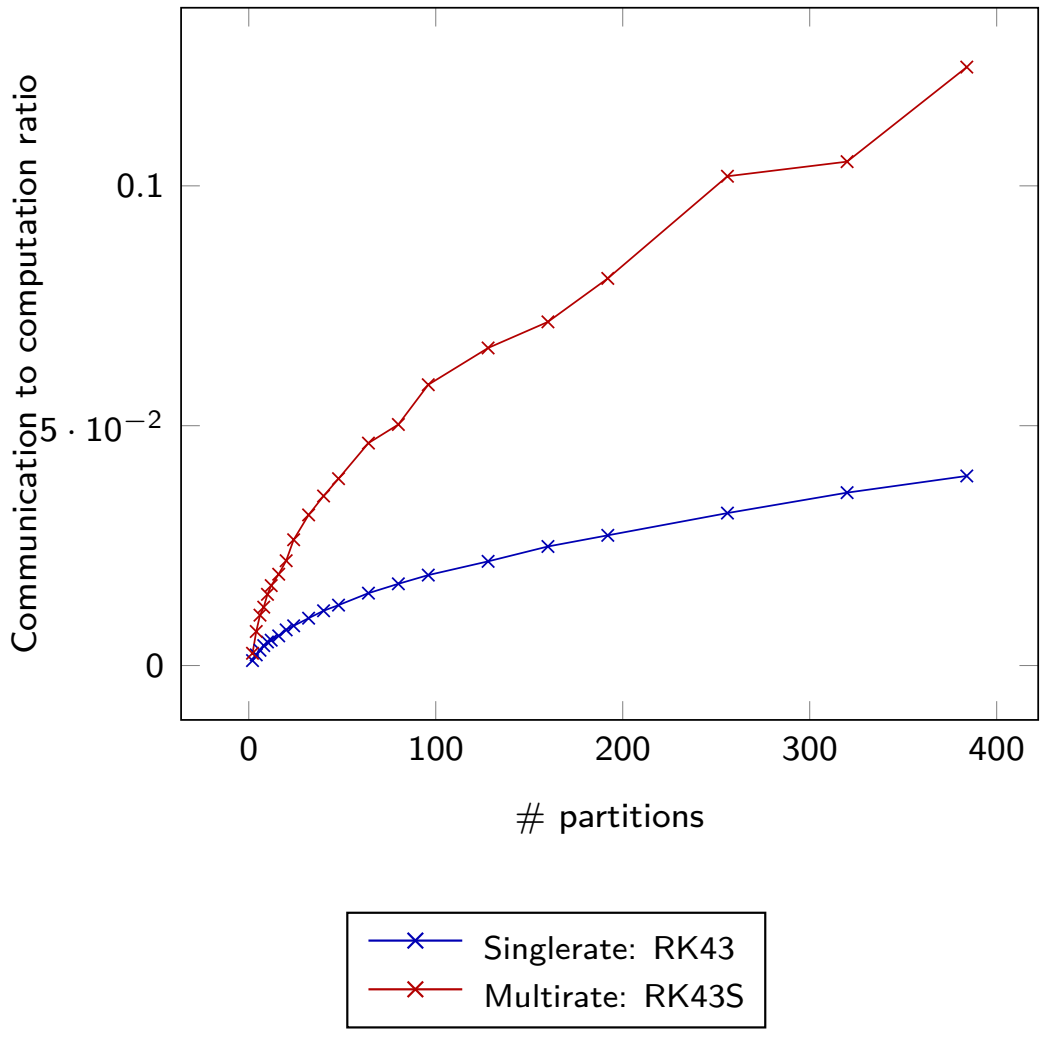
- 1757467 P_3^{DG} triangles
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Tsunami: parallel performance (1)

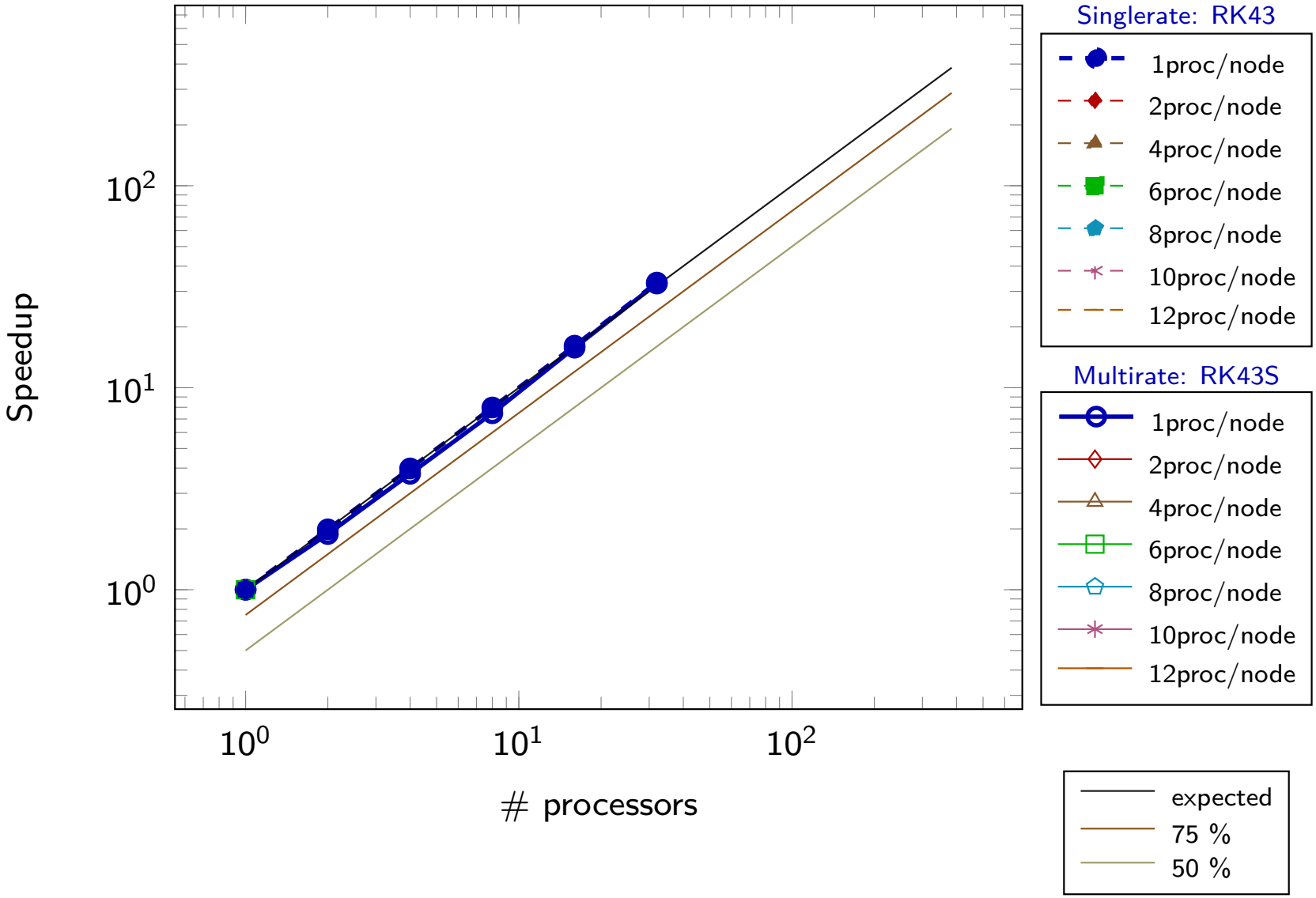
Quality of mesh partitionings

- 1757467 P_3^{DG} triangles
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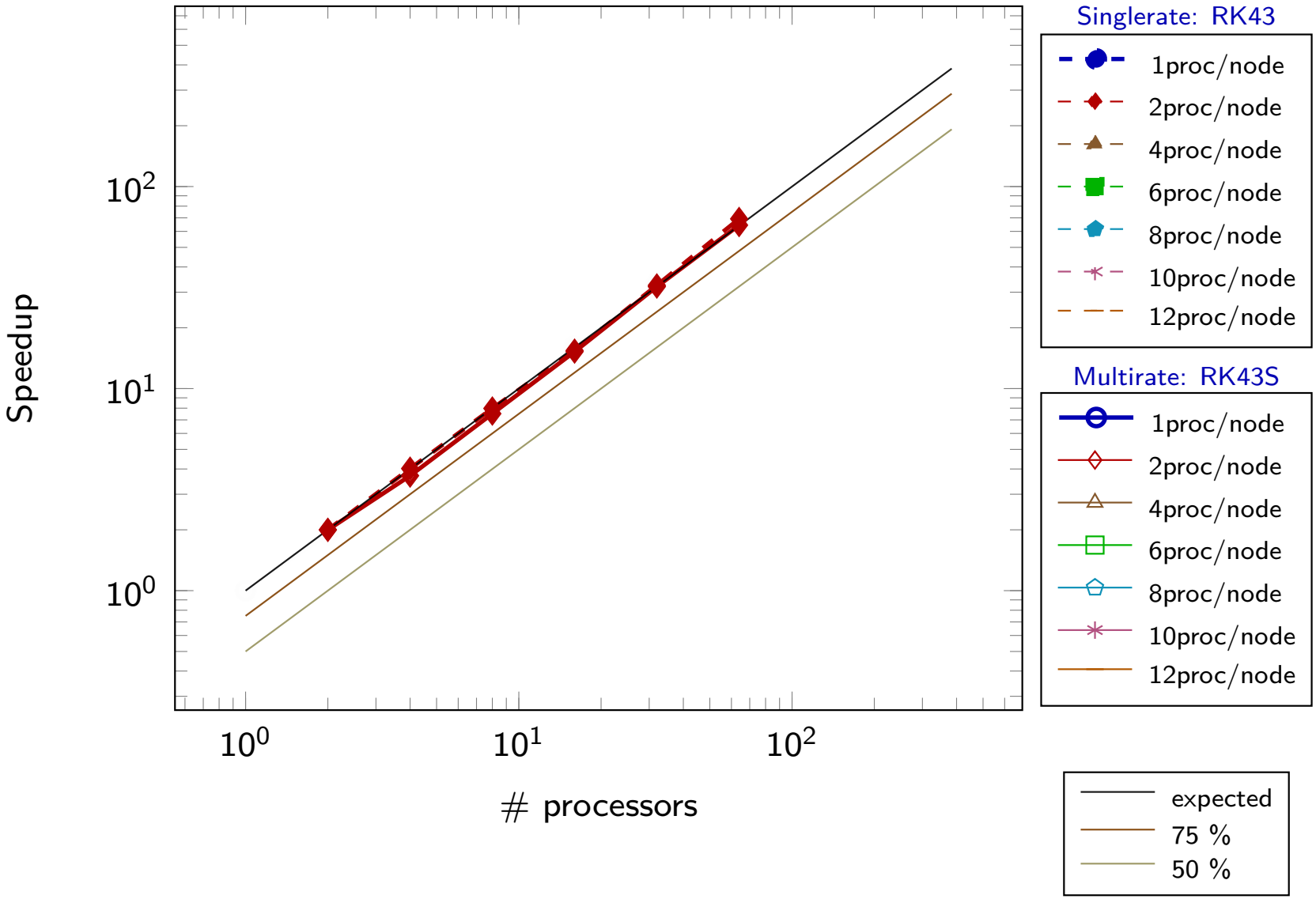
Tsunami: parallel performance (2)

Strong scaling: parallel speedup



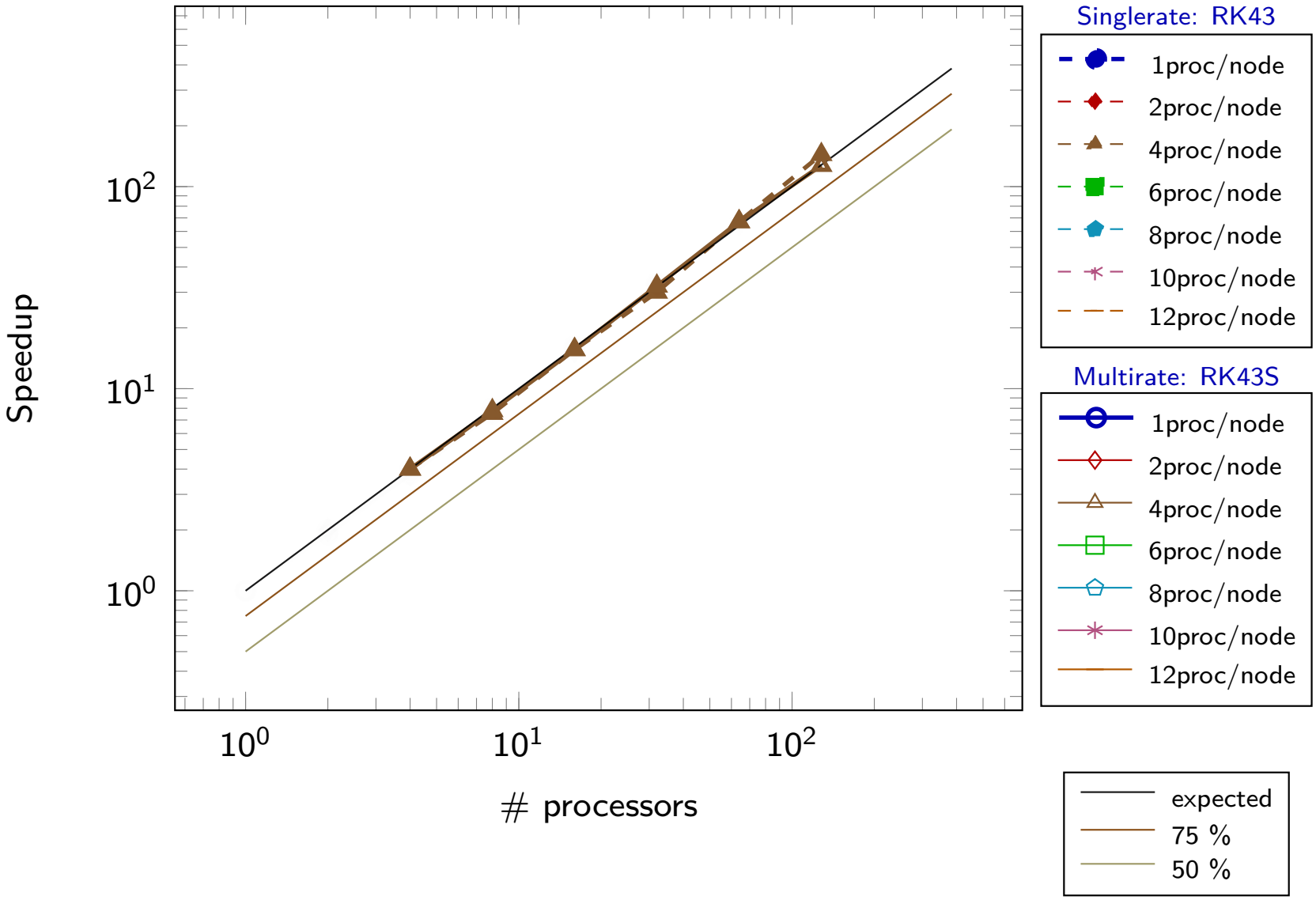
Tsunami: parallel performance (2)

Strong scaling: parallel speedup



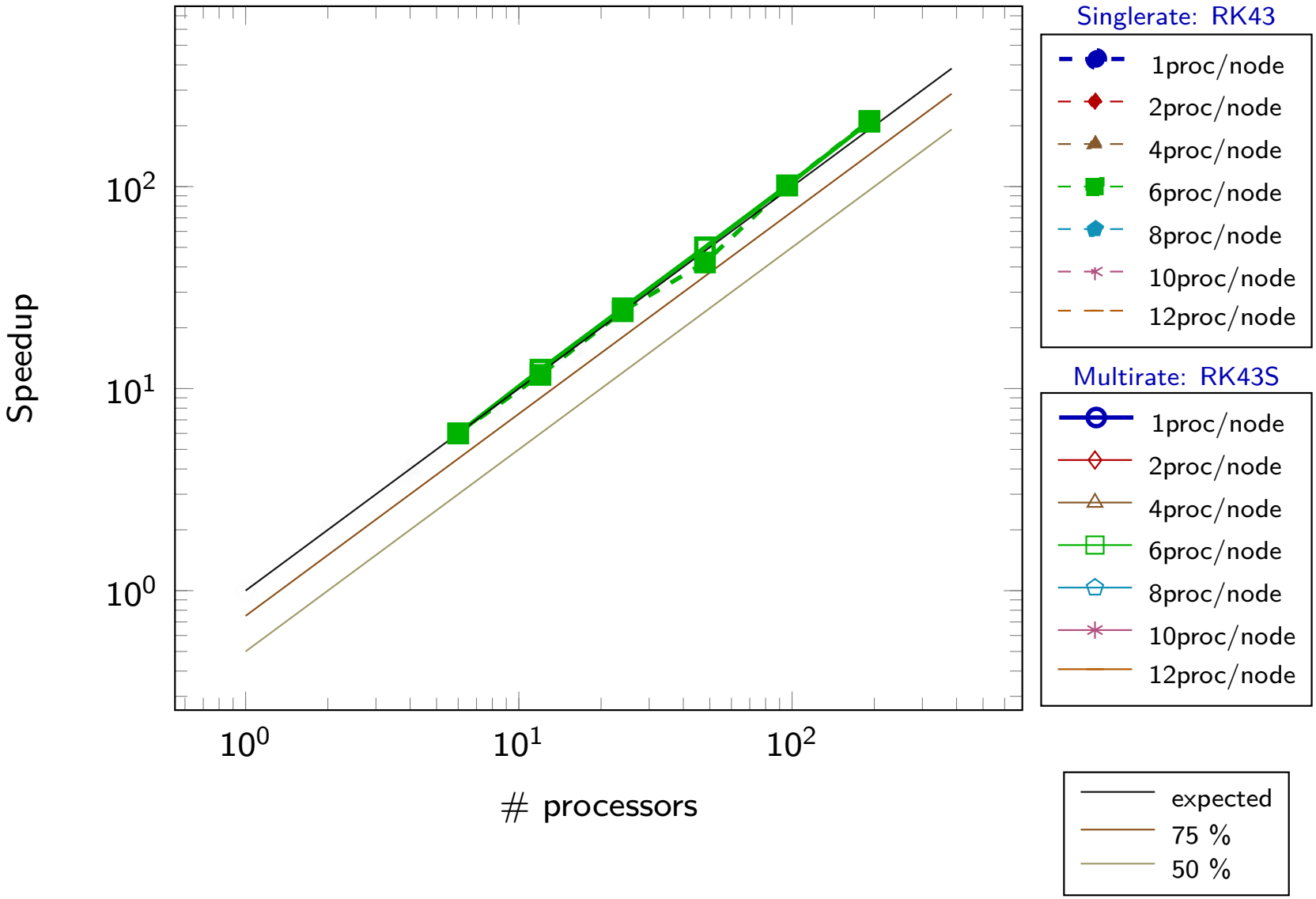
Tsunami: parallel performance (2)

Strong scaling: parallel speedup



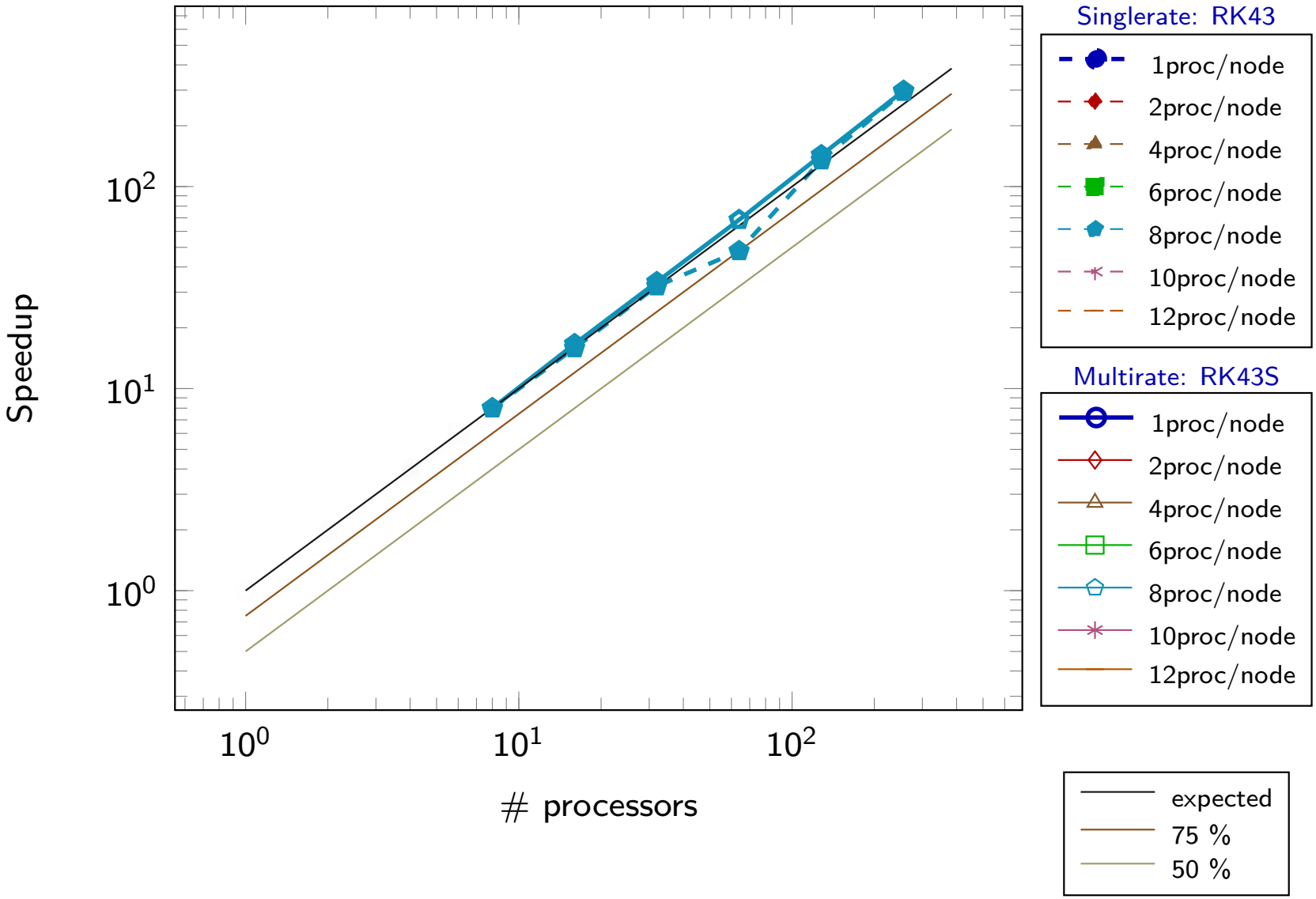
Tsunami: parallel performance (2)

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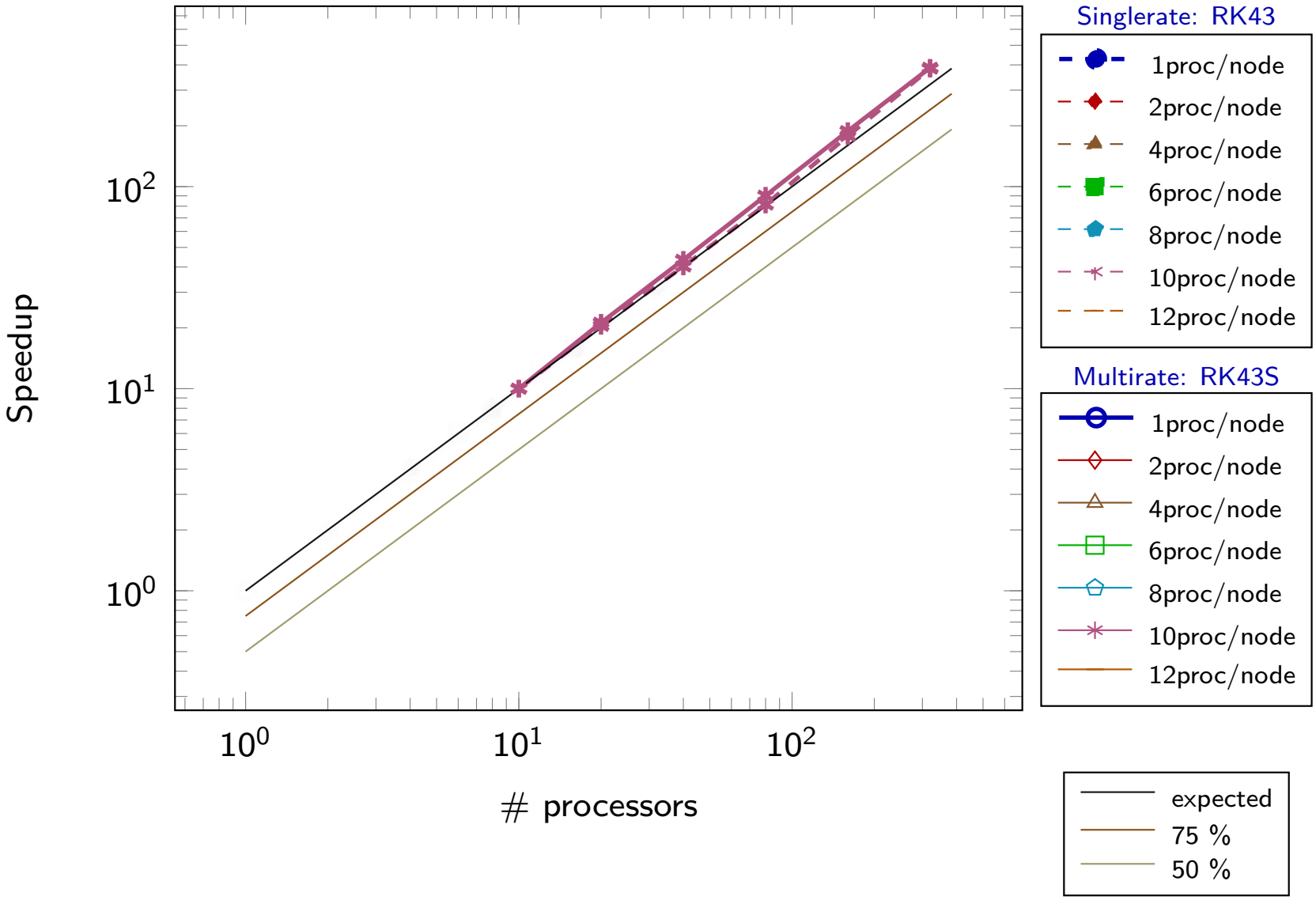
Tsunami: parallel performance (2)

