Existence of a competitive equilibrium in one sector growth model with heterogeneous agents and irreversible investment.[1]

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Summary: We prove existence of a competitive equilibrium in a version of a Ramsey (one sector) model in which agents are heterogeneous and gross investment is constrained to be non negative. We do so by converting the infinite-dimensional fixed point problem stated in terms of prices and commodities into a finite-dimensional Negishi problem involving individual weights in a social value function. This method allows us to obtain detailed results concerning the properties of competitive equilibria. Because of the simplicity of the techniques utilized our approach is amenable to be adapted by practitioners in analogous problems often studied in macroeconomics.

Keywords: One sector growth model, Pareto-optimum, Competitive equilibriun, Heterogeneous agents, Non negative gross investment.

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1 Introduction

This paper addresses the question of existence of a competitive equilibrium in a Ramsey economy in which different agents evaluate the future differently and investment is irreversible. Since we consider an infinite horizon growth model the setting is formally for an economy with infinitely many commodities. Debreu (1954) was the first who extended the equilibrium analysis to such economies. Following his early work many methods have been used to prove existence of competitive equilibria in infinite dimensional spaces: core equivalence (e.g. Peleg and Yaari (1970)), limit of equilibria of finite dimensional economies (e.g. Bewley (1972)), demand approaches (e.g. Florenzano (1983)), Negishi approaches, either in its topological version (e.g. Magill (1981), Dana, Le Van and Magnien (1997), Aliprantis, Border, and Burkinshaw (1997)), or in its dual version using the weight system associated with a Pareto-optimum (e.g. Dana and Le Van (1991), Kehoe, Levin and Romer (1991), Hadji and Le Van (1994), Dana and Le Van (2000), Duran and Le Van (2001)). Aliprantis, Border, and Burkinshaw (1990) and Becker and Boyd (1997) contain modern expositions of these approaches.

Our strategy for tackling the question of existence relies on exploiting the link between Pareto-optima and competitive equilibria. In that respect our proof is in the line of Dana and Le Van (1991), Kehoe, Levin and Romer (1991), Hadji and Le Van (1994), Duran and Le Van (2001). We first study the Pareto-optimum problem involving individual weights in a social value function. We next show that with any optimal path (k^*, c^*) one can associate a price system p for the consumption good and a price r for the initial capital stock such that (k^*, c^*, p, r) constitute a price equilibrium with transfers. The final step to obtain an equilibrium is to prove that there exists a set of welfare weights such that these transfers equal to zero. By doing so, we convert the infinite-dimensional fixed point problem stated in terms of prices and commodities into a finite-dimensional fixed point problem involving individual weights in a social value function.

Our paper is in the line of Dana and Le Van (1991) for a second aspect: in our model agents are heterogeneous. But observe that the model in Dana and Le Van (1991) is more complicated with many sectors and recursive preferences. In our model individuals' preferences are additively separable and there is one sector. The counter-part is that the proofs are much simpler and we obtain more properties for the optimal and equilibrium paths. Our model is a generalization of Duran and Le Van (2001) because we allow heterogeneous agents, but as in their model we constrain gross investment to be non negative (this constraint is not imposed in the usual Ramsey model of Kehoe, Levine and Romer (1991), Dana and Le Van (1991). Intertemporal models with irreversibility, i.e.nonnegative gross investment, have been studied by Mitra (1983) and Mitra and Ray (1983). But these papers do not deal with the problem of existence of equilibrium, and the time horizon is finite). We emphasize that our paper might be useful for macroeconomists who work on heterogeneity and do not want to use sophisticated mathematical tools.

As we said before, this strategy allows us to obtain detailed results concerning the properties of competitive equilibria. In particular, we show that in case where all agents have the same discount factor (i.e. the problem is stationary) the optimal trajectory converges to a steady state: some $k^s > 0$ which is determined by the common discount factor. The proof of this result is a simple modification of existing proofs (e.g. Benhabib and Nishimura, (1985)) and is based on monotonicity of the optimal capital sequence.

When we allow heterogeneous discount factors proving convergence of the optimal path is not so simple. The complications arise largely from the fact that the Pareto-optimum problem is now nonstationary, so it can not have a steady state. Hence, one cannot conclude that the optimal path is monotonic. Nevertheless, by exploiting additional properties of optimal paths, we are able to prove that the optimal capital sequence has a unique accumulation point: some $k^s > 0$ which is the steady state for the stationary problem in which every agent has a discount factor equal to the maximum one. In addition to the convergence result we are able to give a partial characterization for the dynamics of the optimal capital sequence. We show that there exists an integer T (large enough) such that the optimal sequence (k^*_{T+t}) either converges decreasingly to k^s or it converges to k^s with $k^*_{T+t} \le k^s$ for all $t \ge 0$.

Finally, using the Inada condition for the instantaneous utility functions, we show that the consumption paths of all agents with a discount factor equal to the maximum one converge to strictly positive stationary consumptions while the consumption paths of the remaining agents converge to zero.

These results are related to the ones obtained by Becker (1980) and Bewley (1982). Becker also proves that the long-run equilibrium capital stock is determined by the maximum discount factor while Bewley proves that there exists some date T such that beyond this date the consumption of the agents with a discount factor less than the maximum one will be equal to zero (but in his proof implicitly assumes that the marginal utilities are bounded above).

The paper is organized as follows: In section 2 we set up a simple one sector multi-agent economy. Section 3 provides a characterization of the competitive equilibrium for this economy. Section 4 describes the Pareto-optimum problem and proves existence of optimal paths. Section 5 analyzes properties of optimal paths. The existence of a competitive equilibrium is proven in section 6. A conclusion is given in section 7.

2 The model

We consider an intertemporal one sector model with $m \geq 1$ consumers and one firm. The preferences of each consumer take the usual additively separable form, $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t})$, where $0 < \beta_i < 1$ is the discount factor and $c_{i,t}$ denotes the quantity which agent i consumes at date t. Production possibilities are represented by a gross production function F and a physical depreciation rate $0 < \delta < 1$. The initial endowment of capital, the single reproducible productive factor, is $k_0 \geq 0$ and $\theta_i > 0$ is the share owned by consumer i. Obviously, $\sum_{i=1}^{m} \theta_i = 1$ and $\theta_i k_0$ is the endowment of consumer i. Consumers also share the profit of the firm in each period; $\alpha_i > 0$ is the share owned by consumer i; and $\sum_{i=1}^{m} \alpha_i = 1$. Formally, the economy is described by the list,

$$E = \{\mathbb{R}_+^{\infty};\ u_i, i = 1, \ldots, m;\ (\alpha_i, \theta_i), i = 1, \ldots, m;\ \mathbb{R}_+^{\infty};\ k_0;\ F;\ \delta\}.$$

We introduce now some notation. For any initial condition $k_0 \geq 0$, when $k = (k_1, k_2, \ldots)$ is such that $0 \leq (1-\delta)k_t \leq k_{t+1} \leq F(k_t) + (1-\delta)k_t$ for all t, we say it is feasible from k_0 and we denote the set of all feasible accumulation paths by $\Pi(k_0)$. Let $c_t = (c_{1,t}, c_{2,t}, \ldots, c_{m,t})$ denote the $m-$vector of consumptions of all agents at date t. A consumption sequence $c = (c_1, c_2, \ldots)$ is feasible from $k_0 \geq 0$ when there exists $k \in \Pi(k_0)$ such that $0 \leq \sum_{i=1}^{m} c_{i,t} \leq F(k_t) + (1-\delta)k_t - k_{t+1}$ for all t. The set of feasible from k_0 consumption sequences is denoted by $\Sigma(k_0)$. We next specify the properties assumed for the preferences and the technology.

Assumption 1 For $i = 1, \ldots, m$, $u_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous, strictly concave, strictly increasing and twice differentiable. Moreover, $u_i(0) = 0$ and $u_i'(0) = +\infty$.

Assumption 2 The gross production function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous, strictly concave, strictly increasing and twice differentiable. Moreover, $F(0) = 0$, $F'(0) > \frac{1}{\min_i \beta_i} - 1 + \delta$ and $F'(\infty) = 0$.

If we define the interest rate by $r = \frac{1}{\min_i \beta_i} - 1$ then $F'(0) > \frac{1}{\min_i \beta_i} - 1 + \delta$ means that at the origin the marginal productivity of capital is greater than the sum of interest rate and depreciation rate. Since this sum is the cost of investment, zero capital stock is not optimal for investment. Moreover, $F'(\infty) = 0$ rules out a sustained growth of the stock of physical capital.

Since F' is differentiable and $F'(0) > \delta$; for all $k_0 > 0$ there exists some $0 < k' \leq k_0$ such that $F(k') + (1 - \delta)k' > k'$. Hence, for all $k_0 > 0$ there is a feasible, interior, stationary accumulation-consumption plan described by k' and c' such that $\sum_{i=1} c'_i = F(k') - \delta k'$. Further, $F'(\infty) < \delta$ implies the existence of a maximum sustainable capital stock: some $\bar{k} > 0$ for which $F(k) + (1 - \delta)k < k$ iff $k > \bar{k}$; and $F(\bar{k}) + (1 - \delta)\bar{k} = \bar{k}$. In order to save notation we define $f(k) = F(k) + (1 - \delta)k$. Observe that under the previous assumptions we have $f'(0) > \frac{1}{\min_i \beta_i}$ and $f'(\infty) < 1$.

3 Characterization of Equilibrium

A competitive equilibrium for this model consists of a sequence $(p_0, p_1, \ldots) \in \ell_\infty^+ \setminus \{0\}$ of prices for the consumption good, a price $r > 0$ for the initial capital stock, a consumption allocation $c_i = (c_{i,0}, c_{i,1}, \ldots)$ for each consumer i and a sequence of capital stocks $k = (k_1, k_2, \ldots)$ such that

(a) For every i, c_i solves the consumer's problem

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \\ \text{s.t.} \quad & \sum_{t=0}^{\infty} p_t c_{i,t} \leq \theta_i r k_0 + \alpha_i \pi \end{aligned}$$

where π is the profit of the single firm. The maximum is taken over ℓ_∞^+.

(b) k yields the maximal profit π for the firm over production plans $(k_0, k) \in \mathbb{R}_+ \times \ell_\infty^+$ subject to the feasibility constraints

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} p_t [f(k_t) - k_{t+1}] - r k_0 \\ \text{s.t.} \quad & (1 - \delta)k_t \leq k_{t+1} \leq f(k_t); \ \forall t \geq 0 \\ & k_0 \geq 0, \text{ is given.} \end{aligned}$$

(c) Markets clear

$$\sum_{i=1} c_{i,t} + k_{t+1} = f(k_t), \ \forall t \geq 0.$$

To prove existence of a competitive equilibrium we follow the Negishi approach: we first study the Pareto-optimal paths and then show that there exists a Pareto-optimum the transfer payments of which equal zero. The next section describes the Pareto-optimum problem and proves existence of optimal paths.

4 The Pareto-optimum problem

4.1 Existence of solutions

Let $\Delta = \left\{ \lambda_1, \ldots, \lambda_m \mid \lambda_i \geq 0 \text{ and } \sum_{i=1} \lambda_i = 1 \right\}$. Given nonnegative welfare weights $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Delta$ we maximize a weighted sum of the individual consumers' utilities subject to feasibility constraints

$$\begin{aligned}
\max \quad & \sum_{i=1} \lambda_i \sum_{t=0} \beta_i^t u_i(c_{i,t}) \\
s.t. \quad & \sum_{i=1} c_{i,t} + k_{t+1} \leq f(k_t), \ \forall t \geq 0 \\
& (1 - \delta)k_t \leq k_{t+1}, \ \forall t \geq 0 \\
& k_0 \geq 0, \text{ is given.}
\end{aligned}$$

Define $U(\lambda, k, c) = \sum_{i=1} \lambda_i \sum_{t=0} \beta_i^t u_i(c_{i,t})$, where $(\lambda, k, c) \in \Delta \times \Pi(k_0) \times \Sigma(k_0)$. To prove existence of an optimal path we follow the classical method using continuity of both u_i and f. While the latter will ensure that $\Pi(k_0)$ and $\Sigma(k_0)$ are compact the former will ensure that U is continuous in which case Weirstrass Theorem applies.

