Chapter 1

Introduction

Malgré moi l'infini me tourmente.
- Alfred de Musset

Le silence éternel de ces espaces infinis m'effraie.
- Blaise Pascal

Willst du ins Unendliche schreiten, Geh nur im Endlichen nach allen seiten.
Si tu veux progresser vers l'infini, explore le fini dans toutes les directions.
- Johan Wolfgang von Goethe

1.1 Motivation

Many interesting problems appearing in physics, biology and other natural sciences have solutions consisting of waves. Most wave propagation problems are posed on unbounded spatial domains. Wave problems which have infinite geometry are encountered in a wide variety of engineering and mechanic applications such as acoustics and elastodynamics. The propagation of noise generated by a traffic route, seismic wave propagation, resource exploration, interaction of vibrating structures submerged in an infinite acoustics fluid, sea–wave propagation, nondestructive testing, represent selected topics that requires considering wave propagation in unbounded domains.

Wave equations are time dependent. When the dependent variables are harmonic or oscillatory functions of time, waves are called *steady-state* otherwise they are *transient*. If the solution of a problem with a steady state boundary condition is known, the solution of the same problem with a transient boundary condition can (in principle) be determined by use of the Fourier transform. This representation imagines the transient signal decomposed into an infinite number of time-harmonic components. This powerful used procedure is called *Fourier superposition.*

Acoustic wave propagation, with time-harmonic behavior, is described by the Helmholtz equation (also called the reduced wave equation),

$$\Delta p + k^2 p = 0, \tag{1.1}$$

where the physical parameter is the wavenumber k and p is the acoustic pressure.

In elastic media, waves propagate in the form of oscillations of the stress field, and are described by the elastodynamic Helmholtz equation,

$$\nabla \cdot \boldsymbol{\sigma} + \rho\omega^2 \mathbf{u} = 0. \tag{1.2}$$

where $\mathbf{u}$ is the displacement field, ρ is the density of the elastic medium, and $\boldsymbol{\sigma}$ is the stress tensor.

To simulate that no waves enter the physical domain from infinity we need to supplement (1.1) and (1.2) with *Sommerfeld radiation condition* [201] to ensure uniqueness of solution. This condition selects the physical outgoing wave and does not allow energy to be radiated from infinity towards the structure.

The need of accurate methods for modelling mechanical wave fields in exterior domains is evident. To simulate the infiniteness computationally we must, artificially, bound the computational domain because we cannot discretize an unbounded domain with finite elements. Numerical methods for such problems have been developed since the 1970s, see the Givoli's monograph [98] and the review paper by Tsynkov [214].

The main types of numerical methods used to obtain the solution of that partial differential equations are the difference methods, the integral methods and the domain-based (finite element) methods.

1.2 State of Art

A wide variety of methods have been proposed to deal with this problem, ranging from the fast evaluation of exact, nonlocal conditions to the construction of special damping regions.

The major four families of schemes that have emerged since the last three decades are: boundary integral methods, infinite element methods, non-reflecting boundary condition (NRBC) methods and absorbing layer methods.

To ensure well-posedness of the problem in the exterior domain, it must be supplied with the *Sommerfeld radiation condition* at infinity, which guarantees that waves are purely outgoing and decaying as they approach infinity.

1.2.1 Boundary Integral Method (*BIM*)

In *Boundary Integral Method* the problem is first reformulated as a Fredholm integral equation* on the boundary of the spatial domain, and then this integral equation is solved numerically. This boundary reduction has the advantage of diminishing the number of space dimension by one and of the capability to handle problems involving infinite domains [192, 54, 182].

*An inhomogeneous Fredholm equation of the first kind is an integral equation of the form

$$f(x) = \int_a^b \mathcal{K}(x,t)\,\vartheta(t)\,\mathrm{d}t$$

where $\mathcal{K}$ is the kernel, f is a given function and ϑ is an unknown function to be solved for.

The reformulation as an integral equation can be done by either representing the solution in terms of a surface potential function (the indirect method) or by an application of Green's theorem to get a relationship between the acoustic pressure and its normal derivative (the direct method) [98].

For the numerical treatment of obtained boundary integral equations, one needs a discretization of the boundary charge functions. In the *Boundary Element Method* trial functions are chosen as finite elements on the boundary, the boundary elements. Gauss quadrature rules can be used to evaluate the obtained boundary integrals [121, 181]. These methods, in general, render asymmetric and full matrices and, for certain frequencies, the solution to integral equation formulations either does not exist or is non-unique.