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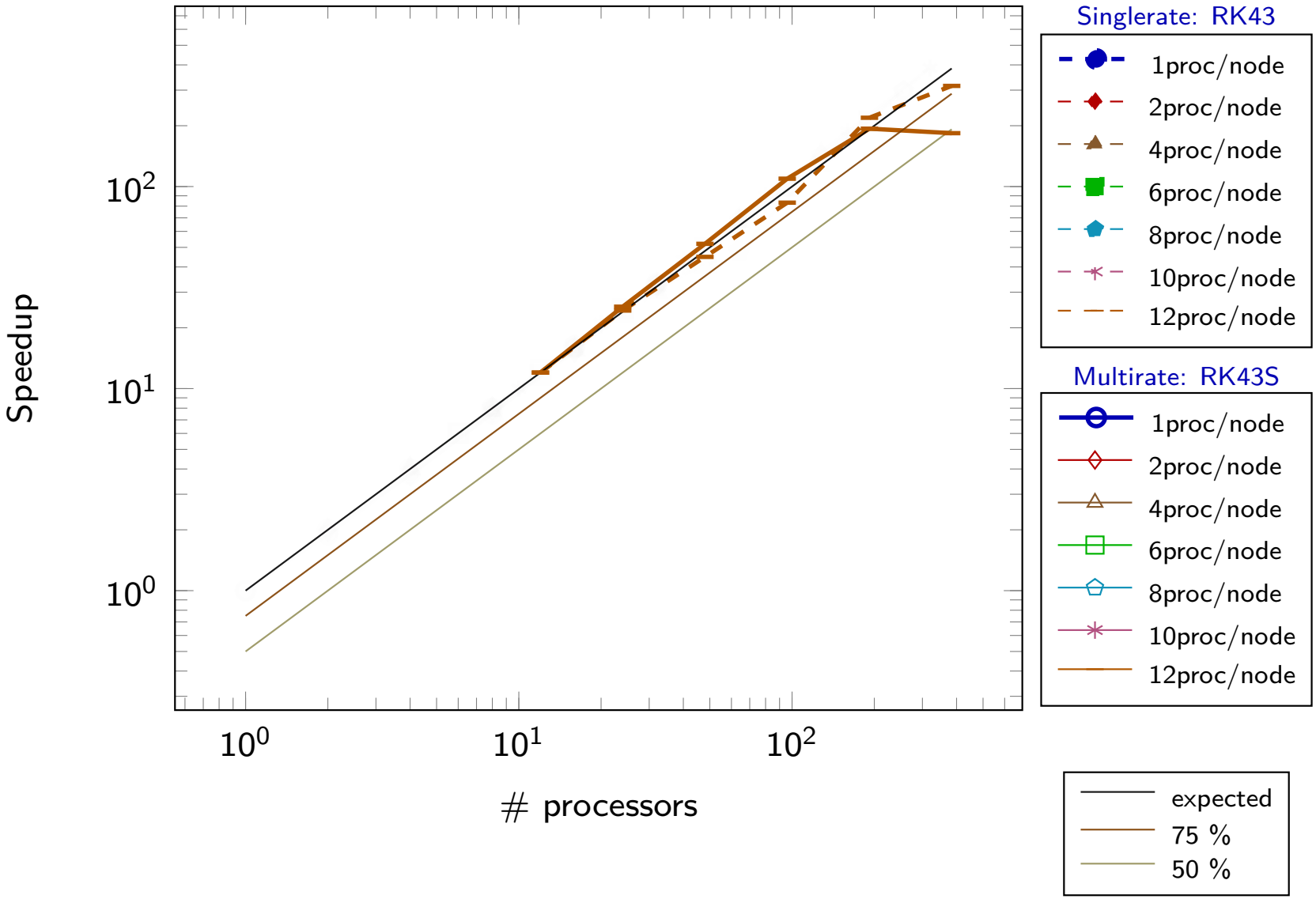
Tsunami: parallel performance (2)

Strong scaling: parallel speedup



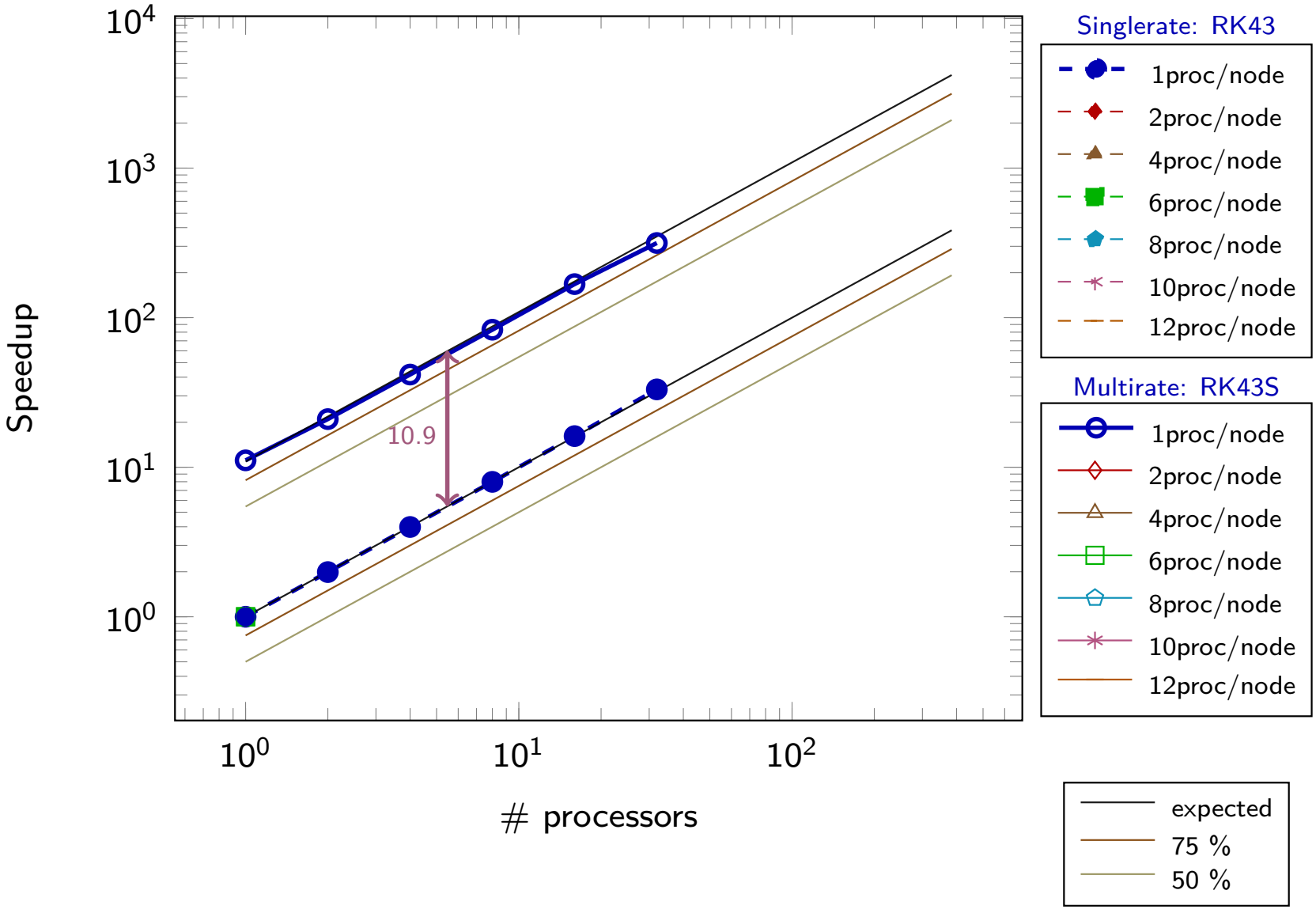
Tsunami: parallel performance (2)

Strong scaling: parallel speedup



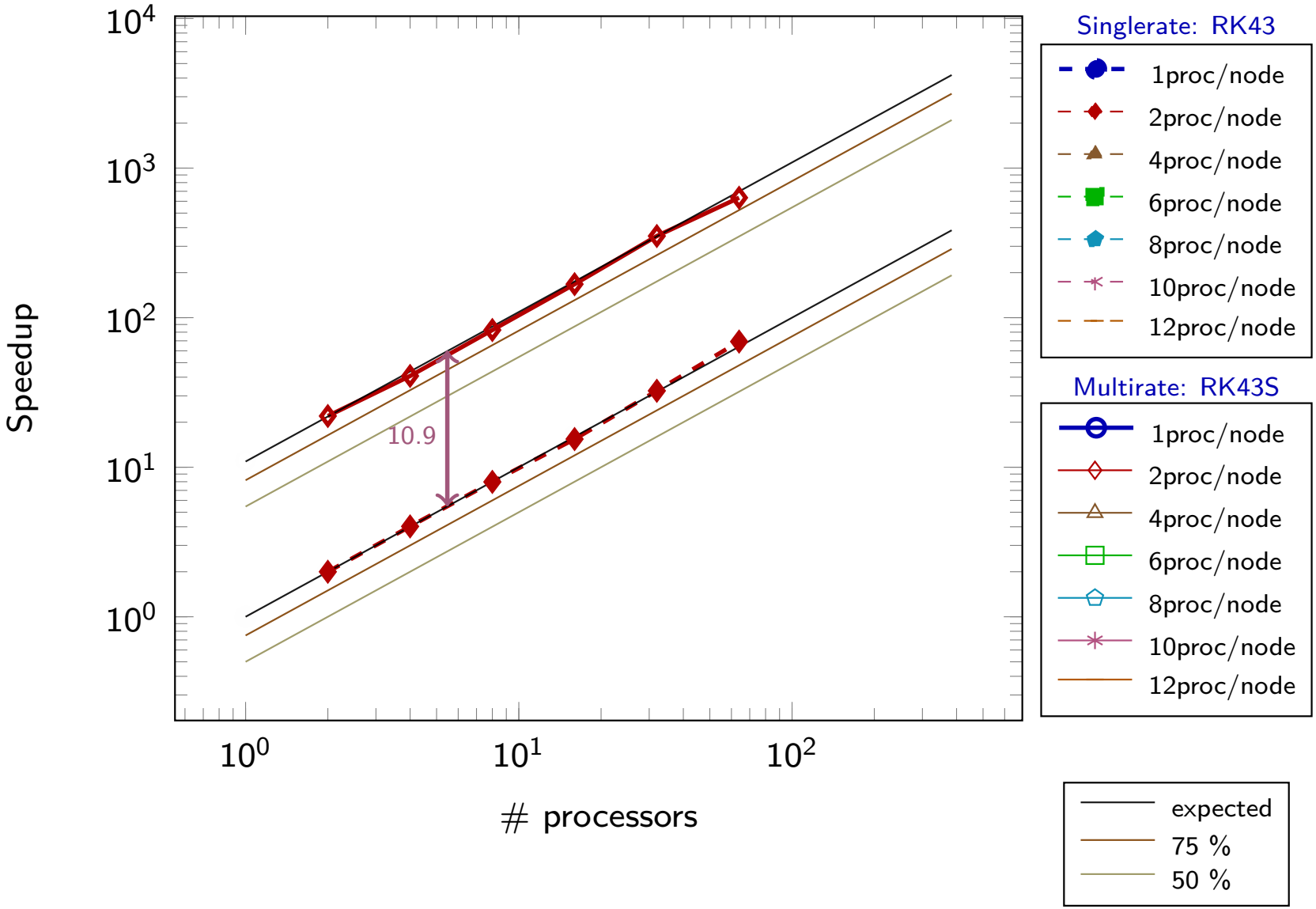
Tsunami: parallel performance (2)

Strong scaling: cumulated parallel + multirate speedup



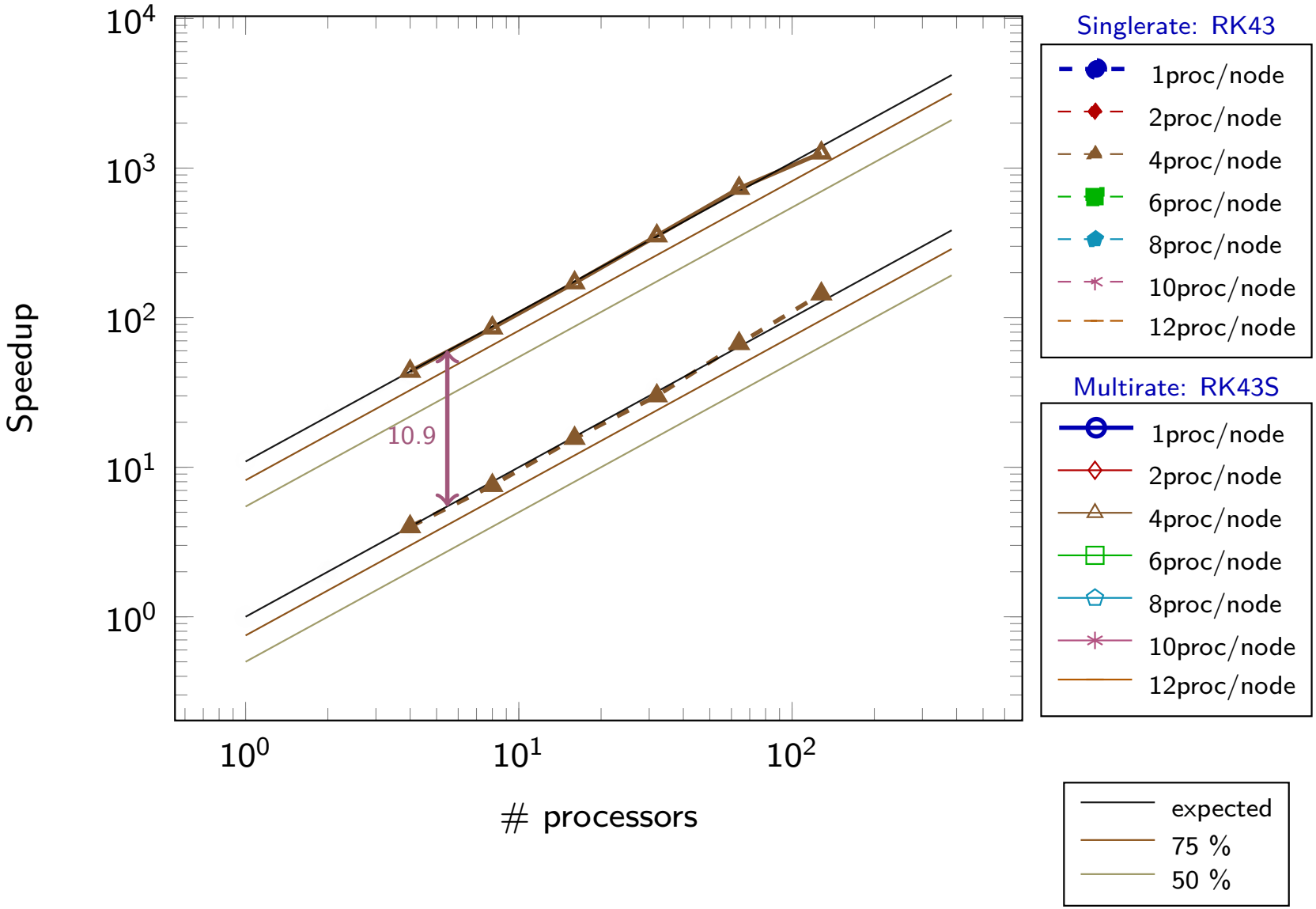
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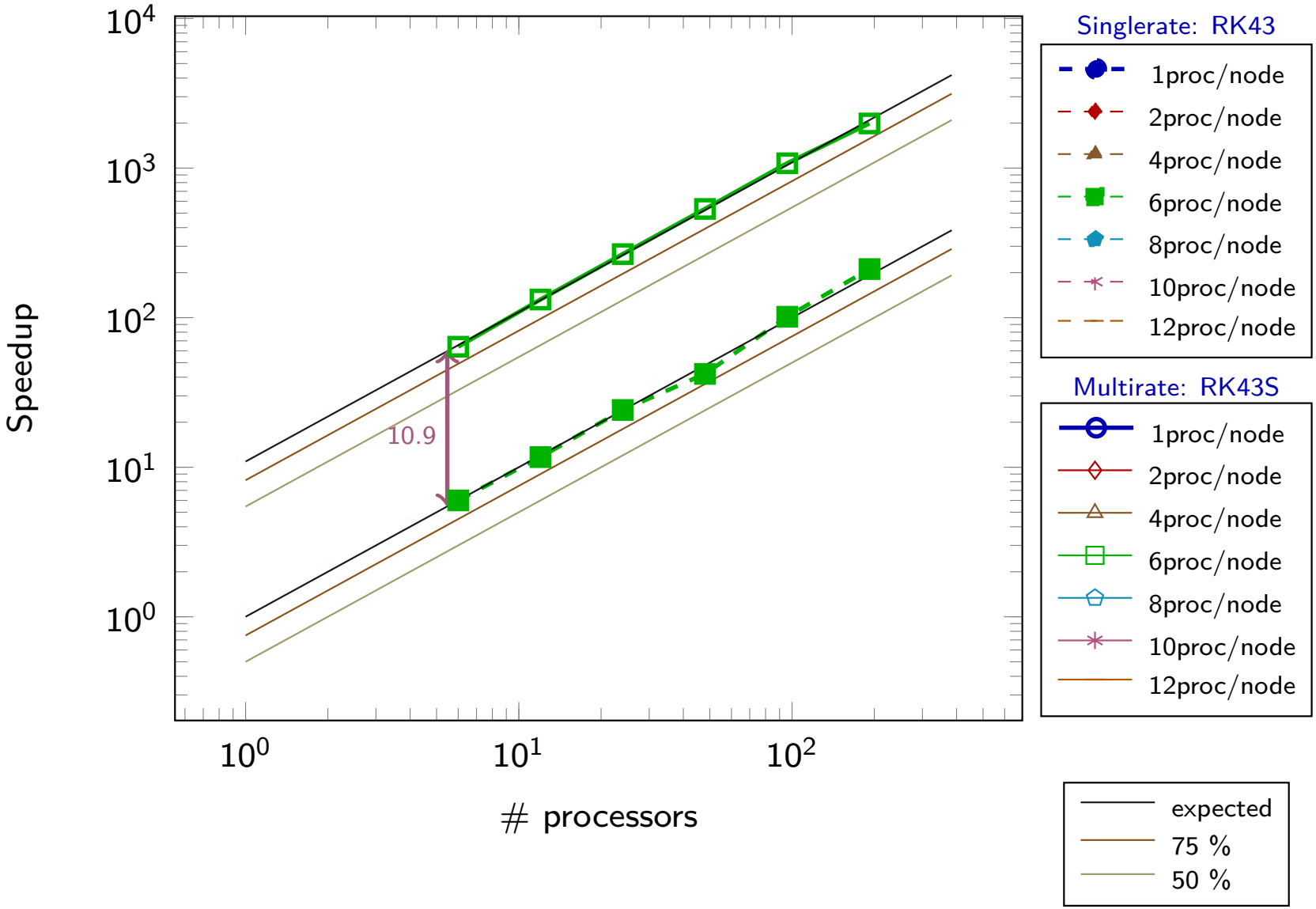
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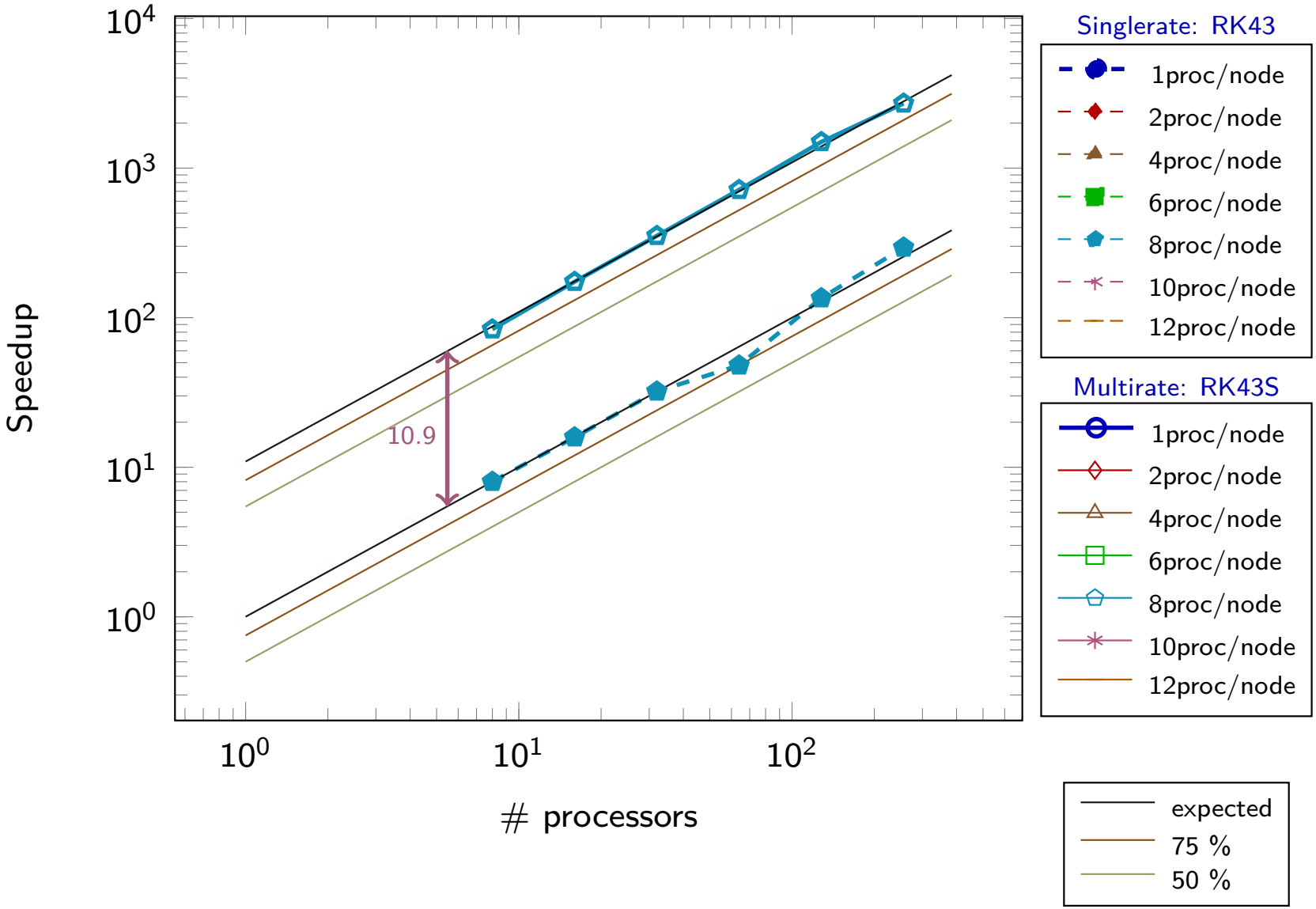
Tsunami: parallel performance (2)