Lemma 1 For all $k_0 \geq 0$, a) there exists $A(k_0)$ such that $k \in \Pi(k_0)$ implies $k_t \leq A(k_0)$, $\forall t$; b) $\Pi(k_0)$ and $\Sigma(k_0)$ are compact in the product topology, c) $0 \leq u_i(c_{i,t}) \leq \overline{B}(k_0)$, $\forall i, \forall t$, where $\overline{B}(k_0)$ is an upper bound.

Proof: (a) follows for $A(k_0) = \max\{k_0, \overline{k}\}$, where $\overline{k}$ is the maximum sustainable capital stock. Then (b) follows from this bound and Tychonov Theorem, while (c) is a consequence of $0 \leq c_{i,t} \leq f(A(k_0)) \leq A(k_0)$, $\forall i, \forall t$. ■

Define the sequence $u_i^n(c_i) = \sum_{t=0}^{n} \beta_i^t u_i(c_{i,t})$. Since this sequence is increasing and bounded it converges and we can write

$$U(\lambda; k; c) = \sum_{i=1}^{m} \lambda_i \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) = \sum_{t=0}^{\infty} \sum_{i=1}^{m} \lambda_i \beta_i^t u_i(c_{i,t}).$$

Lemma 2 For all $k_0 \geq 0$, $U(\cdot)$ is continuous over $\Delta \times \Pi(k_0) \times \Sigma(k_0)$ with respect to the relative product topology.

Proof: Consider a sequence $(\lambda^n; k^n; c^n) \in \Delta \times \Pi(k_0) \times \Sigma(k_0)$ that converges to $(\lambda; k; c) \in \Delta \times \Pi(k_0) \times \Sigma(k_0)$. We just have to show that $U(\lambda^n; k^n; c^n)$ converges to $U(\lambda; k; c)$. Since $(\lambda^n; k^n; c^n) \in \Delta \times \Pi(k_0) \times \Sigma(k_0)$ we have $k_t^n \leq A(k_0)$ and $0 \leq c_{i,t}^n \leq f(A(k_0)) \leq A(k_0)$, $\forall i, \forall n$. Therefore, $0 \leq u_i(c_{i,t}^n) \leq \overline{B}(k_0)$, $\forall i, \forall n$. Note also that

$$\begin{aligned}
|U(\lambda^n; k^n c^n) - U(\lambda; k; c)| &\leq \sum_{t=0}^{T} \sum_{i=1}^{m} \left| \lambda_i^n \beta_i^t u_i(c_{i,t}^n) - \lambda_i \beta_i^t u_i(c_{i,t}) \right| \\
&+ \sum_{t=T+1}^{\infty} \sum_{i=1}^{m} \lambda_i \beta_i^t u_i(c_{i,t}) + \sum_{t=T+1}^{\infty} \sum_{i=1}^{m} \lambda_i^n \beta_i^t u_i(c_{i,t}^n) \\
&\leq \sum_{t=0}^{T} \sum_{i=1}^{m} \left| \lambda_i^n \beta_i^t u_i(c_{i,t}^n) - \lambda_i \beta_i^t u_i(c_{i,t}) \right| \\
&+ \overline{B}(k_0) \sum_{t=T+1}^{\infty} \sum_{i=1}^{m} \beta_i^t + \overline{B}(k_0) \sum_{t=T+1}^{\infty} \sum_{i=1}^{m} \beta_i^t.
\end{aligned}$$

For given T, the continuity of u_i ensures that there exists N such that for any $n \geq N$ the first term is smaller than $\frac{\varepsilon}{2}$. Also, since $0 < \beta_i < 1$ for all i, there exists T such that $2\overline{B}(k_0) \sum_{t=T+1}^{\infty} \sum_{i=1}^{m} \beta_i^t < \frac{\varepsilon}{2}$. ■

Existence of an optimal path is hence ensured since $U(\lambda; \cdot; \cdot)$ is continuous over $\Pi(k_0) \times \Sigma(k_0)$. Moreover, the assumptions made for both u_i and F (strict concavity) imply that the optimal consumption-accumulation path is unique.

Proposition 1 For all $k_0 \geq 0$ there is a unique optimal consumption-accumulation path.

One way to make the analysis of the behavior of optimal programs easier is to introduce the concept of a value function. In what follows, for any $\lambda \in \Delta$, let $I = \{i \mid \lambda_i > 0\}$, $\beta = \max\{\beta_i \mid i \in I\}$, $J = \{i \in I \mid \beta_i = \beta\}$ and $I' = \{i \in I \mid i \notin J\}$.

4.2 Value function, Bellman equation

Given any $\lambda \in \Delta$ and $(k; y)$ such that $0 \le y \le f(k)$; we introduce a time-dependent function V_t; defined by

$$
\begin{aligned}
V_t(\lambda; k; y) = \max \ & \sum_{i \in I} \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u_i(c_i) \\
\text{s.t.} \ & \sum_{i \in I} c_i + y \le f(k)
\end{aligned}
$$

It is easy to check that under assumptions 1 and 2 the Pareto-optimum problem is equivalent to

$$
\begin{aligned}
\max \ & \sum_{t=0}^{\infty} \beta^t V_t(\lambda; k_t; k_{t+1}) \\
\text{s.t.} \ & (1-\delta)k_t \le k_{t+1} \le f(k_t); \ \forall t \ge 0 \\
& k_0 \ge 0; \text{ is given.}
\end{aligned}
$$

As in the traditional one sector growth model we define the value function by

$$
\begin{aligned}
W_0(k_0) = \max \ & \sum_{t=0}^{\infty} \beta^t V_t(\lambda; k_t; k_{t+1}) \\
\text{s.t.} \ & (1-\delta)k_t \le k_{t+1} \le f(k_t); \ \forall t \ge 0 \\
& k_0 \ge 0; \text{ is given.}
\end{aligned}
$$

Recall that in infinite-horizon problems with time-invariant period return functions (stationary problems) the value function is a function of the initial state alone. In the above problem the period return function is time-dependent, so the problem is a nonstationary one. In this case, as the time index on W indicates, time becomes a separate argument of the value function.

The next proposition states formally what is known as the Principle of Optimality.

Proposition 2 The value function satisfies the Bellman equation and for all $k_0 \ge 0$ a feasible path k is optimal if and only if

$$W_t(k_t) = V_t(\lambda; k_t; k_{t+1}) + \beta W_{t+1}(k_{t+1})$$

holds for all $t \ge 0$.

Proof: See Stokey and Lucas (1989, Chapter 4). ■

If we restrict ourselves to the set of agents with a discount factor equal to the maximum one, we can define a time-invariant function $\tilde{v}$ by

$$\tilde{v}(\lambda; k; y) = \max \sum_{i \in J} \lambda_i u_i(c_i)$$
$$\text{s.t.} \quad \sum_{i \in J} c_i + y \leq f(k)$$

Observe that in this case the associated Pareto-optimum problem is stationary

$$\tilde{W}(k_0) = \max \sum_{t=0}^{\infty} \beta^t \tilde{v}(\lambda; k_t; k_{t+1})$$
$$\text{s.t.} \quad (1-\delta)k_t \leq k_{t+1} \leq f(k_t); \ \forall t \geq 0$$
$$k_0 \geq 0; \ \text{is given.}$$

Using lemma 1 it is easy to check that $\forall t$ and $\forall (k; y)$ such that $0 \leq y \leq f(k)$;

$$\tilde{v}(\lambda; k; y) \leq V_t(\lambda; k; y) \leq \left(\frac{\max_{i \in I'} \beta_i}{\beta} \right)^t \bar{C}(k_0) + \tilde{v}(\lambda; k; y)$$

where $\bar{C}(k_0) = \sum_{i \in I'} \bar{B}(k_0)$. Since $\left(\frac{\max_{i \in I'} \beta_i}{\beta} \right) < 1$ it follows that $\forall \varepsilon > 0$ there exists T such that

$$\tilde{v}(\lambda; k; y) \leq V_t(\lambda; k; y) \leq \varepsilon + \tilde{v}(\lambda; k; y); \ \forall t \geq T.$$

Moreover, given any $k_0 \geq 0$; it is easy to check that

$$\tilde{W}(k_0) \leq W_T(k_0) \leq \sum_{t=0}^{\infty} \beta^t \left(\frac{\max_{i \in I'} \beta_i}{\beta} \right)^{t+T} \sum_{i \in I'} \bar{B}(k_0) + \tilde{W}(k_0)$$
$$= \left(\frac{\max_{i \in I'} \beta_i}{\beta} \right)^T \Omega(k_0) + \tilde{W}(k_0)$$

where $\Omega(k_0) = \frac{1}{1 - \max_{i \in I'} \beta_i} \bar{C}(k_0)$: It follows that for any $\varepsilon > 0$ and for all k_t feasible from k_0 there exists T such that

$$\hat{W}(k_t) \le W_t(k_t) \le \varepsilon + \hat{W}(k_t); \ \forall t \ge T.$$

Consider now a feasible capital sequence (k_t) starting from some $k_0 \ge 0$. Using the previous results, for any subsequence (t_n) such that $k_{t_n} \to k \ge 0$ and $k_{t_n+1} \to k' \ge 0$ we have

$$\lim_{n \to \infty} V_{t_n}(\lambda; k_{t_n}; k_{t_n+1}) = \hat{V}(\lambda; k; k') \text{ and } \lim_{n \to \infty} W_{t_n}(k_{t_n}) = \hat{W}(k).$$

5 Properties of optimal paths

In this section we review important properties of optimal paths. It will turn out that these properties are very useful for proving existence of a supporting price system. The main result of this section is Proposition 4, establishing convergence of the optimal accumulation path in case where agents have different discount factors.

Obviously, for any $\lambda \in \Delta$, an optimal consumption-accumulation path will depend on λ. In what follows we suppress λ and denote by $(c^*; k^*)$ any optimal path. The following two lemmas establish the non-nullity of optimal consumption and capital sequences and are stated here for further reference.

Lemma 3 Assume $k_0 > 0$ and let $(c^*; k^*)$ denote the solution to the Pareto-optimum problem. Under assumptions 1 and 2,

a) If $\lambda_i = 0$ then $c^*_{i;t} = 0; \ \forall t \ge 0$.

b) $\sum_{i=1}^{m} c^*_{i;t} > 0; \ \forall t \ge 0$.

c) If $\lambda_i > 0$ then $c^*_{i;t} > 0; \ \forall t \ge 0$.

Proof: See Dana and Le Van (1991, Proposition 3.3, Proposition 3.6). ■

Lemma 4 Let $(c^*; k^*)$ denote the solution to the Pareto-optimum problem. Under assumptions 1 and 2,

a) if $k_0 = 0$ then $k^*_t = 0; \ c^*_t = 0; \ \forall t \ge 0$.

b) if $k_0 > 0$ then $k^*_t > 0; \ \forall t \ge 0$.

Proof: See Dana and Le Van (1991, Proposition 3.6). ■

Lemma 5 Let the function $V_t(\lambda;k;y)$ be defined as in section 4.2, i.e. given any $\lambda \in \Delta$ and $(k;y)$ such that $0 \le y \le f(k)$;

$$V_t(\lambda;k;y) = \max \sum_{i\in I} \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u_i(c_i)$$
$$\text{s.t.} \quad \sum_{i\in I} c_i + y \le f(k).$$

Under assumptions 1 and 2,

a) If $0 < y < f(k)$ then

$$\frac{\partial V_t(\lambda;k;y)}{\partial k} = \mu_t f'(k)$$
$$\frac{\partial V_t(\lambda;k;y)}{\partial y} = -\mu_t$$

where $\mu_t = \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u_i'(c_i^*)$; $\forall i \in I$:

b) If $0 < y < f(k)$ then $\frac{\partial^2 V_t}{\partial k \partial y} > 0$.

c) If $\beta_i = \beta$ for all $i \in I$ and k^* is an optimal path starting from some $k_0 \ge 0$; then k^* is monotone. Moreover, if $k_0 \le k_0'$ and $k^*; k'$ are optimal paths starting respectively from k_0 and k_0'; then $k_t^* \le k_t'$; $\forall t \ge 0$.