1.2.2 Infinite Element Method (*IEM*)

The concept of *infinite elements* was first introduced by Bettess and Zienkiewicz [225, 39, 48] in the mid 70's and has since then been refined and applied to a variety of problems. Infinite elements have been developed for both separable [76, 11] and non separable [197, 78] convex boundaries.

Infinite element schemes do not truncate the exterior domain but represent it in its entirety by using elements of infinite extent. It is very similar to a finite element, except for the element extending towards infinity in one direction, and the corresponding shape functions being integrable over the infinite element. The fundamental idea of infinite elements is to accurately represent the asymptotic behavior of radiating waves within the element shape functions.

A distinction is made between conjugated (Petrov-Galerkin) infinite elements (or wave envelope elements) [13, 17, 61, 204] and unconjugated (Bubnov-Galerkin) infinite elements [55, 56, 57], depending on whether the weighting functions in the integral formulation are, respectively, the complex conjugate of the shape functions or identical to the shape functions.

With the conjugated formulation, the resulting system matrices are frequency independent and require (a relatively simple) integration of polynomial functions but are not symmetric, while the unconjugated formulation yields matrices that are symmetric but frequency dependent and whose coefficients result from more tedious numerical integrations. The integration in the exterior is understood in the sense of the *Cauchy principal value*.

The Astley-Leis or mapped wave envelope elements currently seem to be the most promising infinite elements for acoustics and related physical situations.

1.2.3 Non Reflecting Boundary Conditions (*NRBC*)

A rational and logic approach for modelling the infinite medium is to divide it into the following two parts. The first part is the near filed, which contains the nonlinear behavior of the materials and the geometrical irregularities. The second part is the far field, which represents the rest of the infinite medium. This class of methods have been referred to in the literature by a variety of names:

1. One-way Boundary Conditions
2. Absorbing Boundary Conditions
3. Radiating Boundary Conditions
4. Transparent Boundary Conditions
5. Transmitting Boundary Conditions
6. Free-space Boundary Conditions
7. Open Boundary Conditions

8. Silent Boundary Conditions

9. On Surface Boundary Conditions

10. Pulled-back Boundary Conditions

11. Paraxial Boundary Conditions,

all these nomenclatures can be resumed as *Non Reflecting Boundary Condition* (*NRBC*). The term is used with the understanding that it means boundary conditions that prevent spurious reflections from being generated at the boundary they are applied at, while not affecting the true physical reflections that are part of the actual solution.

An artificial boundary method consists in the subsequent steps:

1. Introduce an artificial boundary $\mathscr{B}$, which divides the original unbounded domain $\mathscr{R}$ into two domains: a finite computational domain Ω and an infinite residual domain $\mathcal{D}$.

2. By analyzing the problem in $\mathcal{D}$, obtain a relation on $\mathscr{B}$ (exact or approximate) involving the unknown function and its derivatives.

3. Use this relation as a boundary condition on $\mathscr{B}$, to obtain a well-posed problem in Ω.

4. Use a numerical method (e.g. spectral element method) to solve the problem in Ω.

The set of NRBCs can be divided into two subsets: local NRBCs, which involve differential operators, and non-local NRBCs, which involve integral operators.

Engquist and Majda [80, 81] have developed approximate NRBC of progressively increasing order in 1977 for some wave propagation problems. Their approach is based on representing the solution as superposition of waves and eliminating all the incoming waves from the solution at the artificial boundary. This method is called one-way wave equation because the boundary condition as result of this method is in the form of a differential equation, which permits wave propagation in one certain direction only.

Trefethen and Halpern [213] approximate the factor $\sqrt{1-s^2}$ on interval $[-1, 1]$ using rational functions like Chebyshev, Chebyshev-Padé, Newman, L^2 and L^∞ approximants.

Bayliss et al. [26, 25] performed an asymptotic expansion of the scattered field far from the scatterer and obtained a sequence of NRBCs with axial and spherical symmetries at large distances.

Feng [84] proposed the asymptotic radiation conditions for the reduced wave equation by using the asymptotic approximation of Hankel functions (Green function method). The expression obtained is exact, nonlocal and integral.

Higdon [122] proposed an improved boundary condition based on Clayton and Engquist's absorbing boundary condition, which successfully eliminates the artificial reflections from arbitrary angles of incidence. Higdon [123] showed that any local boundary condition involving a differential operator eliminates spurious reflections at certain angles of incidence but not at others. The Higdon's NRBCs are, for certain choices of the parameters, equivalent to NRBCs that are derived from symmetric rational approximation of the dispersion relation.