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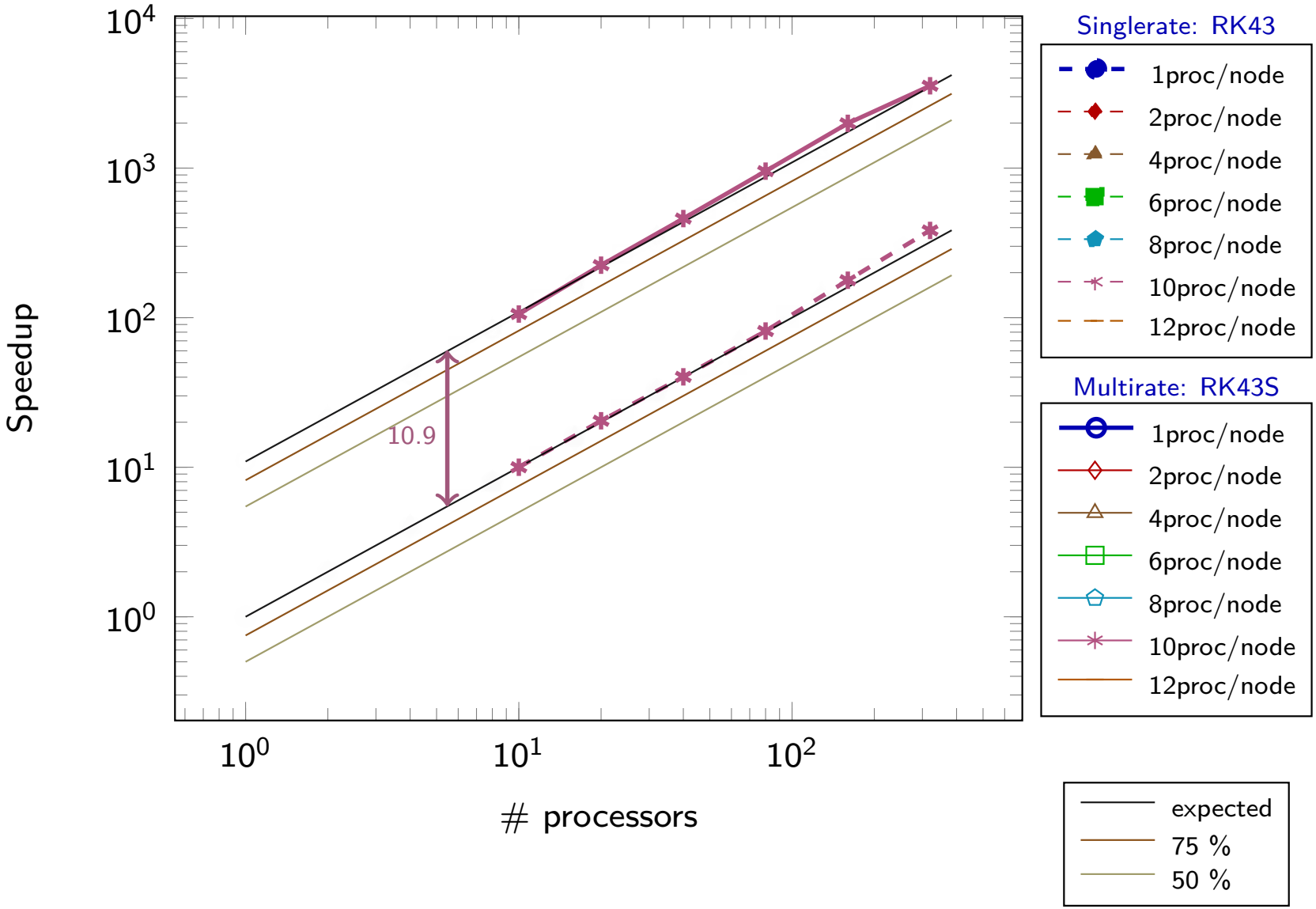
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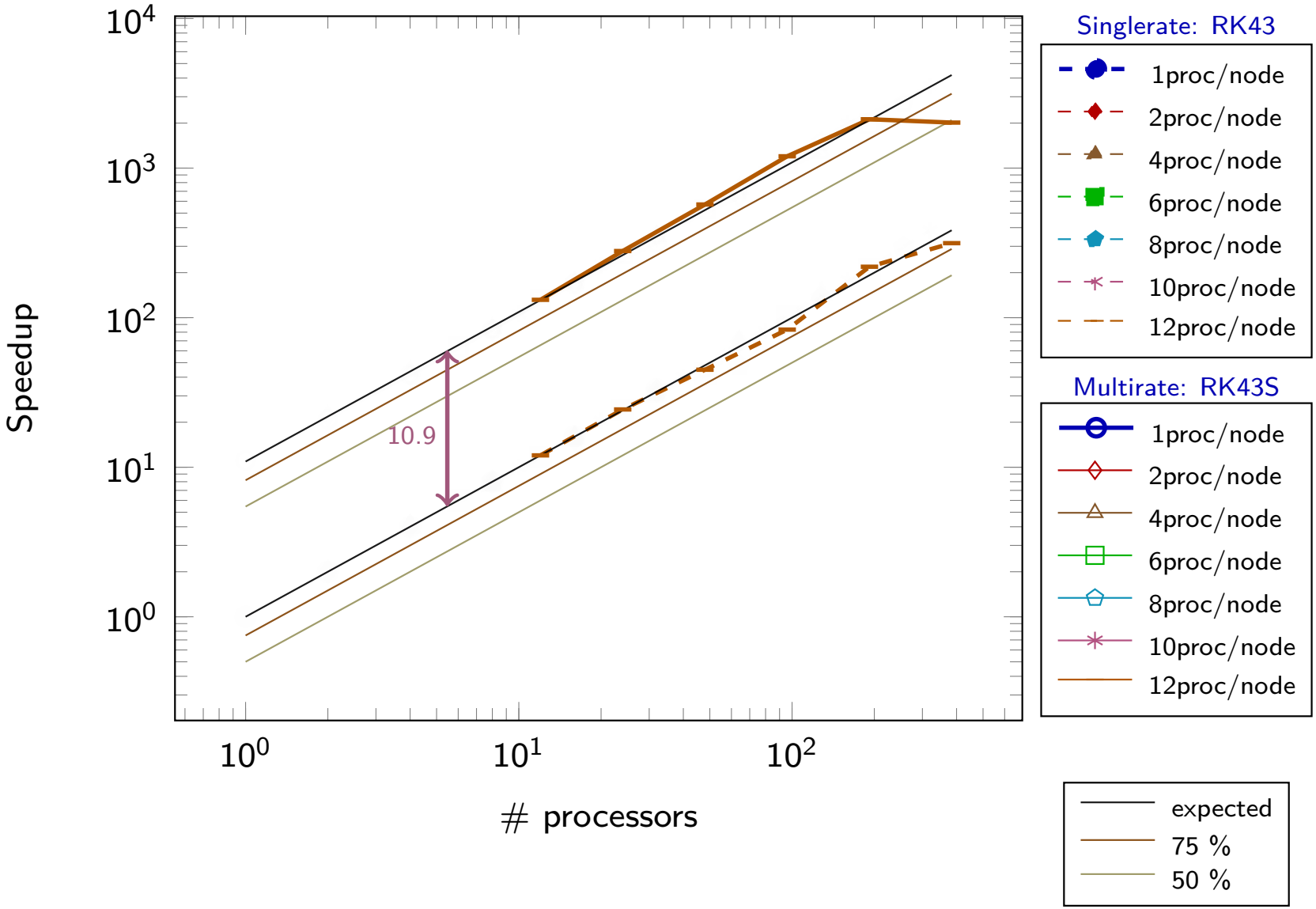
Tsunami: parallel performance (2)

Strong scaling: cumulated parallel + multirate speedup

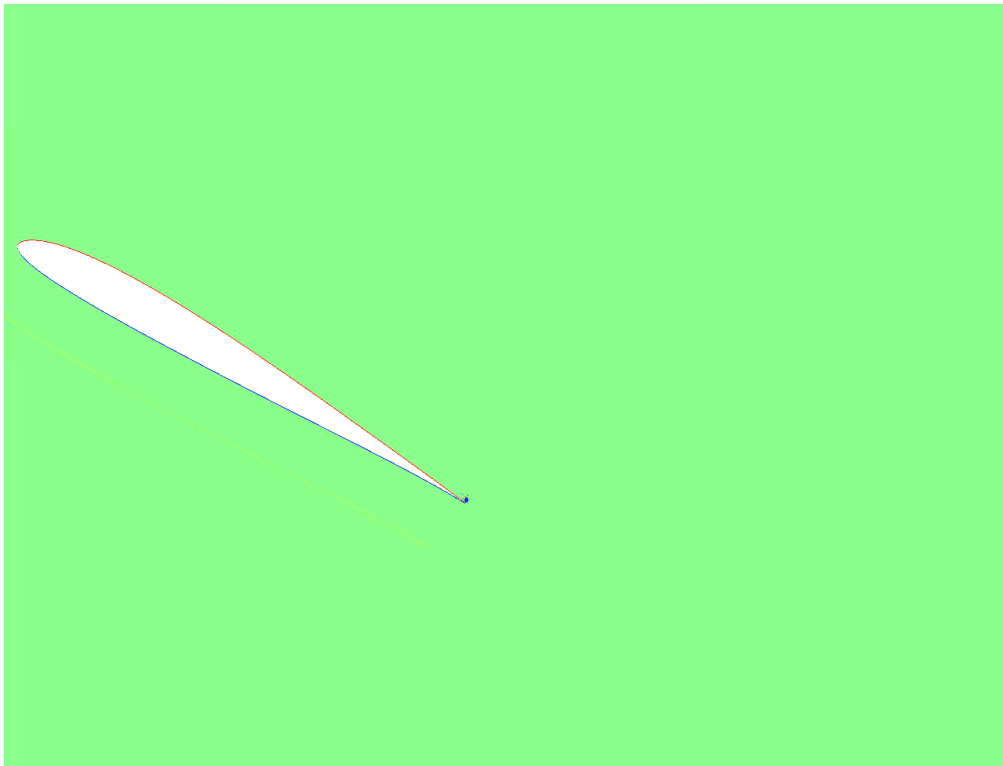


Tsunami: parallel performance (2)

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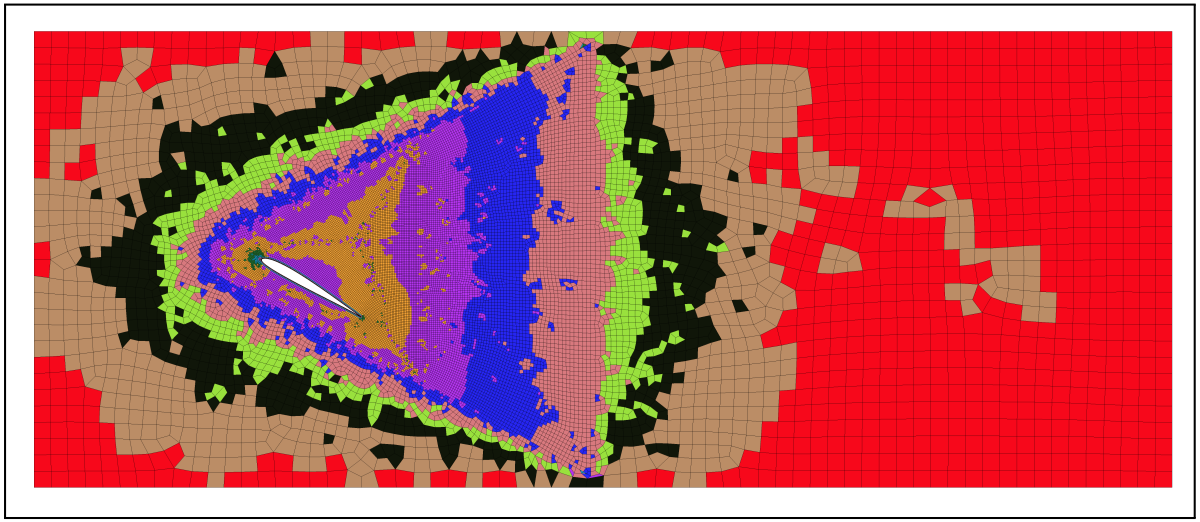
Flow pas the SD7003 airfoil (1)



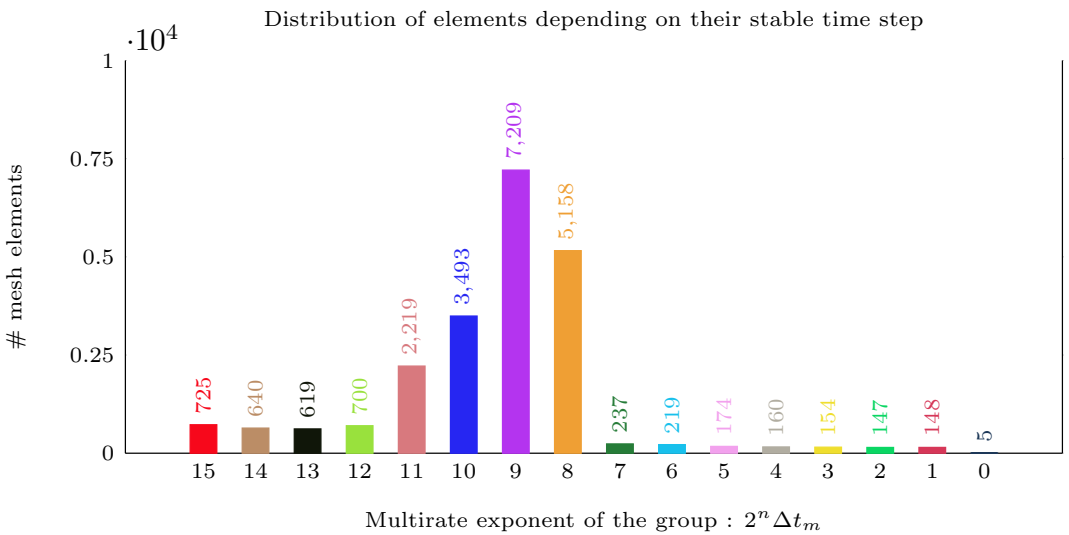
- Re $= 64000$
- Ma $= 0.1$
- $\alpha = 30°$

Flow pas the SD7003 airfoil (2)

Stable explicit time steps of mesh elements

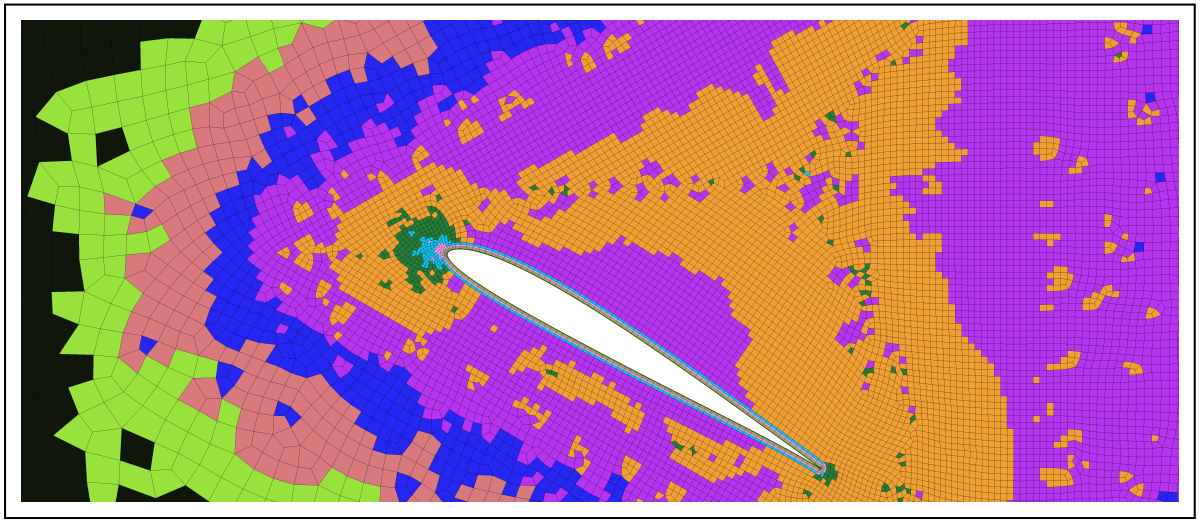


22007 P_3^{DG} □

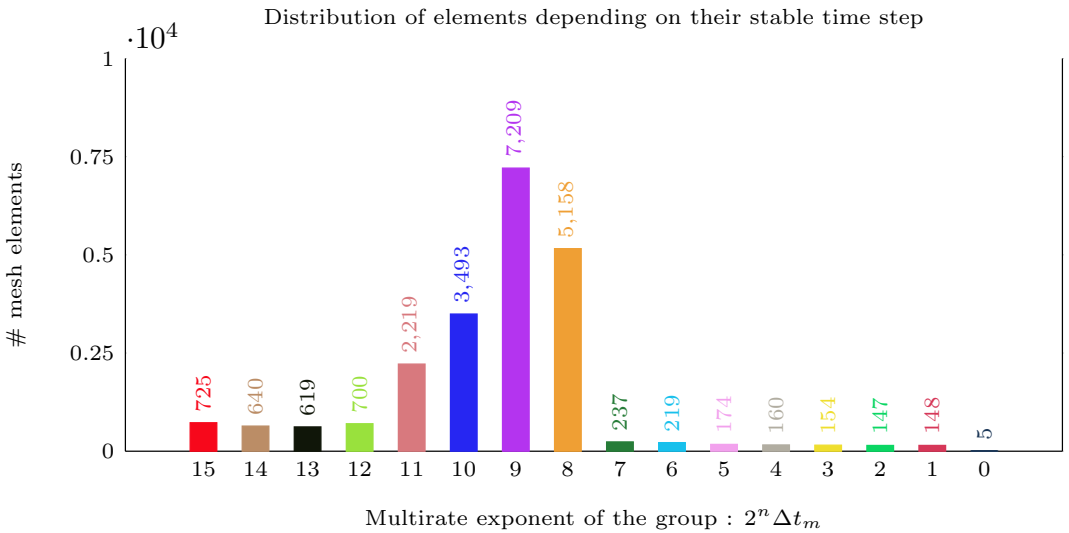


Flow pas the SD7003 airfoil (2)

Stable explicit time steps of mesh elements

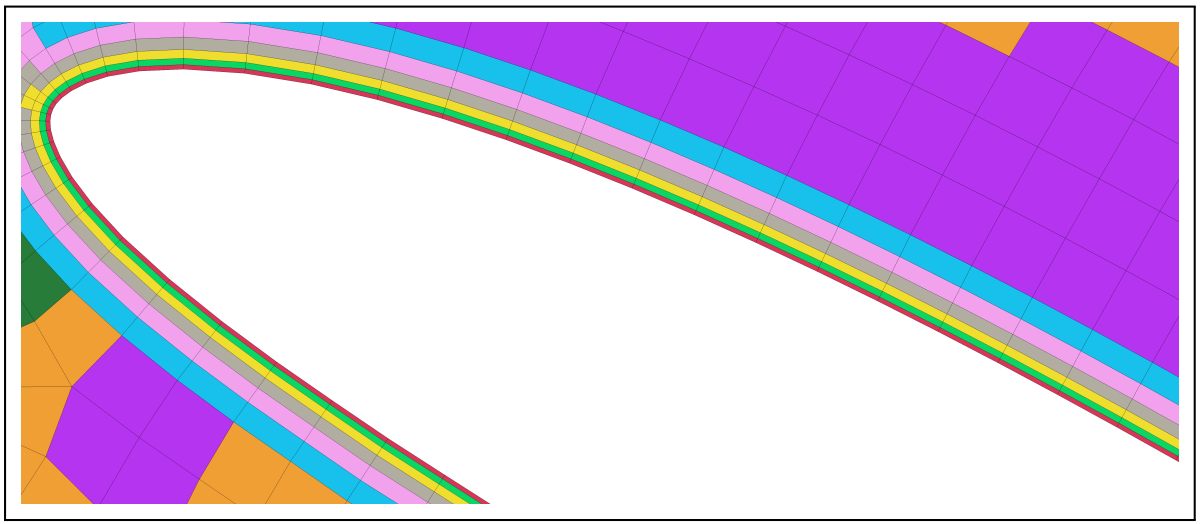


22007 P_3^{DG} □

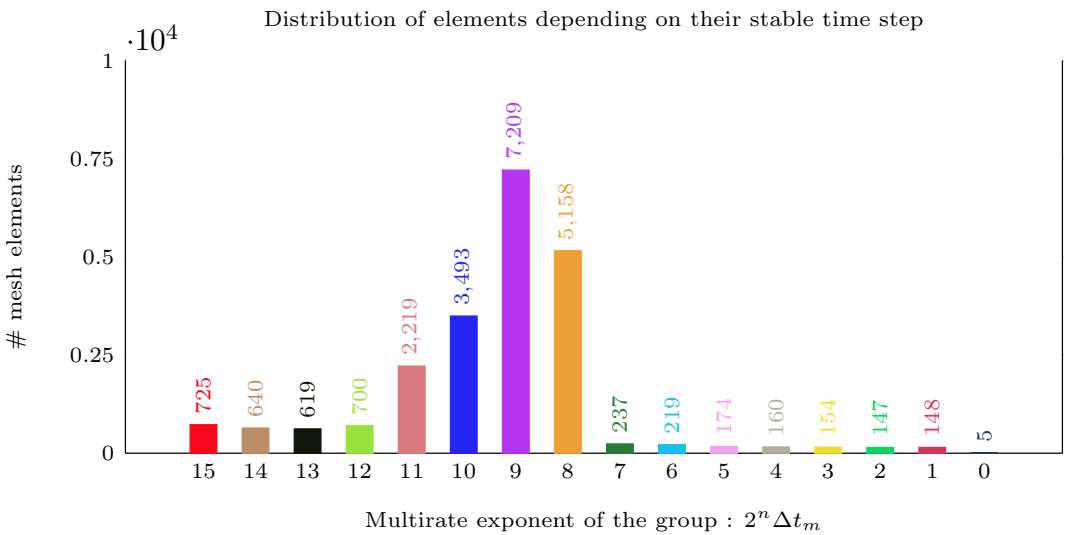


Flow pas the SD7003 airfoil (2)

Stable explicit time steps of mesh elements

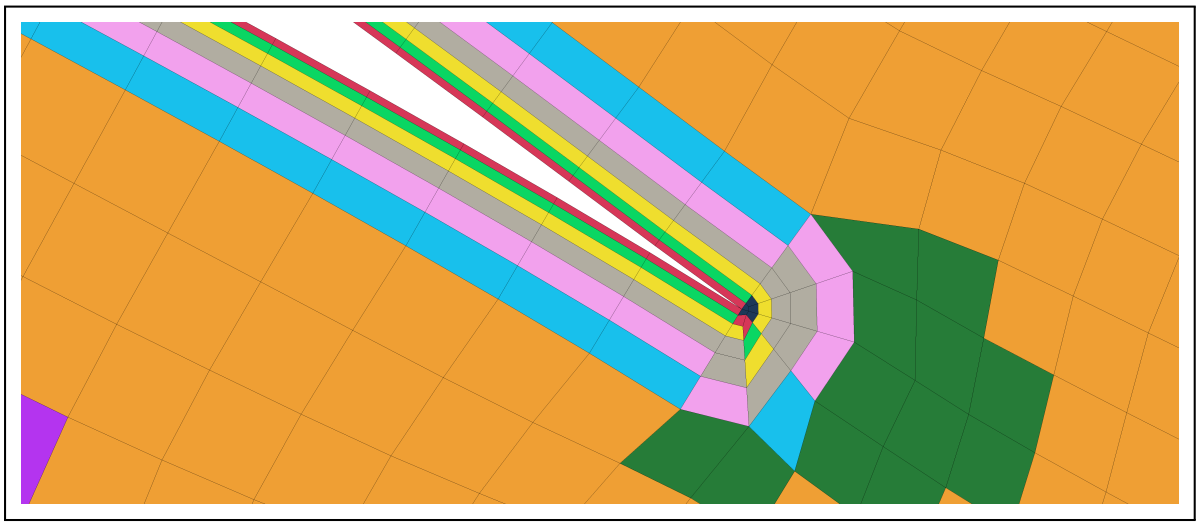


22007 P_3^{DG} $\square$

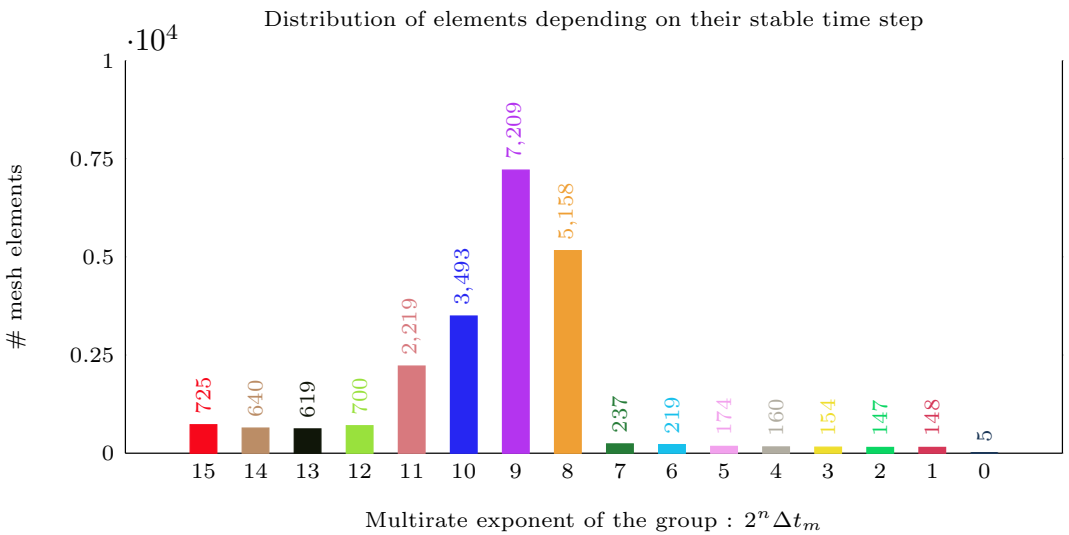


Flow pas the SD7003 airfoil (2)