Proof: a) Let $c_i^* = (c_1^*;\ldots;c_I^*)_{I\,m}$ denote a solution for the maximization problem. Notice that if we let $c_i = \frac{\varepsilon}{\# I}$ for all $i \in I$; where $\varepsilon > 0$ is choosen such that $\varepsilon + y < f(k)$; the Slater condition is verified. Hence, there exists a multiplier $\mu_t \in R$ such that $(c_i^*; \mu_t)$ maximizes the associated Lagrangian. The Kuhn -Tucker first order conditions are

$$\lambda_i \left(\frac{\beta_i}{\beta}\right)^t u_i'(c_i^*) = \mu_t; \ \forall i \in I$$
$$\mu_t \ge 0; \ \mu_t \left[\sum_{i\in I} c_i^* + y - f(k)\right] = 0.$$

Since $u_i' > 0$; $\mu_t > 0$ and $\sum_{i\in I} c_i^* + y = f(k)$. Moreover, the strict concavity of u_i and f implies that the solution $c_i^* = (c_1^*\ldots c_I^*)_{I\,m}$ is unique. Hence, μ_t is unique. If we define $f(k) - y = \alpha$; it can be easily shown (see Corollary 7.3.1 in Florenzano, LeVan and Gourdel, 2001) that $\frac{\partial V_t(\lambda;k;y)}{\partial \alpha} = \mu_t$. Thus

$$\frac{\partial V_t(\lambda;k;y)}{\partial k} = \frac{\partial V_t(\lambda;k;y)}{\partial \alpha}\frac{\partial \alpha}{\partial k} = \mu_t f'(k)$$
$$\frac{\partial V_t(\lambda;k;y)}{\partial y} = \frac{\partial V_t(\lambda;k;y)}{\partial \alpha}\frac{\partial \alpha}{\partial k} = -\mu_t$$

b) We know that

$$\lambda_i \left(\beta_i\right)^t u_i'(c_i^*) = \mu_t; \ \forall i \in I$$
$$\sum_{i \in I} c_i^* + y - f(k) = 0.$$

Differentiation of the above equations gives

$$\lambda_i \left(\beta_i\right)^t u_i''(c_i^*) dc_i^* - d\mu_t = 0; \ \forall i \in I$$
$$\sum_{i \in I} dc_i^* + dy - f'(k) dk = 0.$$

If we write these equations in a matrix form we get

$$\underbrace{\begin{pmatrix} \lambda_1 \left(\beta_1\right)^t u_1''(c_1^*) & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_I \left(\beta_I\right)^t u_I''(c_I^*) & -1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} dc_1^* \\ \vdots \\ dc_I^* \\ d\mu_t \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f'(k_t) dk - dy \end{pmatrix}.$$

Take a vector $x = (x_1, \ldots, x_{I+1})$ and assume that $Ax = 0$. Then

$$x_{I+1} = x_1 \lambda_1 \left(\beta_1\right)^t u_1''(c_1^*) = \cdots = x_I \lambda_I \left(\beta_I\right)^t u_I''(c_I^*);$$
$$x_1 + \cdots + x_I = 0.$$

Combining these equations we get

$$x_{I+1} \left(\frac{1}{\lambda_1 \left(\beta_1\right)^t u_1''(c_1^*)} + \cdots + \frac{1}{\lambda_I \left(\beta_I\right)^t u_I''(c_I^*)} \right) = 0.$$

Since $\lambda_i \left(\beta_i\right)^t u_i''(c_i^*) < 0$ it must be that $x_{I+1} = x_1 = \cdots = x_I = 0$. Thus A is invertible and

$$dc_1^* = \frac{d\mu_t}{\lambda_1 \left(\beta_1\right)^t u_1''(c_1^*)} = \frac{d\mu_t}{\omega_1}$$
$$\vdots$$
$$dc_I^* = \frac{d\mu_t}{\lambda_I \left(\beta_I\right)^t u_I''(c_I^*)} = \frac{d\mu_t}{\omega_I}$$
$$d\mu_t \left(\frac{1}{\omega_1} + \cdots + \frac{1}{\omega_I} \right) = f'(k) dk - dy$$

The last equation implies that $\frac{\partial \mu_t}{\partial y} = -\frac{1}{\sum_{i \in I} \frac{1}{\omega_i}} > 0$. Hence,

$$\frac{\partial^2 V_t}{\partial k \partial y} = \frac{\partial \mu_t}{\partial y} f'(k) > 0.$$

c) If $k_0 = 0$ then Lemma 4 implies that $k_t^* = 0$, $\forall t$. Assume that $k_0 > 0$. Since we have shown that $\frac{\partial^2 \widehat{V}}{\partial k \partial y} > 0$, one may use (sligtly adapted since in our model investment is irreversible) the proof in Benhabib and Nishimura (1985, Theorem 2, pp 293-295). ■

Since in our model investment is irreversible i.e. $(1 - \delta)k_t \le k_{t+1}$, $\forall t$, we face the possibility this constraint being binding at certain periods. However, as the following lemma establishes, the constraint cannot be always binding in the long-run.

Lemma 6 Let $k_0 > 0$. If k^* is an optimal path starting from k_0 there cannot be an integer T such that $(1 - \delta)k_t^* = k_{t+1}^*$ for all $t \ge T$.

Proof: See Appendix. ■

An immediate consequence of the last lemma is that it allows us to prove that an optimal sequence k^* cannot converge to zero.

Lemma 7 Let $k_0 > 0$. If k^* is an optimal path starting from k_0 then k_t^* cannot converge to zero.

Proof: See Appendix. ■

Let us now consider the Pareto-optimum problem involving only agents in J. The next result shows that in this case the optimal capital sequence converges monotonically to a steady state.

Proposition 3 Let k^* denote the opimal trajectory for the Pareto-optimum problem involving only agents in J. There is some $k^s > 0$ with $f(k^s) - k^s > 0$ and $\beta f'(k^s) = 1$ such that for all $k_0 > 0$, $k_t^* \to k^s$.

Proof: Lemma 1 together with the monotonicity of optimal paths (Lemma 5c) imply that $k_t^* \to k^s \ge 0$. However, Lemma 7 established that k_t^* can not converge to zero. Hence, $k^s > 0$. By the principle of optimality

$$\widehat{W}(k_t^*) = \widehat{V}(\lambda; k_t^*; k_{t+1}^*) + \beta \widehat{W}(k_{t+1}^*); \ \forall t \ge 0.$$

Taking the limits we obtain $f(k^s) - k^s > 0$ and since $\sum_{i \in J} c^*_{i,t} \to \sum_{i \in J} c^*_i = f(k^s) - k^s$, there exists some $j \in J$ such that $c^*_j > 0$. Along the optimal consumption path we have

$$\frac{u_i'(c^*_{i,t})}{u_j'(c^*_{j,t})} = \frac{\lambda_j}{\lambda_i} > 0, \ \forall i, j \in J.$$

Thus, if $c^*_{i,t} \to 0$ for some $i \in J$, then $c^*_{j,t} \to 0$, $\forall j \in J$, a contradiction. Hence, $c^*_i > 0$, $\forall i \in J$.

Since $k^*_t \to k^s > 0$ there exists T such that $(1-\delta)k^*_t < k^*_{t+1} < f(k^*_t)$, $\forall t \geq T$. Thus, for all $t \geq T$ the Euler equation holds,

$$\begin{aligned} & \frac{\partial V_t(\lambda, k^*_t, k^*_{t+1})}{\partial y} + \beta \frac{\partial V_{t+1}(\lambda, k^*_{t+1}, k^*_{t+2})}{\partial k} = 0 \\ \Leftrightarrow \quad & \mu_t = \beta \mu_{t+1} f'(k^*_{t+1}) \\ \Leftrightarrow \quad & u_i'(c^*_{i,t}) = \beta u_i'(c^*_{i,t+1}) f'(k^*_{t+1}), \ \forall i \in J. \end{aligned}$$

Taking the limits in Euler equation gives $\beta f'(k^s) = 1$. ■

The following lemma implies that there cannot be a subsequence (t_n) such that $k^*_{t_n} \to 0$. It will turn out that this property is crucial in order to prove convergence of the optimal path in case where agents have different discount factors.

Lemma 8 For any $k_0 > 0$ and k^* optimal from k_0 there exists $\gamma > 0$ such that $k^*_t \geq \gamma$, $\forall t \geq 0$.

Proof: See Appendix. ■

The next result allows for heterogeneous discount factors and uses the above properties, specially Lemma 8, to prove convergence of the optimal capital sequence.

Proposition 4 Let $k_0 > 0$. If k^* denotes an optimal path starting from k_0, then $k^*_t \to k^s$, where k^s is determined by $\beta f'(k^s) = 1$.

Proof: If $\beta_i = \beta$ for every i, then it follows from Proposition 3 that the optimal path converges to k^s with $\beta f'(k^s) = 1$. Consider now the case where there exists i with $\beta_i < \beta$. Assume that there exists an integer T such that the sequence (k^*_{t+T})

is monotonic. In this case, Lemma 1 and Lemma 7 imply that $k^*_{t+T} \to k > 0$. By the principle of optimality

$$W_{t+T}(k^*_{t+T}) = V_{t+T}(\lambda, k^*_{t+T}, k^*_{t+T+1}) + \beta W_{t+T+1}(k^*_{t+T+1}), \ \forall t \geq 0.$$

Taking the limits we obtain

$$\hat{W}(k) = \hat{V}(\lambda, k, k) + \beta \hat{W}(k).$$

If k satisfies the above equation Proposition 3 implies that $k = k^s$, where $\beta f'(k^s) = 1$.

Assume now that for any integer T there exists $t \geq T$ such that either $k^*_t \leq k^*_{t-1}$ and $k^*_t < k^*_{t+1}$ or $k^*_t \geq k^*_{t-1}$ and $k^*_t > k^*_{t+1}$. In this case there exist subsequences (T_k) and (T'_k) such that

$$\begin{array}{lll} k^*_{T_k} & \leq & k^*_{T_k-1} \text{ and } k^*_{T_k} < k^*_{T_k+1}, \ \forall k \in N \\ k^*_{T'_k} & \geq & k^*_{T'_k-1} \text{ and } k^*_{T'_k} > k^*_{T'_k+1}, \ \forall k \in N \\ k^*_{T_k} & < & k^*_{T'_k}, \ \forall k \in N. \end{array}$$

Let $k_0 > 0$ and without loss of generality assume that $T_1 < T'_1$ and $k_0 > k^*_{T_1}$. This case is depicted in Fgure 1. Since (k^*_t) is bounded it has an accumulation point which is denoted by k. That is, there exists a subsequence (t_n) such that $\lim_{n \to \infty} k^*_{t_n} = k$. Observe that $\forall n$ there exist T_{k_n}, T'_{k_n} and T_{k_n+1} such that either $k^*_{T_{k_n}} \leq k^*_{t_n} \leq k^*_{T'_{k_n}}$ or $k^*_{T_{k_n+1}} \leq k^*_{t_n} \leq k^*_{T'_{k_n}}$.