The accuracy of global *exact* boundary conditions can be easily increased to a desired level, but they are often restricted to artificial boundaries with relatively simple shapes. Furthermore, all degrees of freedom on the artificial boundary are coupled, requiring special procedures for implementation and parallelization, as well as usually affecting computational cost and storage.

The probably best-known (global) non-reflecting boundary condition is the so-called *Dirichlet-to-Neumann* map (DtN map). This boundary condition is extensively reviewed by Givoli [98]. The DtN method uses an integro-differential operator, the *DtN Map*, that maps the values of the unknown function on the artificial exterior boundary, $\mathscr{B}$, to the normal derivative of this function on $\mathscr{B}$ [97].

In 2002 Givoli and Patlashenko [101] introduced localized NRBCs for the Helmholtz equation that are optimal in a certain sense and may be directly coupled to symmetric C^0-continuous finite elements. The scheme can be of an arbitrarily high order and the FE formulation does not involve any high-order derivatives.

1.2.4 Perfectly Matched Layer (*PML*)

One approach to decreasing reflections is to introduce a viscous region near the boundary, or to introduce a sponge layer. The idea is to introduce an exterior layer at an artificial interface such that outgoing plane waves are totally absorbed. Berenger [33, 34] introduced the PML (*Perfectly Matched Layer*) as a way of absorbing waves at artificial surfaces for the time dependent Maxwell equations. The interface and the PML are usually formulated in rectilinear Cartesian coordinates. The PML concept has also be generalized to spherical and cylindrical [209, 154] and general curvilinear coordinates [71, 155].

Abarbanel and Gottlieb [1] showed that the Berenger's approach was not well posed and several other approaches have since been suggested. Chew and Weedon [64] and Chew and Liu [63] interpreted the PML as a complex coordinate stretching. Roden and Gedney [190] present an implementation based on the stretched coordinate form of the PML, a recursive convolution, and the use of complex frequency shifted (CFS)PML parameters. For the elastodynamics case, other studies reformulate the classical PML condition in the context of numerical schemes based on the wave equation written as a second-order system in displacement [145, 49].

1.3 Objective and Original Contributions

A main goal of this work is the development and the implementation of effective approaches to the numerical solution of the wave propagation in exterior domain. We will concentrate on domain based methods, particularly the Spectral Element Method (SEM). The core of this work is to extend the spectral element method to exterior acoustic and elastodynamic problems in the frequency domain in agreement with the *DtN*, *PML* and *IEM* schemes.

We present analysis of Legendre and Chebyshev SEM coupled to the previous schemes to simulate the unboundedness of the domain. The implementation will be done in MATLAB® and many applications of the problems with known analytic solutions will be made. Accuracy and convergence, computational cost and conditioning are then compared for the different methods and the interpolation orders.

1.4 Outline of the Dissertation

In chapter 2 we give a brief introduction to Chebyshev and Legendre spectral element method.

The rest of this thesis is organized in two main parts.

- The first part concerns the acoustic wave propagation. Chapter 3 reviews the fundamental theory of exterior acoustics. Assuming a time harmonic, separable solution of the wave equation, the Helmholtz equation is found.

In chapter 4 the DtN boundary condition is derived when the artificial boundary $\mathscr{B}$ is a circle. The spectral element scheme combined with the DtN method is constructed.

Chapter 5 is devoted to the formulation of the PML scheme formulated in rectilinear Cartesian coordinates. The spectral element discretization of the obtained equations is performed.

The *Spectral Element - Wave Envelope Method* for the two-dimensional Helmholtz equation is introduced in chapter 6.

In chapter 7 we will compare the previous methods (*DtN*, *PML* and *IEM*) for different parameters. The relative errors, the condition numbers and the computational costs are compared for various interpolation orders.

- The second part deals with elastodynamic wave propagation.

  Chapter 8 presents the statement of general elastodynamic problem in an unbounded domain including the governing equation, boundary conditions and radiation condition.

  In chapter 9 we investigate *Dirichlet-to-Neumann* boundary conditions with a spectral element scheme for time-harmonic elastic waves.

  In chapter 10 we formulate the elastodynamics system equations with a PML boundary condition using the stretched coordinates. We obtain a weak form which is identical to the form of the classical elasticity equation, but with anisotropic, inhomogeneous medium and complex-valued material properties.

  Chapter 11 is devoted to comparing the performances of the DtN and PML techniques.

Finally, in chapter 12, we present brief conclusions and describe possible future research directions.