Stable explicit time steps of mesh elements

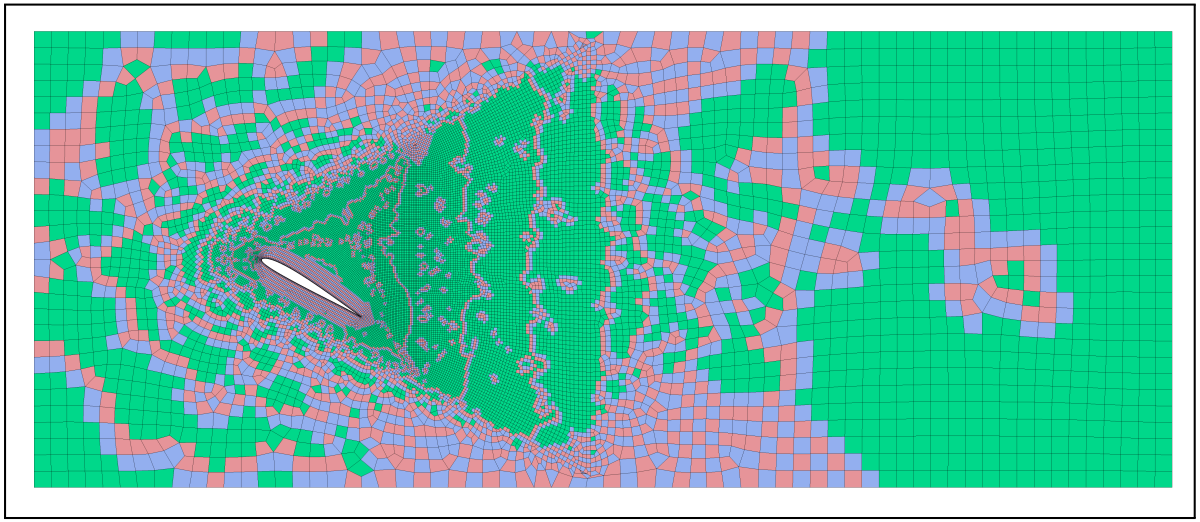


22007 P_3^{DG} $\square$

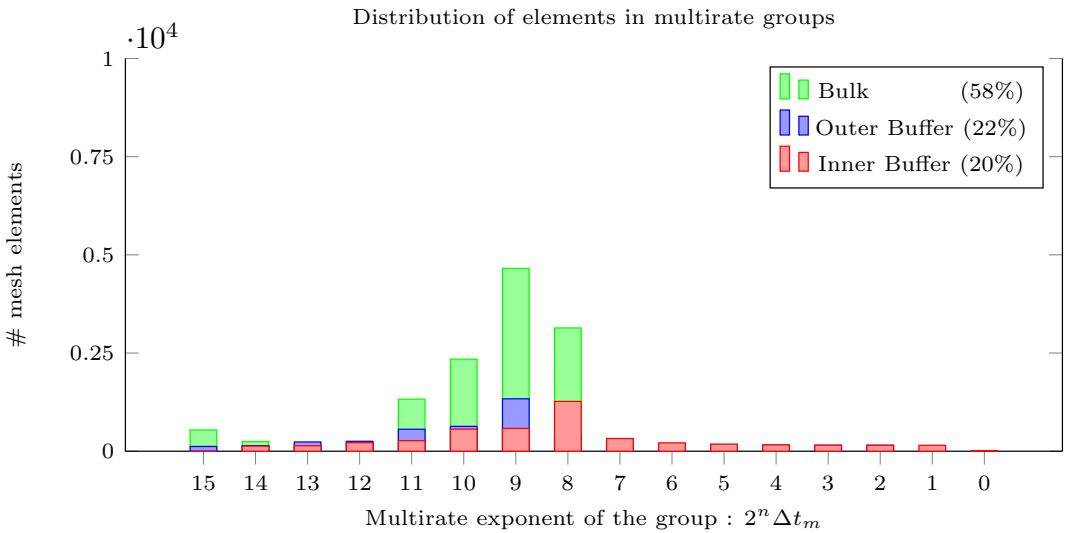


Flow pas the SD7003 airfoil (2)

Multirate setup for RK43S

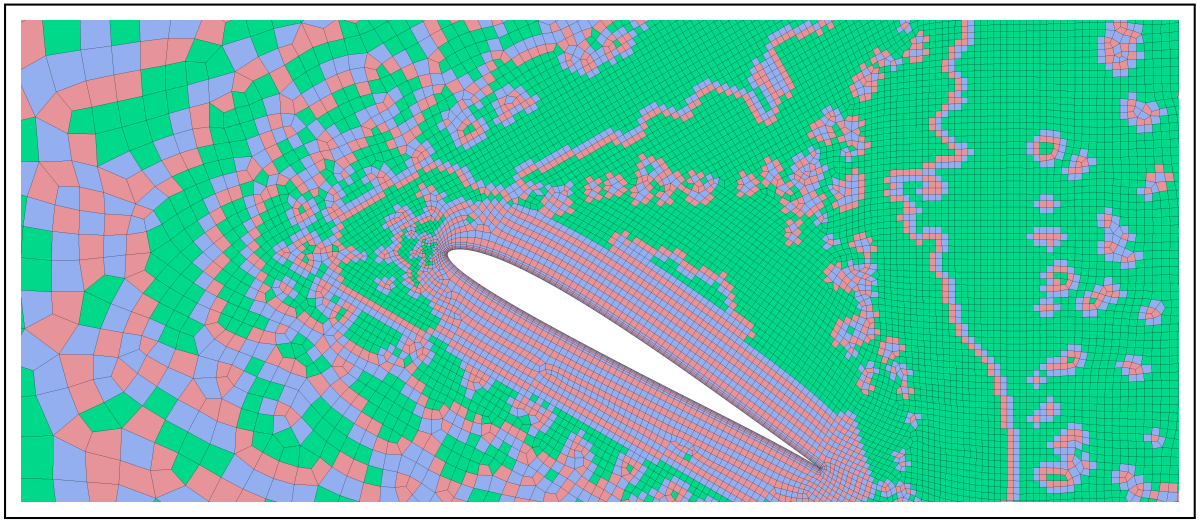


22007 P_3^{DG} □

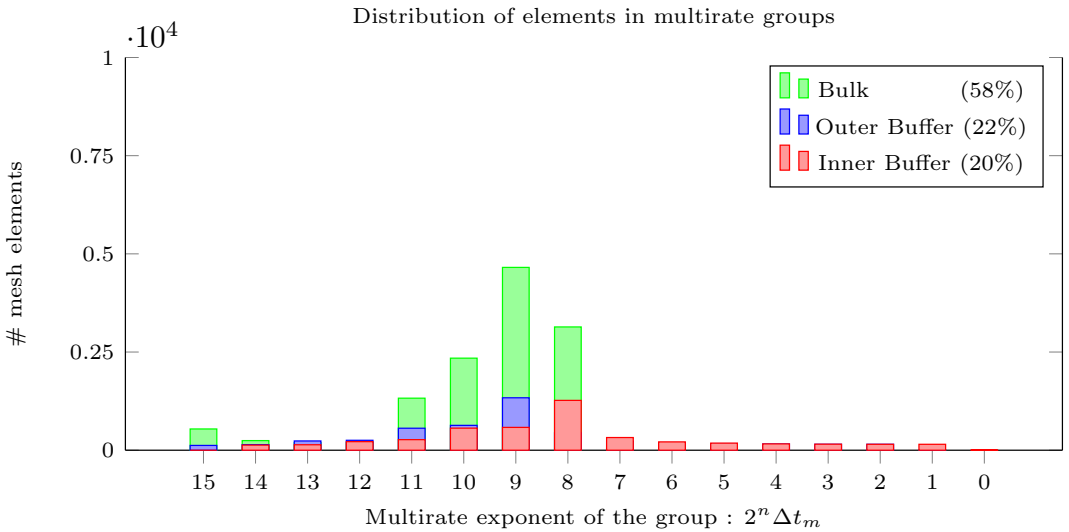


Flow pas the SD7003 airfoil (2)

Multirate setup for RK43S

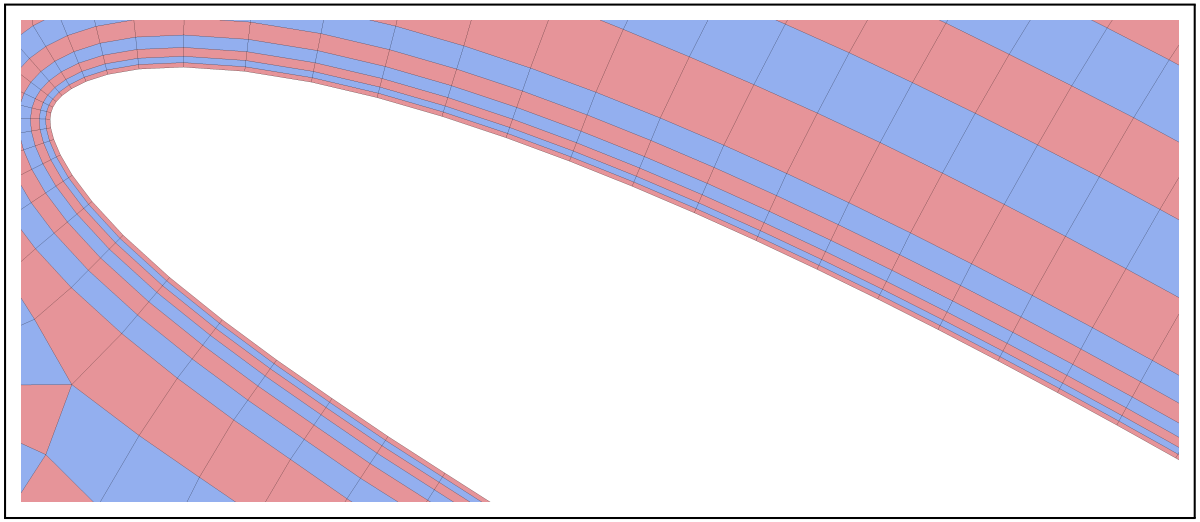


22007 P_3^{DG} $\square$

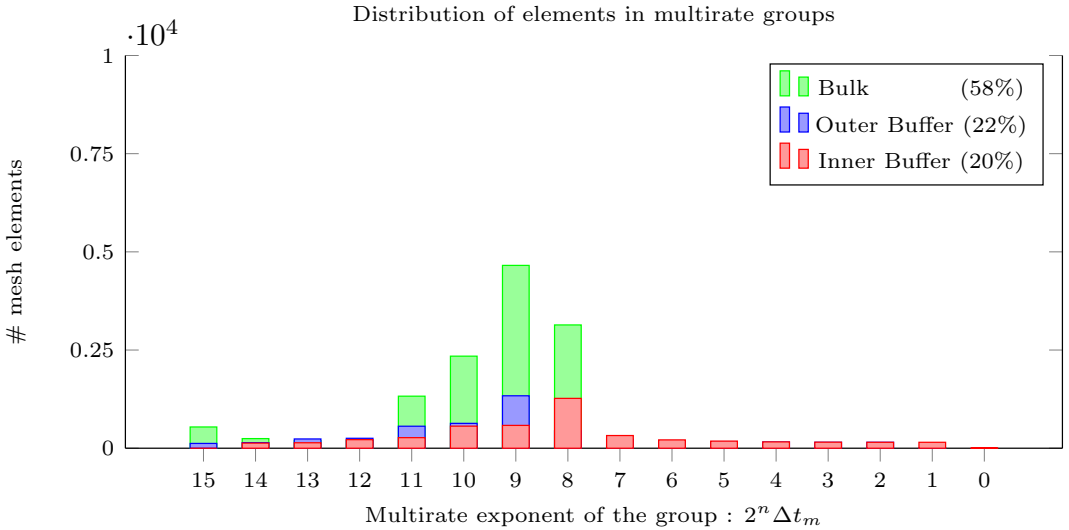


Flow pas the SD7003 airfoil (2)

Multirate setup for RK43S

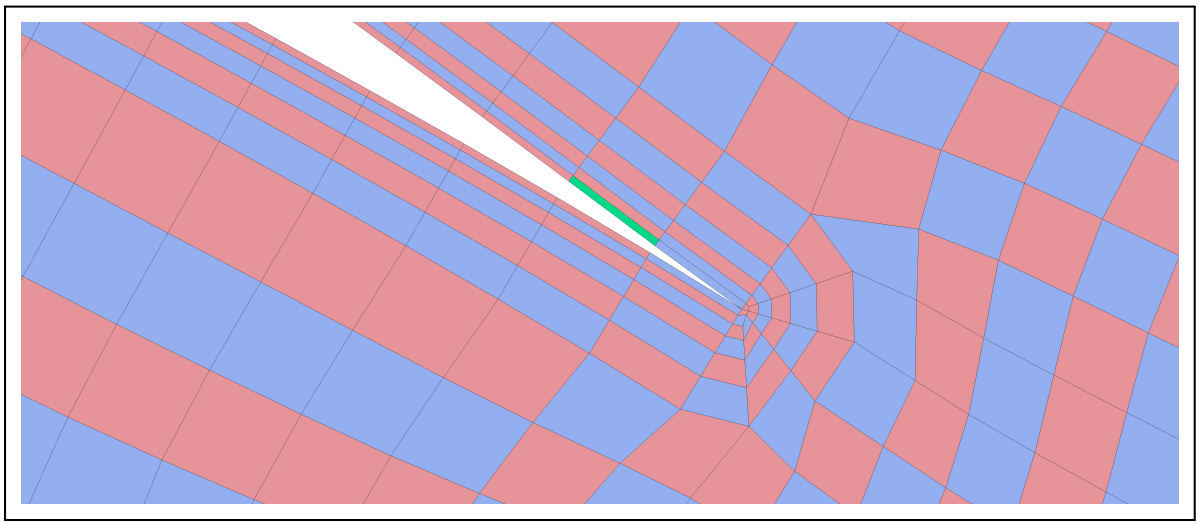


22007 P_3^{DG} $\square$

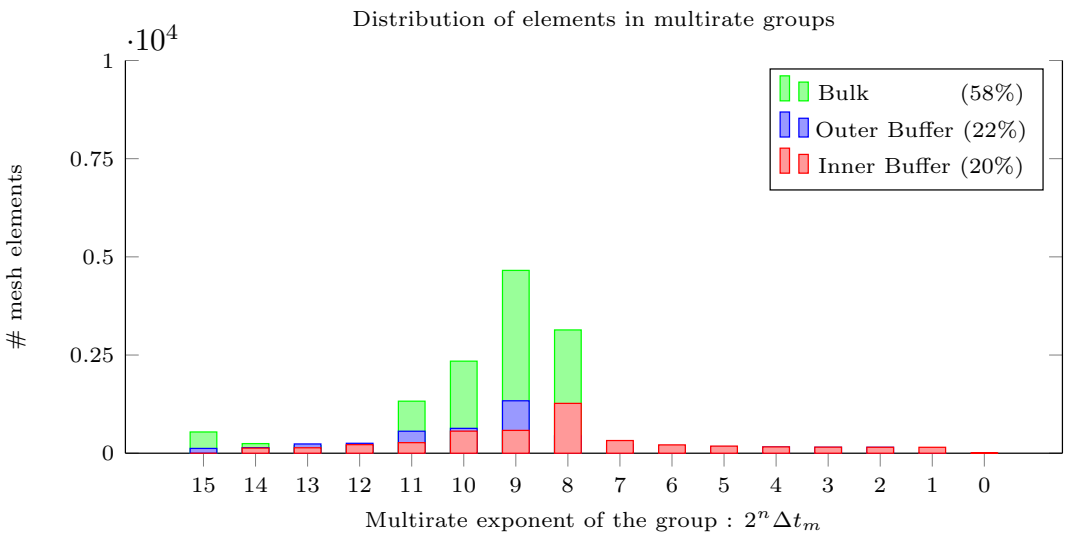


Flow pas the SD7003 airfoil (2)

Multirate setup for RK43S

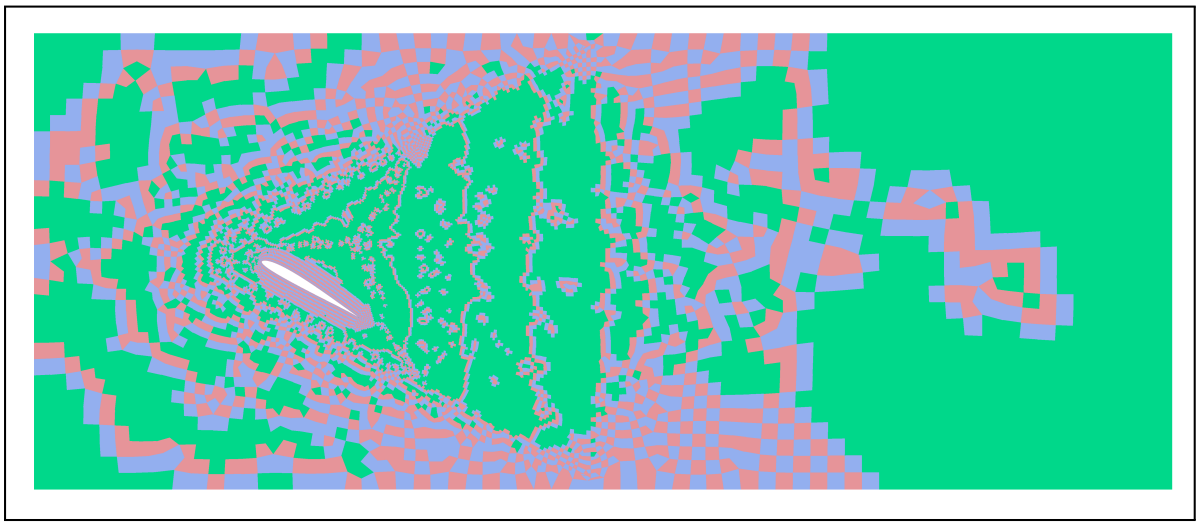


22007 P_3^{DG} $\square$

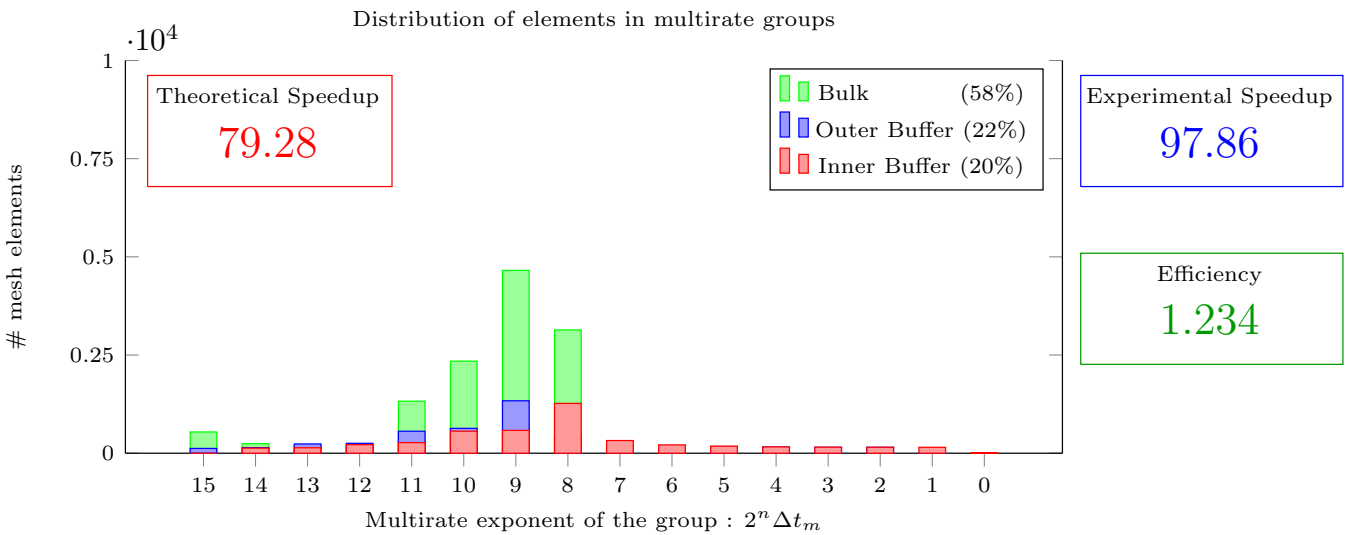


Flow pas the SD7003 airfoil (2)

Effect of the number of MR groups

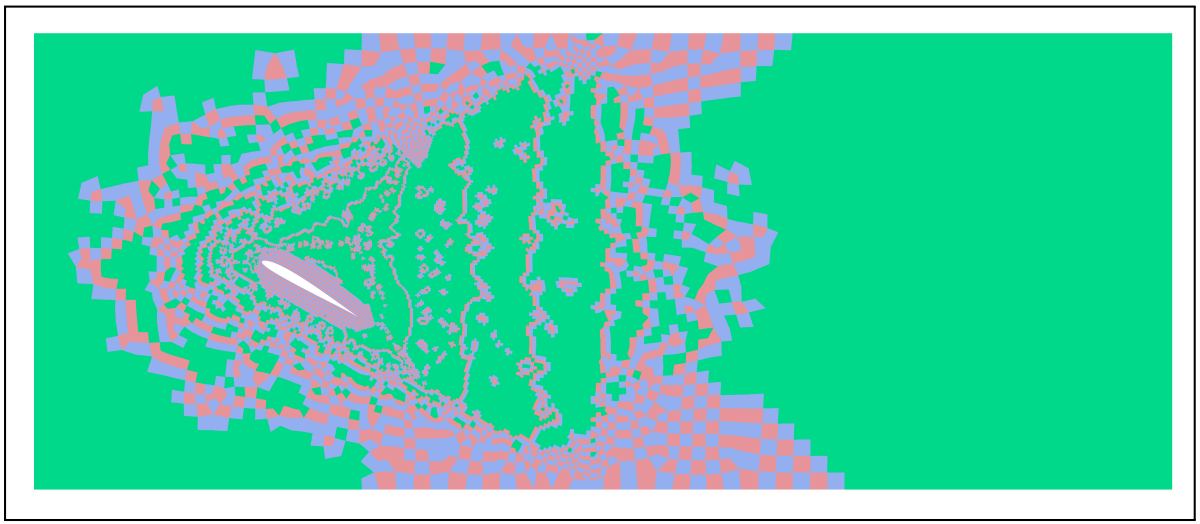


22007 P_3^{DG} □

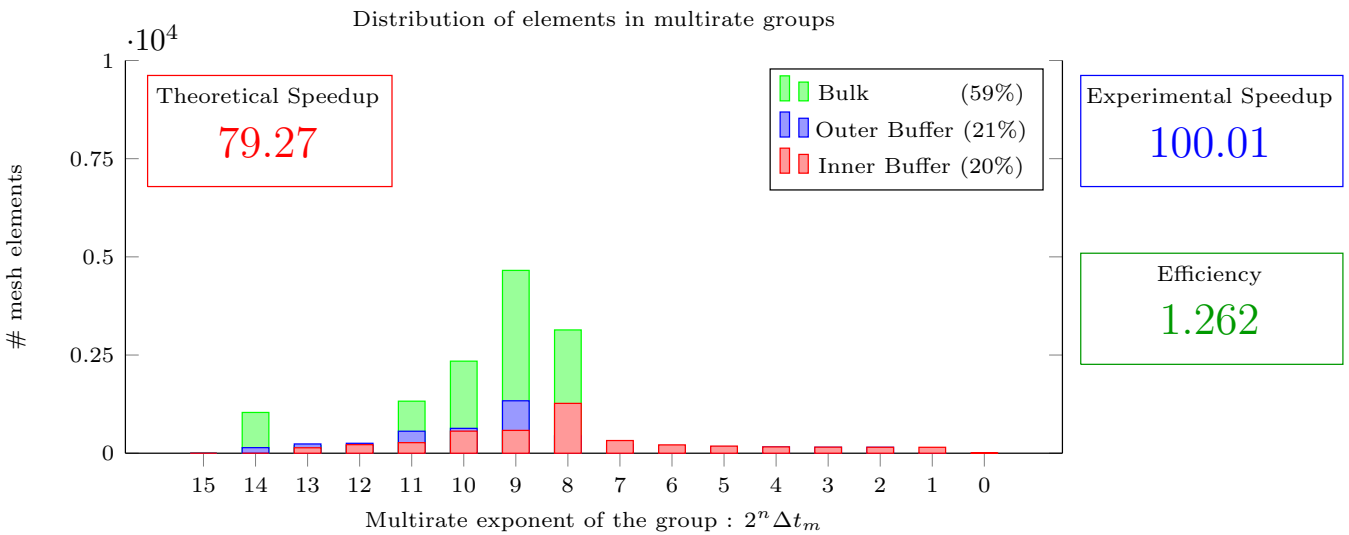


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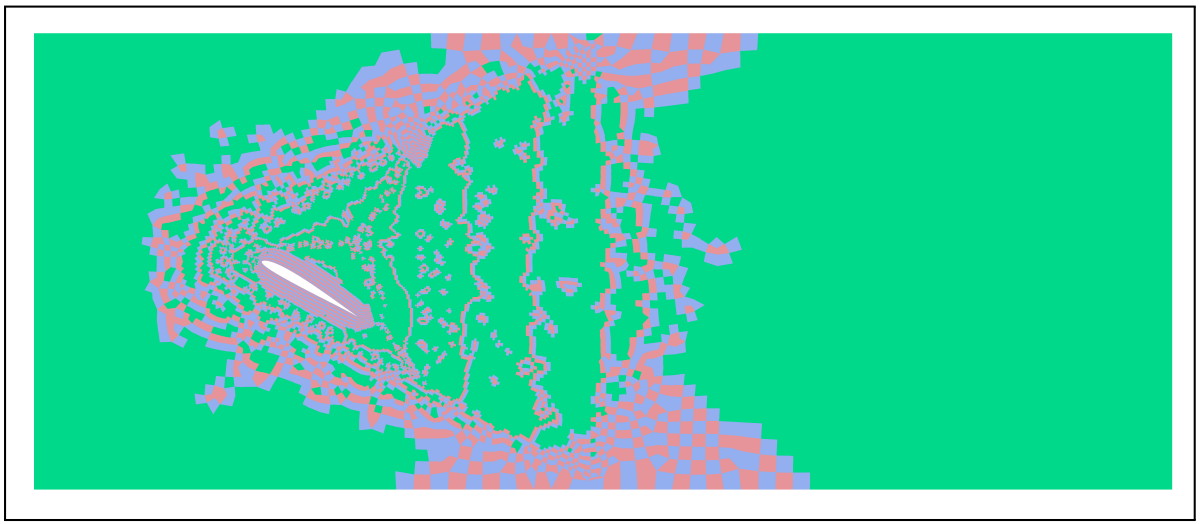