Consider now an eventual subsequence of $(k^*_{t_n})$, denoted by $(k^*_{t_m})$, such that $k^*_{T_{k_m}} \leq k^*_{t_m} \leq k^*_{T'_{k_m}}$ for all m and $k^*_{T_{k_m}} \to k_{min}$, $k^*_{T_{k_m}+1} \to k'_{min}$, $k^*_{T_{k_m}-1} \to k''_{min}$, $k^*_{T'_{k_m}} \to k_{max}$, $k^*_{T'_{k_m}+1} \to k'_{max}$, $k^*_{T'_{k_m}-1} \to k''_{max}$. By the principle of optimality

$$\begin{array}{rcl} W_{T_{k_m}-1}(k^*_{T_{k_m}-1}) & = & V_{T_{k_m}-1}(\lambda, k^*_{T_{k_m}-1}, k^*_{T_{k_m}}) + \beta W_{T_{k_m}}(k^*_{T_{k_m}}), \ \forall m \\ W_{T_{k_m}}(k^*_{T_{k_m}}) & = & V_{T_{k_m}}(\lambda, k^*_{T_{k_m}}, k^*_{T_{k_m}+1}) + \beta W_{T_{k_m}+1}(k^*_{T_{k_m}+1}), \ \forall m. \end{array}$$

Taking the limits we get

$$\begin{array}{rcl} \hat{W}(k''_{min}) & = & \hat{V}(\lambda, k''_{min}, k_{min}) + \beta \hat{W}(k_{min}) \\ \hat{W}(k_{min}) & = & \hat{V}(\lambda, k_{min}, k'_{min}) + \beta \hat{W}(k'_{min}). \end{array}$$

This means that for the stationary optimal problem associated with the value function $\hat{W}$, k'_{min} is optimal from k_{min}, and k_{min} is optimal from k''_{min}. Since $k^*_{T_{k_m}+1} > k^*_{T_{k_m}}$ and $k^*_{T_{k_m}-1} \geq k^*_{T_{k_m}}$ for all m, we have $k'_{min} \geq k_{min}$ and $k''_{min} \geq k_{min}$.

By Lemma 5 (see the statement c), $k''_{min} \geq k_{min}$ implies $k_{min} \geq k'_{min}$. Thus, $k'_{min} = k_{min}$ which in turn implies that either $k_{min} = 0$ or $k_{min} = k^s$ with $\beta f'(k^s) = 1$ (see Proposition 3). But $k_{min} = 0$ is ruled out by Lemma 8 and since $k^*_{T_{k_m}} \leq k^*_{t_m}$ we have $k^s \leq k$. Following a similar argument one can easily establish that $k_{max} = k^s$. Since $k^*_{t_m} \leq k^*_{T'_{t_m}}$ we have $k^s \geq k$. Combining the two results we obtain $k^s = k$.

Consider now an eventual subsequence of $(k^*_{t_n})$, denoted by $(k^*_{t_m})$, such that $k^*_{T_{k_m+1}} \leq k^*_{t_m} \leq k^*_{T'_{k_m}}$ for all m and $k^*_{T_{k_m+1}} \to k_{min}$, $k^*_{T_{k_m+1}+1} \to k'_{min}$, $k^*_{T_{k_m+1}-1} \to k''_{min}$, $k^*_{T'_{k_m}} \to k_{max}$, $k^*_{T'_{k_m}+1} \to k'_{max}$, $k^*_{T'_{k_m}-1} \to k''_{max}$. Following the same reasoning as before one can prove that $k^s = k$. Summing up we have proved that the optimal sequence (k^*_t) has a unique accumulation point k^s determined by $\beta f'(k^s) = 1$ Thus, (k^*_t) must converge to k^s with $\beta f'(k^s) = 1$. ■

We now show that if we allow for heterogeneous discount factors the limit of the optimal capital sequence is not a steady state.

Proposition 5 [2] If there exists $i \in I$ such that $\beta_i < \beta$ then k^s determined by $\beta f'(k^s) = 1$ is not a steady state.

Proof: Let $k_0 = k^s$ and assume that $k^*_t = k^s, \forall t \geq 1$. Since $(1 - \delta)k^s < k^s < f(k^s)$ the Euler equation holds, so we have

$$
\begin{aligned}
& \frac{\partial V_t(\lambda; k^s; k^s)}{\partial y} + \beta \frac{\partial V_{t+1}(\lambda; k^s; k^s)}{\partial k} = 0 \\
\Leftrightarrow\quad & \mu_t = \beta \mu_{t+1} f'(k^s) \\
\Leftrightarrow\quad & \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u'_i(c^*_{i,t}) = \beta \lambda_i \left(\frac{\beta_i}{\beta}\right)^{t+1} u'_i(c^*_{i,t+1}) f'(k^s) \\
\Leftrightarrow\quad & u'_i(c^*_{i,t}) = \beta_i u'_i(c^*_{i,t+1}) f'(k^s);\ \forall i \in I.
\end{aligned}
$$

If there exists $i \in I$ such that $\beta_i < \beta$ then

$$
u'_i(c^*_{i,t}) = \beta_i u'_i(c^*_{i,t+1}) f'(k^s) < u'_i(c^*_{i,t+1});\ \forall t.
$$

But in this case $c^*_{i,t+1} < c^*_{i,t}, \forall i \in I'$ while $c^*_{i,t+1} = c^*_{i,t}, \forall i \in J$. As a result, $\sum_{i \in I} c^*_{i,t+1} < \sum_{i \in I} c^*_{i,t}$, $\forall t$, contradicting the optimality of $k^*_t = k^s$ for all t. ■

[2] This proposition was suggested to us by Tapan Mitra.

Remark 1 The above proposition implies that in case where agents have different discount factors and the economy starts at $k_0 = k^s$ any optimal path (k_t^*) converges to k^s with $k_1^* \neq k_s$. As a result, the optimal path may exhibit fluctuations at least for the beginning periods.

We can now show that the Euler equation do hold from some period on.

Proposition 6 If $k_0 > 0$ and k^* is an optimal path starting from k_0, there exists T such that $(1-\delta)k_t^* < k_{t+1}^* < f(k_t^*)$, $\forall t \geq T$.

Proof: Since $k_t^* \to k^s > 0$, the result follows immediatelly. ■

The next result provides a partial characterization for the dynamics of the optimal capital sequence.

Proposition 7 Let k^* denote the optimal capital sequence starting from some $k_0 > 0$. There exists T such that (k_{T+t}^*) either converges decreasingly to k^s or it converges to k^s with $k_{T+t}^* \leq k^s$, $\forall t \geq 0$.

Proof: Choose T such that for all $t \geq T-1$ the Euler equation holds.

i) Assume that $k_T^* > k^s$. We will show that we cannot have $k_{T-1}^* \leq k_T^*$ and $k_T^* > k_{T+1}^*$. The Euler equation implies that along the optimal path

$$\begin{aligned} u_i'(c_{i,T-1}^*) &= \beta_i u_i'(c_{i,T}^*) f'(k_T^*), \ \forall i \in I' \\ u_i'(c_{i,T-1}^*) &= \beta u_i'(c_{i,T}^*) f'(k_T^*), \ \forall i \in J. \end{aligned}$$

Since $k_T^* > k^s$, $\beta_i f'(k_T^*) < 1$ and $\beta f'(k_T^*) < 1$. From the Euler equations we have $c_{i,T-1}^* > c_{i,T}^*$, $\forall i \in I'$ and $c_{i,T-1}^* > c_{i,T}^*$, $\forall i \in J$. Thus, $\sum_{i \in I} c_{i,T}^* < \sum_{i \in I} c_{i,T-1}^*$. But $f(k_T^*) - k_{T+1}^* > f(k_{T-1}^*) - k_T^*$: a contradiction.

Consider now the case where $k_{T-1}^* \geq k_T^*$ and $k_T^* < k_{T+1}^*$. Let $T_1 > T$ be the first date such that $k_{T_1-1}^* \leq k_{T_1}^*$ and $k_{T_1+1}^* < k_{T_1}^*$ (this date exists since (k_t^*) converges to k^s). Using the Euler equations one can show that $\sum_{i \in I} c_{i,T_1}^* < \sum_{i \in I} c_{i,T_1-1}^*$. But $f(k_{T_1}^*) - k_{T_1+1}^* > f(k_{T_1-1}^*) - k_{T_1}^*$: a contradiction. As a result we conclude that (k_{T+t}^*) converges decreasingly to k^s.

ii) Assume that $k_T^* < k^s$. We claim that there cannot be a $T_1 > T$ such that $k^s < k_{T_1}^*$ and $k_{T_1-1}^* \leq k_{T_1}^*$, $k_{T_1+1}^* < k_{T_1}^*$. If this is not true one obtains as before that $\sum_{i \in I} c_{i,T_1}^* < \sum_{i \in I} c_{i,T_1-1}^*$. But $f(k_{T_1}^*) - k_{T_1+1}^* > f(k_{T_1-1}^*) - k_{T_1}^*$: a contradiction. As a result (k_{T+t}^*) converges to k^s with $k_{T+t}^* \leq k^s$ for all t. ■

Remark 2 Proposition 7 implies that the optimal capital sequence cannot fluctuate around the steady state in the long run (no-crossing property). For T large enough the optimal capital sequence either converges decreasingly to k^s or if it crosses k^s it remains below it.

The next proposition shows that the consumption path of the agents with a discount factor equal to the maximum one converges to a strictly positive stationary consumption, while the consumption path of the remaining agents converges to zero.

Proposition 8 Let c^* denote the optimal consumption path. Then,

a) $c^*_{i,t}$ converges to zero, $\forall i \in I'$.

b) $c^*_{i,t}$ converges to some $c^*_i > 0$, $\forall i \in J$.

Proof: a) Lemma 1 implies that along the optimal consumption path

$$u'_i(c^*_{i,t}) = \frac{\lambda_j}{\lambda_i}\left(\frac{\beta}{\beta_i}\right)^t u'_j(c^*_{j,t}) \geq \frac{\lambda_j}{\lambda_i}\left(\frac{\beta}{\beta_i}\right)^t u'_j(A(k_0)); \ \forall i \in I'; \forall j \in J.$$

Since $\left(\frac{\beta}{\beta_i}\right) > 1$ we must have $c^*_{i,t} \to 0; \forall i \in I'$.

b) By Proposition 4 we know that $\sum_{i \in I} c^*_{i,t} \to \sum_{i \in I} c^*_i = f(k^s) - k^s > 0$. Since $c^*_{i,t} \to 0; \forall i \in I'$; there must exist some $j \in J$ such that $c^*_j > 0$. Along the optimal consumption path

$$\frac{u'_i(c^*_{i,t})}{u'_j(c^*_{j,t})} = \frac{\lambda_j}{\lambda_i} > 0; \ \forall i, j \in J.$$

Thus if $c^*_{i,t} \to 0$ for some $i \in J$; then $c^*_{j,t} \to 0; \forall j \in J$: a contradiction. Hence $c^*_i > 0; \forall i \in J$. ■

6 Existence of a competitive equilibrium

In this section we want to prove:

i) with the optimal path $c^*(\lambda); k^*(\lambda)$ one can associate a sequence of prices $p(\lambda)$ defined as $p_t(\lambda) = \beta^t \lambda_t$ for all t and a price $r(\lambda) = p_0(\lambda)F'(k_0)$ of the initial stock such that $(c^*(\lambda); k^*(\lambda); p(\lambda); r(\lambda))$ is a price equilibrium with transfers,

ii) there exists a set of welfare weights such that these transfers equal to zero. As in the previous section we suppress λ wherever it is possible.

Lemma 9 The sequence of prices $p(\lambda)$, defined as $p_t(\lambda) = \beta^t \mu_t$ for all t, is a sequence which belongs to $l_\infty^+ \setminus \{0\}$.