22007 P_3^{DG} □

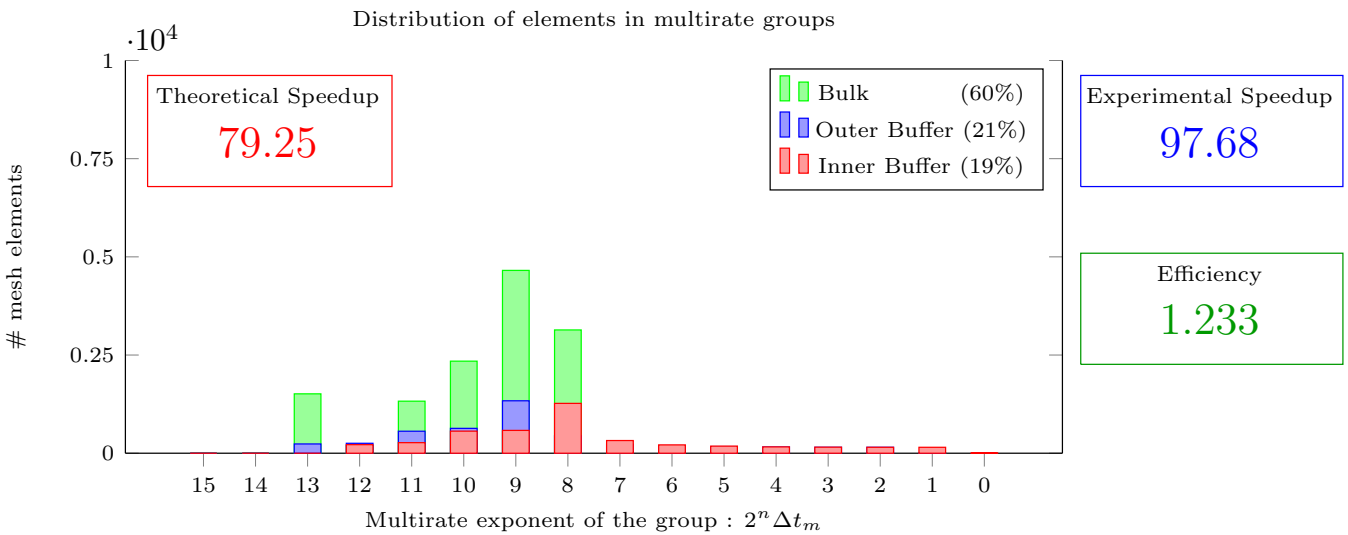


Flow pas the SD7003 airfoil (2)

Effect of the number of MR groups

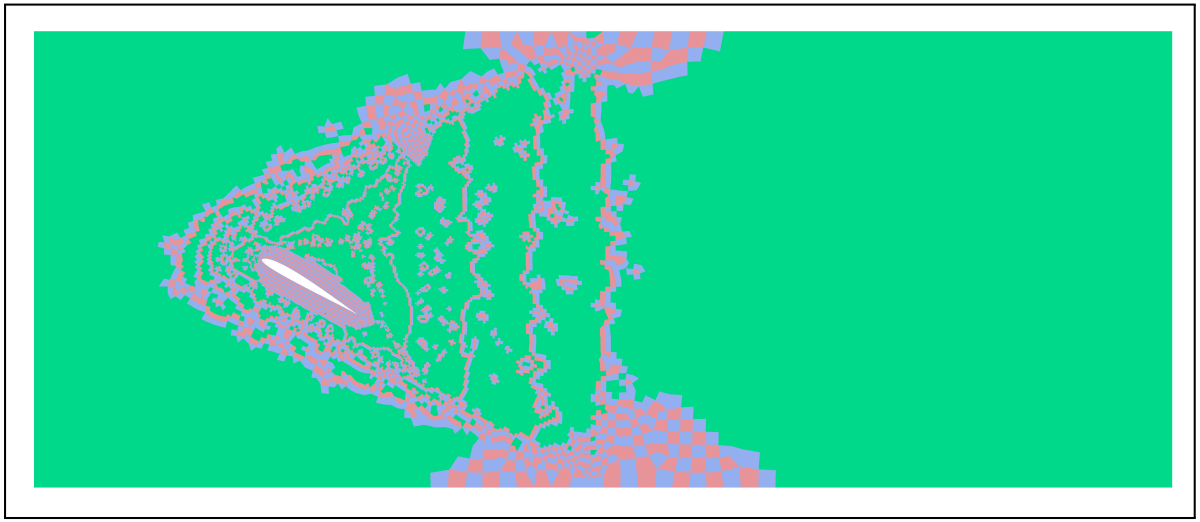


22007 P_3^{DG} $\square$

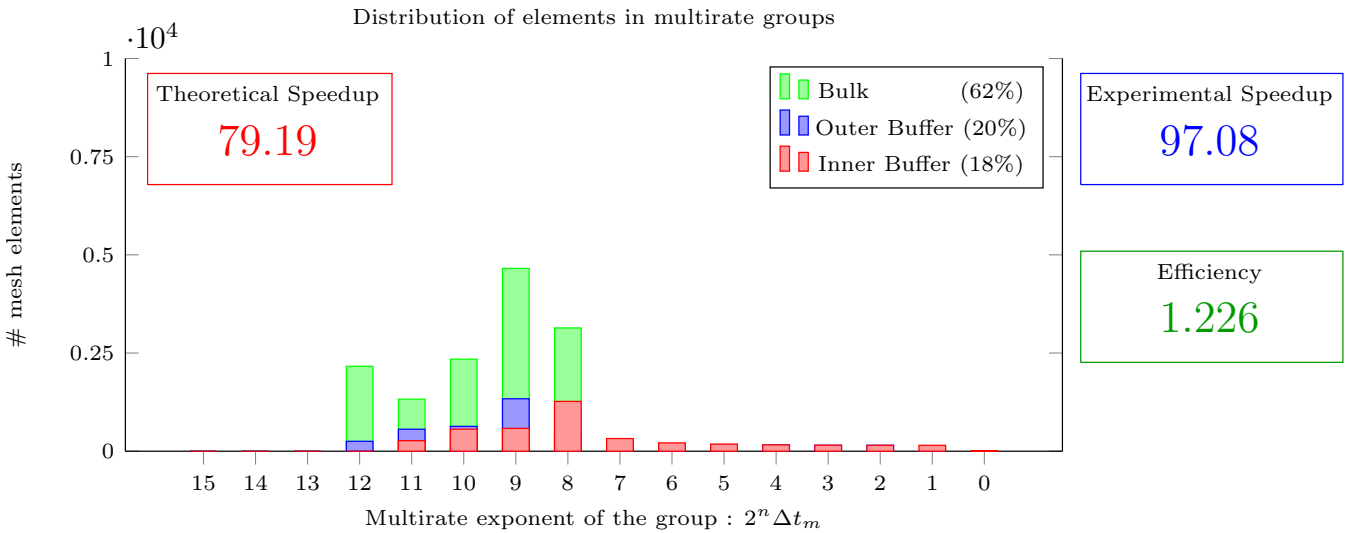


Flow pas the SD7003 airfoil (2)

Effect of the number of MR groups

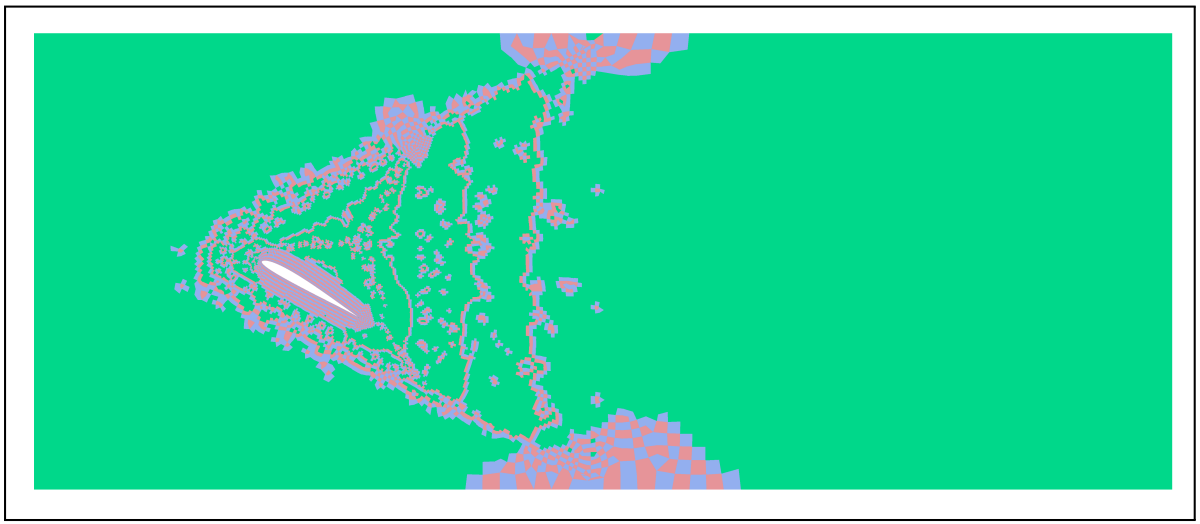


22007 P_3^{DG} □

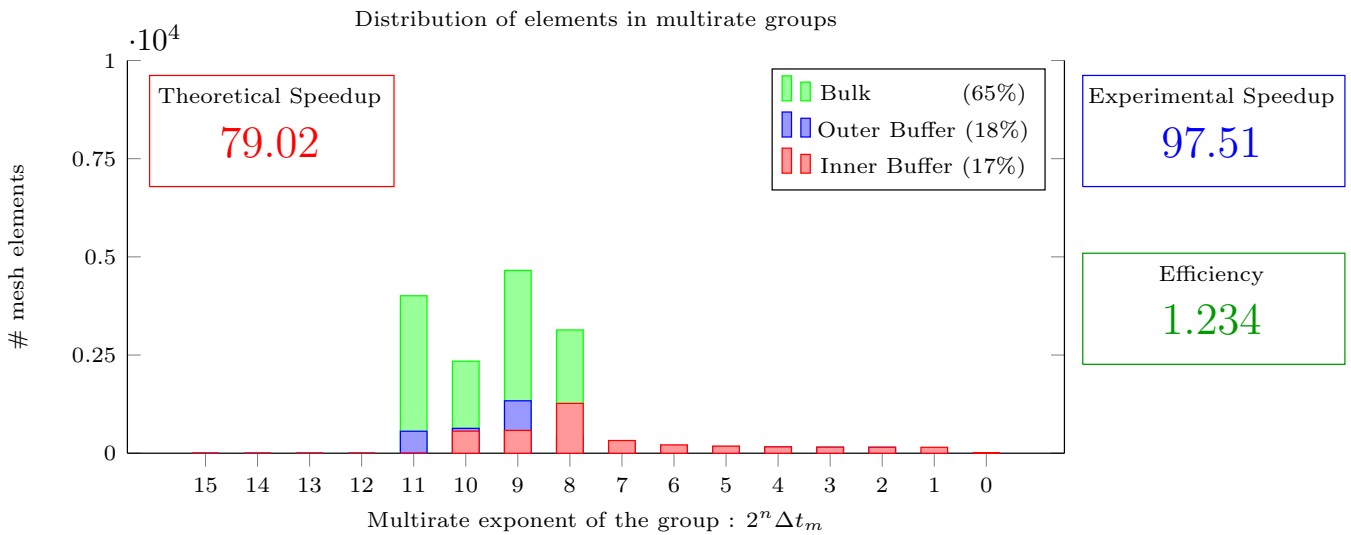


Flow pas the SD7003 airfoil (2)

Effect of the number of MR groups

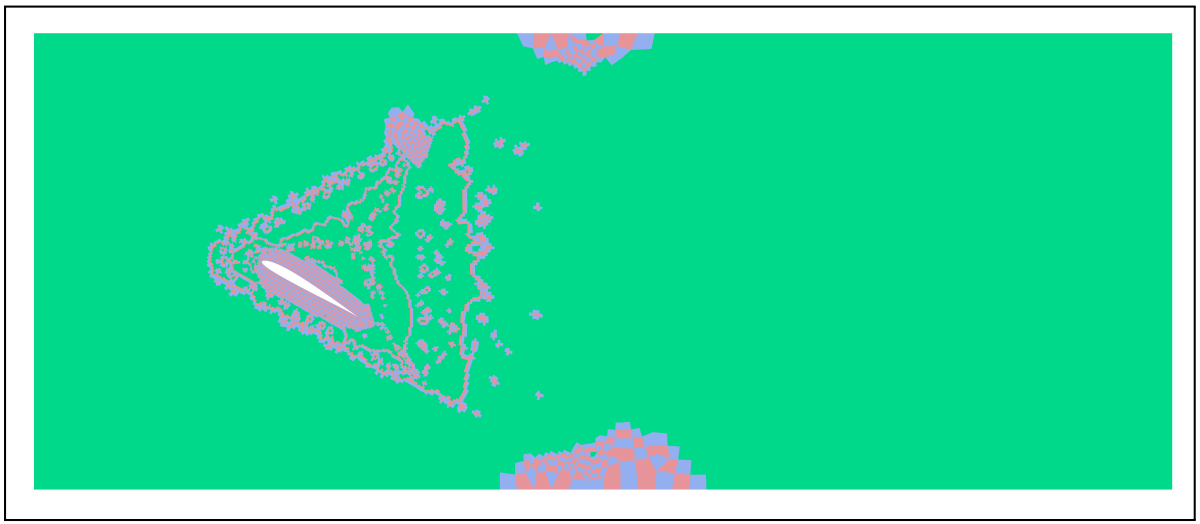


22007 P_3^{DG} $\square$

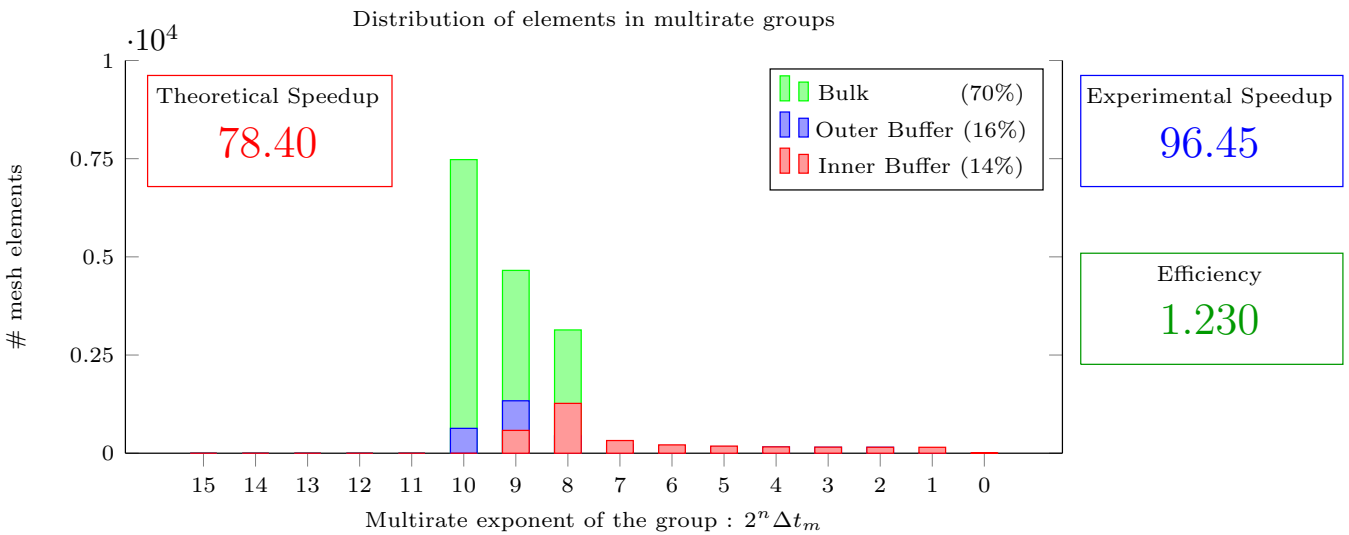


Flow pas the SD7003 airfoil (2)

Effect of the number of MR groups

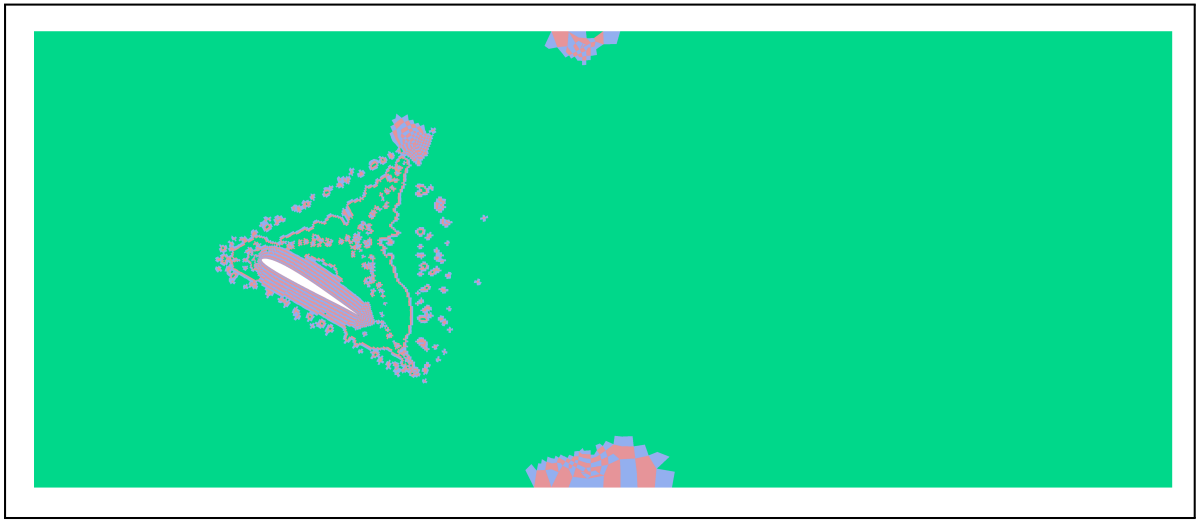


22007 P_3^{DG} $\square$

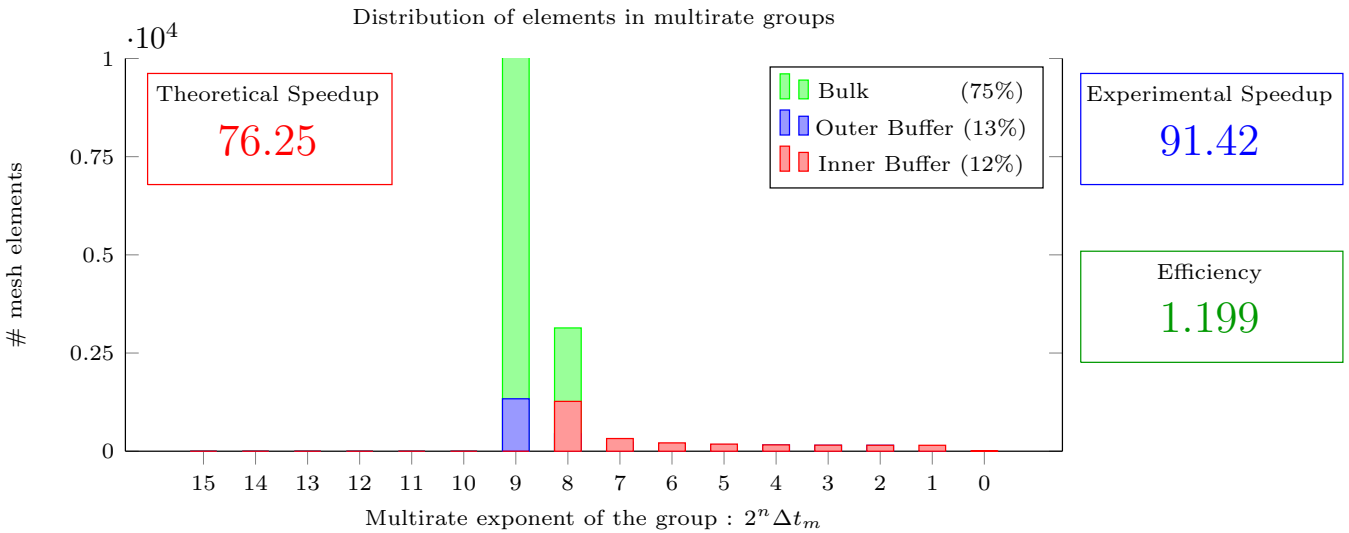


Flow pas the SD7003 airfoil (2)

Effect of the number of MR groups

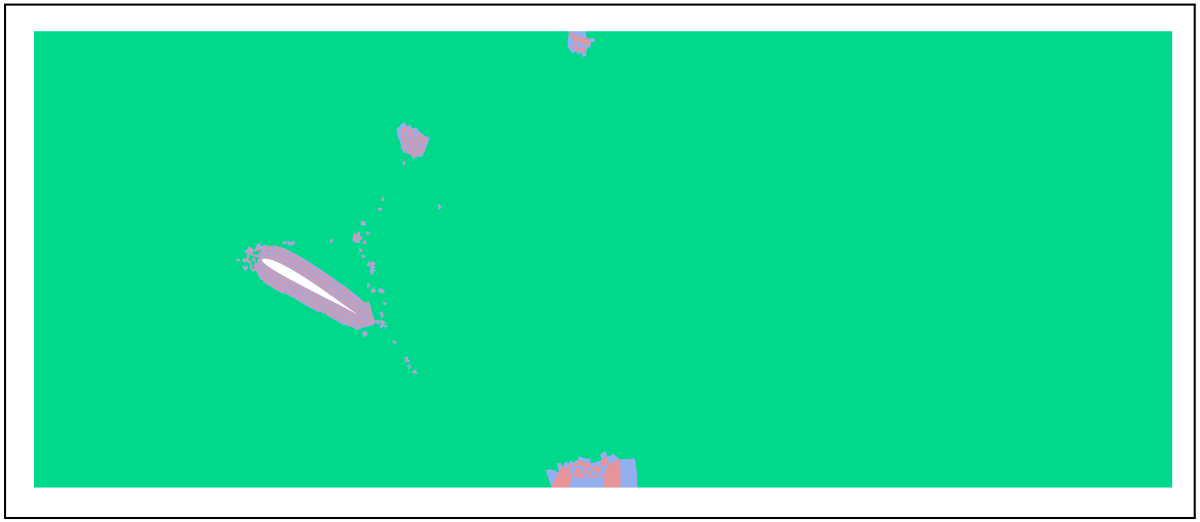


22007 P_3^{DG} $\square$

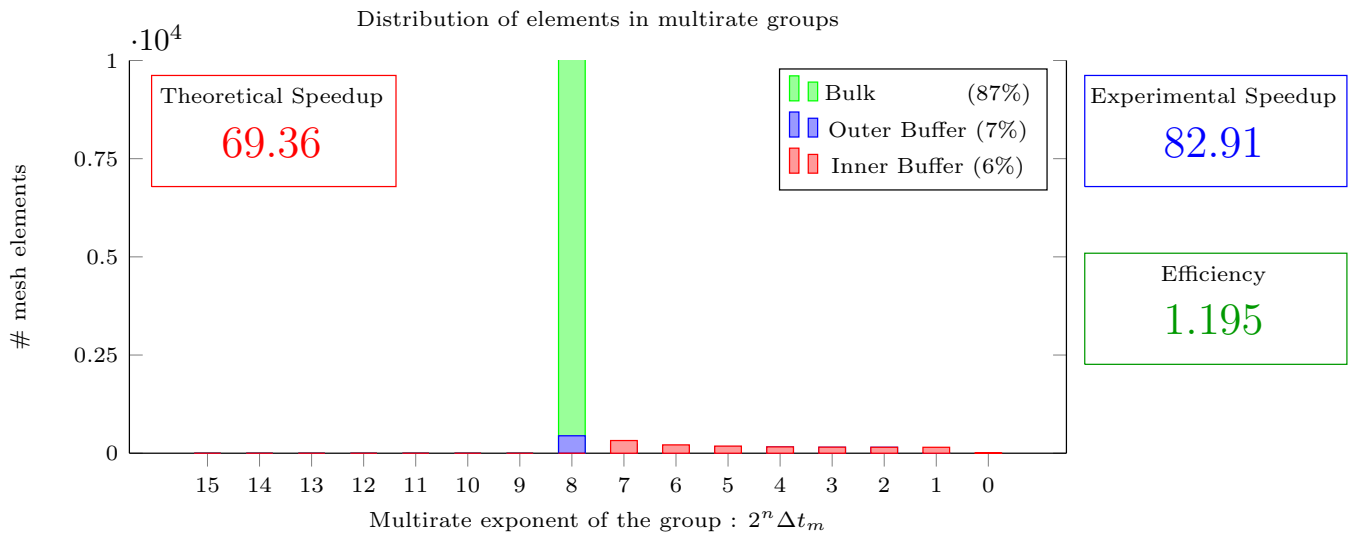


Flow pas the SD7003 airfoil (2)