Proof: Take $j \in J$. Since $c^*_{j,t} > 0$, $\forall t$ and $c^*_{j,t} \to c^*_j > 0$, there exists $a > 0$ such that $c^*_{j,t} \geq a$, $\forall t$. Thus $p_t(\lambda) = \beta^t \mu_t = \lambda_j \beta_j^t u_j'(c^*_{j,t}) \leq \beta_j^t u_j'(a)$, $\forall t$ and therefore

$$\sum_{t=0}^{\infty} p_t(\lambda) \leq u_j'(a) \sum_{t=0}^{\infty} \beta_j^t < \infty.$$

■

Theorem 1 Let $k_0 > 0$. Then $c^*(\lambda), k^*(\lambda)$ optimal from k_0, $p(\lambda)$ defined as $p_t(\lambda) = \beta^t \mu_t$ for all t, and $r(\lambda) = p_0(\lambda) F'(k_0)$ is a price equilibrium with transfers.

Proof: An allocation $c^*(\lambda), k^*(\lambda)$, a price sequence $p(\lambda) \in l_\infty^+ \setminus \{0\}$ for the consumption good, and a price $r(\lambda)$ for the initial capital stock constitute a price equilibrium with transfers if

a) For every i, $c_i^*(\lambda) = (c^*_{i,0}, c^*_{i,1}, \ldots)$ solves

$$\max \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t})$$
$$\text{s.t.} \sum_{t=0}^{\infty} p_t(\lambda) c_{i,t} \leq p_t(\lambda) c^*_{i,t}.$$

b) $k^*(\lambda)$ solves the firm's problem

$$\max \sum_{t=0}^{\infty} p_t(\lambda)[f(k_t) - k_{t+1}] - r(\lambda) k_0$$
$$\text{s.t.} \quad (1-\delta)k_t \leq k_{t+1} \leq f(k_t), \; \forall t \geq 0.$$
$$k_0 \geq 0 \text{ is given.}$$

c) Markets clear

$$\sum_{i=1}^{I} c^*_{i,t} + k^*_{t+1} = f(k^*_t), \; \forall t \geq 0.$$

The concavity of the instantaneous utility function u_i implies that $c_i^*(\lambda)$ solves the consumer's problem. It only remains to prove that the production plan indeed solves the firm's problem.

Proposition 6 establishes that there exists T such that $(1-\delta)k_t^* < k_{t+1}^* < f(k_t^*)$; $\forall t \geq T$: Since $k^*(\lambda)$ is optimal, $(k_1^*; ::::; k_T^*)$ must solve

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{T} \beta^t V_t(\lambda; k_t; k_{t+1}) \\
s.t.: \quad & (1-\delta)k_t \leq k_{t+1} \leq f(k_t); \; \forall t = 0; :::::; T \\
& k_{T+1} = k_{T+1}^*
\end{aligned}
$$

By lemma 3, $k_{T+1}^* < f(k_T^*)$; so the Slater condition is verified. Hence, there are multipliers $\rho_t; \gamma_t \in R$ associated with the above constraints such that $(k_t^*; \rho_t; \gamma_t)_{t=0}^{T}$ maximizes the associated Lagrangian. By Lemma 3, $\gamma_t = 0$ for all $t = 0,....,T$. For $t = 0; ::::; T-1$ the Kuhn -Tucker first order conditions are

$$
\begin{aligned}
& \beta^t \frac{\partial V_t(\lambda; k_t^*; k_{t+1}^*)}{\partial y} + \beta^{t+1} \frac{\partial V_{t+1}(\lambda; k_{t+1}^*; k_{t+2}^*)}{\partial k} + \rho_t - \rho_{t+1}(1-\delta) = 0 \\
& \rho_t \geq 0; \; \rho_t[(1-\delta)k_t^* - k_{t+1}^*] = 0
\end{aligned}
$$

while for $t \geq T$ the Euler equation implies

$$
\begin{aligned}
& \frac{\partial V_t(\lambda; k_t^*; k_{t+1}^*)}{\partial y} + \beta \frac{\partial V_{t+1}(\lambda; k_{t+1}^*; k_{t+2}^*)}{\partial k} = 0 \\
\Leftrightarrow \quad & \mu_t = \beta \mu_{t+1} f'(k_t^*):
\end{aligned}
$$

For any $k(\lambda) \in \Pi(k_0)$ and any $T' \geq T$ define

$$
\begin{aligned}
\varphi(T'; k(\lambda)) &= \sum_{t=0}^{T'} p_t(\lambda)[f(k_t^*) - k_{t+1}^*] - \sum_{t=0}^{T'} p_t(\lambda)[f(k_t) - k_{t+1}] \\
&= \sum_{t=0}^{T'} \beta^t \mu_t[(f(k_t^*) - k_{t+1}^*) - (f(k_t) - k_{t+1})]
\end{aligned}
$$

We want to prove that $\lim_{T' \to \infty} \varphi(T'; k(\lambda)) \geq 0$: Using the concavity of f and rearranging terms we get

$$
\begin{aligned}
\varphi(T';k(\lambda)) &\geq \sum_{t=0}^{T'} \beta^t \mu_t \Big[f'(k_t^*)(k_t^* - k_t) - (k_{t+1}^* - k_{t+1}) \Big] \\
&= \mu_0 \Big[f'(k_0)(k_0 - k_0) - (k_1^* - k_1) \Big] \\
&+ \beta \mu_1 \Big[f'(k_1^*)(k_1^* - k_1) - (k_2^* - k_2) \Big] \\
&\vdots \qquad\qquad \vdots \\
&+ \beta^{T'} \mu_{T'} \Big[f'(k_{T'}^*)(k_{T'}^* - k_{T'}) - (k_{T'+1}^* - k_{T'+1}) \Big] \\
&= \Big[-\mu_0 + \beta \mu_1 f'(k_1^*) \Big] (k_1^* - k_1) \\
&+ \Big[-\beta \mu_1 + \beta^2 \mu_2 f'(k_2^*) \Big] (k_2^* - k_2) \\
&\vdots \qquad\qquad \vdots \\
&+ \Big[-\beta^{T'-1} \mu_{T'-1} + \beta^{T'} \mu_{T'} f'(k_{T'}^*) \Big] (k_{T'}^* - k_{T'}) \\
&- \beta^{T'} \mu_{T'} (k_{T'+1}^* - k_{T'+1}).
\end{aligned}
$$

Since the Euler equation holds for $t \geq T$, the terms between T and T' vanish. Moreover, using the Kuhn-Tucker conditions we have

$$
\begin{aligned}
\varphi(T';k(\lambda)) &\geq -\beta^{T'} \mu_{T'} (k_{T'+1}^* - k_{T'+1}) \\
&+ [-\rho_0 + \rho_1(1-\delta)](k_1^* - k_1) \\
&+ [-\rho_1 + \rho_2(1-\delta)](k_2^* - k_2) \\
&\vdots \qquad\qquad \vdots \\
&+ [-\rho_{T-2} + \rho_{T-1}(1-\delta)](k_{T-1}^* - k_{T-1}) \\
&+ [-\rho_{T-1} + \rho_T(1-\delta)](k_T^* - k_T) \\
&= -\beta^{T'} \mu_{T'} (k_{T'+1}^* - k_{T'+1}) - \rho_0 k_1^* + \rho_0 k_1 \\
&+ \rho_1[(1-\delta)k_1^* - k_2^*] + \rho_1[k_2 - (1-\delta)k_1] \\
&\vdots \qquad\qquad \vdots \\
&+ \rho_{T-1}[(1-\delta)k_{T-1}^* - k_T^*] + \rho_{T-1}[k_T - (1-\delta)k_{T-1}] \\
&+ \rho_T[(1-\delta)k_T^* - (1-\delta)k_T].
\end{aligned}
$$

Since $\rho_t[(1-\delta)k_t^* - k_{t+1}^*] = 0$ for $t \leq T-1$; $k_{t+1} - (1-\delta)k_t \geq 0; \forall t$ and $\rho_T = 0$ (because $(1-\delta)k_T^* < k_{T+1}^*$); we obtain

$$
\begin{aligned}
\varphi(T';k(\lambda)) &\geq -\beta^{T'}\mu_{T'}(k^*_{T'+1} - k_{T'+1}) - \rho_0 k^*_1 + \rho_0 k_1 \\
&= -\beta^{T'}\mu_{T'}(k^*_{T'+1} - k_{T'+1}) \\
&+ \rho_0(1-\delta)k_0 - \rho_0 k^*_1 + \rho_0 k_1 - \rho_0(1-\delta)k_0 \\
&= -\beta^{T'}\mu_{T'}(k^*_{T'+1} - k_{T'+1}) + \rho_0[k_1 - (1-\delta)k_0] \\
&\geq -\beta^{T'}\mu_{T'}(k^*_{T'+1} - k_{T'+1}) \\
&\geq -\beta^{T'}\mu_{T'}k^*_{T'+1}.
\end{aligned}
$$

But $\mu_{T'}$ and $k^*_{T'+1}$ are bounded from above while $\beta^{T'} \to 0$ as $T' \to \infty$. Then, $\varphi(\infty;k(\lambda)) \geq 0$ as was to be shown. ■

The appropriate transfer to each consumer is the amount that just allows the consumer to afford the consumption stream allocated by the social optimization problem. Thus, for given weights $\lambda \in \Delta$, the required transfers are

$$
\Phi_i(\lambda) = \sum_{t=0}^{\infty} p_t(\lambda)c^*_{i,t}(\lambda) - \alpha_i\pi(\lambda) - \theta_i r(\lambda)k_0; \ \forall i
$$

where $\pi(\lambda) = \sum_{t=0}^{\infty} p_t(\lambda)[f(k^*_t(\lambda)) - k^*_{t+1}(\lambda)] - r(\lambda)k_0$.

A competitive equilibrium for this economy corresponds to a set of welfare weights $\lambda \in \Delta$ such that these transfers equal to zero. The next two lemmas will allow us to use a fixed point argument to prove that such a λ exists.

Lemma 10 For every i, $\Phi_i(\cdot)$ is a continuous function of λ.

Proof: Lemma 2 shows that, given $\lambda \in \Delta$, $U(\lambda;c;k)$ is continuous over $\Pi(k_0) \times \Sigma(k_0)$. Since $\Pi(k_0)$ and $\Sigma(k_0)$ are compact a direct application of Berge's Theorem implies that $c^*(\lambda)$ and $k^*(\lambda)$ are continuous functions of λ in the product topology. By lemma 1, for any $\lambda \in \Delta$, we have

$$
\begin{aligned}
\beta^t(\#I)\overline{B}(k_0) &= \sum_{i\in I} \beta^t \overline{B}(k_0) \\
&\geq \sum_{i\in I} \lambda_i \beta_i^t u_i\left(c^*_{i;t}(\lambda)\right) - \sum_{i\in I} \lambda_i \beta_i^t u_i(0) \\
&\geq \sum_{i\in I} \lambda_i \beta_i^t u_i'\left(c^*_{i;t}(\lambda)\right) c^*_{i;t}(\lambda) \\
&= p_t(\lambda)\sum_{i\in I} c^*_{i;t}(\lambda) \geq p_t(\lambda)c^*_{i;t}(\lambda); \ \forall t; \forall i = 1;\ldots;m
\end{aligned}
$$

(because if $i \notin I$, $c^*_{i,t}(\theta) = 0$, $\forall t$).

As a result $\forall \varepsilon > 0$, there exists T such that $\forall \theta \in \Theta$,

$$\sum_{t=T}^{\infty} p_t(\theta) c^*_{i,t}(\theta) \le \sum_{t=T}^{\infty} \overline{D}(k_0)\beta^t < \frac{\varepsilon}{3}, \ \forall i$$

where $\overline{D}(k_0) = (\# I)\overline{B}(k_0)$.

Consider a sequence $\theta^n \in \Theta$ that converves to $\theta \in \Theta$. We want to show that $\Phi_i(\theta^n) \to \Phi_i(\theta)$. Observe that $\forall i, \forall n$ we have

$$\left| \sum_{t=0}^{\infty} p_t(\theta^n) c^*_{i,t}(\theta^n) - \sum_{t=0}^{\infty} p_t(\theta) c^*_{i,t}(\theta) \right| \le \sum_{t=0}^{T} \left| p_t(\theta^n) c^*_{i,t}(\theta^n) - p_t(\theta) c^*_{i,t}(\theta) \right| + \sum_{t=T}^{\infty} p_t(\theta^n) c^*_{i,t}(\theta^n) + \sum_{t=T}^{\infty} p_t(\theta) c^*_{i,t}(\theta).$$

Observe also that $p_t(\theta^n)$ converges to $p_t(\theta)$. Indeed, we have $p_t(\theta) = \theta_i \beta^t u_i'\left(c^*_{i,t}(\theta)\right)$ for some $i \in I$. Since $c^*_{i,t}(\theta^n)$ converges to $c^*_{i,t}(\theta) > 0$, we have that $u_i'\left(c^*_{i,t}(\theta^n)\right) \to u_i'\left(c^*_{i,t}(\theta)\right)$.

Let $\varepsilon > 0$. Using the previous results there exists T such that $\sum_{t=T}^{\infty} p_t(\theta^n) c^*_{i,t}(\theta^n) + \sum_{t=T}^{\infty} p_t(\theta) c^*_{i,t}(\theta) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$. Moreover, given T, the continuity of $p_t(\theta)$ and $c^*_{i,t}(\theta)$ implies that there exists N such that for any $n \ge N$ the first term is smaller than $\frac{\varepsilon}{3}$. As a result, for any i, $\sum_{t=0}^{\infty} p_t(\theta) c^*_{i,t}(\theta)$ is continuous with respect to θ.

Note also that for any $\theta \in \Theta$

$$\begin{aligned} \beta^t (\# I)\overline{B}(k_0) &\ge p_t(\theta) \sum_{i \in I} c^*_{i,t}(\theta) \\ &= p_t(\theta)[f(k^*_t(\theta)) - k^*_{t+1}(\theta)], \ \forall t. \end{aligned}$$

Following the same reasoning it can be easily shown that $p_t(\theta)[f(k^*_t(\theta)) - k^*_{t+1}(\theta)]$ is continuous with respect to θ. Since $r(\theta) = p_0(\theta) F'(k_0) = \theta_i u_i'\left(c^*_{i,0}(\theta)\right) F'(k_0)$ it follows that $r(\theta)$ is also a continuous function of θ. As a result, for any i, $\alpha_i \pi(\theta) + \vartheta_i r(\theta) k_0$ is continuous with respect to θ. ■

Lemma 11 Let $k_0 > 0$. Then, for any $\theta \in \Theta$, $\pi(\theta) > 0$.

Proof: Take the feasible sequence k defined by $(1 - \delta)k_t = k_{t+1}$, $\forall t \ge 1$. Since

$\pi(\lambda)$ is the maximum profit we have

$$\begin{aligned}\pi(\lambda) &\geq \sum_{t=0}^{\infty} p_t(\lambda)[f(k_t) - k_{t+1}] - r(\lambda)k_0 \\ &= \sum_{t=0}^{\infty} p_t(\lambda)F(k_t) - r(\lambda)k_0 \\ &> p_0(\lambda)[F(k_0) - F'(k_0)k_0] > 0.\end{aligned}$$

■

Theorem 2 Let $k_0 > 0$. Under the assumptions made about the preferences and the technology there exists $\lambda \in \Delta$ such that $\Phi_i(\lambda) = 0$, $\forall i$, i.e. there exists an equilibrium.

Proof: The proof is a direct application of Brouwer's fixed point theorem. Let $T : \Delta \longrightarrow \Delta$, where $T(\lambda) = (T_1(\lambda), \ldots, T_m(\lambda))$ and $T_i(\lambda)$ defined as

$$T_i(\lambda) = \frac{\lambda_i + \Phi'_i(\lambda)}{1 + \sum_{i=1}^{m} \Phi'_i(\lambda)}$$

with $\Phi'_i(\lambda) = -\Phi_i(\lambda)$ if $\Phi_i(\lambda) < 0$ and $\Phi'_i(\lambda) = 0$ if $\Phi_i(\lambda) \geq 0$. T is a continuous mapping from the simplex into itself. By the Brouwer fixed point theorem there exists $\bar{\lambda} \in \Delta$ such that $T(\bar{\lambda}) = \bar{\lambda}$. We have

$$\bar{\lambda}_i = \frac{\bar{\lambda}_i + \Phi'_i(\bar{\lambda})}{1 + \sum_{i=1}^{m} \Phi'_i(\bar{\lambda})} \Leftrightarrow \bar{\lambda}_i \sum_{i=1}^{m} \Phi'_i(\bar{\lambda}) = \Phi'_i(\bar{\lambda}) \qquad (1)$$

If $\bar{\lambda}_i = 0$, Lemma 3 implies $c^*_{i,t}(\lambda) = 0$ for all t, so we have $\Phi_i(\bar{\lambda}) < 0$ and $\Phi'_i(\bar{\lambda}) > 0$: a contradiction with (1). Thus, $\bar{\lambda}_i > 0$, $\forall i$. If $\sum_{i=1}^{m} \Phi'_i(\bar{\lambda}) > 0$ then $\Phi'_i(\bar{\lambda}) > 0$, $\forall i$. From the definition of $\Phi'_i(\lambda)$ this implies $\Phi_i(\lambda) < 0$, $\forall i$. But this contradicts Walras' Law which says $\sum_{i=1}^{m} \Phi_i(\bar{\lambda}) = 0$. Thus, $\sum_{i=1}^{m} \Phi'_i(\bar{\lambda}) = 0$ which implies $\Phi'_i(\bar{\lambda}) = 0$, $\forall i$. But in this case we have $\Phi_i(\bar{\lambda}) \geq 0$, $\forall i$. From Walras' Law we have $\Phi_i(\bar{\lambda}) = 0$, $\forall i$. ■

7 Conclusions

This paper proves existence of a competitive equilibrium in a version of a Ramsey (one sector) model in which agents are heterogeneous and investment is irreversible. The analysis is carried out by exploiting the link between Pareto-optima and competitive equilibria (Negishi method). This method allows us to obtain detailed results concerning the properties of competitive equilibria, with most important the convergence of the optimal capital trajectory to a limit point: some $k^s > 0$ determined by the maximum discount factor. In contrast to the traditional one sector growth model, our proof of convergence does not rely on the monototnicity property simply because such a property does not exist if one allows different discount factors. In addition to the convergence result we are able to give a partial characterization for the dynamics of the optimal capital sequence: in the long-run the optimal capital trajectory exhibits a "no-crossing" property in the sense that it cannot fluctuate around the steady state. Finally, using the Inada condition for the instantaneous utility functions, we are able to show that the consumption paths of all agents with a discount factor equal to the maximum one converge to strictly positive stationary consumptions, while the consumption paths of the remaining agents converge to zero.

Appendix

Proof of lemma 6: Let $k_0 > 0$ but assume that such T exists. Since $k^*_t \to 0$ we can choose some integer $T' \geq T$ such that $F'(k^*_{T'+1}) > \frac{1}{\min_i \beta_i} - 1 + \delta$. Lemma 3 implies that $k^*_{t+1} < F(k^*_t) + (1-\delta)k^*_t$ for all t; so there is $\varepsilon > 0$ small enough to verify

$$(1-\delta)k^*_{T'} < k^*_{T'+1}(1+\varepsilon) < F(k^*_{T'}) + (1-\delta)k^*_{T'}.$$

Let k' be an alternative accumulation path defined as $k'_t = k^*_t$ for $t = 1, \ldots, T'$ and $k'_t = k^*_t(1+\varepsilon)$ for $t \geq T'+1$. Up to date $T'+1$ the path k' is feasible in regard of the choice of ε. For $t \geq T'+2$ we have,

$$(1-\delta)k'_t = (1-\delta)(1+\varepsilon)k^*_t = (1+\varepsilon)k^*_{t+1} = k'_{t+1}$$

where the second equality holds because $(1-\delta)k^*_t = k^*_{t+1}, \forall t \geq T$. Since the same equality imlies that

$$k'_{t+1} = (1-\delta)k'_t < F(k'_t) + (1-\delta)k'_t$$

the path k' is feasible. We next show that k' dominates k^* for some $\varepsilon > 0$ small enough. Define $\varphi(\varepsilon)$ as

$$
\begin{aligned}
\varphi(\varepsilon) &= \sum_{t=0}^{\infty} \beta^t V_t(\lambda; k'_t; k'_{t+1}) - \sum_{t=0}^{\infty} \beta^t V_t\left(\lambda; k^*_t; k^*_{t+1}\right) \\
&= \beta^{T'}\left[V_{T'}(\lambda; k^*_{T'}; k'_{T'+1}) - V_{T'}(\lambda; k^*_{T'}; k^*_{T'+1})\right] \\
&+ \beta^{T'+1}\left[V_{T'+1}(\lambda; k'_{T'+1}; k'_{T'+2}) - V_{T'+1}(\lambda; k^*_{T'+1}; k^*_{T'+2})\right] \\
&+ \sum_{t>T'+1}^{\infty} \beta^t \left[V_t(\lambda; k'_t; k'_{t+1}) - V_t(\lambda; k^*_t; k^*_{t+1})\right]. \qquad (1)
\end{aligned}
$$

Using the concavity of V_t we obtain

$$
\begin{aligned}
V_{T'}(\lambda; k^*_{T'}; k'_{T'+1}) - V_{T'}(\lambda; k^*_{T'}; k^*_{T'+1}) &\geq \frac{\partial V_{T'}(\lambda; k^*_{T'}; k'_{T'+1})}{\partial y}(k'_{T'+1} - k^*_{T'+1}) \\
&= \frac{\partial V_{T'}(\lambda; k^*_{T'}; k'_{T'+1})}{\partial y} k^*_{T'+1}\varepsilon. \qquad (2)
\end{aligned}
$$

For $t \geq T' + 1$;

$$
\begin{aligned}
\sum_{i \in I} c'_{i,t} &= f(k'_t) - k'_{t+1} = F(k'_t) + (1-\delta)k'_t - k'_{t+1} = F(k'_t) = F(k^*_t(1+\varepsilon)) \\
\sum_{i \in I} c^*_{i,t} &= f(k^*_t) - k^*_{t+1} = F(k^*_t) + (1-\delta)k^*_t - k^*_{t+1} = F(k^*_t)
\end{aligned}
$$

where $(c'_{i,t})_{i \in I}$ are such that $V_t(\lambda; k'_t; k'_{t+1}) = \sum_{i \in I} \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u_i(c'_{i,t})$. Using the concavity of u_i and F we get

$$
\begin{aligned}
& V_{T'+1}(\lambda; k'_{T'+1}; k'_{T'+2}) - V_{T'+1}(\lambda; k^*_{T'+1}; k^*_{T'+2}) \\
= & \sum_{i \in I} \lambda_i \left(\frac{\beta_i}{\beta}\right)^{T'+1} \left[u_i(c'_{i,T'+1}) - u_i(c^*_{i,T'+1})\right] \\
\geq & \sum_{i \in I} \lambda_i \left(\frac{\beta_i}{\beta}\right)^{T'+1} u'_i(c'_{i,T'+1})(c'_{i,T'+1} - c^*_{i,T'+1}) \\
= & \mu_{T'+1,1+\varepsilon} \sum_{i=1}^{I} (c'_{i,T'+1} - c^*_{i,T'+1}) \\
\geq & \mu_{T'+1,1+\varepsilon} \left[F\left(k^*_{T'+1}(1+\varepsilon)\right) - F(k^*_{T'+1})\right] \\
\geq & \mu_{T'+1,1+\varepsilon} F'\left(k^*_{T'+1}(1+\varepsilon)\right) k^*_{T'+1}\varepsilon \qquad (3)
\end{aligned}
$$

where $\mu_{T'+1,1+\varepsilon} = \lambda_i \left(\frac{\beta}{\beta_i}\right)^{T'+1} u_i'(c'_{i,T'+1})$, $\forall i \in I$. Similarly, for $t > T'+1$, the concavity of u_i and f implies

$$V_t(\lambda; k'_t, k'_{t+1}) - V_t(\lambda; k^*_t, k^*_{t+1}) \geq \mu_{t,1+\varepsilon} F'((1+\varepsilon)k^*_t)k^*_t\varepsilon.$$