Effect of the number of MR groups

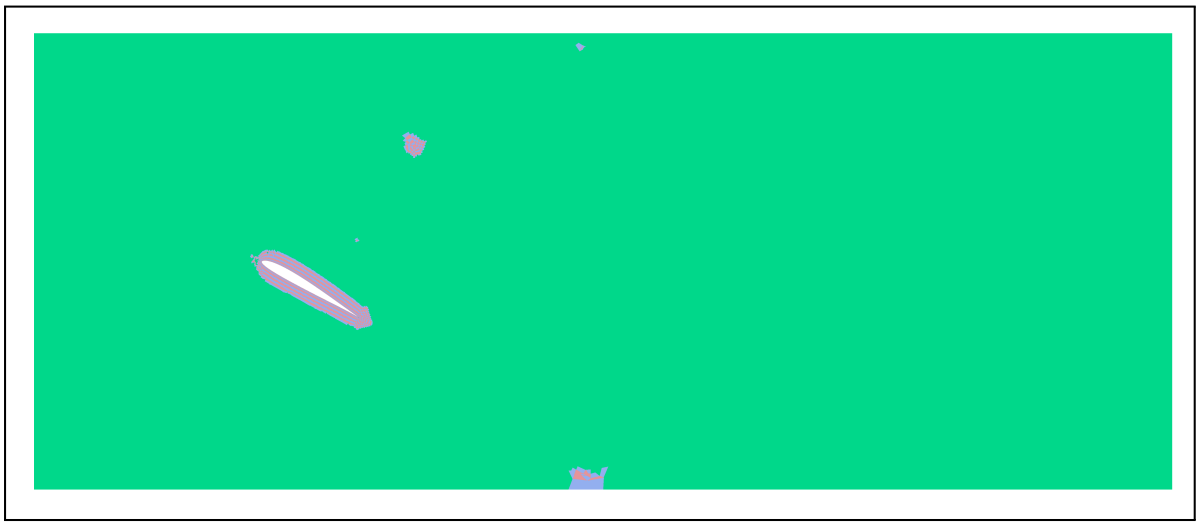


22007 P_3^{DG} □

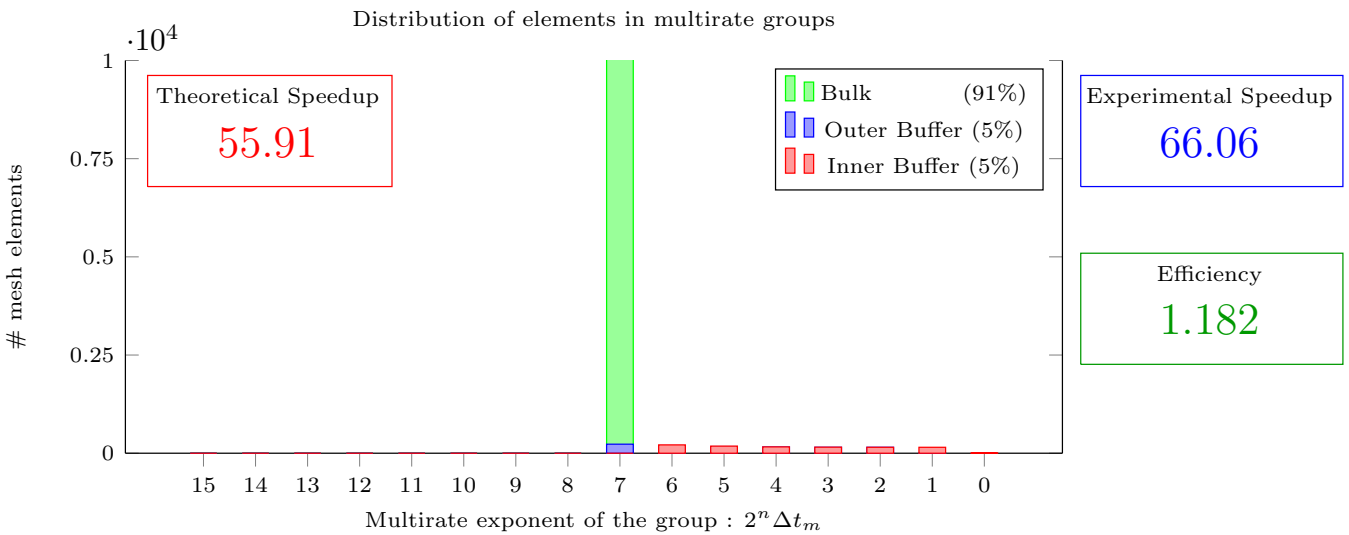


Flow pas the SD7003 airfoil (2)

Effect of the number of MR groups

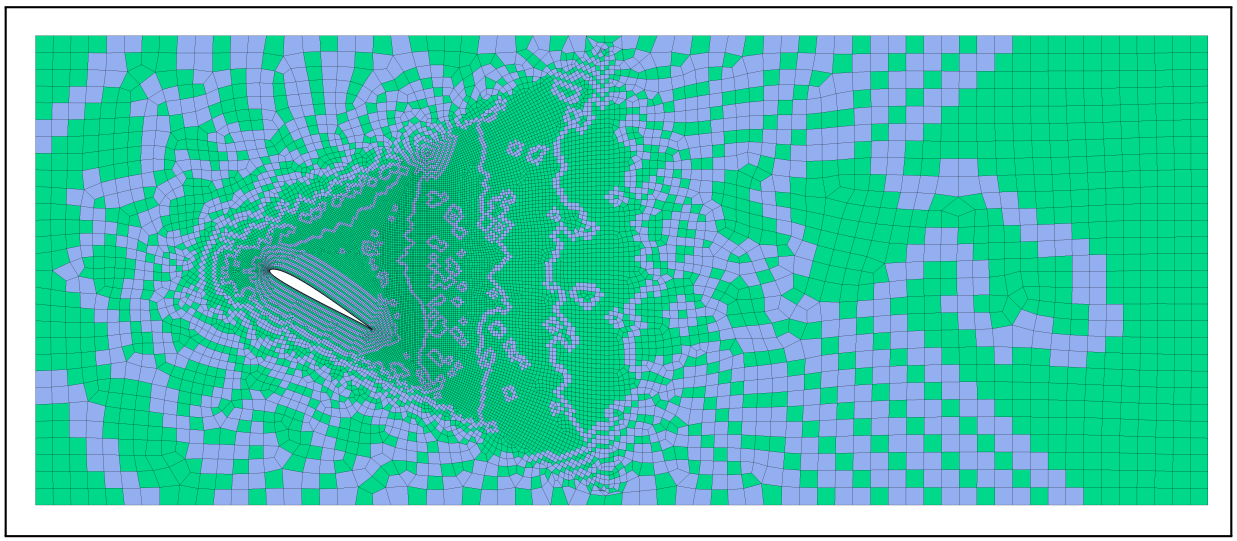


22007 P_3^{DG} $\square$



Flow pas the SD7003 airfoil (3)

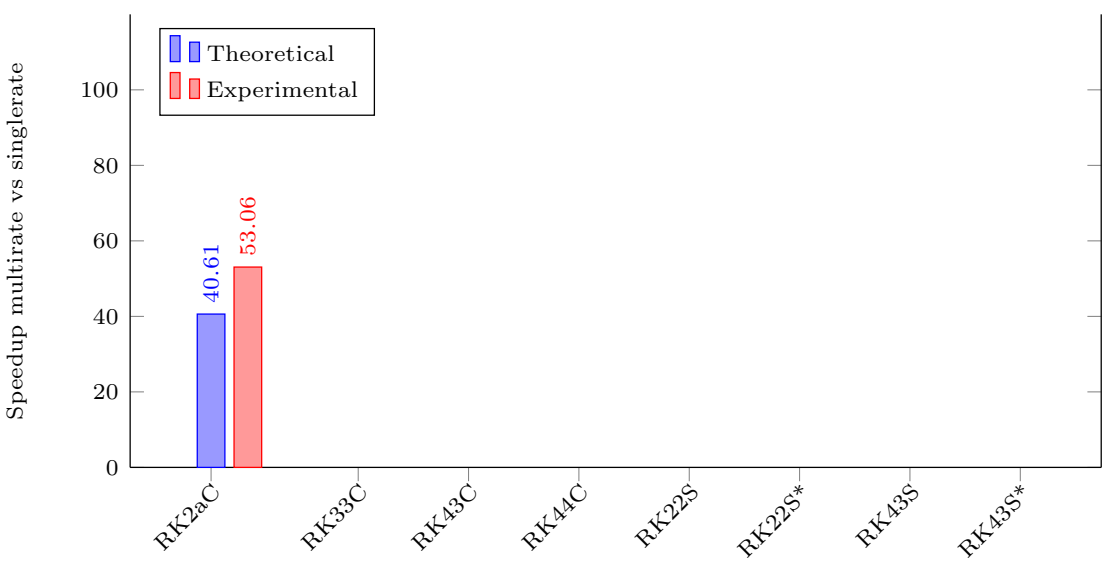
Comparison with other multirate strategies



Bulk (65%)

I. Buffer (0%)

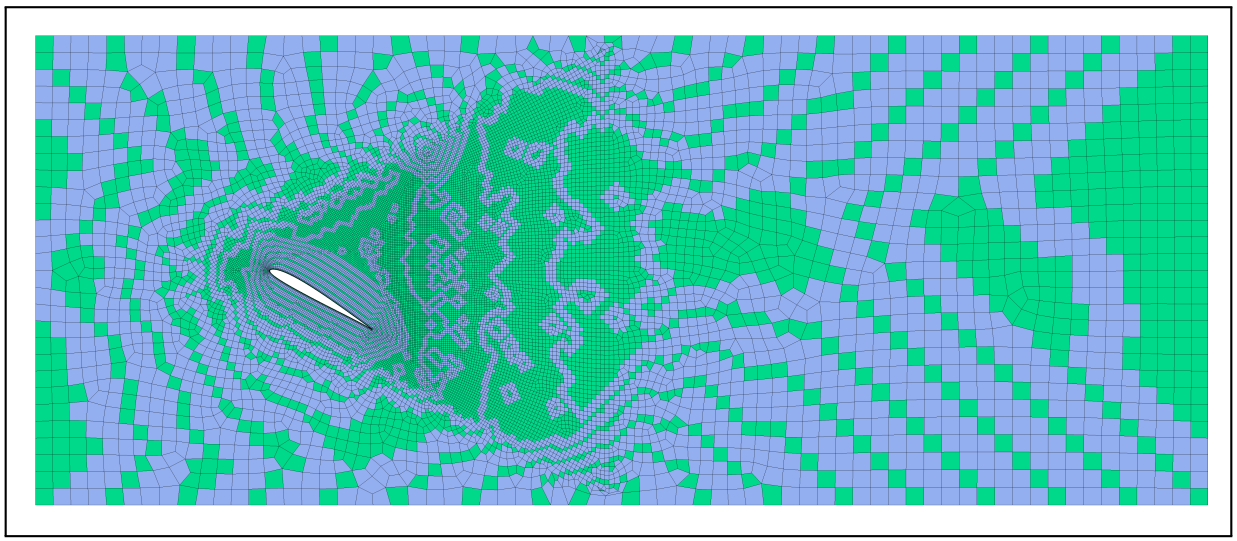
O. Buffer (35%)



	conservative	order
RK2aC	$\mathcal{V}$	2

Flow pas the SD7003 airfoil (3)

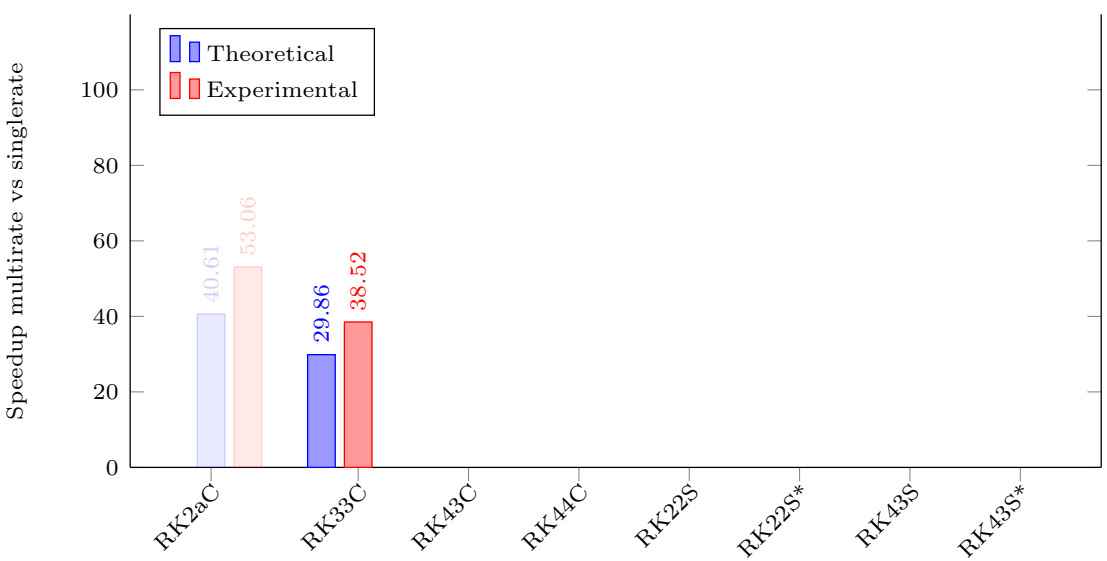
Comparison with other multirate strategies



Bulk (52%)

I. Buffer (0%)

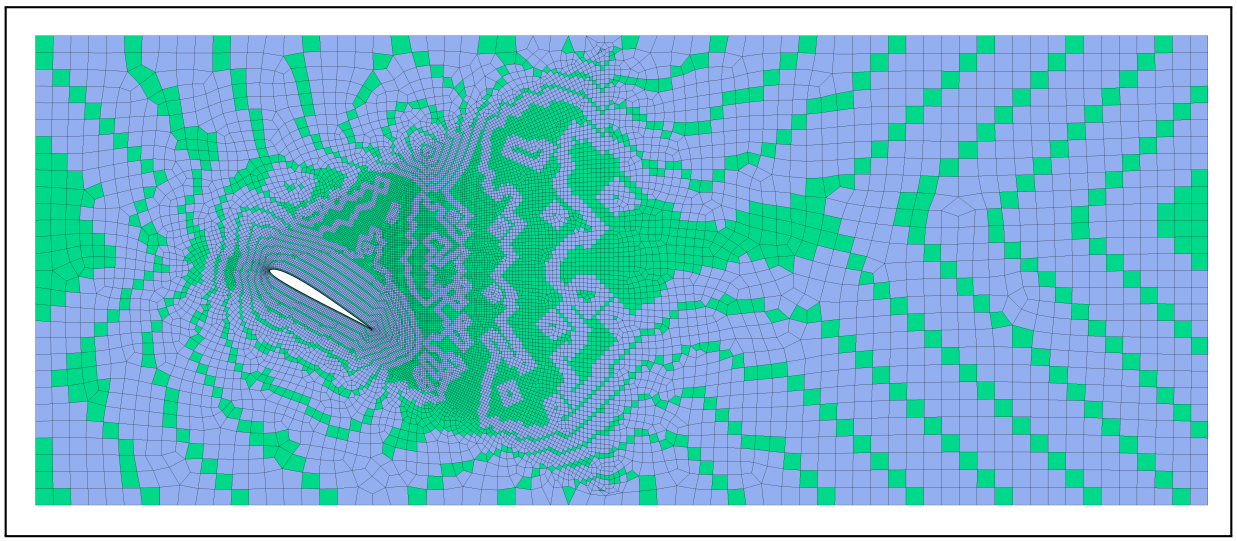
O. Buffer (48%)



	conservative	order
RK2aC	$\mathcal{V}$	2
RK33C	$\mathcal{V}$	2

Flow pas the SD7003 airfoil (3)

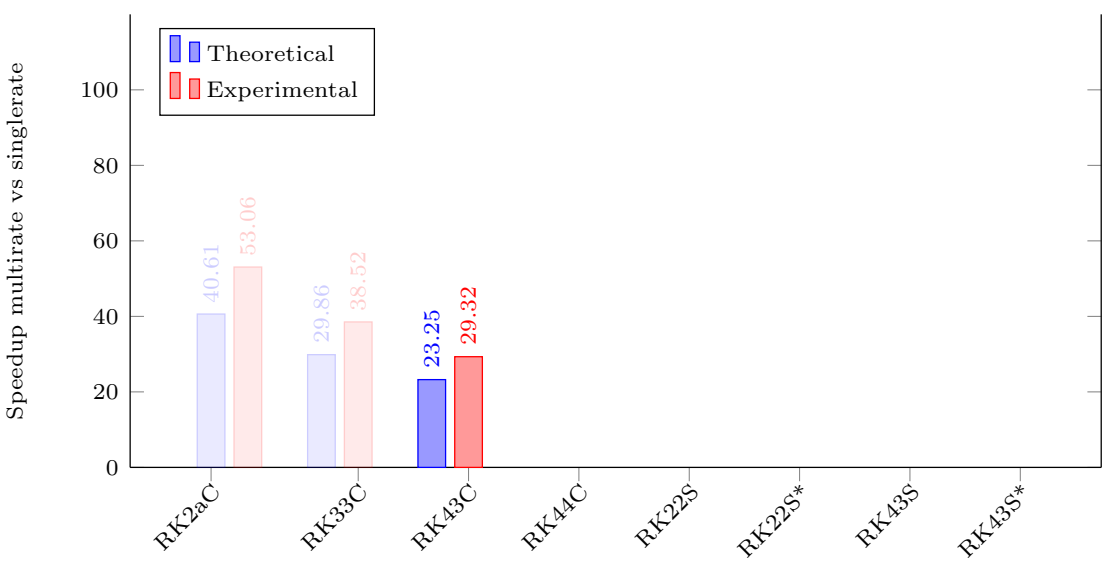
Comparison with other multirate strategies



Bulk (41%)

I. Buffer (0%)

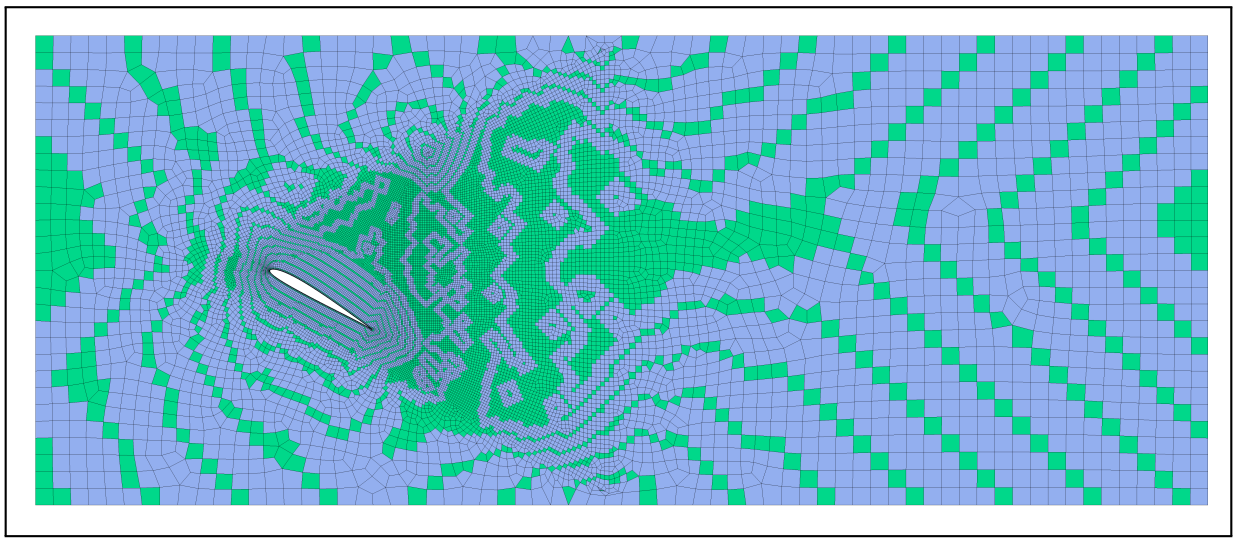
O. Buffer (59%)



	conservative	order
RK2aC	$\mathcal{V}$	2
RK33C	$\mathcal{V}$	2
RK43C	$\mathcal{V}$	2

Flow pas the SD7003 airfoil (3)

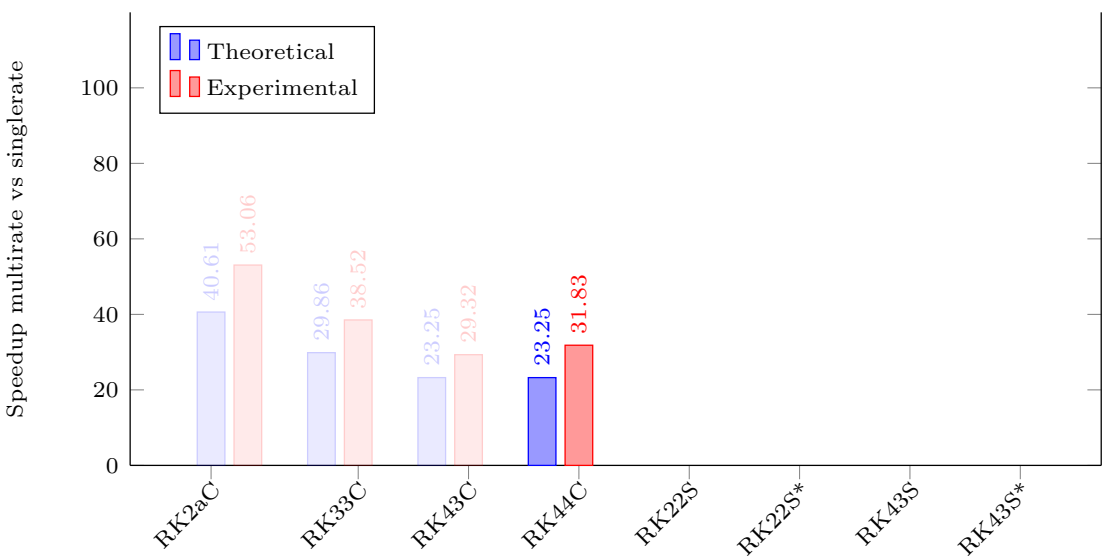
Comparison with other multirate strategies



Bulk (41%)

I. Buffer (0%)

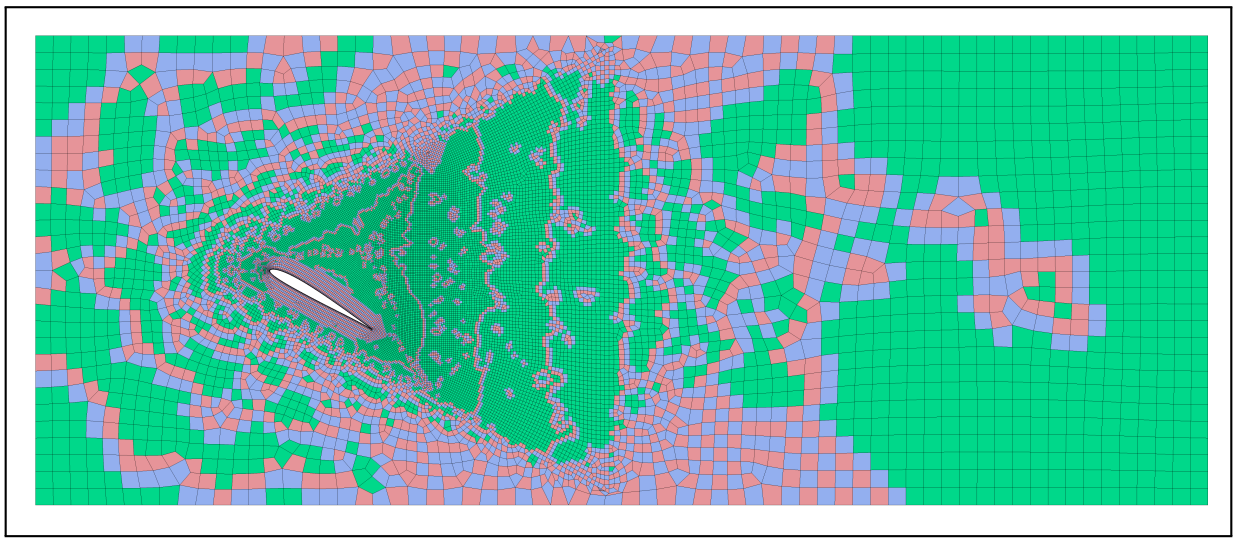
O. Buffer (59%)



	conservative	order
RK2aC	$\mathcal{V}$	2
RK33C	$\mathcal{V}$	2
RK43C	$\mathcal{V}$	2
RK44C	$\mathcal{V}$	2

Flow pas the SD7003 airfoil (3)

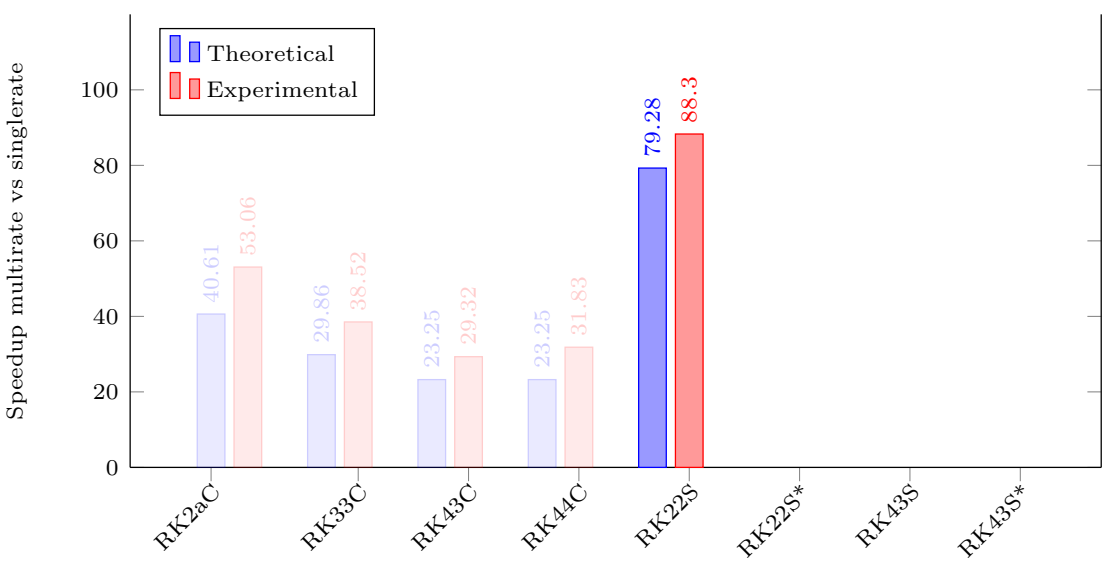
Comparison with other multirate strategies