Note that $k^*_t = (1-\delta)^{t-T'-1}k^*_{T'+1}$, $\forall t > T'+1$. Thus $k'_t = (1+\varepsilon)k^*_t < k'_{T'+1} = (1+\varepsilon)k^*_{T'+1}$ and $\sum_{i\in I} c'_{i,t} = F(k'_t) < \sum_{i\in I} c'_{i,T'+1} = F(k'_{T'+1})$, $\forall t > T'+1$. Therefore, $\forall t > T'+1$, there exists some $i \in I$ such that $c'_{i,t} < c'_{i,T'+1}$. But this implies

$$\mu_{t,1+\varepsilon} = \lambda_i \left(\frac{\beta}{\beta_i}\right)^t u_i'(c'_{i,t}) > \mu_{T'+1,1+\varepsilon} = \lambda_i \left(\frac{\beta}{\beta_i}\right)^t u_i'(c'_{i,T'+1}), \ \forall t > T'+1.$$

Using the above inequalities we obtain

$$\begin{aligned}
&\sum_{t>T'+1}^{\infty} \beta^t \Big[V_t(\lambda; k'_t, k'_{t+1}) - V_t(\lambda; k^*_t, k^*_{t+1})\Big] \\
\geq\ & \sum_{t>T'+1}^{\infty} \beta^t \Big[\mu_{t,1+\varepsilon} F'(k^*_t(1+\varepsilon))k^*_t\varepsilon\Big] \\
\geq\ & \sum_{t>T'+1}^{\infty} \beta^t \Big[\mu_{T'+1,1+\varepsilon} F'\left(k^*_{T'+1}(1+\varepsilon)\right)(1-\delta)^{t-T'-1}k^*_{T'+1}\varepsilon\Big] \\
=\ & \mu_{T'+1,1+\varepsilon} F'\left(k^*_{T'+1}(1+\varepsilon)\right) \frac{k^*_{T'+1}\varepsilon}{(1-\delta)^{T'+1}} \sum_{t>T'+1}^{\infty} \beta^t (1-\delta)^t \\
=\ & \mu_{T'+1,1+\varepsilon} F'\left(k^*_{T'+1}(1+\varepsilon)\right) \frac{\beta^{T'+2}(1-\delta)}{1-\beta(1-\delta)} k^*_{T'+1}\varepsilon. \qquad (4)
\end{aligned}$$

Combining (1), (2), (3) and (4) we get

$$\begin{aligned}
\varphi(\varepsilon) =\ & \sum_{t=0}^{\infty} \beta^t V_t(\lambda; k'_t, k'_{t+1}) - \sum_{t=0}^{\infty} \beta^t V_t(\lambda; k^*_t, k^*_{t+1}) \\
\geq\ & \beta^{T'} \frac{\partial V_{T'}(\lambda; k^*_{T'}, k'_{T'+1})}{\partial y} \varepsilon k^*_{T'+1} + \beta^{T'+1} \mu_{T'+1,1+\varepsilon} F'\left(k^*_{T'+1}(1+\varepsilon)\right) k^*_{T'+1}\varepsilon \\
+\ & \beta^{T'+2} \mu_{T'+1,1+\varepsilon} F'\left(k^*_{T'+1}(1+\varepsilon)\right) \frac{(1-\delta)}{1-\beta(1-\delta)} k^*_{T'+1}\varepsilon \\
=\ & \beta^{T'} k^*_{T'+1}\varepsilon \left\{ \frac{\partial V_{T'}(\lambda; k^*_{T'}, k'_{T'+1})}{\partial y} + \beta \mu_{T'+1,1+\varepsilon} F'\left(k^*_{T'+1}(1+\varepsilon)\right) \left[1 + \frac{\beta(1-\delta)}{1-\beta(1-\delta)}\right] \right\}.
\end{aligned}$$

When $\varepsilon \to 0$ the term inside the brackets converges to

$$-\mu_{T'} + \beta\mu_{T'+1}F'(k^*_{T'+1})\frac{1}{1-\beta(1-\delta)}.$$

We will show that this term is strictly positive. Note that

$$\sum_{i\in I} c^*_{i,T'} = f(k^*_{T'}) - k^*_{T'+1} = f(k^*_{T'}) - (1-\delta)k^*_{T'};$$
$$\sum_{i\in I} c^*_{i,T'+1} = f(k^*_{T'+1}) - k^*_{T'+2} = f\left((1-\delta)k^*_{T'}\right) - (1-\delta)^2k^*_{T'}.$$

Subtracting and using the concavity of f we get

$$\begin{aligned}
& f(k^*_{T'}) - (1-\delta)k^*_{T'} - f\left((1-\delta)k^*_{T'}\right) + (1-\delta)^2k^*_{T'} \\
= \; & f(k^*_{T'}) - f\left((1-\delta)k^*_{T'}\right) - \delta(1-\delta)k^*_{T'} \\
\geq \; & f'(k^*_{T'})\delta k^*_{T'} - \delta(1-\delta)k^*_{T'} = \delta k^*_{T'}\left[f'(k^*_{T'}) - (1-\delta)\right] \\
= \; & \delta k^*_{T'}F'(k^*_{T'}) > 0.
\end{aligned}$$

Thus, there must exist some $i \in I$ such that $c^*_{i,T'} > c^*_{i,T'+1}$ and hence $u_i'(c^*_{i,T'}) < u_i'(c^*_{i,T'+1})$. But in this case

$$\mu_{T'} = \lambda_i\left(\frac{\beta_i}{\beta}\right)^{T'} u_i'(c^*_{i,T'}) < \lambda_i\left(\frac{\beta_i}{\beta}\right)^{T'} u_i'(c^*_{i,T'+1})\frac{\beta}{\beta_i} < \mu_{T'+1}\frac{\beta}{\beta_i}.$$

Since

$$F'(k^*_{T'+1})\beta_i \geq F'(k^*_{T'+1})\min_i \beta_i > 1 - \min_i \beta_i(1-\delta) > 1-\beta(1-\delta)$$

we have

$$\mu_{T'} < \mu_{T'+1}\frac{\beta}{\beta_i} < \mu_{T'+1}\frac{\beta}{\beta_i}\frac{F'(k^*_{T'+1})\beta_i}{1-\beta(1-\delta)} = \beta\mu_{T'+1}F'(k^*_{T'+1})\frac{1}{1-\beta(1-\delta)}.$$

In short $\varphi(0) = 0$ and $\varphi(\varepsilon) > 0$ for $\varepsilon > 0$ small enough: a contradiction. ■

Proof of Lemma 7: Assume the contrary: $k_0 > 0$ and k^* is optimal but $k^*_t \to 0$. The rest of the proof follows in two steps.

Step 1: We claim that there is some T with $(1-\delta)k^*_t < k^*_{t+1}$ for all $t \geq T$. Suppose the claim is false. Then for any integer T there exists $T' > T$ such that $(1-\delta)k^*_{T'-1} = k^*_{T'}$. Note that lemma 6 implies that T' can be choosen such that $(1-\delta)k^*_{T'} < k^*_{T'+1}$. Moreover, since $k^*_t \to 0$, T' can be choosen such that $F'(k^*_{T'}) > \frac{1}{\min_i \beta_i} - 1 + \delta$.

By lemma 3 $k^*_{T'} < f(k^*_{T'-1})$, so we can choose $\varepsilon > 0$ small enough such that $k^*_{T'} + \varepsilon < f(k^*_{T'-1})$ and $(1-\delta)(k^*_{T'} + \varepsilon) < k^*_{T'+1}$. Consider now the accumulation path k' defined by $k'_t = k^*_t$ for all $t \neq T'$ and $k'_{T'} = k^*_{T'} + \varepsilon$. Since

$$(1-\delta)k^*_{T'-1} < k^*_{T'} + \varepsilon < f(k^*_{T'-1})$$

k' is feasible. We next show that k' dominates k^* for some $\varepsilon > 0$ small enough. Define $\varphi(\varepsilon)$ as

$$\begin{aligned}
\varphi(\varepsilon) &= \sum_{t=0}^{\infty} \beta^t V_t(\lambda, k'_t, k'_{t+1}) - \sum_{t=0}^{\infty} \beta^t V_t(\lambda, k^*_t, k^*_{t+1}) \\
&= \beta^{T'-1}\left[V_{T'-1}(\lambda, k^*_{T'-1}, k'_{T'}) - V_{T'-1}(\lambda, k^*_{T'-1}, k^*_{T'})\right] \\
&\quad + \beta^{T'}\left[V_{T'}(\lambda, k'_{T'}, k^*_{T'+1}) - V_{T'}(\lambda, k^*_{T'}, k^*_{T'+1})\right].
\end{aligned}$$

Using the concavity of V we have

$$\begin{aligned}
\varphi(\varepsilon) &\geq \beta^{T'-1}\frac{\partial V_{T'-1}(\lambda, k^*_{T'-1}, k'_{T'})}{\partial y}\varepsilon + \beta^{T'}\frac{\partial V_{T'}(\lambda, k'_{T'}, k^*_{T'+1})}{\partial k}\varepsilon \\
&= \beta^{T'-1}\varepsilon\left\{\frac{\partial V_{T'-1}(\lambda, k^*_{T'-1}, k'_{T'})}{\partial y} + \beta\frac{\partial V_{T'}(\lambda, k'_{T'}, k^*_{T'+1})}{\partial k}\right\} \\
&= \beta^{T'-1}\varepsilon\left[-\mu_{T'-1,\varepsilon} + \beta\mu_{T',\varepsilon} f'(k^*_{T'} + \varepsilon)\right].
\end{aligned}$$

When $\varepsilon \to 0$ the term inside the brackets converges to $-\mu_{T'-1} + \beta\mu_{T'} f'(k^*_{T'})$. We want to show that this term is strictly positive. Note that

$$\begin{aligned}
\sum_{i\in I} c^*_{i,T'-1} &= f(k^*_{T'-1}) - k^*_{T'}, \\
\sum_{i\in I} c^*_{i,T'} &= f(k^*_{T'}) - k^*_{T'+1}
\end{aligned}$$

and

$$\begin{aligned}
& f(k^*_{T'}) - k^*_{T'+1} \\
&= F(k^*_{T'}) + (1-\delta)k^*_{T'} - k^*_{T'+1} \\
&< F(k^*_{T'}) < F(k^*_{T'-1}) = f(k^*_{T'-1}) - (1-\delta)k^*_{T'-1} \\
&= f(k^*_{T'-1}) - k^*_{T'}.
\end{aligned}$$

Thus, there must exist some $i \in I$ such that $c^*_{i,T'-1} > c^*_{i,T'}$ and $u'_i(c^*_{i,T'-1}) < u'_i(c^*_{i,T'})$. But in this case

$$\mu_{T'-1} = \lambda_i \left(\frac{\beta_i}{\beta}\right)^{T'-1} u'_i(c^*_{i,T'-1}) < \lambda_i \left(\frac{\beta_i}{\beta}\right)^{T'-1} u'_i(c^*_{i,T'})\frac{\beta_i}{\beta}\frac{\beta}{\beta_i} < \mu_{T'}\frac{\beta}{\beta_i}.$$

Since

$$F'(k^*_{T'}) + (1-\delta) = f'(k^*_{T'}) > \frac{1}{\min_i \beta_i} \geq \frac{1}{\beta_i}$$

we have $\mu_{T'-1} < \beta\mu_{T'} f'(k^*_{T'})$. In short $\varphi(0) = 0$ and $\varphi(\varepsilon) > 0$ for $\varepsilon > 0$ small enough, a contradiction.