Bulk (58%)

I. Buffer (20%)

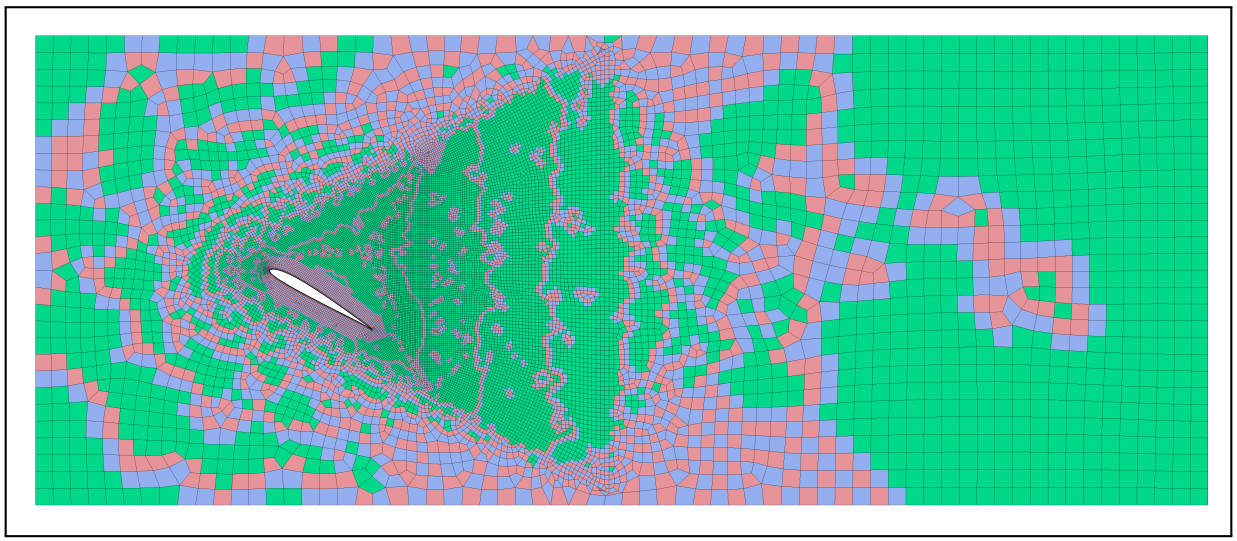
O. Buffer (22%)



	conservative	order
RK2aC	✓	2
RK33C	✓	2
RK43C	✓	2
RK44C	✓	2
RK22S	×	2

Flow pas the SD7003 airfoil (3)

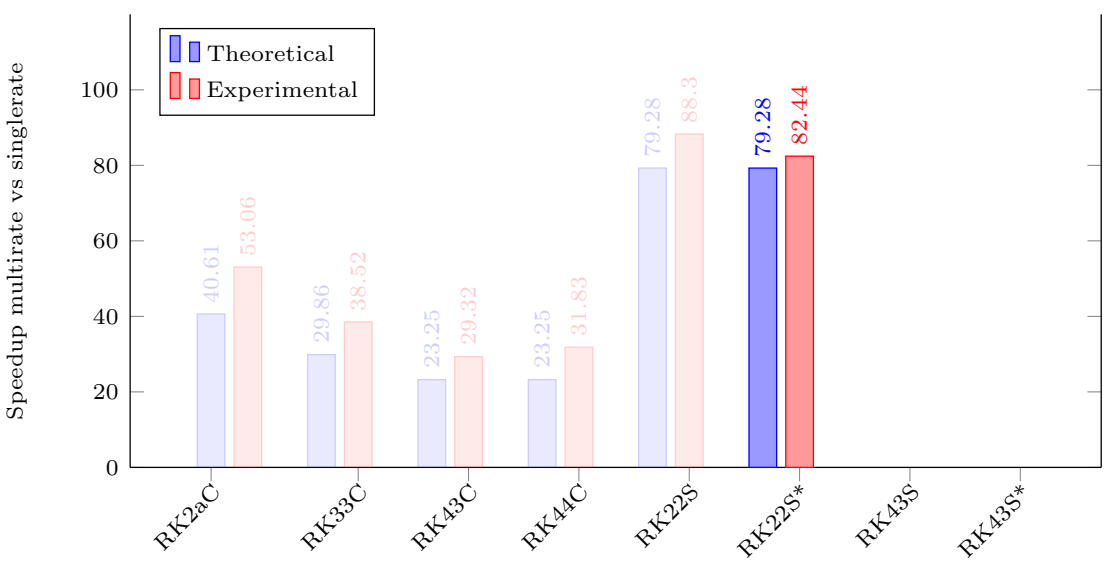
Comparison with other multirate strategies



Bulk (58%)

I. Buffer (20%)

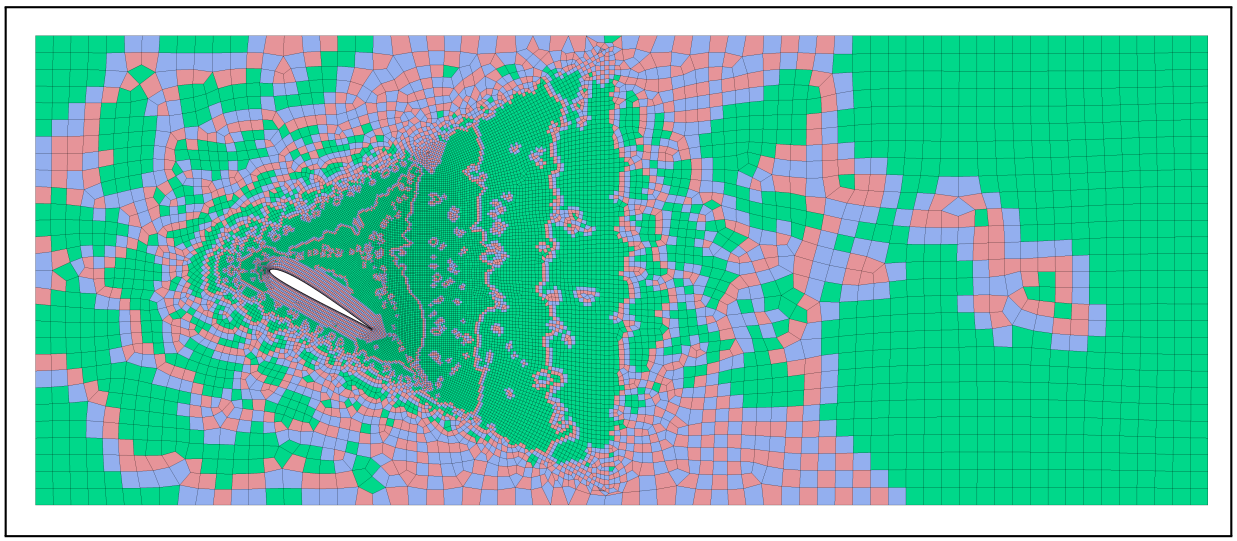
O. Buffer (22%)



	conservative	order
RK2aC	$\mathcal{V}$	2
RK33C	$\mathcal{V}$	2
RK43C	$\mathcal{V}$	2
RK44C	$\mathcal{V}$	2
RK22S	$\times$	2
RK22S*	$\mathcal{V}$	2

Flow pas the SD7003 airfoil (3)

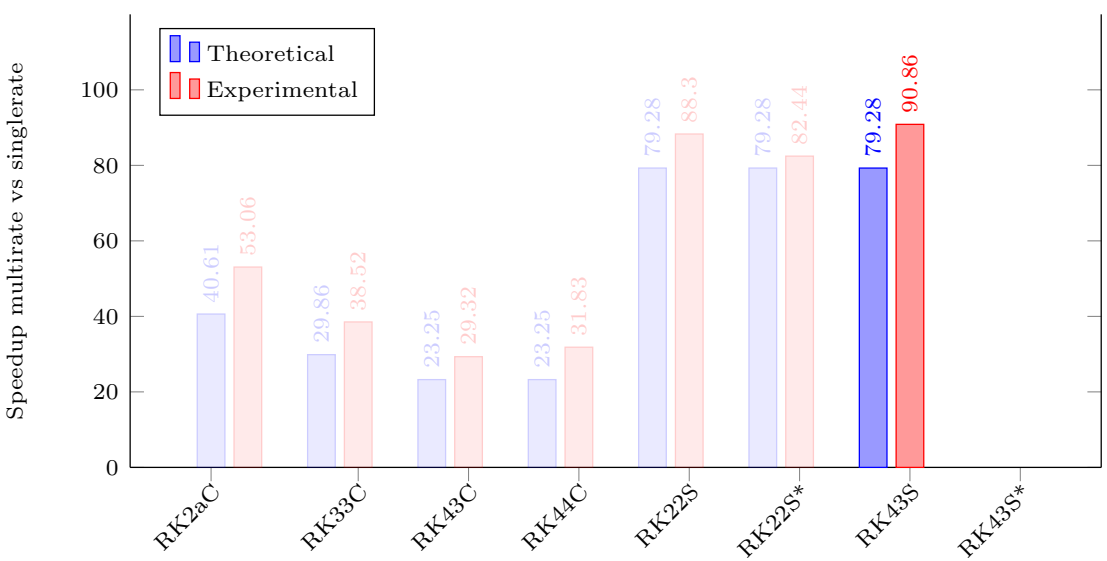
Comparison with other multirate strategies



Bulk (58%)

I. Buffer (20%)

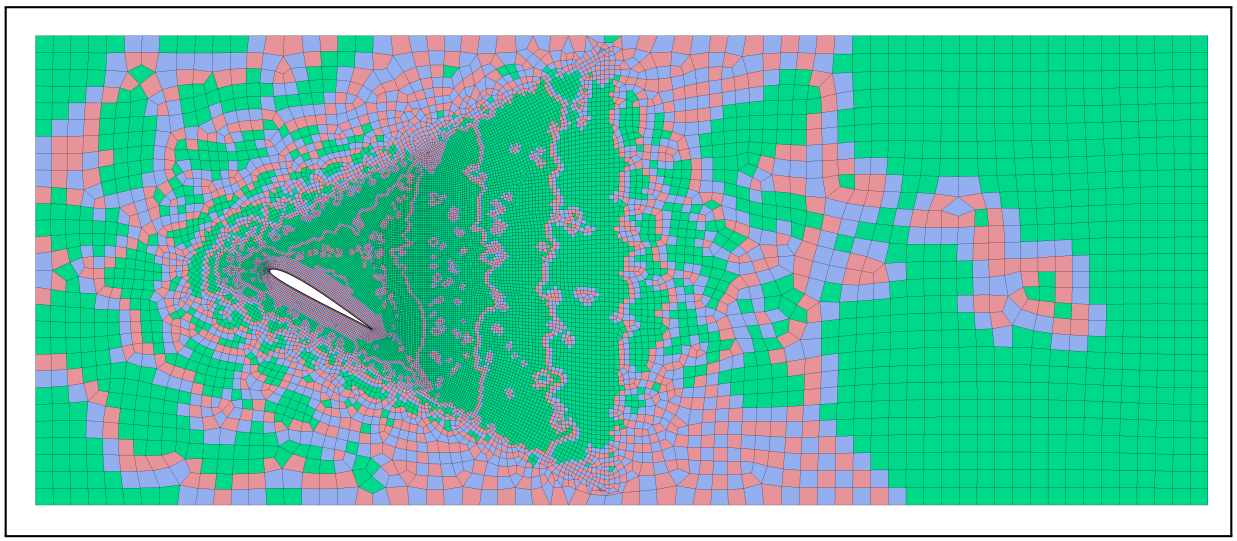
O. Buffer (22%)



	conservative	order
RK2aC	✓	2
RK33C	✓	2
RK43C	✓	2
RK44C	✓	2
RK22S	×	2
RK22S*	✓	2
RK43S	×	3

Flow pas the SD7003 airfoil (3)

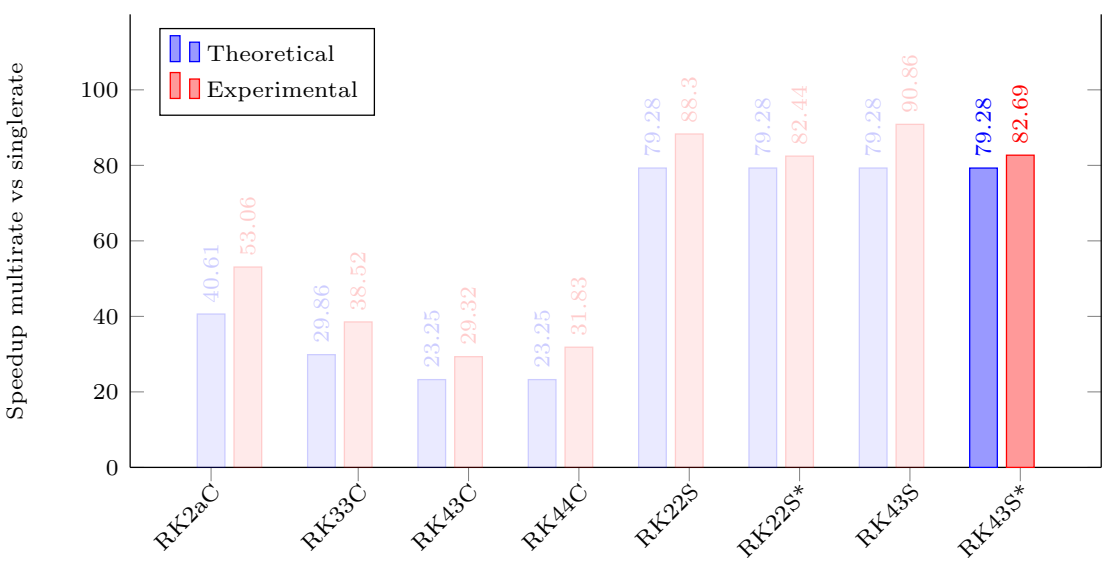
Comparison with other multirate strategies



Bulk (58%)

I. Buffer (20%)

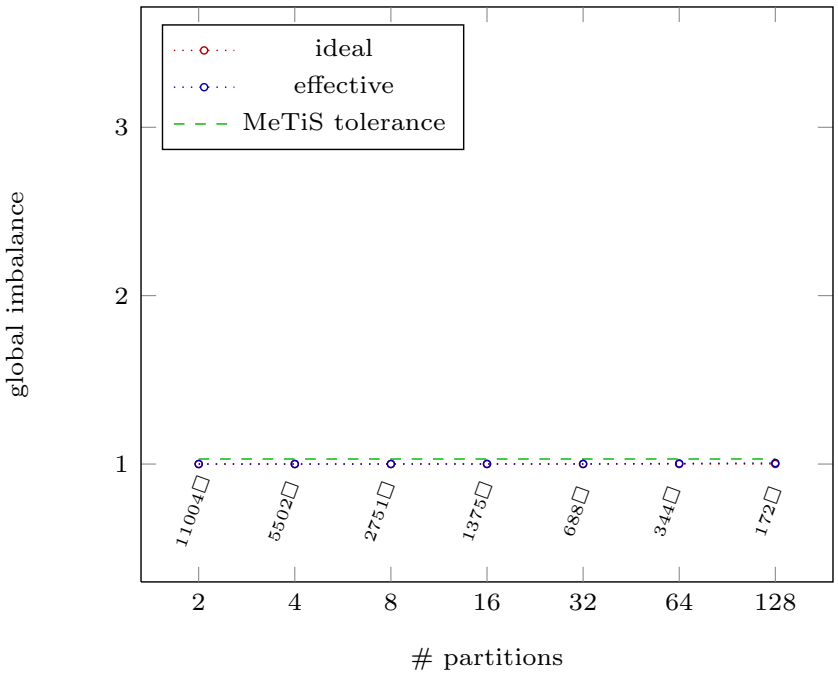
O. Buffer (22%)

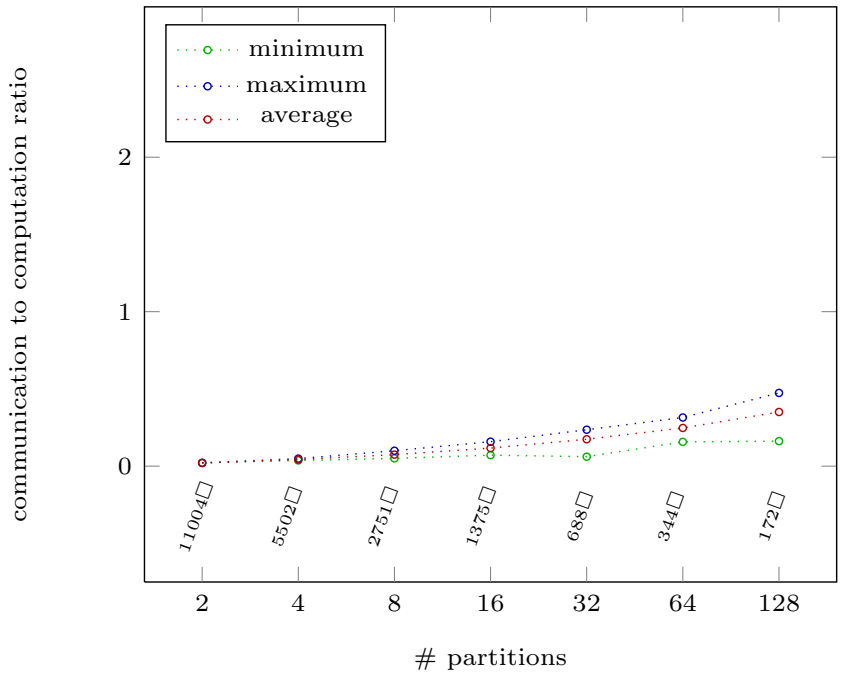


	conservative	order
RK2aC	$\mathcal{V}$	2
RK33C	$\mathcal{V}$	2
RK43C	$\mathcal{V}$	2
RK44C	$\mathcal{V}$	2
RK22S	$\times$	2
RK22S*	$\mathcal{V}$	2
RK43S	$\times$	3
RK43S*	$\mathcal{V}$	3

Flow pas the SD7003 airfoil (4)

Quality of multi-constraint partitioning

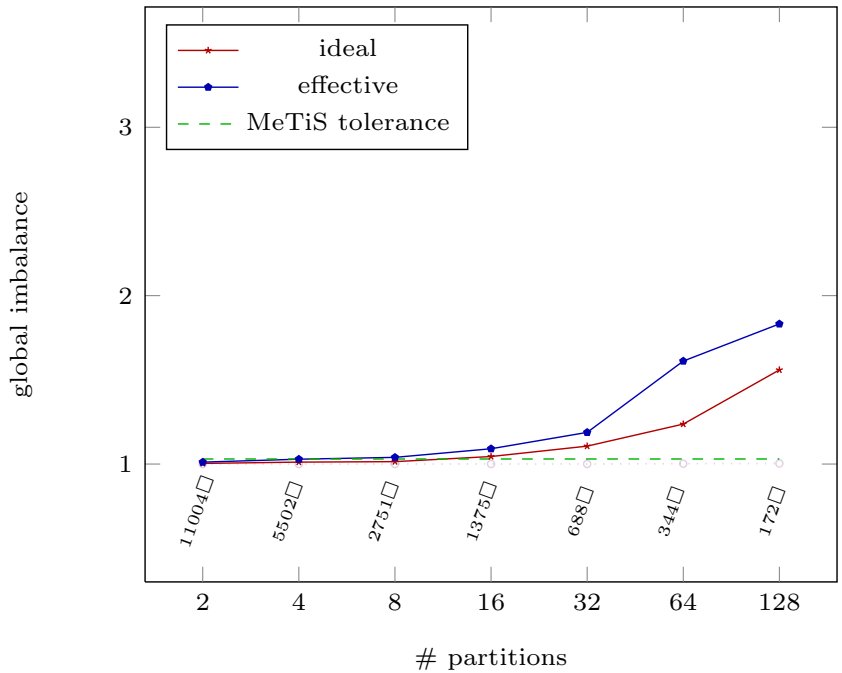


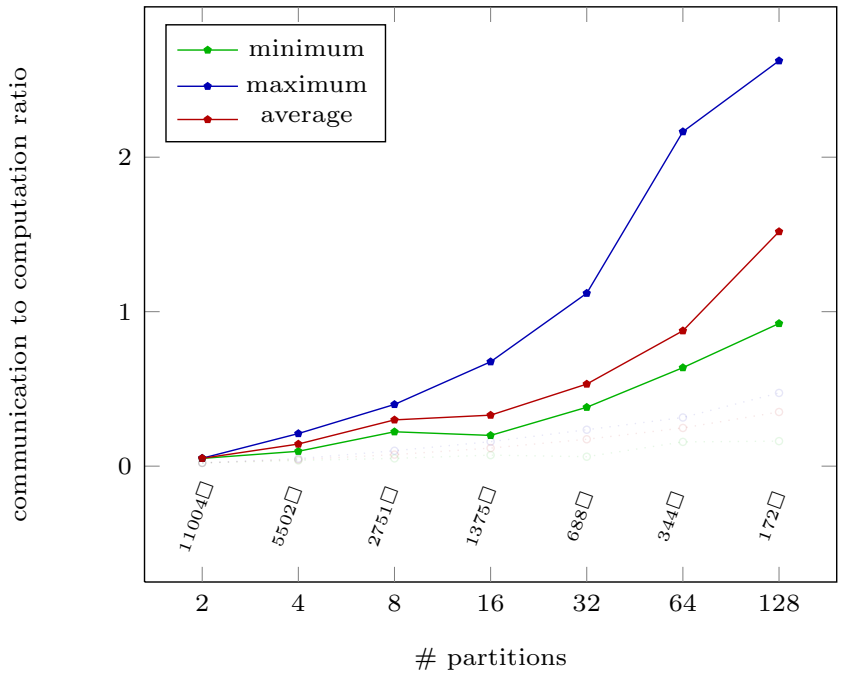


Maximum number of constraints for MeTiS partitioner: 15

Flow pas the SD7003 airfoil (4)

Quality of multi-constraint partitioning





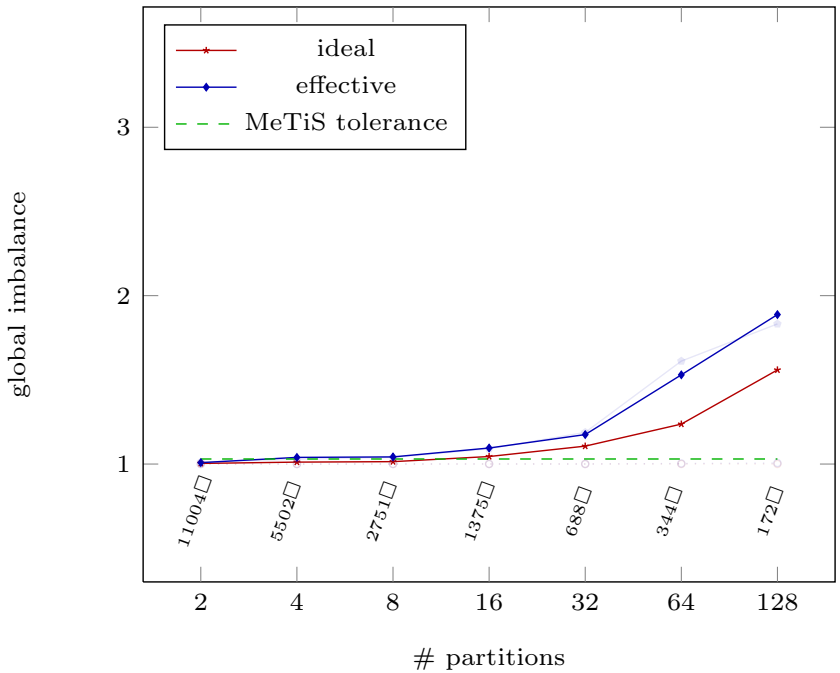
15 constraints, partitioning for 91% of the computational load

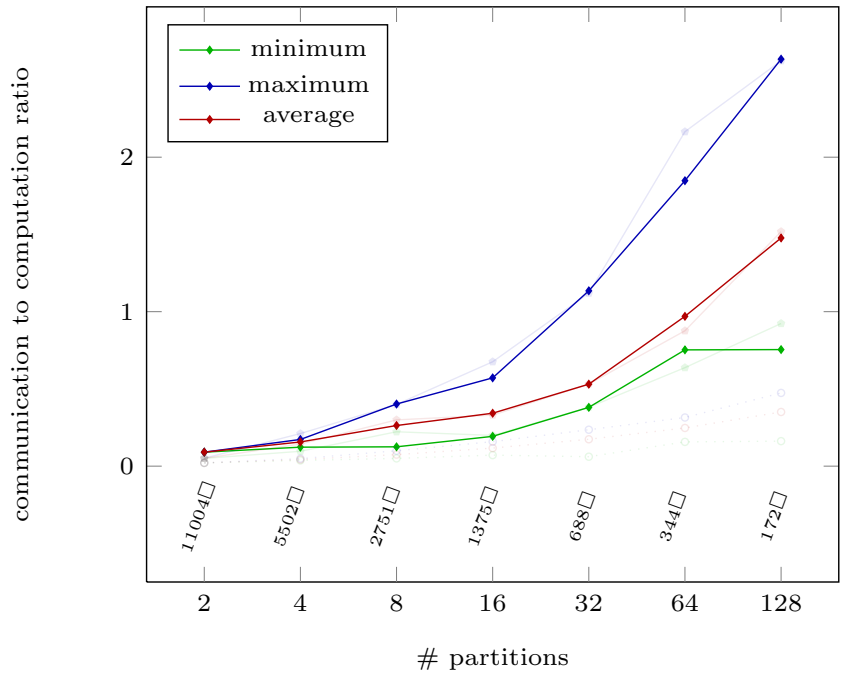
constraint number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
computational weight	0.1441	0.1360	0.1051	0.0869	0.0743	0.0687	0.0615	0.0406	0.0360	0.0341	0.0263	0.0258	0.0244	0.0231	0.0198	0.0174
constraint number	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
computational weight	0.0153	0.0135	0.0121	0.0086	0.0085	0.0062	0.0045	0.0023	0.0019	0.0012	0.0012	0.0003	0.0001	0.0001	0.0000	

Maximum number of constraints for MeTiS partitioner: 15

Flow pas the SD7003 airfoil (4)

Quality of multi-constraint partitioning





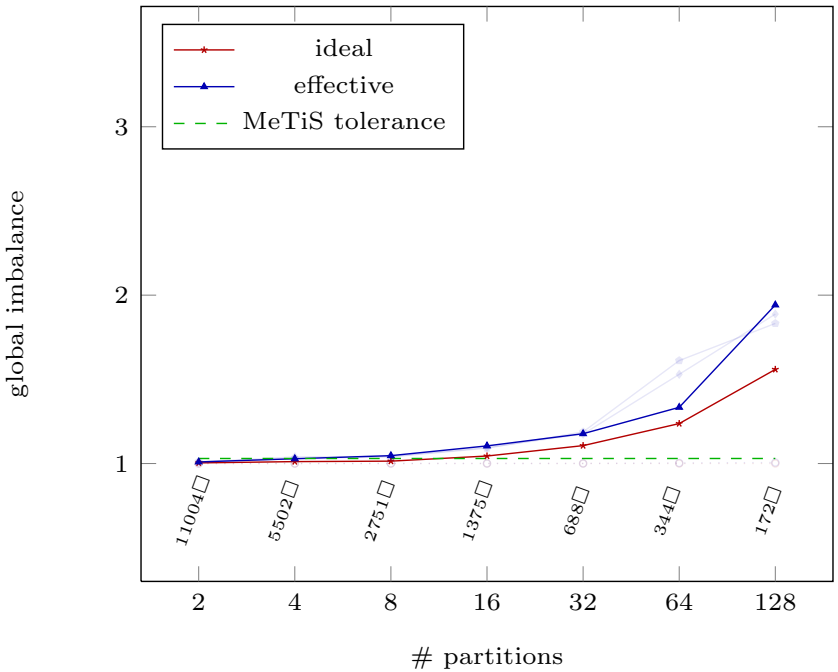
14 constraints, partitioning for 89% of the computational load

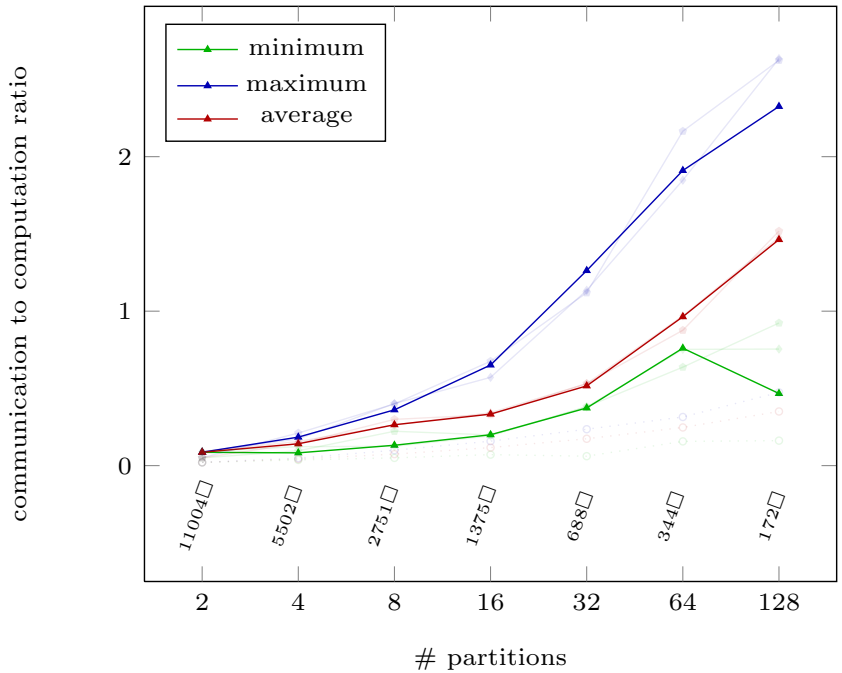
constraint number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
computational weight	0.1441	0.1360	0.1051	0.0869	0.0743	0.0687	0.0615	0.0406	0.0360	0.0341	0.0263	0.0258	0.0244	0.0231	0.0198	0.0174
constraint number	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
computational weight	0.0153	0.0135	0.0121	0.0086	0.0085	0.0062	0.0045	0.0023	0.0019	0.0012	0.0012	0.0003	0.0001	0.0001	0.0000	

Maximum number of constraints for MeTiS partitioner: 15

Flow pas the SD7003 airfoil (4)

Quality of multi-constraint partitioning





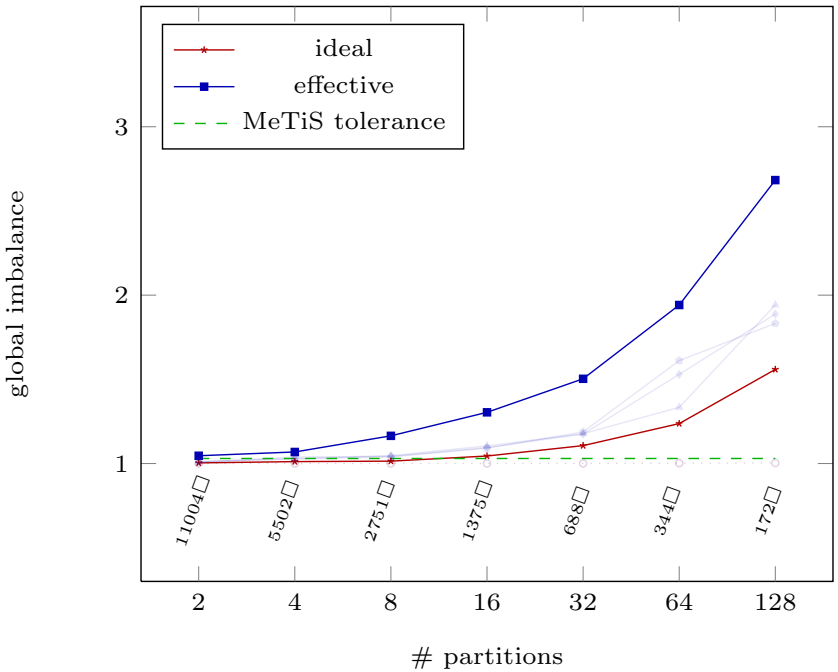
13 constraints, partitioning for 86% of the computational load

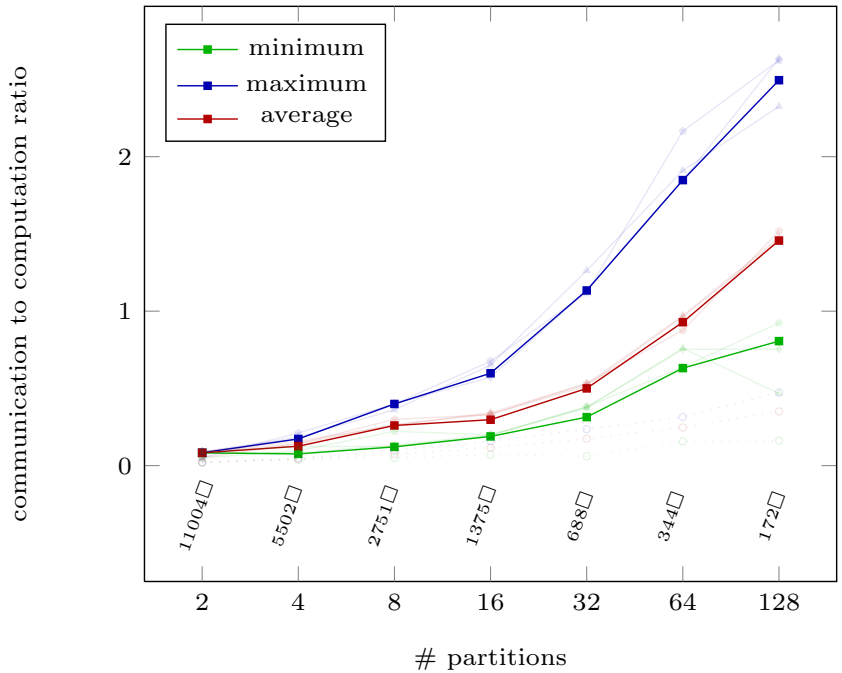
constraint number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
computational weight	0.1441	0.1360	0.1051	0.0869	0.0743	0.0687	0.0615	0.0406	0.0360	0.0341	0.0263	0.0258	0.0244	0.0231	0.0198	0.0174
constraint number	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
computational weight	0.0153	0.0135	0.0121	0.0086	0.0085	0.0062	0.0045	0.0023	0.0019	0.0012	0.0012	0.0003	0.0001	0.0001	0.0000	

Maximum number of constraints for MeTiS partitioner: 15

Flow pas the SD7003 airfoil (4)

Quality of multi-constraint partitioning





9 constraints, partitioning for 75% of the computational load

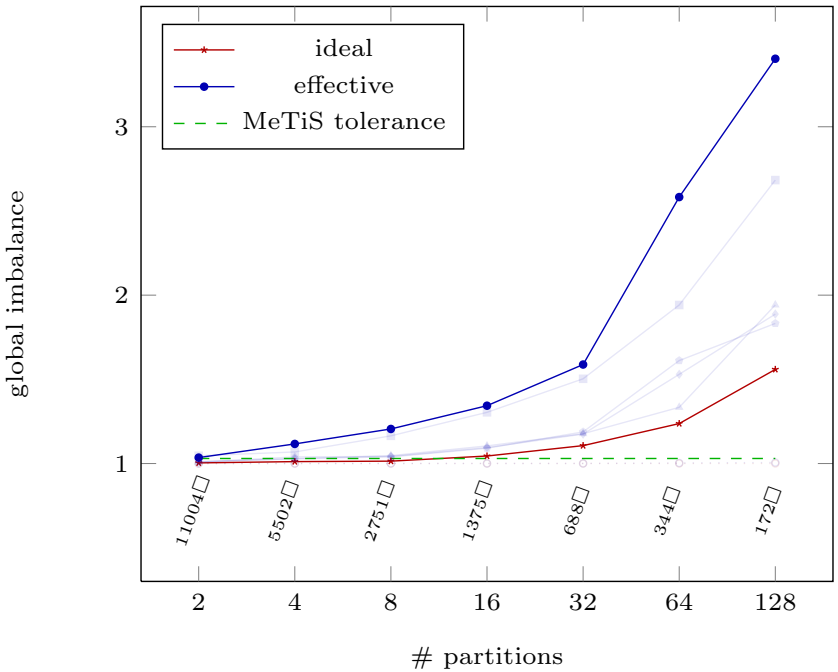
constraint number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
computational weight	0.1441	0.1360	0.1051	0.0869	0.0743	0.0687	0.0615	0.0406	0.0360	0.0341	0.0263	0.0258	0.0244	0.0231	0.0198	0.0174

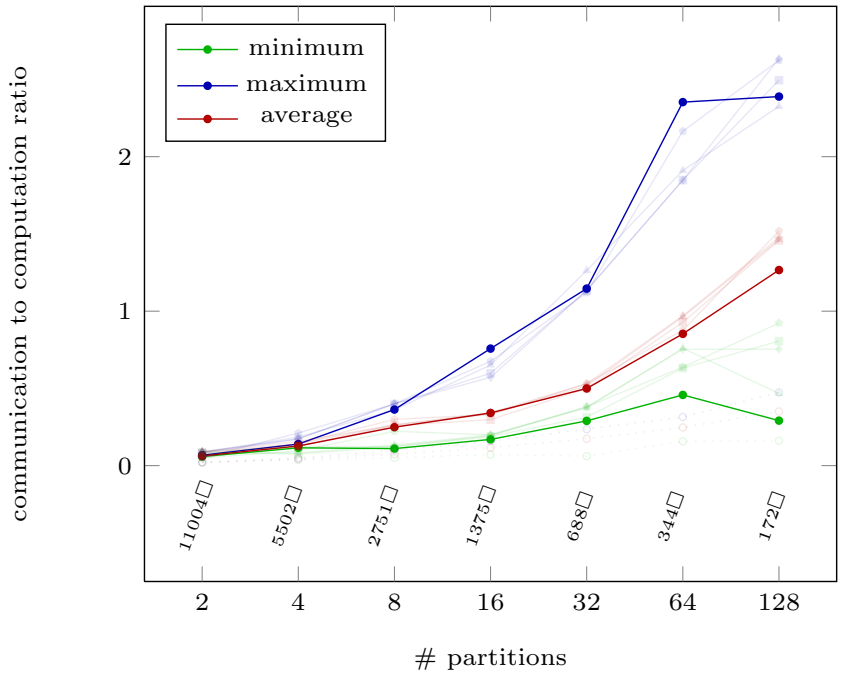
constraint number	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
computational weight	0.0153	0.0135	0.0121	0.0086	0.0085	0.0062	0.0045	0.0023	0.0019	0.0012	0.0012	0.0003	0.0001	0.0001	0.0000	

Maximum number of constraints for MeTiS partitioner: 15

Flow pas the SD7003 airfoil (4)

Quality of multi-constraint partitioning





5 constraints, partitioning for 55% of the computational load

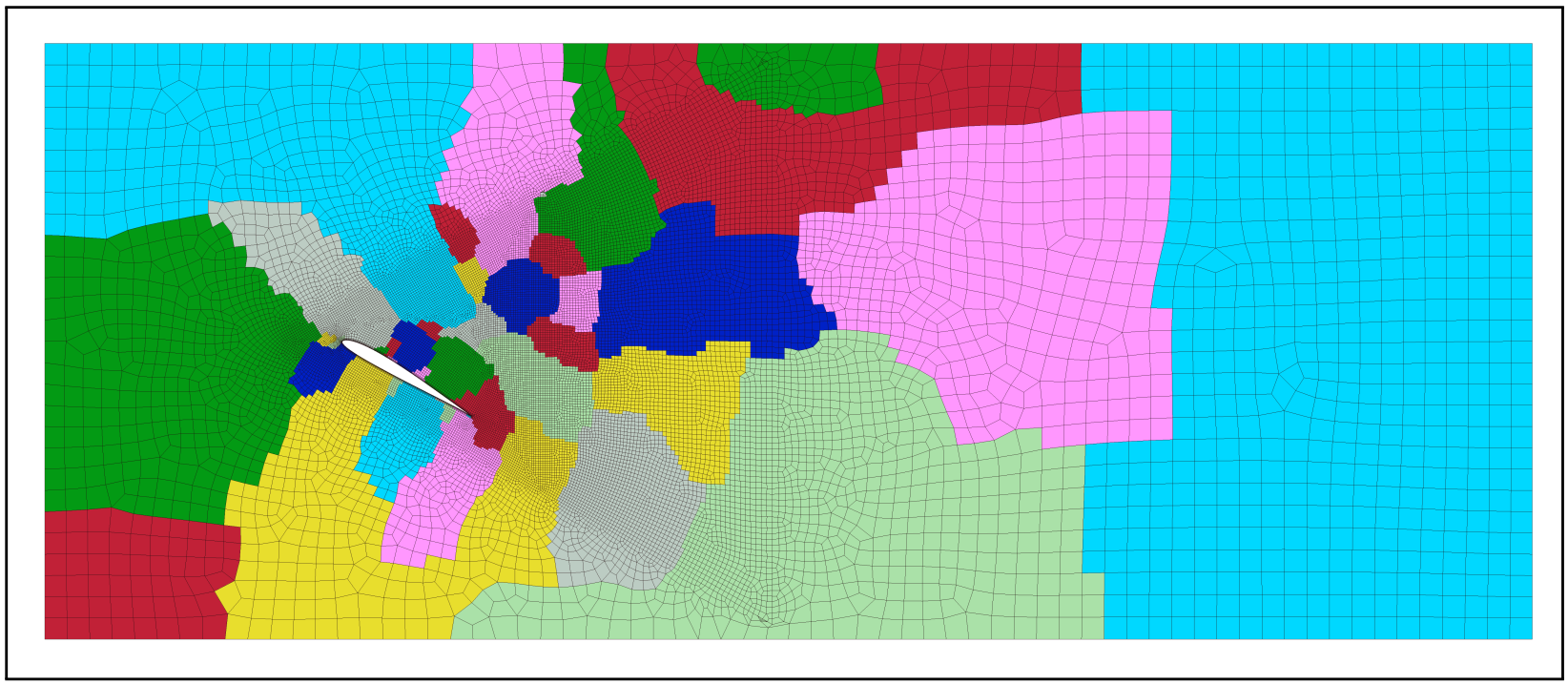
constraint number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
computational weight	0.1441	0.1360	0.1051	0.0869	0.0743	0.0687	0.0615	0.0406	0.0360	0.0341	0.0263	0.0258	0.0244	0.0231	0.0198	0.0174

constraint number	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
computational weight	0.0153	0.0135	0.0121	0.0086	0.0085	0.0062	0.0045	0.0023	0.0019	0.0012	0.0012	0.0003	0.0001	0.0001	0.0000	

Maximum number of constraints for MeTiS partitioner: 15

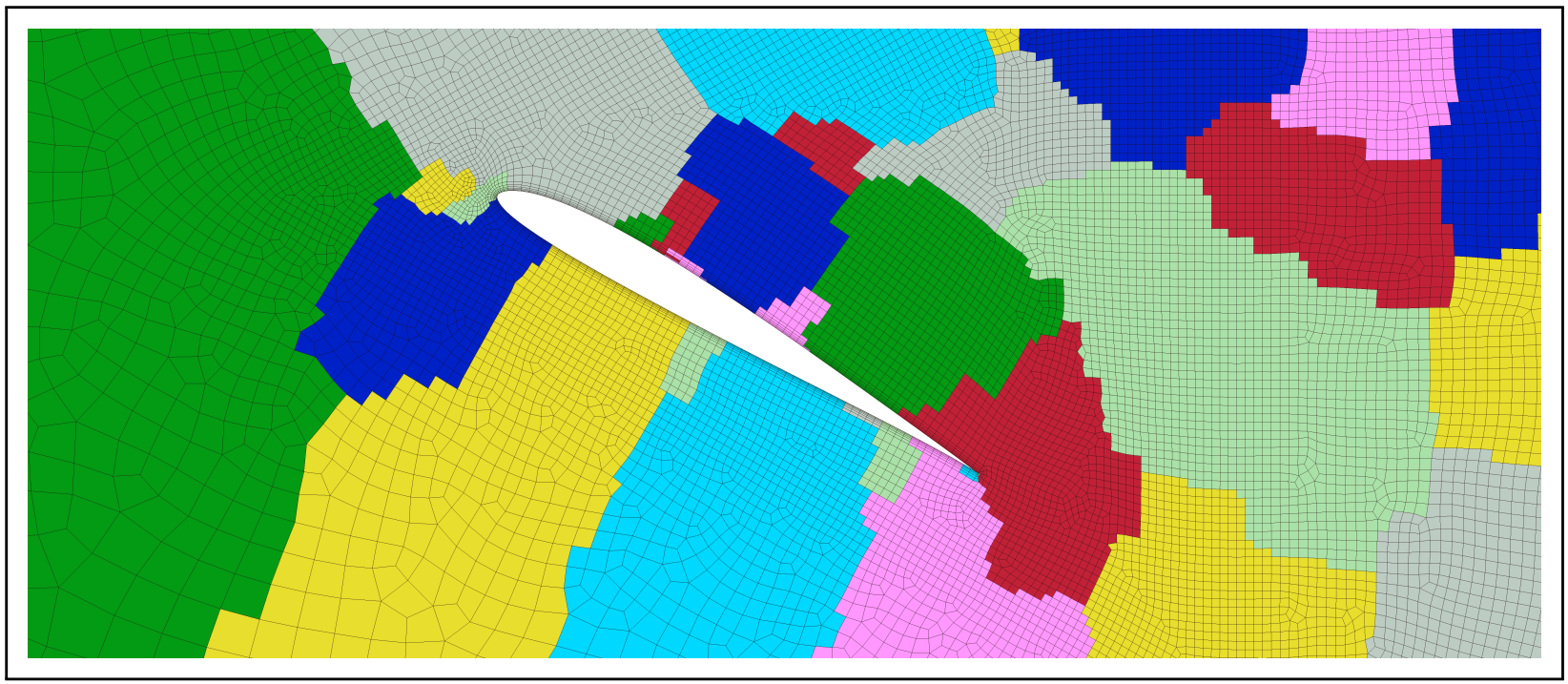
Flow pas the SD7003 airfoil (5)

Multi-constraint partitioning (8 partitions)



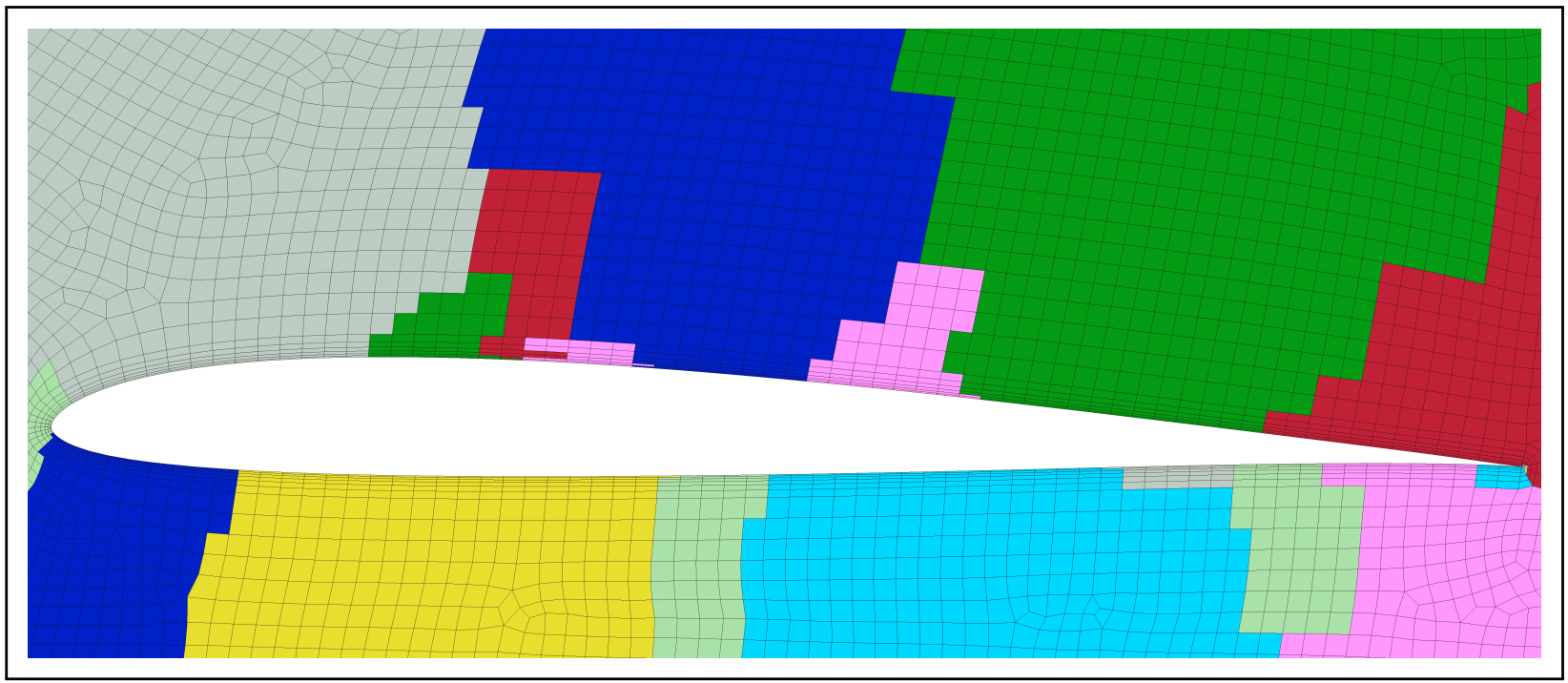
Flow pas the SD7003 airfoil (5)

Multi-constraint partitioning (8 partitions)



Flow pas the SD7003 airfoil (5)

Multi-constraint partitioning (8 partitions)



Conclusion

- Multirate methods are well suited for DG on unstructured meshes
- Considerable speedups are observed for realistic applications
- The multi-constraint partitioning with MeTiS yields acceptable domain decompositions
- The parallel strategy yields satisfying results up to a significant number of processors
- Global accuracy is comparable to singlerate for low spatial discretization orders

Perspectives

- Dynamic multirate and load-balancing
- Improve scalability: memory access and storage, intra-node contiguity, ...

Any Questions ?

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