Step 2: From Step 1 and lemma 3 we know that there exists T such that $(1-\delta)k^*_T < k^*_{T+1} < F(k^*_T)$, $\forall t \geq T$. Thus, for all $t \geq T$ the Euler equation holds

$$\begin{aligned} & \frac{\partial V_t(\lambda, k^*_t, k^*_{t+1})}{\partial y} + \beta \frac{\partial V_{t+1}(\lambda, k^*_{t+1}, k^*_{t+2})}{\partial k} = 0 \\ \Leftrightarrow\quad & \mu_t = \beta\mu_{t+1} f'(k^*_{t+1}) \\ \Leftrightarrow\quad & \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u'_i(c^*_{i,t}) = \beta\lambda_i \left(\frac{\beta_i}{\beta}\right)^{t+1} u'_i(c^*_{i,t+1}) f'(k^*_{t+1}) \\ \Leftrightarrow\quad & u'_i(c^*_{i,t}) = \beta_i u'_i(c^*_{i,t+1}) f'(k^*_{t+1}),\ \forall i \in I. \end{aligned}$$

If $k^*_t \to 0$ there exists $T' \geq T$ such that $\beta_i f'(k^*_{t+1}) \geq (\min_i \beta_i) f'(k^*_{t+1}) > 1$, $\forall t \geq T'$. The Euler equation implies $u'_i(c^*_{i,t}) > u'_i(c^*_{i,t+1})$, $\forall t \geq T'$. But in this case $c^*_{i,t} < c^*_{i,t+1}$, $\forall t \geq T'$ and in particular $c^*_{i,t} > c^*_{i,T'} > 0$, $\forall i \in I$ $\forall t \geq T'+1$. However, $k^*_t \to 0$ implies $c^*_{i,t} \to 0$ by feasibility, a contradiction. ■

Proof of Lemma 8: Let α be such that $f'(\alpha) = \frac{1}{\min_i \beta_i}$. We consider two cases:

Case 1: Assume $k_0 > \alpha$. In this case we show that $k^*_t \geq \alpha$ for all t, so we let $\gamma = \alpha$.

Assume the contrary and denote by t_0 the first date such that $k^*_{t_0} < \alpha \leq k^*_{t_0-1}$. The rest of the proof follows in two steps.

Step 1: We claim that there exists T such that $k^*_{t_0+T} < \alpha$ and $k^*_{t_0+T} < k^*_{t_0+T-1}$, $k^*_{t_0+T} \leq k^*_{t_0+T+1}$. To prove this we proceed by induction. If $k^*_{t_0+1} \geq k^*_{t_0}$ we let $T = 0$. If not we have $k^*_{t_0+1} < k^*_{t_0}$. In the same way, if $k^*_{t_0+2} \geq k^*_{t_0+1}$ we let $T = 1$. If not we have $k^*_{t_0+2} < k^*_{t_0+1}$ and so on.

Observe that if $k^*_{t_0+T+1} < k^*_{t_0+T} < \alpha$, $\forall T \geq 0$, Lemma 7 implies that $\lim_{T\to\infty} k^*_{t_0+T} = k > 0$. By the principle of optimality

$$W_{t_0+T}(k^*_{t_0+T}) = V_{t_0+T}(\lambda, k^*_{t_0+T}, k^*_{t_0+T+1}) + \beta W_{t_0+T+1}(k^*_{t_0+T+1}),\ \forall T.$$

Taking the limits we get

$$\tilde{W}(k) = \tilde{V}(\theta; k; k) + \beta \tilde{W}(k)$$

If k satisfies the above equation Proposition 3 implies that $k = k^s$ with $\beta f'(k^s) = 1$. But $k^s < \alpha$, so we have $\frac{1}{\beta} = f'(k^s) > f'(\alpha) = \frac{1}{\min_i \beta_i}$: a contradiction. Thus there exists T such that $k^*_{t_0+T} < \alpha$ and $k^*_{t_0+T} < k^*_{t_0+T-1}$, $k^*_{t_0+T} \leq k^*_{t_0+T+1}$.

<u>Step 2</u>: For simplicity denote $T_0 = t_0 + T$. Step 1 established that there exists T_0 such that $k^*_{T_0} < \alpha$, and $k^*_{T_0} < k^*_{T_0-1}$, $k^*_{T_0} \leq k^*_{T_0+1}$. We also have

$$(1-\delta)k^*_{T_0-1} \leq k^*_{T_0} < f(k^*_{T_0-1});$$
$$(1-\delta)k^*_{T_0} < k^*_{T_0} \leq k^*_{T_0+1} < f(k^*_{T_0}).$$

Consider now an alternative capital path defined by $k'_t = k^*$, $\forall t \neq T_0$ and $k'_{T_0} = k^*_{T_0} + \varepsilon$. Note that ε can be choosen such that k' is feasible i.e.

$$(1-\delta)k^*_{T_0-1} < k^*_{T_0} + \varepsilon < f(k^*_{T_0-1});$$
$$(1-\delta)k^*_{T_0} < k^*_{T_0} \leq k^*_{T_0+1} < f(k^*_{T_0}).$$

We now show that k' dominates k^* in which case we arrive at a contradiction. Define $\varphi(\varepsilon)$ as

$$\begin{aligned}
\varphi(\varepsilon) &= \sum_{t=0}^{\infty} \beta^t V_t(\theta; k'_t; k'_{t+1}) - \sum_{t=0}^{\infty} \beta^t V_t(\theta; k^*_t; k^*_{t+1}) \\
&= \beta^{T_0-1}\left[V_{T_0-1}(\theta; k^*_{T_0-1}; k'_{T_0}) - V_{T_0-1}(\theta; k^*_{T_0-1}; k^*_{T_0})\right] \\
&+ \beta^{T_0}\left[V_{T_0}(\theta; k'_{T_0}; k^*_{T_0+1}) - V_{T_0}(\theta; k^*_{T_0}; k^*_{T_0+1})\right].
\end{aligned}$$

Using the concavity of V_t we have

$$\begin{aligned}
\varphi(\varepsilon) &\geq \beta^{T_0-1}\frac{\partial V_{T_0-1}(\theta; k^*_{T_0-1}; k'_{T_0})}{\partial y}\varepsilon + \beta^{T_0}\frac{\partial V_{T_0}(\theta; k'_{T_0}; k^*_{T_0+1})}{\partial k}\varepsilon \\
&= \beta^{T_0-1}\varepsilon\left\{\frac{\partial V_{T_0-1}(\theta; k^*_{T_0-1}; k'_{T_0})}{\partial y} + \beta\frac{\partial V_{T_0}(\theta; k'_{T_0}; k^*_{T_0+1})}{\partial k}\right\} \\
&= \beta^{T_0-1}\varepsilon\left[-\mu_{T_0-1;\varepsilon} + \beta\mu_{T_0;\varepsilon} f'(k^*_{T_0} + \varepsilon)\right].
\end{aligned}$$

When $\varepsilon \to 0$ the term inside the brackets converges to $-\mu_{T_0-1} + \beta\mu_{T_0} f'(k^*_{T_0})$. We want to show that this term is strictly positive. Note that

$$\sum_{i \in I} c^*_{i;T_0-1} = f(k^*_{T_0-1}) - k^*_{T_0};$$
$$\sum_{i \in I} c^*_{i;T_0} = f(k^*_{T_0}) - k^*_{T_0+1}.$$

Since $f(k^*_{T_0-1}) - k^*_{T_0} > f(k^*_{T_0}) - k^*_{T_0+1}$ there exists some $i \in I$ such that $c^*_{i,T_0-1} > c^*_{i,T_0}$. Thus

$$\mu_{T_0-1} = \lambda_i \left(\frac{\beta_i}{\bar{\beta}}\right)^{T_0-1} u_i'(c^*_{i,T_0-1}) < \lambda_i \left(\frac{\beta_i}{\bar{\beta}}\right)^{T_0-1} u_i'(c^*_{i,T_0})\frac{\beta_i}{\bar{\beta}}\frac{\bar{\beta}}{\beta_i} < \mu_{T_0}\frac{\bar{\beta}}{\beta_i}.$$

But in this case $\frac{1}{\beta_i} \le \frac{1}{\min_i \beta_i} = f'(\alpha) < f'(k^*_{T_0})$; so we have $\mu_{T_0-1} < \bar{\beta}\mu_{T_0} f'(k^*_{T_0})$.

Case 2: $0 < k_0 \le \alpha$. In this case we distinguish between two subcases.

a) Let $0 < k_0 \le \alpha$ but assume that there exists t_0 such that $k^*_{t_0} \ge \alpha$. Repeating the argument applied in case 1 one can show that $k^*_t \ge \alpha$, $\forall t \ge t_0$, so we let $\gamma = \min\left(\alpha, \min\{k^*_1, \ldots, k^*_{t_0}\}\right)$.

b) Let $0 < k_0 \le \alpha$ but assume $k^*_t < \alpha$, $\forall t$. We show that $k^*_t \ge k_0$, $\forall t$, and in that case we let $\gamma = k_0$.

Assume that $k^*_1 < k_0 \le \alpha$. We claim that there exist $T_0 \ge 1$ such that $k^*_{T_0} < \alpha$ and $k^*_{T_0} < k^*_{T_0-1}$, $k^*_{T_0} \le k^*_{T_0+1}$. If the claim is false, then one can show (see step 1 in case 1) that k^*_t converges decreasingly to k^s. Since $k^s < \alpha$, we have $\frac{1}{\bar{\beta}} = f'(k^s) > f'(\alpha) = \frac{1}{\min_i \beta_i}$: a contradiction.

Next consider an alternative accumulation path k', defined by $k'_t = k^*_t$, $\forall t \neq T_0$ and $k'_{T_0} = k^*_{T_0} + \varepsilon$. One can show (see step 2 in case 1) that, for proper choice of ε, k' is feasible and dominates k^*. But this contradicts the optimality of k^*, so we must have $k^*_1 \ge k_0$. Applying the same reasoning one can show that $k^*_2 \ge k^*_1$. Continuing in that way one can establish that k^*_t is increasing and therefore $k^*_t \ge k_0$, $\forall t$. ■

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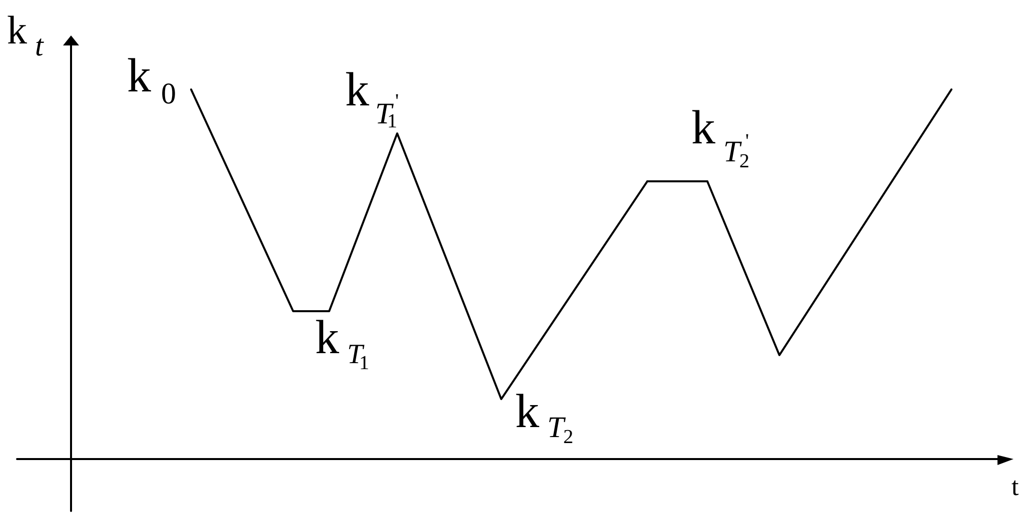

Figure 1: