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Limiting Behavior of Minimizing p-Harmonic Maps in 3d as p Goes to 2 with Finite Fundamental Group

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Abstract

We study the limiting behavior of minimizing p-harmonic maps from a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ to a compact connected Riemannian manifold without boundary and with finite fundamental group as $p \nearrow 2$. We prove that there exists a closed set S_* of finite length such that minimizing p-harmonic maps converge to a locally minimizing harmonic map in $\Omega \setminus S_*$. We prove that locally inside Ω the singular set S_* is a finite union of straight line segments, and it minimizes the mass in the appropriate class of admissible chains. Furthermore, we establish local and global estimates for the limiting singular harmonic map. Under additional assumptions, we prove that globally in $\overline{\Omega}$ the set S_* is a finite union of straight line segments, and it minimizes the mass in the appropriate class of admissible chains, which is defined by a given boundary datum and Ω.

Contents

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1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$ and a compact connected smooth Riemannian manifold $\mathcal{N}$ without boundary and of finite dimension, which will be assumed, thanks to Nash's embedding theorem, to be isometrically embedded into the Euclidean space $\mathbb{R}^\nu$ for some $\nu \in \mathbb{N} \setminus \{0\}$, we are interested in the problem of finding a minimizing harmonic map with boundary datum $g : \partial\Omega \to \mathcal{N}$, that is, find a minimizer of the Dirichlet integral

$$\int_\Omega \frac{|Du|^2}{2}\,\mathrm{d}x \tag{1.1}$$

among all Sobolev mappings $u \in W^{1,2}(\Omega, \mathcal{N})$ that have g as a trace on $\partial\Omega$.

If there exists some $v \in W^{1,2}(\Omega, \mathcal{N})$ having g as a trace on $\partial\Omega$, standard methods in the calculus of variations show the existence of a minimizer $u \in W^{1,2}(\Omega, \mathcal{N})$ of the energy (1.1). Following the classical works of Schoen and Uhlenbeck, the map u is continuous outside a discrete set of points in Ω [52]; moreover, the solution u inherits further regularity of the boundary datum: for example, if $g \in C^{2,\alpha}(\partial\Omega, \mathcal{N})$, then u has the same regularity in a neighborhood of the boundary [53].

When the target manifold $\mathcal{N}$ is simply connected, or equivalently has a trivial fundamental group (i.e., $\pi_1(\mathcal{N}) \simeq \{0\}$), the space of admissible traces is the fractional Sobolev space $W^{1/2,2}(\partial\Omega, \mathcal{N})$ [34], similarly to the trace theory of linear spaces [31], giving then rise to a nice theory of harmonic extension of mappings into $\mathcal{N}$.

When the manifold $\mathcal{N}$ is not simply connected, the admissible traces form a strict subset of $W^{1/2,2}(\partial\Omega, \mathcal{N})$ due to several obstruction phenomena. First, one has global topological obstructions created by the combination of the topologies of $\partial\Omega$ and $\mathcal{N}$ [12, Theorem 5] if $\Omega \simeq B_1^2 \times \mathbb{S}^1$ is a solid torus, taking $\gamma \in C^\infty(\mathbb{S}^1, \mathcal{N})$, which is not homotopic to a constant, the function $g \in C^\infty(\partial\Omega, \mathcal{N})$ defined by $g(x', x'') = \gamma(x')$ for $(x', x'') \in \mathbb{S}^1 \times \mathbb{S}^1 \simeq \partial\Omega$ has no extension in $W^{1,2}(\Omega, \mathcal{N})$. Next, one has local topological obstructions with boundary data that behave like $g(x) = \gamma(x'/|x'|)$ [12, Theorem 4; 34, Section 6.3]. Finally, there are some analytical obstructions coming from the infiniteness of $\pi_1(\mathcal{N})$ [9], and since the integrability exponent 2 is an integer, from the nontriviality of $\pi_1(\mathcal{N})$ [44, Theorem 1.10] these obstructions arise as limits of juxtaposition of boundary data, which have arbitrarily large lower bounds on the extension energy.

These obstructions to trace extension disappear if one relaxes the problem and considers the problem of minimizing, for $1 < p < 2$, the p-energy

$$\int_\Omega \frac{|Du|^p}{p} \,\mathrm{d}x.$$

Indeed, given $g \in W^{1/2,2}(\partial\Omega, \mathcal{N}) \subset W^{1-1/p,p}(\partial\Omega, \mathcal{N})$, there exists $v \in W^{1,p}(\Omega, \mathcal{N})$ such that $\operatorname{tr}_{\partial\Omega}(v) = g$. One way then to find a good replacement for solutions to the problem of minimizing harmonic maps is to define u_p to be a minimizing p-harmonic map under the boundary condition $u = g$ on $\partial\Omega$ and study the asymptotics of the solutions u_p of this p-harmonic relaxation as $p \nearrow 2$.

Such an approach has been developed successfully for planar domains $\Omega \subset \mathbb{R}^2$ when $\mathcal{N}$ is a circle [35] or a general compact Riemannian manifold [56], where, up to a subsequence, u_p converges as $p \nearrow 2$ then to a harmonic map outside a finite set of singular points whose positions and topological charges minimize a suitable renormalized energy [10,46]. Similar results hold for p-energy minimizing maps from $\Omega \subset \mathbb{R}^m$ to $\mathbb{S}^{m-1}$ as $p \nearrow m$ [36].

In higher dimensions, this problem has been studied for maps taking their values in $\mathbb{S}^n$: Hardt and Lin [41, Section 3] have proved that the p-energy measures associated to the energy densities of p-energy minimizing maps from an m-dimensional domain to $\mathbb{S}^n$ as $p \nearrow n < m$ converge, up to subsequences, to a limit measure whose support coincides with some area minimizing current of dimension $m - n$; Stern [54] has obtained the convergence of the p-energy measures associated to the energy densities of stationary p-harmonic maps from a closed oriented Riemannian manifold to the circle $\mathbb{S}^1$ as $p \nearrow 2$ to the weight measure of a stationary varifold (we recall that the most-studied class of stationary p-harmonic maps (for arbitrary target manifolds) are the p-energy minimizers).

In three dimensions, these asymptotics have been analyzed for maps from a closed oriented Riemannian manifold into the circle $\mathbb{S}^1$ [41,54].

Our first result is the following.

Theorem 1.1. *If $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, $\pi_1(\mathcal{N})$ is finite, $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$, $p_n \nearrow 2$ as $n \to +\infty$ and if for each $n \in \mathbb{N}$, $u_n \in W^{1,p_n}(\Omega, \mathcal{N})$ is a minimizing p_n-harmonic map with boundary datum g, then*

$$\limsup_{n\to+\infty} (2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x \leqq C|g|^2_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)},$$

where $C = C(\partial\Omega, \mathcal{N}) > 0$. Furthermore, there exists a closed set $S_* \subset \overline{\Omega}$, at most a countable union of segments $(L_j)_{j\in J}$ lying in $\overline{\Omega}$, mappings $(\gamma_j)_{j\in J}$ in $C(\mathbb{S}^1, \mathcal{N})$, a map $u_* : \Omega \to \mathcal{N}$ such that $S_* \cap \Omega = \bigcup_{j\in J} L_j \cap \Omega$ and, up to a subsequence (not relabeled), the following assertions hold.

(i) $u_n \to u_*$ *almost everywhere in* Ω.

(ii) For each $q \in [1, 2)$, $u_* \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathcal{N})$, $u_n \to u_*$ *in* $W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^\nu)$ *and*

$$\limsup_{n\to+\infty} \int_\Omega (\mathrm{dist}(x, \partial\Omega)|Du_n(x)|)^q \,\mathrm{d}x \leqq C|\Omega| + \frac{C}{2-q} \limsup_{n\to+\infty}(2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x,$$

where $C = C(\mathrm{diam}(\Omega), \mathcal{N}) > 0$.

(iii) Assume that there exists $r_0 = r_0(\Omega) > 0$ *such that if* $x \in \mathbb{R}^3 \setminus \Omega$ *and* $\mathrm{dist}(x, \partial\Omega) \leqq r_0$, *there exists a unique point* $P(x) \in \partial\Omega$ *such that* $\mathrm{dist}(x, \partial\Omega) = |x - P(x)|$. *If g is a C^1 map outside a finite set of points around each of which g is homotopically nontrivial and whose Euclidean norm of the gradient behaves like the distance to the power of -1 close to these points, then for each* $q \in [1, 2)$, $u_* \in W^{1,q}(\Omega, \mathcal{N})$, $u_n \to u_*$ *in* $W^{1,q}(\Omega, \mathbb{R}^\nu)$ *and*

$$\limsup_{n\to+\infty} \int_\Omega |Du_n|^q \,\mathrm{d}x \leqq C + \frac{C}{2-q} \limsup_{n\to+\infty}(2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x,$$

where $C = C(g, \Omega, \mathcal{N}) > 0$.

(iv) Any point of Ω has a neighborhood that only intersects finitely many L_j.

(v) $u_* \in W^{1,2}_{\mathrm{loc}}(\Omega\backslash S_*, \mathcal{N})$ *is a locally minimizing harmonic map in* $\Omega\backslash S_*$: *if* $K \Subset \Omega\backslash S_*$ *is an open set and* $v \in W^{1,2}(K, \mathbb{R}^\nu)$ *satisfies* $v - u_* \in W^{1,2}_0(K, \mathbb{R}^\nu)$, *then*

$$\int_K \frac{|Du_*|^2}{2} \,\mathrm{d}x \leqq \int_K \frac{|Dv|^2}{2} \,\mathrm{d}x.$$

Furthermore, there exists at most a countable and locally finite subset S_0 of $\Omega\backslash S_*$ such that $u_* \in C^\infty(\Omega \setminus (S_* \cup S_0), \mathcal{N})$.

(vi) The varifold $V_* = (\mathrm{Id}, A_*)_\# \left(\sum_{j\in J} \mathcal{E}^{\mathrm{sg}}_2(\gamma_j)\mathcal{H}^1 \llcorner (L_j \cap \Omega)\right)$ *supported by* $\bigcup_{j\in J} L_j \cap \Omega$ *and with a multiplicity* $\mathcal{E}^{\mathrm{sg}}_2(\gamma_j)$ *is a stationary varifold which is the limit of the rescaled stress-energy tensors and*

$$V_*(\Omega \times \mathrm{G}(3, 1)) = \sum_{j\in J} \mathcal{E}^{\mathrm{sg}}_2(\gamma_j)\mathcal{H}^1(L_j \cap \Omega) \leqq \liminf_{n\to+\infty}(2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x,$$

where $\mathrm{Id} : \mathbb{R}^3 \to \mathbb{R}^3$ *is the identity map,* $A_*(x) \in \mathbb{R}^3 \otimes \mathbb{R}^3$ *represents the orthogonal projection onto the approximate tangent plane* $T_x S_*$ *to* S_* *at* x *and* $\mathrm{G}(3, 1)$ *is the Grassmann manifold.*

(vii) S_* minimizes the mass among all admissible chains locally inside Ω, namely, locally inside Ω it solves a homotopical Plateau problem in codimension 2.

(viii) If $\partial\Omega$ is homeomorphic to $\mathbb{S}^2$ and $g \in W^{1,2}(\partial\Omega, \mathcal{N})$, then $S_* = \emptyset$ and $u_* \in C^\infty(\Omega \setminus S_0, \mathcal{N})$ is a minimizing harmonic extension of g to Ω, where $S_0 \subset \Omega$ is a discrete set.

The singular energy $\mathcal{E}_2^{\mathrm{sg}}(\gamma)$, appearing in the definition of the varifold V_* in Theorem 1.1, is defined for $\gamma \in C(\mathbb{S}^1, \mathcal{N})$ as the infimum of $\sum_{i=1}^m \int_{\mathbb{S}^1} \frac{|f_i'|^2}{2} \, \mathrm{d}\mathcal{H}^1$ for $f_1, \ldots, f_m \in C^1(\mathbb{S}^1, \mathcal{N})$ such that γ has a continuous extension to a domain consisting of the 2-disk B_1^2 from which m small 2-disks have been removed whose restriction on the boundary of the i-th disk is a rescaled version of the mapping f_i.

The finite assumption on the fundamental group in Theorem 1.1 makes it inapplicable to the case of the unit circle $\mathcal{N} = \mathbb{S}^1$; the framework of our results is thus completely disjoint from Stern's [54]. Among the situations covered by our theorem, we point out the case where $\mathcal{N}$ is the real projective plane $\mathbb{RP}^2$ motivated by liquid crystals, and the case where $\mathcal{N}$ is the space of attitudes of the cube $SO(3)/O$, where O is the octahedral group, which appears as framefields in meshing problems. We point out that the finiteness assumption on the fundamental group of the compact manifold $\mathcal{N}$ is equivalent to a compactness assumption on the universal covering $\widetilde{\mathcal{N}}$ of the manifold $\mathcal{N}$.

When $\mathcal{N}$ is the real projective plane $\mathbb{RP}^2$, Theorem 1.1 is a counterpart of Canevari's result for Landau-de Gennes models of nematic liquid crystals [21].

The strategy of our proof of Theorem 1.1 is inspired by Canevari's η-compactness approach for Landau-de Gennes [21]. Whereas Canevari works with a perturbation involving no derivatives and with a domain enlarged to a full linear Sobolev space, in our setting we have a perturbation involving the first-order derivatives defined on a nonlinear space with the same manifold constraint but enlarged by lower integrability requirements. This results in different constructions to deal with the different structures and different analytical arguments because the space in which the minimization is performed depends on the relaxation parameter p.

We also emphasize that in our results we obtain the uniform interior $W^{1,q}$ estimates and the local minimization property outside the singular set S_* for u_*, in contrast to a minimization result restricted to balls by Canevari, and we prove that locally inside Ω, S_* solves a homotopical Plateau problem in codimension 2.

At a more technical point, we could identify the stationary varifold structure of the limit of the stress-energy tensors directly through Arroyo-Rabasa, De Philippis, Hirsch and Rindler's work on rectifiability of measures satisfying some differential constraints [6].

It is also worth noting that the smaller the fractional seminorm of the boundary datum g, the closer to $\partial\Omega$ the singular set S_* of Theorem 1.1 is located (see Proposition 7.1).

Under additional assumptions we get global properties of the limit.

Theorem 1.2. *With the assumptions and notations of Theorem 1.1 if we assume moreover that Ω is of class C^2, $\overline{\Omega}$ is strongly convex at every point of $\partial\Omega$, g is a C^1 map outside a finite set of points $\{a_1, \ldots, a_m\} \subset \partial\Omega$ around each of which g is homotopically nontrivial and whose Euclidean norm of the gradient behaves like the distance to the power of -1 close to these points and if for each $n \in \mathbb{N}$, Du_n is continuous on $\partial\Omega \setminus \{a_1, \ldots, a_m\}$, then the following assertions hold.*

(i) $S_* \cap \partial\Omega = \{a_1, \ldots, a_m\}$ *and* S_* *lies in the convex hull of* $\{a_1, \ldots, a_m\}$.
(ii) The set J *is finite and* $S_* = \bigcup_{j \in J} L_j$.
(iii) The quantity

$$\sum_{j \in J} \mathcal{E}_2^{\mathrm{sg}}(\gamma_j)\mathcal{H}^1(L_j) \quad \textit{minimizes} \quad \sum_{i \in I} \mathcal{E}_2^{\mathrm{sg}}(f_i)\mathcal{H}^1(K_i)$$

among all configurations of closed straight line segments $(K_i)_{i \in I}$ *and topological charges* $(f_i)_{i \in I} \subset C(\mathbb{S}^1, \mathcal{N})$ *satisfying the following conditions:* I *is finite; each* K_i *has a positive length; the interiors of the segments (i.e., the segments without their endpoints) are pairwise disjoint;* $K = \bigcup_{i \in I} K_i$ *is a compact subset of* $\overline{\Omega}$ *and* $K \cap \partial\Omega$ *is a finite set of points;* K *has no endpoints in* Ω*, namely, if* K_i *is a segment having an endpoint* $x \in \Omega$*, then there exists another segment* K_j *emanating from* x*; there exists a map* $v \in C(\Omega \backslash (K \cup F), \mathcal{N})$*, with* F *being a locally finite subset of* $\Omega \backslash K$*, such that* $\operatorname{tr}_{\partial\Omega}(v) = g$ *and* v *restricted to small circles transversal to* K_i *is, up to orientation, homotopic to* f_i*. Namely,* S_* *solves a homotopical Plateau problem in codimension 2.*

The additional ingredient in the proof of Theorem 1.2 is the exploitation of the Pohozaev-type identity combined with White's maximum principle for minimizing varifolds [57].

Although our results and proofs are written in the framework of domains of the Euclidean space $\mathbb{R}^3$, most of the analysis can be transferred to compact 3-dimensional Riemannian manifolds. The adaptation of Theorem 1.1 should be completely straightforward, provided line segments are replaced by geodesics. Transferring Theorem 1.2 to manifolds would require a suitable use of the strong convexity of the boundary of submanifolds.

Finally, we point out that if $\overline{\Omega}$ is strongly convex at every point of $\partial\Omega$ and $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$, then the singular set S_* is a minimal connection (see Definition 7.24 and Proposition 7.25). Thus, in this case, we exclude the appearance of closed loops (i.e., homeomorphic images of $\mathbb{S}^1$) in S_*. In fact, the singular sets of minimizing p_n-harmonic maps u_n could have disclination loops [25, Section 2.4], however, in the limit, as $n \to +\infty$, each of these loops, which is contractible in Ω, should become unstable, shrink and disappear. We also suspect that the singular sets of the corresponding minimizing p_n-harmonic maps converge, up to a subsequence, to the singular set S_* in the Hausdorff distance, but this question is beyond the scope of our paper.

2. Preliminaries

2.1. Conventions and Notation

Conventions: in this paper, we say that a value is positive if it is strictly greater than zero, and a value is nonnegative if it is greater than or equal to zero. Euclidean spaces are endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$. A set will be called a domain whenever it is open and connected. Throughout this paper, Ω, unless otherwise stated, will denote a bounded (locally) Lipschitz domain in $\mathbb{R}^3$, and $\mathcal{N}$ will denote a finite-dimensional compact connected smooth Riemannian manifold without boundary and with finite fundamental group, which will be assumed, thanks to Nash's embedding theorem, to be isometrically embedded into the Euclidean space $\mathbb{R}^\nu$ for some $\nu \in \mathbb{N} \setminus \{0\}$. The Hausdorff measures, which we shall use, coincide in terms of normalization with the appropriate outer Lebesgue measures. We shall say that a property holds μ-*almost everywhere* (or simply *almost everywhere*) for a measure μ (abbreviated μ-a.e.) if it holds outside a set of zero μ-measure.

Notation: if A, B, M are real matrices, we denote the inner product (also known as Frobenius or Hilbert-Schmidt) between A and B by $A : B = \operatorname{tr}(A^{\mathrm{T}}B)$ and the Euclidean norm of M by $|M| = \sqrt{M : M}$. We denote by $B^l_r(x)$, $\overline{B}^l_r(x)$, and $\partial B^l_r(x)$, respectively, the open ball in $\mathbb{R}^l$, the closed ball in $\mathbb{R}^l$, and its boundary the $(l-1)$-sphere with center x and radius r, where $l \in \mathbb{N}\setminus\{0\}$. If the center is at the origin 0, we write B^l_r, $\overline{B}^l_r$ and ∂B^l_r the corresponding balls and the $(l-1)$-sphere. We shall sometimes write $\mathbb{S}^{m-1}$ instead of ∂B^m_1. We denote by $\operatorname{dist}(x, A)$ and $|A|$, respectively, the Euclidean distance from $x \in \mathbb{R}^l$ to $A \subset \mathbb{R}^l$, and the l-dimensional Lebesgue measure of A. We shall denote by $\mathcal{H}^l(A)$ the l-dimensional Hausdorff measure of A. If $U \subset \mathbb{R}^l$ is Lebesgue measurable, then for $p \in [1, +\infty) \cup \{+\infty\}$, $L^p(U)$ will denote the space consisting of all real measurable functions on U that are p^{th}-power integrable on U if $p \in [1, +\infty)$ and are essentially bounded if $p = +\infty$; $L^p(U; \mathbb{R}^k)$ is the respective space of functions with values in $\mathbb{R}^k$. If $p \in [1, +\infty)$ and $E \subset \mathbb{R}^m$ is a Borel-measurable set of Hausdorff dimension $l \leqq m$, then $L^p(E, \mathcal{H}^l)$ is the space of real measurable functions on E that are p^{th}-power integrable on E with respect to the $\mathcal{H}^l$-measure; $L^p(E, \mathbb{R}^k, \mathcal{H}^l)$ is the respective space of functions with values in $\mathbb{R}^k$. By $L^1_{\text{loc}}(U)$ we denote the space of functions u such that $u \in L^1(V)$ for all $V \Subset U$; if $E \subset \mathbb{R}^m$ is a Borel-measurable set of Hausdorff dimension $l \leqq m$, $L^1_{\text{loc}}(E, \mathbb{R}^k, \mathcal{H}^l)$ denote the space of measurable functions on E with values in $\mathbb{R}^k$ such that $u \in L^1(F, \mathbb{R}^k, \mathcal{H}^l)$ for all $F \Subset E$. The norm on $L^p(U)$ ($L^p(U; \mathbb{R}^k)$) is denoted by $\|\cdot\|_{L^p(U)}$ ($\|\cdot\|_{L^p(U;\mathbb{R}^k)}$) or $\|\cdot\|_p$ when appropriate. We use the standard notation for Sobolev spaces. If $L : \mathbb{R}^m \to \mathbb{R}^l$ is a linear map, $L[h] \in \mathbb{R}^l$ will denote the value of L at the vector $h \in \mathbb{R}^m$. For each $x \in \mathcal{N}$ and for each $r > 0$, we shall denote by $B^{\mathcal{N}}_r(x)$ the geodesic ball in $\mathcal{N}$ with center x and radius r, namely $B^{\mathcal{N}}_r(x) = \{y \in \mathcal{N} : d_{\mathcal{N}}(y, x) \leqq r\}$, where $d_{\mathcal{N}} : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ is the geodesic distance in $\mathcal{N}$. If $U \subset \mathbb{R}^l$ is a bounded Lipschitz domain, then $\operatorname{tr}_{\partial U}$ will denote the trace operator (the reader may consult, for instance, [28, Section 4.3]). We denote by $|\cdot|_{W^{s,p}(E,\mathbb{R}^k)}$ (or $|\cdot|_{W^{s,p}(E)}$ when appropriate) the canonical seminorm on $W^{s,p}(E, \mathbb{R}^k)$: if E is a Borel-measurable set of Hausdorff dimension l, then $|u|^p_{W^{s,p}(E,\mathbb{R}^k)} = \int_E \int_E \frac{|u(x)-u(y)|^p}{|x-y|^{l+sp}} \, \mathrm{d}\mathcal{H}^l(x) \, \mathrm{d}\mathcal{H}^l(y)$. By $\#X$ we shall denote the cardinality of X.

2.2. Embedding and Nearest Point Retraction

For each $\lambda \in (0, +\infty)$, we define the neighborhood

$$\mathcal{N}_\lambda = \{x \in \mathbb{R}^\nu : \operatorname{dist}(x, \mathcal{N}) < \lambda\}.$$

The next lemma describes the nearest point retraction of a neighborhood of $\mathcal{N}$ onto $\mathcal{N}$.

Lemma 2.1. *There exists* $\lambda_{\mathcal{N}} > 0$ *such that the nearest point retraction* $\Pi_{\mathcal{N}} : \mathcal{N}_{\lambda_{\mathcal{N}}} \to \mathcal{N}$ *characterized by*

$$|x - \Pi_{\mathcal{N}}(x)| = \operatorname{dist}(x, \mathcal{N})$$

is a well-defined and smooth map. Moreover, if the mappings $P^{\top}_{\mathcal{N}} : \mathcal{N} \to \operatorname{Lin}(\mathbb{R}^{\nu}, \mathbb{R}^{\nu})$ *and* $P^{\perp}_{\mathcal{N}} : \mathcal{N} \to \operatorname{Lin}(\mathbb{R}^{\nu}, \mathbb{R}^{\nu})$ *are defined for each* $x \in \mathcal{N}$ *by setting* $P^{\top}_{\mathcal{N}}(x)$ *and* $P^{\perp}_{\mathcal{N}}(x)$ *as the orthogonal projections on* $T_x\mathcal{N}$ *and* $(T_x\mathcal{N})^{\perp}$*, identified as linear subspaces of* $\mathbb{R}^{\nu}$*, then for each* $x \in \mathcal{N}_{\lambda_{\mathcal{N}}}$ *and* $v \in \mathbb{R}^{\nu}$*,*

$$\left(1 - \frac{\operatorname{dist}(x, \mathcal{N})}{\lambda_{\mathcal{N}}}\right) |D\Pi_{\mathcal{N}}(x)[v]|^2 \leqq |P^{\top}_{\mathcal{N}}(\Pi_{\mathcal{N}}(x))[v]|^2 \leqq C|D\Pi_{\mathcal{N}}(x)[v]|^2,$$

where $C = C(\mathcal{N}, \nu) > 0$*. Furthermore, for each* $x \in \mathcal{N}_{\lambda_{\mathcal{N}}} \setminus \mathcal{N}$*,*

$$|D\operatorname{dist}(x, \mathcal{N})| = 1.$$

Proof. For a proof, we refer to [45, Lemma 2.1]. □

2.3. Sobolev Mappings into Manifolds

Let $\mathcal{X}$ and $\mathcal{Y}$ be compact Riemannian manifolds. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are isometrically embedded into Euclidean spaces, which is always possible by Nash's isometric embedding theorem [49]. Notice that these embeddings are bilipschitz. In fact, if $\mathcal{M}$ is a compact Riemannian manifold, which is isometrically embedded into a Euclidean space, then, clearly, the Euclidean distance between two points of $\mathcal{M}$ is less than or equal to the geodesic distance between these points, which is induced by the Riemannian metric on $\mathcal{M}$. On the other hand, there exists a constant $L = L(\mathcal{M}) > 0$ such that the geodesic distance on $\mathcal{M}$ is less than or equal to the Euclidean distance on $\mathcal{M}$ times L (the idea behind the proof is that each compact Riemannian manifold $\mathcal{M}$ has a positive normal injectivity radius, thus, one can construct a smooth nearest point retraction from a tubular neighborhood of $\mathcal{M}$ onto $\mathcal{M}$ (see Lemma 2.1) and appropriately use the fact that for each open cover of $\mathcal{M}$ there is a finite subcover). Thus, identifying $\mathcal{X}$ and $\mathcal{Y}$ to their embedding's images, for each $s \in (0, 1)$ and $p \in [1, +\infty)$, we can define the fractional Sobolev space $W^{s,p}(\mathcal{X}, \mathcal{Y})$ as

$$\begin{aligned} W^{s,p}(\mathcal{X}, \mathcal{Y}) = \{&u \in L^p(\mathcal{X}, \mathbb{R}^k, \mathcal{H}^l) : u(x) \in \mathcal{Y} \text{ for a.e. } x \in \mathcal{X} \\ &\text{and } |u|_{W^{s,p}(\mathcal{X}, \mathbb{R}^k)} < +\infty\}, \end{aligned}$$

where $\mathbb{R}^k$ is the Euclidean space into which $\mathcal{Y}$ is embedded and $l \in \mathbb{N}\setminus\{0\}$ is the Hausdorff dimension of $\mathcal{X}$ (the reader may also consult [23, Section 3.1]). This definition does not depend on the choice of the isometric Euclidean embedding of $\mathcal{Y}$. We shall consider in $W^{1,p}(\mathcal{X}, \mathbb{R}^k)$ its strongly closed subset

$$W^{1,p}(\mathcal{X}, \mathcal{Y}) = \{u \in W^{1,p}(\mathcal{X}, \mathbb{R}^k) : u(x) \in \mathcal{Y} \text{ for a.e. } x \in \mathcal{X}\}.$$

This definition does not depend on the choice of the isometric Euclidean embedding of $\mathcal{Y}$ (see, for instance, [15, Proposition 2.1] and [38, Section 3.1]; for intrinsic definitions of Sobolev spaces for Riemannian manifolds, see [38, Section 3.2]).

For convenience, we recall the definition of weak differentiability of mappings defined on bilipschitz images of open sets.

Definition 2.2. Let $U \subset \mathbb{R}^l$ be open and $\Phi : E \to U$ be bilipschitz. A map $u \in L^1_{\text{loc}}(E, \mathbb{R}^k, \mathcal{H}^l)$ is said to be weakly differentiable if the map $u \circ \Phi^{-1}$ is weakly differentiable.

Since the weak differentiability is preserved under bilipschitz maps (see [58, Theorem 2.2.2]), the above definition is independent of the bilipschitz parametrization of E. Taking into account that E is bilipschitz homeomorphic to an open set $U \subset \mathbb{R}^l$, through a bilipschitz homeomorphism $\Phi : E \to U$, E admits the approximate tangent plane $T_x E$ for $\mathcal{H}^l$-a.e. $x \in E$ (for the definition of the approximate tangent plane, see, for example, [30, Section 3.2.16] and [29, Definition 4.3]). Thus, we can define the notion of the tangential derivative of Φ for $\mathcal{H}^l$-a.e. $x \in E$, as the linear map from the tangent plane $T_x E$ to $\mathbb{R}^l$. If $u \in L^1_{\text{loc}}(E, \mathbb{R}^k, \mathcal{H}^l)$ is weakly differentiable, the weak (tangential) derivative of u at $\mathcal{H}^l$-a.e. $x \in E$ is defined by $D_\top u(x) = D(u \circ \Phi^{-1})(\Phi(x)) \circ D_\top \Phi(x)$, where $D_\top \Phi(x)$ is the tangential derivative of Φ at x (which is defined for $\mathcal{H}^l$-a.e. $x \in E$).

We shall denote by $W^{1,1}_{\text{loc}}(E, \mathbb{R}^k)$ the space of all weakly differentiable mappings on E with values in $\mathbb{R}^k$. If $p \in [1, +\infty)$ and E is a bilipschitz image of an open set in $\mathbb{R}^l$, then we define

$$W^{1,p}(E, \mathbb{R}^k) = \{u \in W^{1,1}_{\text{loc}}(E, \mathbb{R}^k) : |u|, |D_\top u| \in L^p(E, \mathcal{H}^l)\}.$$

Next, we define Sobolev spaces of mappings on bilipschitz images of open sets into a compact Riemannian manifold.

Definition 2.3. Let $p \in [1, +\infty)$ and E be a bilipschitz image of an open set in $\mathbb{R}^l$ with $l \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{Y}$ be a compact Riemannian manifold isometrically embedded into $\mathbb{R}^k$. Then we define the Sobolev space $W^{1,p}(E, \mathcal{Y})$ by

$$W^{1,p}(E, \mathcal{Y}) = \{u \in W^{1,p}(E, \mathbb{R}^k) : u(x) \in \mathcal{Y} \text{ for a.e. } x \in E\}.$$

2.4. Topological Resolution of the Boundary Datum

Since $\mathcal{N}$ is a connected Riemannian manifold, it is path-connected and its fundamental group, namely $\pi_1(\mathcal{N}, x_0)$, where $x_0 \in \mathcal{N}$, is, up to isomorphism, independent of the choice of a basepoint x_0 (see [37, Proposition 1.5]) and denoted by $\pi_1(\mathcal{N})$. Moreover, there is a one-to-one correspondence between the set of conjugacy classes in $\pi_1(\mathcal{N})$ and the set $[\mathbb{S}^1, \mathcal{N}]$ of free homotopy classes of maps $\mathbb{S}^1 \to \mathcal{N}$, that is, homotopy class without any condition on some basepoints (see Exercise 6 in [37, Section 1.1, p. 38]). In general, there is no group structure on $[\mathbb{S}^1, \mathcal{N}]$ (there exist groups G with the functor $L : G \to LG$ sending G to its set of conjugacy classes, which cannot be lifted to a group-valued functor; for instance, one can find an inclusion of groups $H \to G$ implying an inclusion of conjugacy classes $LH \to LG$ such that the order of LH does not divide the order of LG). After all, hereinafter, when we talk about "homotopy", we are talking about *free* homotopy (with no conditions on base points). We shall denote by $\#[\mathbb{S}^1, \mathcal{N}]$ the number of equivalence classes of $[\mathbb{S}^1, \mathcal{N}]$.

Next, for convenience, we recall the definition of a bounded Lipschitz domain.

Definition 2.4. A bounded domain $U \subset \mathbb{R}^l$, $l \geqq 2$ and its boundary ∂U are said to be (locally) Lipschitz if there exist a radius $r_{\partial U}$ and a constant $\delta > 0$ such that for every point $x \in \partial U$ and every radius $s \in (0, r_{\partial U})$, up to a rotation of coordinates, it holds

$$U \cap B^l_s(x) = \{(y', y_l) \in \mathbb{R}^{l-1} \times \mathbb{R} : y_l > \varphi(y')\} \cap B^l_s(x)$$

for some Lipschitz function $\varphi : \mathbb{R}^{l-1} \to \mathbb{R}$ satisfying $\|D\varphi\|_{L^\infty(\mathbb{R}^{l-1})} \leqq \delta$.

Given $k \in \mathbb{N}\backslash\{0\}$ and a bounded Lipschitz domain $U \subset \mathbb{R}^2$, we denote by $\mathrm{Conf}_k U$ the configuration space of k ordered points of U, namely

$$\mathrm{Conf}_k U = \{(a_1, \dots, a_k) \in U^k : a_i \neq a_j \text{ if } i \neq j\}. \tag{2.1}$$

Given $(a_1, \dots, a_k) \in \mathrm{Conf}_k U$, we define the quantity

$$\begin{aligned} \bar{\varrho}(a_1, \dots, a_k) &= \min\left\{\left\{\frac{|a_i - a_j|}{2} : i, j \in \{1, \dots, k\} \text{ and } i \neq j\right\}\right. \\ &\quad \left.\cup \left\{\mathrm{dist}(a_i, \partial U) : i \in \{1, \dots, k\}\right\}\right\}, \end{aligned} \tag{2.2}$$

so that if $\varrho \in (0, \bar{\varrho}(a_1, \dots, a_k))$, we have $\overline{B}^2_\varrho(a_i) \cap \overline{B}^2_\varrho(a_j) = \emptyset$ for each $i, j \in \{1, \dots, k\}$ such that $i \neq j$ and $\overline{B}^2_\varrho(a_i) \subset U$ for each $i \in \{1, \dots, k\}$, and hence the set $U \backslash \bigcup_{i=1}^k \overline{B}^2_\varrho(a_i)$ is connected and has a Lipschitz boundary

$$\partial(U \backslash \bigcup_{i=1}^k \overline{B}^2_\varrho(a_i)) = \partial U \cup \bigcup_{i=1}^k \partial B^2_\varrho(a_i).$$

Definition 2.5. We say that $(\gamma_1, \dots, \gamma_k) \in C(\mathbb{S}^1, \mathcal{N})^k$ is a *topological resolution* of $g \in C(\partial U, \mathcal{N})$ whenever there exist $(a_1, \dots, a_k) \in \mathrm{Conf}_k U, \varrho \in (0, \bar{\varrho}(a_1, \dots, a_k))$ and $u \in C(U \backslash \bigcup_{i=1}^k \overline{B}^2_\varrho(a_i), \mathcal{N})$ such that $u|_{\partial U} = g$ and for each $i \in \{1, \dots, k\}$, $u(a_i + \varrho\cdot)|_{\mathbb{S}^1} = \gamma_i$.

The definition of topological resolution is independent of the order of the curves in the k-tuple $(\gamma_1, \dots, \gamma_k)$. Furthermore, if $(b_1, \dots, b_k) \in \mathrm{Conf}_k U$ and $r \in (0, \bar{\varrho}(b_1, \dots, b_k))$, then there exists a homeomorphism between $U \backslash \bigcup_{i=1}^k \overline{B}^2_\varrho(a_i)$ and $U \backslash \bigcup_{i=1}^k \overline{B}^2_r(b_i)$, which shows that the statement of the previous definition is independent of the choice of the points and of the radius. Similarly, if for each $i \in \{1, \dots, k\}$ the map $\widetilde{\gamma}_i$ is freely homotopic to γ_i and if $\widetilde{g}$ is freely homotopic to g, then, by definition of homotopy and the fact that an annulus is homeomorphic to a finite cylinder, we have that $(\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_k)$ is also a topological resolution of $\widetilde{g}$. Thus, the property of being a topological resolution is invariant under homotopies.

Definition 2.5 can be extended to the case when $g : \partial U \to \mathcal{N}$ is a map of vanishing mean oscillation, namely $g \in \mathrm{VMO}(\partial U, \mathcal{N})$ and $(\gamma_1, \dots, \gamma_k) \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})^k$ (see [18,19]). Indeed, if $\Gamma \simeq \mathbb{S}^1$ is a closed curve, path-connected components of the space $\mathrm{VMO}(\Gamma, \mathcal{N})$ are known to be the closure in $\mathrm{VMO}(\Gamma, \mathcal{N})$ of path-connected components of $C(\Gamma, \mathcal{N})$ (see [18, Lemma A.23]). Since $W^{1/2,2}(\partial U, \mathcal{N}) \subset \mathrm{VMO}(\partial U, \mathcal{N})$, Definition 2.5 extends also to the case when $g \in W^{1/2,2}(\partial U, \mathcal{N})$.

Definition 2.6. We say that $(\gamma_1, \dots, \gamma_k) \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})^k$ is a *topological resolution* of a map $g \in \mathrm{VMO}(\partial U, \mathcal{N})$ whenever $(\gamma_1, \dots, \gamma_k)$ is homotopic in $\mathrm{VMO}(\mathbb{S}^1, \mathcal{N})^k$ to a *topological resolution* $(\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_k)$ of $\widetilde{g} \in C(\partial U, \mathcal{N})$ which is homotopic to g in $\mathrm{VMO}(\partial U, \mathcal{N})$.

Since Definition 2.5 is invariant under homotopies, and continuous maps are homotopic in VMO if and only if they are homotopic through a continuous homotopy, Definition 2.6 generalizes Definition 2.5.

2.5. *Singular p-Energy*

First, we define the minimal length in the homotopy class of a map $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ by

$$\lambda(\gamma) = \inf \left\{ \int_{\mathbb{S}^1} |\widetilde{\gamma}'| \,\mathrm{d}\mathcal{H}^1 : \widetilde{\gamma} \in C^1(\mathbb{S}^1, \mathcal{N}) \text{ and } \gamma \text{ are homotopic in } \mathrm{VMO}(\mathbb{S}^1, \mathcal{N}) \right\}, \tag{2.3}$$

where $|\cdot|$ is the norm induced by the Riemannian metric $g_{\mathcal{N}}$ of the manifold $\mathcal{N}$ on each fiber of the tangent bundle $T\mathcal{N}$, and since $\mathcal{N}$ is isometrically embedded into $\mathbb{R}^\nu$, $|\cdot|$ is the Euclidean norm.

For each $p \in [1, +\infty)$, using the parametrization by arc length and Hölder's inequality if $p \in (1, +\infty)$, we have

$$\inf \left\{ \int_{\mathbb{S}^1} \frac{|\widetilde{\gamma}'|^p}{p} \,\mathrm{d}\mathcal{H}^1 : \widetilde{\gamma} \in C^1(\mathbb{S}^1, \mathcal{N}) \text{ and } \gamma \text{ are homotopic in } \mathrm{VMO}(\mathbb{S}^1, \mathcal{N}) \right\} = \frac{\lambda(\gamma)^p}{(2\pi)^{p-1} p}.$$

In particular, if γ is a minimizing closed geodesic, then

$$\lambda(\gamma) = \int_{\mathbb{S}^1} |\gamma'| \,\mathrm{d}\mathcal{H}^1 = \left((2\pi)^{p-1} p \int_{\mathbb{S}^1} \frac{|\gamma'|^p}{p} \,\mathrm{d}\mathcal{H}^1 \right)^{\frac{1}{p}} = 2\pi \|\gamma'\|_\infty,$$

since $|\gamma'|$ is constant for a minimizing geodesic.

Recall that the *systole* of the compact manifold $\mathcal{N}$ is the length of the shortest closed nontrivial geodesic on $\mathcal{N}$, namely,

$$\mathrm{sys}(\mathcal{N}) = \inf\{\lambda(\gamma) : \gamma \in C^1(\mathbb{S}^1, \mathcal{N}) \text{ is not homotopic to a constant}\}. \tag{2.4}$$

If $[\mathbb{S}^1, \mathcal{N}] \simeq \{0\}$, we set $\mathrm{sys}(\mathcal{N}) = +\infty$. Notice that, if $[\mathbb{S}^1, \mathcal{N}] \not\simeq \{0\}$, then for each map $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$, it holds $\lambda(\gamma) \in \{0\} \cup [\mathrm{sys}(\mathcal{N}), +\infty)$.

We now define the singular p-energy of a map $g \in \mathrm{VMO}(\partial U, \mathcal{N})$, where $U \subset \mathbb{R}^2$ is a bounded Lipschitz domain. This notion was previously introduced for $p = 2$ in [46] and generalized in [56] to $p \in [1, +\infty)$.

Definition 2.7. If $p \in [1, +\infty)$, $U \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $g \in \mathrm{VMO}(\partial U, \mathcal{N})$, we define the *singular p-energy* of g by

$$\mathcal{E}_p^{\mathrm{sg}}(g) = \inf \Bigg\{ \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^p}{(2\pi)^{p-1} p} : k \in \mathbb{N} \setminus \{0\} \\ \text{and } (\gamma_1, \ldots, \gamma_k) \text{ is a topological resolution of } g \Bigg\}. \tag{2.5}$$

Observe that $\mathcal{E}_p^{\mathrm{sg}}$ is invariant under homotopies and depends continuously on p. For each $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$, $\mathcal{E}_p^{\mathrm{sg}}(g) = 2\pi |\deg(g)|^p / p$, where $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$. If $\mathcal{N} = \mathbb{S}^2$, then $\pi_1(\mathcal{N})$ is trivial and for each $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^2)$, $\mathcal{E}_p^{\mathrm{sg}}(g) = 0$. If $\mathcal{N} = \{n \otimes n - \frac{1}{3}\mathrm{Id} : n \in \mathbb{S}^2\}$ and $g(t) = v(t) \otimes v(t) - \frac{1}{3}\mathrm{Id}$, where $v(t) = (\cos(t/2), \sin(t/2), 0) \in \mathbb{S}^2$ for each $t \in [0, 2\pi]$, then $\mathcal{N}$ is diffeomorphic to $\mathbb{RP}^2$, $g \in C^\infty(\mathbb{S}^1, \mathcal{N})$ and $\mathcal{E}_p^{\mathrm{sg}}(g) = 2^{\frac{2-p}{2}} \pi / p$ (see [21, Lemma 19]). The reader may consult [46, Section 9.3] for other interesting examples in the case where $p = 2$.

Lemma 2.8. *Let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then for each $g \in \mathrm{VMO}(\partial U, \mathcal{N})$, the function $p \in [1, +\infty) \mapsto \mathcal{E}_p^{\mathrm{sg}}(g)$ is locally bounded and locally Lipschitz continuous. In particular, if $\mathcal{N}$ is simply connected, then $\mathcal{E}_p^{\mathrm{sg}}(g) = 0$ for each $p \in [1, +\infty)$. Furthermore, the infimum in (2.5) is actually a minimum and the set $\{\mathcal{E}_p^{\mathrm{sg}}(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\}$ is finite.*

Proof. If $\mathcal{N}$ is simply connected, then each $\gamma \in \mathrm{VMO}(\partial U, \mathcal{N})$ is homotopic to a constant map, which yields that $\lambda(\gamma) = 0$ and hence $\mathcal{E}_p^{\mathrm{sg}}(g) = 0$.

Otherwise, $[\mathbb{S}^1, \mathcal{N}] \neq \{0\}$ and then $\mathrm{sys}(\mathcal{N}) \in (0, +\infty)$. This comes, since $\mathcal{N}$ is compact, and hence the length spectrum (i.e., the set of nonnegative real numbers $\{\lambda(\gamma) : \gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})\}$) is discrete (for a proof, see [45, Proposition 3.2]). Then, according to [56, Lemma 2.10], $p \mapsto \mathcal{E}_p^{\mathrm{sg}}(g)$ is locally bounded and locally Lipschitz continuous. The fact that the infimum in (2.5) is a minimum comes from the discreteness of the length spectrum. Finally, since $\mathcal{E}_p^{\mathrm{sg}}$ is invariant under homotopies and $\#[\mathbb{S}^1, \mathcal{N}] < +\infty$ by our assumption on $\mathcal{N}$, the set $\{\mathcal{E}_p^{\mathrm{sg}}(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\}$ is finite. This completes our proof of Lemma 2.8. □

It is also worth recalling the notion of a *minimal topological resolution* (when $p = 2$, see [46, Definition 2.7]).

Definition 2.9. A topological resolution $(\gamma_1, \ldots, \gamma_k)$ of g is said to be p-minimal if

$$\mathcal{E}_p^{\mathrm{sg}}(g) = \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^p}{(2\pi)^{p-1} p}$$

and $\lambda(\gamma_i) > 0$ for each $i \in \{1, \ldots, k\}$.

2.6. Renormalized Energy of Renormalizable Maps

We recall the definition of a renormalizable map (see [46, Definition 7.1]).

Definition 2.10. Given an open set $U \subset \mathbb{R}^2$, a map $u : U \to \mathcal{N}$ is said to be *renormalizable* whenever there exists a finite set $\mathcal{S} \subset U$ such that for each sufficiently small radius $\varrho > 0$, $u \in W^{1,2}(U\setminus \bigcup_{a\in\mathcal{S}} \overline{B}^2_\varrho(a), \mathcal{N})$ and its *renormalized* energy $\mathcal{E}^{\mathrm{ren}}_2(u)$ is finite, where

$$\mathcal{E}^{\mathrm{ren}}_2(u) = \liminf_{\varrho\to 0+} \int_{U\setminus\bigcup_{a\in\mathcal{S}}\overline{B}^2_\varrho(a)} \frac{|Du|^2}{2}\,\mathrm{d}x - \sum_{a\in\mathcal{S}} \frac{\lambda(\mathrm{tr}_{\partial B^2_\varrho(a)}(u))^2}{4\pi} \ln\left(\frac{1}{\varrho}\right). \tag{2.6}$$

We denote by $W^{1,2}_{\mathrm{ren}}(U, \mathcal{N})$ the set of renormalizable maps.

For each renormalizable map $u : U \to \mathcal{N}$, there exists the *minimal* set $\mathcal{S}_0$ for which it holds $u \in W^{1,2}(U\setminus \bigcup_{a\in\mathcal{S}_0} \overline{B}^2_\varrho(a), \mathcal{N})$ for each small enough $\varrho > 0$. If $a \in U\setminus\mathcal{S}_0$, then $\lambda(\mathrm{tr}_{\partial B^2_\varrho(a)}(u)) = 0$ for $\varrho > 0$ small enough. Thus, the lim inf defining $\mathcal{E}^{\mathrm{ren}}_2$ in (2.6) does not depend on the set $\mathcal{S}$. If $\mathcal{S}_0 = \emptyset$, then $u \in W^{1,2}(U, \mathcal{N})$ and $\mathcal{E}^{\mathrm{ren}}_2(u) = \int_U \frac{|Du|^2}{2}\,\mathrm{d}x$. If $\mathcal{S}_0 = \{a_1, \dots, a_k\}$ with $k \in \mathbb{N}\setminus\{0\}$ and $(a_1, \dots, a_k) \in \mathrm{Conf}_k(U)$ (see (2.1)), then, according to [46, Lemma 2.11], the map

$$\varrho \in (0, \bar\varrho(a_1, \dots, a_k)) \mapsto \int_{U\setminus\bigcup_{i=1}^k\overline{B}^2_\varrho(a_i)} \frac{|Du|^2}{2}\,\mathrm{d}x - \sum_{i=1}^k \frac{\lambda(\mathrm{tr}_{\partial B^2_\varrho(a_i)}(u))^2}{4\pi} \ln\left(\frac{1}{\varrho}\right)$$

is nonincreasing, where $\bar\varrho(a_1, \dots, a_k) \in (0, +\infty)$ is defined in (2.2). The main ingredients of the proof of [46, Lemma 2.11] are the use of the additivity of the integral and the estimate of the energy on the annulus. The latter comes from the application of the special case of the coarea formula and the comparison of the angular energy taking into account the definition of the minimal length in the homotopy class of a map $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ (see (2.3)).

Let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $u \in W^{1,2}_{\mathrm{ren}}(U, \mathcal{N})$ and $\mathcal{S}_0 = \{a_1, \dots, a_k\}$ for some $k \in \mathbb{N}\setminus\{0\}$, where $(a_1, \dots, a_k) \in \mathrm{Conf}_k U$. By [46, Theorem 5.1] (see also [56, Corollary 5.13]), $Du \in L^{2,\infty}(U, \mathbb{R}^\nu \otimes \mathbb{R}^2) \subset L^p(U, \mathbb{R}^\nu \otimes \mathbb{R}^2)$ for each $p \in [1, 2)$ (see, for instance, [24, Theorem 5.9]), where $L^{2,\infty}(U, \mathbb{R}^\nu \otimes \mathbb{R}^2)$ denotes the Marcinkiewicz space or endpoint Lorentz space. In addition, we can define the p-renormalized energy of u by

$$\mathcal{E}^{\mathrm{ren}}_p(u) = \int_U \frac{|Du|^p}{p}\,\mathrm{d}x - \sum_{i=1}^k \frac{\lambda(\mathrm{tr}_{\partial B^2_\varrho(a_i)}(u))^p}{(2-p)(2\pi)^{p-1}p}, \tag{2.7}$$

which is finite and tends to $\mathcal{E}^{\mathrm{ren}}_2(u)$ as $p \nearrow 2$.

Proposition 2.11. *Let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain. If $u \in W^{1,2}_{\mathrm{ren}}(U, \mathcal{N})$, then*

$$\mathcal{E}^{\mathrm{ren}}_p(u) \to \mathcal{E}^{\mathrm{ren}}_2(u) \tag{2.8}$$

as $p \nearrow 2$.

Proof. For a proof, see [56, Proposition 2.17]. □

Lemma 2.12. *Let $\gamma \in C^1(\mathbb{S}^1, \mathcal{N})$ be a minimizing geodesic in its (free) homotopy class. Then there exists a renormalizable map $v \in W^{1,2}_{\mathrm{ren}}(B_1^2, \mathcal{N})$ satisfying the following conditions.*

(i) $\mathrm{tr}_{\mathbb{S}^1}(v) = \gamma$. *If γ is a constant map, then v is the constant map.*
(ii) *For each $p \in [1, 2)$, the following estimate holds*

$$\int_{B_1^2} \frac{|Dv|^p}{p}\,\mathrm{d}x = \mathcal{E}_p^{\mathrm{ren}}(v) + \frac{\mathcal{E}_p^{\mathrm{sg}}(\gamma)}{2-p}.$$

Proof. If γ is a constant map, then we define v to be the constant map taking the same value in $\mathcal{N}$ as γ. Clearly, in this case, the assertion (ii) holds.

If γ is not homotopic to a constant, we apply [46, Proposition 8.3]. A key step for the proof of [46, Proposition 8.3] is to use the estimate of [46, Lemma 2.11], which is based on the comparison of the angular energy. In particular, let $(\gamma_1, \ldots, \gamma_k)$ be a 2-minimal topological resolution of γ (see Definition 2.9) such that for each $j \in \{1, \ldots, k\}$, $\gamma_j \in C^1(\mathbb{S}^1, \mathcal{N})$ is a minimizing geodesic. Then, using [46, Proposition 8.3], together with (2.7), we deduce that there exists $v \in W^{1,2}_{\mathrm{ren}}(B_1^2, \mathcal{N})$ satisfying

$$\mathcal{E}_p^{\mathrm{ren}}(v) = \int_{B_1^2} \frac{|Dv|^p}{p}\,\mathrm{d}x - \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^p}{(2-p)(2\pi)^{p-1}p} = \int_{B_1^2} \frac{|Dv|^p}{p}\,\mathrm{d}x - \frac{\mathcal{E}_p^{\mathrm{sg}}(\gamma)}{2-p},$$

which completes our proof of Lemma 2.12. □

2.7. Several Extension Results

We obtain an extension of the boundary datum with a *linear*-type estimate.

Lemma 2.13. *Let $m \in \mathbb{N}\backslash\{0, 1\}$, $p \in [1, 2]$, $\widetilde{g} \in W^{1,p}(\partial B_r^m(x_0), \widetilde{\mathcal{N}})$ and $g = \pi \circ \widetilde{g}$, where $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the universal covering of $\mathcal{N}$. Then there exists a map $w \in W^{1,p}(B_r^m(x_0), \mathcal{N})$ such that $\mathrm{tr}_{\partial B_r^m(x_0)}(w) = g$ and*

$$\|Dw\|^p_{L^p(B_r^m(x_0))} \leqq Cr\|D_\top g\|^p_{L^p(\partial B_r^m(x_0))}, \tag{2.9}$$

where $C = C(m, \mathcal{N}) > 0$. Moreover, $w \in W^{1,2}(B_r^m(x_0), \mathcal{N})$.

Remark 2.14. If $m \geqq 3$, then w can be chosen as the homogeneous extension of g. In this case, (2.9) holds with the constant $C = \frac{1}{m-p} \leqq 1$ and without assuming that $\pi_1(\mathcal{N})$ is finite and g has a lifting. It is also worth mentioning that for each $p \in (1, 2)$, the estimate (2.9) holds without assuming that g has a lifting, but the constant C in (2.9) then behaves like $C \sim \frac{1}{2-p}$.

Proof of Lemma 2.13. Since the manifold $\mathcal{N}$ is compact and its fundamental group $\pi_1(\mathcal{N})$ is finite, its covering space $\widetilde{\mathcal{N}}$ is compact. Notice that $g \in W^{1,p}(\partial B_r^m(x_0), \mathcal{N})$, since $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a local isometry. According to Nash's embedding theorem [49], $\widetilde{\mathcal{N}}$ can be isometrically embedded into $\mathbb{R}^k$ for some $k \in \mathbb{N} \setminus \{0\}$. If $p = 1$, we define $w(x) := g(x_0 + r(x - x_0)/|x - x_0|)$ for $x \in B_r^m(x_0) \setminus \{x_0\}$ (see [52, Proposition 2.4]). Then $w \in W^{1,1}(B_r^m(x_0), \mathcal{N})$ and

$$\int_{B_r^m(x_0)} |Dw| \,\mathrm{d}x = \frac{r}{m-1} \int_{\partial B_r^m(x_0)} |D_\top g| \,\mathrm{d}\mathcal{H}^{m-1},$$

which proves (2.9) for $p = 1$. If $p \in (1, 2]$, according to [40, Section 6.9], $\widetilde{g}$ admits the extension $\widetilde{v} \in C^\infty(B_r^m(x_0), \mathbb{R}^k)$ satisfying the estimate

$$\|D\widetilde{v}\|^p_{L^p(B_r^m(x_0))} \leqq C|\widetilde{g}|^p_{W^{1-1/p,p}(\partial B_r^m(x_0))}, \tag{2.10}$$

where $C = C(m) > 0$. The construction of the extension $\widetilde{v}$ is standard: since $\partial B_r^m(x_0)$ is smooth, using a partition of unity, the problem reduces to constructing for a given trace $u \in W^{1-1/p,p}(Q)$, where $Q = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < 1,\ i = 1, 2\}$, a real-valued function $U \in W^{1,p}(P)$, where $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_3 < 1,\ |x_i| < 1 - x_3\}$, by the convolution of u with a kernel φ satisfying [40, (2.5.1.1)]. Namely, first, extend u on $\mathbb{R}^2 \setminus Q$ by $u = 0$ and next define

$$U(x) = \frac{1}{x_3^2} \int_{|x'-y'|<x_3} \varphi\left(\frac{x'-y'}{x_3}\right) u(y') \,\mathrm{d}y',$$

where $x = (x', x_3) \in \mathbb{R}^3$, $x_3 > 0$. Then, proceeding as in [34, Theorem 6.2], namely, using Sard's Theorem and choosing an appropriate locally Lipschitz retraction [34, Lemma 6.1], we define $\widetilde{w} \in W^{1,2}(B_r^m(x_0), \mathcal{N})$ (generally speaking, the statement of [34, Theorem 6.2] provides us with a $W^{1,p}$ extension, however, using the fact that $\pi_1(\widetilde{\mathcal{N}})$ is trivial, one observes that the locally Lipschitz retraction coming from [34, Lemma 6.1] belongs to $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^k, \mathcal{N})$ for each $q \in [1, 3)$, which, since $\widetilde{g} \in W^{1/2,2}(\partial B_r^m(x_0), \mathcal{N})$, implies that the extension coming from [34, Theorem 6.2] is actually a $W^{1,2}$ extension in this particular case) such that $\mathrm{tr}_{\partial B_r^m(x_0)}(\widetilde{w}) = \widetilde{g}$ and, in view of (2.10),

$$\|D\widetilde{w}\|^p_{L^p(B_r^m(x_0))} \leqq C'|\widetilde{g}|^p_{W^{1-1/p,p}(\partial B_r^m(x_0))} \leqq C''r\|D_\top\widetilde{g}\|^p_{L^p(\partial B_r^m(x_0))}, \tag{2.11}$$

where $C' = C'(m, \widetilde{\mathcal{N}}) > 0$, $C'' = C''(m, \widetilde{\mathcal{N}}) > 0$ and the last estimate comes from (A.8). Defining $w := \pi \circ \widetilde{w}$ and using (2.11), together with the fact that π is a local isometry, we deduce (2.9) for $p \in (1, 2)$ (since the universal covering is unique up to a diffeomorphism, we can actually assume that C'' depends only on m and $\mathcal{N}$). This completes our proof of Lemma 2.13. □

The next lemma provides us with a *nonlinear*-type estimate on the extension of the boundary datum, in contrast to a *linear*-type estimate provided by Lemma 2.13.

Lemma 2.15. *Let $m \in \mathbb{N}\backslash\{0,1\}$, $p_0 \in (1,2)$, $p \in [p_0, 2)$, $\widetilde{g} \in W^{1,p}(\partial B_r^m(x_0), \widetilde{\mathcal{N}})$, $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the universal covering of $\mathcal{N}$ and $g = \pi \circ \widetilde{g}$. Then there exists a map $w \in W^{1,p}(B_r^m(x_0), \mathcal{N})$ such that $\mathrm{tr}_{\partial B_r^m(x_0)}(w) = g$ and*

$$\|Dw\|^p_{L^p(B_r^m(x_0))} \leqq Cr^{\frac{m-1}{p}} \|D_\top g\|^{p-1}_{L^p(\partial B_r^m(x_0))}, \tag{2.12}$$

where $C = C(p_0, m, \mathcal{N}) > 0$.

Remark 2.16. The estimate (2.12) is based on the fact that $\pi_1(\mathcal{N})$ is finite, since in this case $\widetilde{\mathcal{N}}$ is compact (due to the compactness of $\mathcal{N}$, the manifold $\widetilde{\mathcal{N}}$ is compact if and only if $\pi_1(\mathcal{N})$ is finite). More precisely, to obtain (2.12), we have used that the diameter of the manifold $\widetilde{\mathcal{N}}$ (which is identified with its Euclidean embedding) is finite. We have used the constant p_0 to avoid the dependence of the constant C in (2.12) on p when p is close to 1 (see (A.1)). Notice also that if p is close to 2, the estimate (2.12) is valid without assuming that g has a lifting, but the constant C in (2.12) then behaves like $C \sim \frac{1}{2-p}$.

Proof of Lemma 2.15. Let $\widetilde{w} \in W^{1,2}(B_r^m(x_0), \widetilde{\mathcal{N}})$ be the map as in the proof of Lemma 2.13. Using (2.10) and (A.9), we deduce the estimate

$$\|D\widetilde{w}\|^p_{L^p(B_r^m(x_0))} \leqq Cr^{\frac{m-1}{p}} \|D_\top \widetilde{g}\|^{p-1}_{L^p(\partial B_r^m(x_0))}, \tag{2.13}$$

where $C = C(p_0, m, \mathcal{N}) > 0$. Define $w = \pi \circ \widetilde{w}$. Since π is a local isometry, (2.13) implies (2.12), which concludes our proof of Lemma 2.15. □

For convenience, we recall the next proposition.

Lemma 2.17. *Let $m \in \mathbb{N}\backslash\{0,1\}$, $p \in [1, +\infty)$, $u \in W^{1,p}(B_r^m(x_0), \mathcal{Y})$ and $v \in W^{1,p}(\partial B_r^m(x_0), \mathcal{Y})$, where $\mathcal{Y}$ is either a Euclidean space or a compact Riemannian manifold which is isometrically embedded into $\mathbb{R}^k$ for some $k \in \mathbb{N} \setminus \{0\}$. Then the following assertions hold.*

(i) *For $\mathcal{H}^1$-a.e. $\varrho \in (0, r)$, $\mathrm{tr}_{\partial B_\varrho^m(x_0)}(u) = u|_{\partial B_\varrho^m(x_0)} \in W^{1,p}(\partial B_\varrho^m(x_0), \mathcal{Y})$.*
(ii) *Assume that $E \subset \partial B_r^m(x_0)$ is bilipschitz homeomorphic to an open bounded set in $\mathbb{R}^l$ with $l \leqq m-1$. Then for $\mathcal{H}^{m(m-1)/2}$-a.e. $\omega \in SO(m)$, $\mathrm{tr}_{\omega(E)}(v) = v|_{\omega(E)} \in W^{1,p}(\omega(E), \mathcal{Y})$.*

Proof. The proof follows using the appropriate coarea formula (see [28, Theorem 3.12] for the assertion (i) and [30, Theorem 3.2.48] for the assertion (ii)) and the fact that each map $u \in W^{1,p}(B_r^m(x_0), \mathbb{R}^k)$ and $v \in W^{1,p}(\partial B_r^m(x_0), \mathbb{R}^k)$ can be approximated by smooth maps in $C^\infty(B_r^m(x_0), \mathbb{R}^k)$ and $C^\infty(\partial B_r^m(x_0), \mathbb{R}^k)$, respectively. □

Lemma 2.18. *Let $p \in [1,2)$ and $g \in W^{1,p}(\partial B_r^2(x_0), \mathcal{N})$. Then there exists $u \in W^{1,p}(B_r^2(x_0), \mathcal{N})$ such that $\mathrm{tr}_{\partial B_r^2(x_0)}(u) = g$ and*

$$\int_{B_r^2(x_0)} \frac{|Du|^p}{p}\, \mathrm{d}x \leqq Cr \int_{\partial B_r^2(x_0)} \frac{|D_\top g|^p}{p}\, \mathrm{d}\mathcal{H}^1 + \frac{\mathcal{E}_p^{\mathrm{sg}}(g) r^{2-p}}{2-p}, \tag{2.14}$$

where $C = C(\mathcal{N}) > 0$.

Proof. Up to scaling and translation, we can assume that $r = 1$ and $x_0 = 0$. First, if g were homotopic to a constant map, then we would have $\mathcal{E}_p^{\mathrm{sg}}(g) = 0$ and, according to Lemma 2.13, (2.14) would hold.

Thus, we can assume that g is not homotopic to a constant. Let $\gamma \in C^1(\mathbb{S}^1, \mathcal{N})$ be a minimizing geodesic in the (free) homotopy class of g. Hereinafter in this proof, C will denote a positive constant that can only depend on $\mathcal{N}$ and can be different from line to line. Let $v \in W^{1,2}_{\mathrm{ren}}(B_1^2, \mathcal{N})$ be the map of Lemma 2.12 such that $\mathrm{tr}_{\mathbb{S}^1}(v) = \gamma$. Define $u(x) = v(2x)$ for each $x \in B^2_{1/2}$. Using Lemma 2.12 (ii) and the fact that g is homotopic to γ, we get

$$\int_{B^2_{1/2}} \frac{|Du|^p}{p}\,\mathrm{d}x = 2^{p-2}\int_{B_1^2} \frac{|Dv|^p}{p}\,\mathrm{d}x \leqq \mathcal{E}_p^{\mathrm{ren}}(v) + \frac{\mathcal{E}_p^{\mathrm{sg}}(g)}{2-p} \leqq C + \frac{\mathcal{E}_p^{\mathrm{sg}}(g)}{2-p}, \tag{2.15}$$

where the last estimate comes, since the value $\mathcal{E}_p^{\mathrm{ren}}(v)$ depends only on p and γ and tends to $\mathcal{E}_2^{\mathrm{ren}}(v)$ as $p \nearrow 2$ (see Proposition 2.11), and hence it is bounded by the constant depending only on $\mathcal{N}$, since there are finitely many free homotopy classes and we can fix a geodesic in each one.

Finally, we define u on the annulus $B_1^2 \backslash B^2_{1/2}$. Set $f(x) := \gamma(2x)$ for each $x \in \partial B^2_{1/2}$. In view of the Morrey embedding theorem, we can assume that $g \in C(\mathbb{S}^1, \mathcal{N})$. Then g is continuously homotopic to f. Restricting a continuous homotopy between the mappings g and f to the segment $[\sigma/2, \sigma] \subset \overline{B}_1^2 \backslash B^2_{1/2}$ for some $\sigma \in \mathbb{S}^1$, we obtain a map $h \in C([\sigma/2, \sigma], \mathcal{N})$ such that $h(\sigma/2) = f(\sigma/2)$ and $h(\sigma) = g(\sigma)$. Next, observe that $B_1^2 \setminus \overline{B}^2_{1/2}$ is bilipschitz homeomorphic to $(0,1)^2/\sim$, where $\sim$ is the equivalence relation identifying the sides $[(0,0),(0,1)]$ and $[(1,0),(1,1)]$. Thus, we obtain the map $w_0 \in C(\partial(0,1)^2, \mathcal{N})$ taking the values of g on $[(0,0),(1,0)]$, of f on $[(0,1),(1,1)]$ and of h on $[(0,0),(0,1)] \cup [(1,0),(1,1)]$. Then w_0 is continuously homotopic to a constant, and the classical theory of liftings says that there exists $\widetilde{w}_0 \in C(\partial(0,1)^2, \widetilde{\mathcal{N}})$ such that $w_0 = \pi \circ \widetilde{w}_0$, where $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the universal covering map. Let $\widetilde{c} \in C^1([0,1], \widetilde{\mathcal{N}})$ be a minimizing geodesic joining $\widetilde{w}_0((0,0))$ and $\widetilde{w}_0((0,1))$. Since $\pi_1(\mathcal{N})$ is finite and $\mathcal{N}$ is compact, $\widetilde{\mathcal{N}}$ is compact and hence there exists $L = L(\mathcal{N}) > 0$ such that $\max\{\int_0^1 |\widetilde{c}'|^p\,\mathrm{d}t : p \in [1,2]\} \leqq L$ (actually, L depends on $\widetilde{\mathcal{N}}$, but the universal covering is unique up to a diffeomorphism, and we can assume that L depends only on $\mathcal{N}$). Now we define the map $\widetilde{w} : \partial(0,1)^2 \to \widetilde{\mathcal{N}}$ taking the values of $\widetilde{w}_0$ on $[(0,0),(1,0)] \cup [(0,1),(1,1)]$ and of $\widetilde{c}$ on $[(0,0),(0,1)] \cup [(1,0),(1,1)]$. Define $w = \pi \circ \widetilde{w}$. Then w takes the values of g on $[(0,0),(1,0)]$, of $\pi \circ \widetilde{c}$ on $[(0,0),(0,1)] \cup [(1,0),(1,1)]$ and the values of f on $[(0,1),(1,1)]$. According to Lemma 2.13 with B_1^2 replaced by $(0,1)^2$ ($(0,1)^2$ is bilipschitz homeomorphic to B_1^2 (see [32])), there exists $W \in W^{1,p}((0,1)^2, \mathcal{N})$ such that

$$\begin{aligned}\int_{(0,1)^2} \frac{|DW|^p}{p}\,\mathrm{d}x &\leqq C\int_{\partial(0,1)^2} \frac{|D_\top w|^p}{p}\,\mathrm{d}\mathcal{H}^1\\ &\leqq C\left(\int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^1 + \int_{\mathbb{S}^1} \frac{|D_\top \gamma|^p}{p}\,\mathrm{d}\mathcal{H}^1 + \int_0^1 \frac{|(\pi\circ\widetilde{c})'|^p}{p}\,\mathrm{d}t\right)\\ &\leqq C\int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^1 + C,\end{aligned}$$

where we have used that π is a local isometry and that $\int_0^1 |\widetilde{c}'|^p\,\mathrm{d}t \leqq C$. Passing to the quotient $(0,1)^2 \to B_1^2\backslash\overline{B}^2_{1/2}$, we obtain the map $\varphi \in W^{1,p}(B_1^2\backslash\overline{B}^2_{1/2}, \mathcal{N})$ such that $\mathrm{tr}_{\mathbb{S}^1}(\varphi) = g$, $\mathrm{tr}_{\partial B^2_{1/2}}(\varphi) = f$ and

$$\int_{B_1^2\backslash\overline{B}^2_{1/2}} \frac{|D\varphi|^p}{p}\,\mathrm{d}x \leqq C\int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^1 + C. \tag{2.16}$$

Next, using (2.4) and if necessary Hölder's inequality, we have

$$\mathrm{sys}(\mathcal{N}) \leqq \int_{\mathbb{S}^1} |D_\top g|\,\mathrm{d}\mathcal{H}^1 \leqq (2\pi)^{1-\frac{1}{p}}\left(\int_{\mathbb{S}^1} |D_\top g|^p\,\mathrm{d}\mathcal{H}^1\right)^{\frac{1}{p}} = (2\pi)^{1-\frac{1}{p}}\|D_\top g\|_{L^p(\mathbb{S}^1)}.$$

This, together with the facts that $p \in [1,2)$ and $\mathrm{sys}(\mathcal{N}) \in (0,+\infty)$ (since g is not nullhomotopic and $\mathcal{N}$ is compact), implies that

$$\int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^1 \geqq \frac{1}{C}. \tag{2.17}$$

Define $u(x) = \varphi(x)$ for each $x \in B_1^2\backslash\overline{B}^2_{1/2}$. Then $u \in W^{1,p}(B_1^2, \mathcal{N})$, $\mathrm{tr}_{\mathbb{S}^1}(u) = g$ and combining (2.15), (2.16) and (2.17), we deduce the following

$$\int_{B_1^2} \frac{|Du|^p}{p}\,\mathrm{d}x \leqq C\int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^1 + \frac{\mathcal{E}_p^{\mathrm{sg}}(g)}{2-p},$$

which completes our proof of Lemma 2.18. □

To lighten the notation, for every $x_0 \in \mathbb{R}^3$, $r > 0$ and $L > 0$, we define the cylinder and its lateral surface, namely

$$\Lambda_{r,L}(x_0) := B_r^2(x_0) \times (-L, L) \quad \text{and} \quad \Gamma_{r,L}(x_0) = \partial B_r^2(x_0) \times (-L, L). \tag{2.18}$$

Remark 2.19. The next construction is very reminiscent the one proposed by Rubinstein and Sternberg (see [51, Section 2]) in the case when the target manifold is the circle $\mathbb{S}^1$, which was extended by Brezis, Li, Mironescu and Nirenberg (see [16, Section 1]) to the case where the targed manifold is compact, connected and oriented. Let $\Gamma = \Gamma_{r,L}(x_0)$ for some $x_0 \in \mathbb{R}^3$, $r, L > 0$. Then, for each $g \in W^{1,2}(\Gamma, \mathcal{N})$, we can define the so-called "homotopy class" of g. Indeed, according to [53, Proposition, p. 267], there exists $(g_n)_{n\in\mathbb{N}} \subset C^\infty(\Gamma, \mathcal{N})$ such that $g_n \to g$

strongly in $W^{1,2}(\Gamma, \mathbb{R}^\nu)$. Then, using the coarea formula and the Sobolev embedding, we observe that for $\mathcal{H}^1$-a.e. $z \in (-L, L)$, $\operatorname{tr}_{\partial B_r^2(x_0)\times\{z\}}(g) = g|_{\partial B_r^2(x_0)\times\{z\}}$ is continuous, belongs to $W^{1,2}(\partial B_r^2(x_0) \times \{z\}, \mathcal{N})$ and

$$g_n|_{\partial B_r^2(x_0)\times\{z\}} \to g|_{\partial B_r^2(x_0)\times\{z\}}$$

uniformly. This implies that, for $\mathcal{H}^1$-a.e. $z_1, z_2 \in (-L, L)$, $g|_{\partial B_r^2(x_0)\times\{z_1\}}$ is continuously homotopic to $g|_{\partial B_r^2(x_0)\times\{z_2\}}$, and we can define the "homotopy class" of g, namely $[g]_\Gamma$, by setting $[g]_\Gamma := [g|_{\partial B_r^2(x_0)\times\{z_1\}}]$. To simplify our notation, hereinafter, we shall always write $[g]_\Gamma$ instead of $[g]_{\Gamma_{r,L}(x_0)}$.

Lemma 2.20. *Let $p \in [1, 2)$ and $g \in W^{1,2}(\Gamma_{r,L}(x_0), \mathcal{N})$. Then there exists $u \in W^{1,p}(\Lambda_{r,L}(x_0), \mathcal{N})$ such that $\operatorname{tr}_{\Gamma_{r,L}(x_0)}(u) = g$ and the following assertions hold.*

(i) *For some $C = C(\mathcal{N}) > 0$, the following estimate holds*

$$\int_{\Lambda_{r,L}(x_0)} \frac{|Du|^p}{p}\,\mathrm{d}x \leqq \frac{2L\mathcal{E}_p^{\mathrm{sg}}([g]_\Gamma)r^{2-p}}{2-p} + C\left(\frac{L^p}{r^{p-1}} + r\right)\int_{\Gamma_{r,L}(x_0)} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2.$$

(ii) *For each $z \in \{-L, L\}$, $\operatorname{tr}_{B_r^2(x_0)\times\{z\}}(u) \in W^{1,p}(B_r^2(x_0) \times \{z\}, \mathcal{N})$ and there exists a constant $C = C(\mathcal{N}) > 0$ such that*

$$\int_{B_r^2(x_0)\times\{z\}} \frac{|D\operatorname{tr}_{B_r^2(x_0)\times\{z\}}(u)|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{\mathcal{E}_p^{\mathrm{sg}}([g]_\Gamma)r^{2-p}}{2-p} + C\left(\left(\frac{L}{r}\right)^{p-1} + \frac{r}{L}\right)\int_{\Gamma_{r,L}(x_0)} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2.$$

Proof. Up to translation, we can assume that $x_0 = 0$. Denote $\Gamma = \Gamma_{r,L}$ and $\Lambda = \Lambda_{r,L}$. Using the coarea formula, we get some $z_0 \in (-L/2, L/2)$ such that $g|_{\partial B_r^2\times\{z_0\}} \in W^{1,2}(\partial B_r^2 \times \{z_0\}, \mathcal{N})$ and

$$\|D_\top g\|^p_{L^p(\partial B_r^2\times\{z_0\})} \leqq \frac{2}{L}\|D_\top g\|^p_{L^p(\Gamma)}. \tag{2.19}$$

First, we define u on $(B_r^2 \setminus B_{r/2}^2) \times (-L, L)$. Let $f(\sigma, z) := g(r\sigma, z)$ for each $\sigma \in \mathbb{S}^1$ and $z \in (-L, L)$. For every $\varrho \in [r/2, r]$, $\sigma \in \mathbb{S}^1$ and $z \in (-L, L)$, we set

$$u(\varrho\sigma, z) := \begin{cases} f(\sigma, \xi(\varrho, z)) & \text{if } \varrho \in [\varrho_0(z), r], \\ f(\sigma, z_0) & \text{if } \varrho \in [r/2, \varrho_0(z)], \end{cases}$$

where

$$\xi(\varrho, z) := \begin{cases} z - \frac{2(r-\varrho)(L-z_0)}{r} & \text{if } z \in [z_0, L], \\ z + \frac{2(r-\varrho)(L+z_0)}{r} & \text{if } z \in [-L, z_0] \end{cases} \quad \text{and}$$

$$\varrho_0(z) := \begin{cases} r - \frac{r(z-z_0)}{2(L-z_0)} & \text{if } z \in [z_0, L], \\ r + \frac{r(z-z_0)}{2(L+z_0)} & \text{if } z \in [-L, z_0]. \end{cases}$$

Notice that $z \mapsto \varrho_0(z)$ is affine on $[-L, z_0]$ and on $[z_0, L]$. Also $\varrho_0(-L) = \varrho_0(L) = r/2$, $\varrho_0(z_0) = r$. Furthermore, $\xi(\varrho, z) \in (-L, L)$ if $z \in (-L, L)$ and $\varrho \in [\varrho_0(z), r]$. Performing the changes of variables, we have, firstly, the splitting

$$\|Du\|^p_{L^p((B^2_r \setminus \overline{B}^2_{r/2}) \times (z_0, L))} = \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} := \int_{r/2}^{\varrho_0(z)} \mathrm{d}\varrho \int_{\mathbb{S}^1} \mathrm{d}\mathcal{H}^1(\sigma) \int_{z_0}^{L} \varrho^{1-p} \left|\frac{\partial f}{\partial \sigma}\right|^p (\sigma, z_0)\, \mathrm{d}z$$

and

$$\begin{aligned}\mathrm{II} := \int_{\varrho_0(z)}^{r} \mathrm{d}\varrho \int_{\mathbb{S}^1} \mathrm{d}\mathcal{H}^1(\sigma) \int_{z_0}^{L} \varrho \Bigg(\left(\frac{4(L - z_0)^2}{r^2} + 1 \right) \left|\frac{\partial f}{\partial z}\right|^2 (\sigma, \xi(\varrho, z)) \\ + \frac{1}{\varrho^2} \left|\frac{\partial f}{\partial \sigma}\right|^2 (\sigma, \xi(\varrho, z)) \Bigg)^{\frac{p}{2}} \mathrm{d}z.\end{aligned}$$

Using the fact that $\frac{1-(1/2)^{2-p}}{2-p} \leqq \ln(2)$, the elementary inequality $(a + b)^{\frac{p}{2}} \leqq a^{\frac{p}{2}} + b^{\frac{p}{2}}$ for each $a, b \geqq 0$ and (2.19), we estimate I by

$$\begin{aligned}\mathrm{I} &\leqq \int_{\mathbb{S}^1} \ln(2)(L - z_0) r^{2-p} \left|\frac{\partial f}{\partial \sigma}\right|^p (\sigma, z_0)\, \mathrm{d}\mathcal{H}^1(\sigma) \\ &\leqq \ln(2)(L - z_0) r \|D_\top g\|^p_{L^p(\partial B^2_r \times \{z_0\})}\end{aligned} \tag{2.20}$$

and II by

$$\begin{aligned}\mathrm{II} &\leqq \frac{r}{2(L - z_0)} \int_{z_0}^{z} \mathrm{d}\xi \int_{\mathbb{S}^1} \mathrm{d}\mathcal{H}^1(\sigma) \int_{z_0}^{L} r \Bigg(\left(\frac{4(L - z_0)^2}{r^2} + 1 \right) \left|\frac{\partial f}{\partial z}\right|^2 (\sigma, \xi) \\ &\quad + \frac{4}{r^2} \left|\frac{\partial f}{\partial \sigma}\right|^2 (\sigma, \xi) \Bigg)^{\frac{p}{2}} \mathrm{d}z \\ &\leqq \frac{r}{2} \left(\frac{4(L - z_0)^2}{r^2} + 4 \right)^{\frac{p}{2}} \int_{z_0}^{L} \mathrm{d}\xi \int_{\mathbb{S}^1} r \left(\left|\frac{\partial f}{\partial z}\right|^2 (\sigma, \xi) + \frac{1}{r^2} \left|\frac{\partial f}{\partial \sigma}\right|^2 (\sigma, \xi) \right)^{\frac{p}{2}} \mathrm{d}\mathcal{H}^1(\sigma) \\ &\leqq 2 \left(\frac{(L - z_0)^p}{r^{p-1}} + r \right) \|D_\top g\|^p_{L^p(\partial B^2_r \times (z_0, L))},\end{aligned}$$

which, together with (2.20), implies that

$$\|Du\|^p_{L^p((B^2_r \setminus \overline{B}^2_{r/2}) \times (z_0, L))} \leqq C \left(\frac{L^p}{r^{p-1}} + r \right) \|D_\top g\|^p_{L^p(\Gamma)}, \tag{2.21}$$

where $C > 0$ is the constant that, hereinafter in this proof, can depend only on $\mathcal{N}$ and can be different from line to line. Similarly, we have

$$\|Du\|^p_{L^p((B^2_r \setminus \overline{B}^2_{r/2}) \times (-L, z_0))} \leqq C \left(\frac{L^p}{r^{p-1}} + r \right) \|D_\top g\|^p_{L^p(\Gamma)}. \tag{2.22}$$

Applying Lemma 2.18, we define u on $B^2_{r/2} \times \{z_0\}$, satisfying

$$\begin{aligned}\int_{B^2_{r/2}\times\{z_0\}} \frac{|Du|^p}{p}\,\mathrm{d}\mathcal{H}^2 &\leqq \frac{\mathcal{E}^{\mathrm{sg}}_p([g]_\Gamma)r^{2-p}}{2-p} + Cr\int_{\partial B^2_r\times\{z_0\}} \frac{|D_\top u|^p}{p}\,\mathrm{d}\mathcal{H}^1 \\ &\leqq \frac{\mathcal{E}^{\mathrm{sg}}_p([g]_\Gamma)r^{2-p}}{2-p} + \frac{Cr}{L}\int_\Gamma \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2,\end{aligned} \tag{2.23}$$

where to obtain the last estimate, we have used (2.19). Since u restricted to $\partial B^2_{r/2} \times (-L, L)$ is independent of the z-variable, we can define $u(\varrho\sigma, z) = u(\varrho\sigma, z_0)$ for every $\varrho \in (0, r/2]$, $\sigma \in \mathbb{S}^1$ and $z \in (-L, L)$. At this point, u is defined on Λ and $u \in W^{1,p}(\Lambda, \mathcal{N})$. Observe that, integrating both sides of (2.23) with respect to $z \in (-L, L)$, we obtain that

$$\int_{B^2_{r/2}\times(-L,L)} \frac{|Du|^p}{p}\,\mathrm{d}x \leqq \frac{2L\mathcal{E}^{\mathrm{sg}}_p([g]_\Gamma)r^{2-p}}{2-p} + Cr\int_\Gamma \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2. \tag{2.24}$$

Combining (2.21) with (2.22) and (2.24), we complete the proof of the assertion (i).

Lastly, looking at the definition of u, one observes that $\mathrm{tr}_{B^2_r\times\{z\}}(u) \in W^{1,p}(B^2_r \times \{z\}, \mathcal{N})$ for each $z \in \{-L, L\}$. In particular, if $z = L$, then, setting

$$\mathrm{III} = \|D\,\mathrm{tr}_{B^2_r\times\{L\}}(u)\|^p_{L^p((B^2_r\setminus\overline{B}^2_{r/2})\times\{L\})},$$

we deduce

$$\begin{aligned}\mathrm{III} &\leqq \int_{r/2}^r \mathrm{d}\varrho \int_{\mathbb{S}^1} \varrho\left(\frac{4(L-z_0)^2}{r^2}\left|\frac{\partial f}{\partial z}\right|^2(\sigma,\xi(\varrho,L)) + \frac{1}{\varrho^2}\left|\frac{\partial f}{\partial\sigma}\right|^2(\sigma,\xi(\varrho,L))\right)^{\frac{p}{2}}\mathrm{d}\mathcal{H}^1(\sigma) \\ &\leqq \frac{r}{2(L-z_0)}\left(\frac{4(L-z_0)^2}{r^2}+4\right)^{\frac{p}{2}}\int_{z_0}^L \mathrm{d}\xi\int_{\mathbb{S}^1} r\left(\left|\frac{\partial f}{\partial z}\right|^2(\sigma,\xi) + \frac{1}{r^2}\left|\frac{\partial f}{\partial\sigma}\right|^2(\sigma,\xi)\right)^{\frac{p}{2}}\mathrm{d}\mathcal{H}^1(\sigma) \\ &\leqq \frac{r}{2(L-z_0)}\left(\frac{4(L-z_0)^p}{r^p}+4\right)\int_\Gamma |D_\top g|^p\,\mathrm{d}\mathcal{H}^2 \\ &\leqq C\left(\left(\frac{L}{r}\right)^{p-1} + \frac{r}{L}\right)\int_\Gamma |D_\top g|^p\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

This, together with (2.23), proves the inequality of the assertion (ii) for $z = L$. Similarly, one obtains the same estimate for $\|D\,\mathrm{tr}_{B^2_r\times\{-L\}}(u)\|^p_{L^p(B^2_r\times\{-L\})}$, which completes our proof of the assertion (ii) and of Lemma 2.20. □

2.8. *Minimizing p-Harmonic Maps*

Given an open bounded set $U \subset \mathbb{R}^3$, recall that a mapping $u_p \in W^{1,p}(U, \mathcal{N})$ is a p-minimizer in U whenever the following holds

$$\int_U \frac{|Du_p|^p}{p}\,\mathrm{d}x = \min\left\{\int_U \frac{|Du|^p}{p}\,\mathrm{d}x : u \in W^{1,p}(U,\mathcal{N}),\ \ u - u_p \in W^{1,p}_0(U,\mathbb{R}^\nu)\right\}.$$

In particular, if U has a Lipschitz boundary, then $u_p \in W^{1,p}(U,\mathcal{N})$ is a p-minimizer in U if and only if

$$\int_U \frac{|Du_p|^p}{p}\,\mathrm{d}x = \min\left\{\int_U \frac{|Du|^p}{p}\,\mathrm{d}x : u \in W^{1,p}(U,\mathcal{N}),\ \ \mathrm{tr}_{\partial U}(u) = \mathrm{tr}_{\partial U}(u_p)\right\}.$$

When appropriate, we shall simply say that u_p is a p-minimizer.

The next proposition says that if $p \in (1,2)$, then for every p-minimizer in Ω which is equal on $\partial\Omega$ to some fixed mapping g lying in $W^{1/2,2}(\partial\Omega,\mathcal{N})$, the p-energy is bounded from above by $C((2-p)^{-1})$, where $C > 0$ is a constant depending only on g, $\partial\Omega$ and $\mathcal{N}$.

Proposition 2.21. *Let $g \in W^{1/2,2}(\partial\Omega,\mathcal{N})$ and $p \in (1,2)$. Then there exists a positive constant C depending only on $\partial\Omega$ and $\mathcal{N}$ such that if u_p is a p-minimizer with $\mathrm{tr}_{\partial\Omega}(u_p) = g$, then*

$$(2-p)\int_\Omega \frac{|Du_p|^p}{p}\,\mathrm{d}x \leqq C|g|^p_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)}.$$

Proof. Inasmuch as $g \in W^{1/2,2}(\partial\Omega,\mathcal{N})$, it holds $g \in W^{1-1/p,p}(\partial\Omega,\mathbb{R}^\nu)$ (the main ingredients in the proof are Hölder's inequality, the Sobolev inequality for fractional Sobolev spaces (the reader may consult [1, Theorem 7.57], [55, Section 3.3] and [14, Theorem 1]) and the fact that $\partial\Omega\times\partial\Omega = \bigcup_{i=1}^n \varphi_i(B_1^2)\times\varphi_i(B_1^2)$ for some $n \in \mathbb{N}\backslash\{0\}$, where φ_i is a bilipschitz map; see also [27, Proposition 2.1]) and

$$|g|_{W^{1-1/p,p}(\partial\Omega,\mathbb{R}^\nu)} \leqq C|g|_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)}. \tag{2.25}$$

According to [34, Theorem 6.2], there exists $v \in W^{1,p}(\Omega,\mathcal{N})$ such that $\mathrm{tr}_{\partial\Omega}(v) = g$ and

$$(2-p)\int_\Omega \frac{|Dv|^p}{p}\,\mathrm{d}x \leqq C'|g|^p_{W^{1-1/p,p}(\partial\Omega,\mathbb{R}^\nu)}. \tag{2.26}$$

Next, using the direct method in the calculus of variations (notice that $p \in (1,2)$), we deduce that there exists a p-minimizer u_p such that $\mathrm{tr}_{\partial\Omega}(u_p) = g$. In view of the minimality of u_p and the estimates (2.26), (2.25), the following holds

$$(2-p)\int_\Omega \frac{|Du_p|^p}{p}\,\mathrm{d}x \leqq C'|g|^p_{W^{1-1/p,p}(\partial\Omega,\mathbb{R}^\nu)} \leqq C''|g|^p_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)} \tag{2.27}$$

for some $C'' = C''(\partial\Omega,\mathcal{N}) > 0$. This completes our proof of Proposition 2.21. □

We also recall the following fact. Let $u \in W^{1,p}(\Omega,\mathcal{N})$ be a p-minimizer. Fix an arbitrary $\xi \in C_c^\infty(\Omega,\mathbb{R}^3)$, which is not identically zero. Define $\varrho_0 = \min\{\mathrm{dist}(x,\partial\Omega) : x \in \mathrm{supp}(\xi)\}$. If $s \in (0,\varrho_0/\|\xi\|_\infty)$ is small enough, then the mapping $\varphi_s(x) = x + s\xi(x)$, $x \in \Omega$ is invertible. For each $s > 0$ small enough, define $u_s(x) = u \circ \varphi_s^{-1}(x)$ for each $x \in \Omega$. Since u minimizes the p-energy, $\frac{\mathrm{d}}{\mathrm{d}s}\int_\Omega |Du_s|^p\,\mathrm{d}x\big|_{s=0} = 0$. This, together with the facts that $D\varphi_s^{-1}\circ\varphi_s = (D\varphi_s)^{-1}$, $(D\varphi_s)^{-\mathrm{T}} = \mathrm{Id} - s(D\xi)^{\mathrm{T}} + O(s^2)$ and $\det(D\varphi_s) = 1 + s\,\mathrm{div}(\xi) + O(s^2)$, implies that the stress-energy tensor $T_u^p = \frac{|Du|^p}{p}\mathrm{Id} - \frac{Du\otimes Du}{|Du|^{2-p}}$ is (row-wise) divergence free. Namely, we have the following integral identity

$$\int_\Omega \sum_{i,j=1}^{3} (|Du|^p \delta_{i,j} - p|Du|^{p-2}\langle D_i u, D_j u\rangle) D_i \xi_j \,\mathrm{d}x = 0 \quad \forall \xi \in C_c^1(\Omega, \mathbb{R}^3), \tag{2.28}$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. A map satisfying (2.28) is called *p-stationary* or simply *stationary*.

3. Extension from Triangulation

3.1. Lower Bounds for the Energy

We begin with the following simple estimate.

Lemma 3.1. *If* $p \in [1, 2]$, $\delta \in (0, +\infty)$ *and* $g \in W^{1,p}(\mathbb{S}^1, \mathbb{R}^k)$, *then, defining* $v(x) := g(x/|x|)$ *for each* $x \in B^2_{1+\delta}\backslash\overline{B}^2_1$, *we have* $v \in W^{1,p}(B^2_{1+\delta}\backslash\overline{B}^2_1, \mathbb{R}^k)$ *and*

$$\int_{B^2_{1+\delta}\backslash\overline{B}^2_1} |Dv|^p \,\mathrm{d}x \leqq \delta \int_{\mathbb{S}^1} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1.$$

Proof. If $p \in [1, 2)$, then $v \in W^{1,p}(B^2_{1+\delta}\backslash\overline{B}^2_1, \mathbb{R}^k)$ and

$$\begin{aligned}\int_{B^2_{1+\delta}\backslash\overline{B}^2_1} |Dv|^p \,\mathrm{d}x &= \int_1^{1+\delta} \varrho^{1-p} \,\mathrm{d}\varrho \int_{\mathbb{S}^1} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1 \\ &= \frac{(1+\delta)^{2-p} - 1}{2-p} \int_{\mathbb{S}^1} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1.\end{aligned} \tag{3.1}$$

Since $1 \leqq p \leqq 2$, the function $t \in [0, +\infty) \mapsto t^{2-p}$ is concave and hence $(1+\delta)^{2-p} \leqq 1 + (2-p)\delta$. This, together with (3.1), completes our proof of Lemma 3.1, since the case $p = 2$ is similar. □

Recall that if $p \in [1, 2)$ and $\pi_1(\mathcal{N}) \not\simeq \{0\}$, then the set $C^\infty(B^2_r(x_0), \mathcal{N})$ is not dense in $W^{1,p}(B^2_r(x_0), \mathcal{N})$. In fact, in the latter case there exists a map $\gamma \in W^{1,p}(\partial B^2_r(x_0), \mathcal{N})$, which is not nullhomotopic. Defining $u(x) = \gamma(x_0 + r(x - x_0)/|x - x_0|)$ for each $x \in B^2_r(x_0)\backslash\{x_0\}$, we have $u \in W^{1,p}(B^2_r(x_0), \mathcal{N})$. Assuming that there exists a sequence of smooth maps $u_n : B^2_r(x_0) \to \mathcal{N}$ such that $u_n \to u$ in $W^{1,p}(B^2_r(x_0), \mathbb{R}^\nu)$, and using the coarea formula together with the Sobolev embedding theorem, we could find some $\varrho \in (0, r)$ such that $u_n|_{\partial B^2_\varrho(x_0)}$ converges uniformly to $\mathrm{tr}_{\partial B^2_\varrho(x_0)}(u) = u|_{\partial B^2_\varrho(x_0)}$. But this would lead to a contradiction, since $u_n|_{\partial B^2_\varrho(x_0)}$ is nullhomotopic contrary to $u|_{\partial B^2_\varrho(x_0)}$. This type of counterexample has its roots in the seminal work of Schoen and Uhlenbeck (see [53, Example]). Nonetheless, as was first observed by Bethuel and Zheng for the circle $\mathbb{S}^1$ (see [13, Theorem 4]) and generalized by Bethuel (see [8, Theorem 2]) to general compact Riemannian manifolds, Sobolev maps in $W^{1,p}(B^2_r(x_0),\mathcal{N})$ can be approximated by continuous maps outside a finite set of points in $B^2_r(x_0)$. For the reader's convenience, we recall Bethuel's theorem, slightly modified for our purposes.

Theorem 3.2. *Let $p \in [1,2)$. Then every map $u \in W^{1,p}(B^2_r(x_0),\mathcal{N}) \cap C(B^2_r(x_0)\setminus B^2_{r/2}(x_0),\mathcal{N})$ can be approximated in $W^{1,p}(B^2_r(x_0),\mathbb{R}^\nu)$ by maps $u_n \in W^{1,p}(B^2_r(x_0),\mathcal{N})$ which are continuous outside some finite set of points $A_n \subset B^2_{r/2}(x_0)$ such that $u_n = u$ on $B^2_r(x_0)\setminus B^2_{3r/4}(x_0)$ and $u_n|_{\partial B^2_\varrho(a)}$ is not nullhomotopic for each $a \in A_n$ and for each sufficiently small $\varrho > 0$ depending on n.*

Sketch of the proof. The proof consists of covering $B^2_{r/2}(x_0)$ with "good" and "bad" balls with respect to the p-energy of u, relying on an argument based on the coarea formula, and by changing u inside each good and bad ball from the covering by a smooth function in the case of a good ball and a smooth beyond the center of the ball in the case of a bad ball; the reader may consult [8, Theorem 2], where the balls are replaced by cubes. We only need to clarify that $u_n|_{\partial B^2_\varrho(a)}$ is not nullhomotopic for each $a \in A_n$ and for each sufficiently small $\varrho > 0$. In fact, otherwise, using the coarea formula, we could modify u_n inside a ball $B^2_\delta(a)$, where $\delta \in (\varrho/2,\varrho)$ for some sufficiently small $\varrho > 0$, with the quality of the approximation controlled by the energy of the original map on the ball $B^2_\varrho(a)$, so that the modified map u_n is continuous on $B^2_\varrho(a)$. □

Proposition 3.3. *Let $p \in (1,2)$, $u \in W^{1,p}(B^2_r(x_0),\mathcal{N})$ and $\operatorname{tr}_{\partial B^2_r(x_0)}(u) \in W^{1,p}(\partial B^2_r(x_0),\mathcal{N})$. Then the following estimate holds.*

$$\int_{B^2_r(x_0)} \frac{|Du|^p}{p}\,\mathrm{d}x + r\int_{\partial B^2_r(x_0)} \frac{|D_\top \operatorname{tr}_{\partial B^2_r(x_0)}(u)|^p}{p}\,\mathrm{d}\mathcal{H}^1 \geqq \frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\operatorname{tr}_{\partial B^2_r(x_0)}(u))r^{2-p}}{2-p}.$$

Proof. Up to scaling and translation, we can assume that $r = 1$ and $x_0 = 0$. Since $p > 1$, in view of Morrey's embedding for Sobolev mappings, without loss of generality, we suppose that $\operatorname{tr}_{\mathbb{S}^1}(u) \in C(\mathbb{S}^1,\mathcal{N})$. Define the mapping $v : B^2_2 \to \mathcal{N}$ by

$$v(x) := \begin{cases} \operatorname{tr}_{\mathbb{S}^1}(u)\left(\frac{x}{|x|}\right) & \text{if} \quad x \in B^2_2 \setminus B^2_1, \\ u(x) & \text{if} \quad x \in B^2_1. \end{cases}$$

Then $v \in W^{1,p}(B^2_2,\mathcal{N}) \cap C(B^2_2\setminus B^2_1,\mathcal{N})$. By Lemma 3.1 applied with $\delta = 1$, we obtain

$$\int_{B^2_2} |Dv|^p\,\mathrm{d}x \leqq \int_{B^2_1} |Du|^p\,\mathrm{d}x + \int_{\mathbb{S}^1} |D_\top \operatorname{tr}_{\mathbb{S}^1}(u)|^p\,\mathrm{d}\mathcal{H}^1. \tag{3.2}$$

By Theorem 3.2, there exists $(v_n)_{n\in\mathbb{N}} \subset W^{1,p}(B^2_2,\mathcal{N})$ such that $v_n \to v$ in $W^{1,p}(B^2_2,\mathbb{R}^\nu)$ and for each $n \in \mathbb{N}$, v_n is continuous in $B^2_2\setminus A_n$, where $A_n \subset B^2_1$ is a finite set of points; $\operatorname{tr}_{\partial B^2_2}(v_n)$ is homotopic to $\operatorname{tr}_{\mathbb{S}^1}(u)$; for each point $a \in A_n$ and for each sufficiently small radius $\varrho > 0$, $v_n|_{\partial B^2_\varrho(a)}$ is homotopically nontrivial. Next, using [56, Corollary 4.4] with $\delta = 1 < \operatorname{dist}(a, \partial B_2^2)$ (in the statement of [56, Corollary 4.4], δ is assumed to satisfy the same assumptions as in [56, Proposition 4.3]) to $v_n \in W^{1,p}(B_2^2, \mathcal{N})$ with the set of singular points A_n, we get

$$\begin{aligned}\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\operatorname{tr}_{\mathbb{S}^1}(u)) &= \mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\operatorname{tr}_{\partial B_2^2}(v_n))\\ &\leqq (2-p)\frac{(p-1)^{p-1}}{((2\pi)/p)^{2-p}}\int_{B_2^2}\frac{|Dv_n|^p}{p}\,\mathrm{d}x\\ &\leqq (2-p)\int_{B_2^2}\frac{|Dv_n|^p}{p}\,\mathrm{d}x,\end{aligned}\tag{3.3}$$

since the singular p-energy is invariant under homotopies and $\operatorname{tr}_{\partial B_2^2}(v_n)$ is homotopic to $\operatorname{tr}_{\mathbb{S}^1}(u)$ and since for $1 < p < 2$, $\frac{(p-1)^{p-1}}{((2\pi)/p)^{2-p}} < 1$. □

Remark 3.4. Adapting the proof of [56, Proposition 4.3], yields a variant of the latter, where the estimate depends on $\operatorname{sys}(\mathcal{N})$. Therefore, one can obtain a variant of Proposition 3.3 with the estimate depending on $\operatorname{sys}(\mathcal{N})$. More precisely, if

$$\int_{B_r^2(x_0)} |Du|^p \,\mathrm{d}x + r\int_{\partial B_r^2(x_0)} |D_\top \operatorname{tr}_{\partial B_r^2(x_0)}(u)|^p \,\mathrm{d}\mathcal{H}^1 > 0,$$

then one can obtain the estimate

$$\begin{aligned}&\frac{2-p}{p}\left(\int_{B_r^2(x_0)} |Du|^p \,\mathrm{d}x + r\int_{\partial B_r^2(x_0)} |D_\top \operatorname{tr}_{\partial B_r^2(x_0)}(u)|^p \,\mathrm{d}\mathcal{H}^1\right)\\ &\geqq C\mathcal{E}^{\mathrm{sg}}_p(\operatorname{tr}_{\partial B_r^2(x_0)}(u))^{p-1} r^{2-p},\end{aligned}$$

where $C = \left(\frac{(\operatorname{sys}(\mathcal{N}))^p}{(2\pi)^{p-1}p}\right)^{2-p}$. This is an upper bound of $\mathcal{E}^{\mathrm{sg}}_p(\operatorname{tr}_{\mathbb{S}^1}(u))$ by the p-energy, which is in one instance better than the control in Proposition 3.3 of $\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\operatorname{tr}_{\mathbb{S}^1}(u))$ by the p-energy. However, the estimate depends on $\operatorname{sys}(\mathcal{N})$. Since we prefer to avoid the use of estimates depending on the systole, we decided to keep the estimation of Proposition 3.3.

3.2. Lipschitz Triangulations of $\mathbb{S}^2$

For a given integer $n \in \mathbb{N} \setminus \{0\}$, we shall call the uniform 1-dimensional grid of step $1/n$ in $\partial[-1,1]^3$ the set of all points $x \in \partial[-1,1]^3$ such that $nx_i \in \mathbb{Z}$ for all $i = 1, 2, 3$ except for at most one.

The uniform 1-dimensional grid of step $1/n$ gives the decomposition of $\partial[-1,1]^3$ into mutually disjoint sets (points, edges and two-dimensional cells), namely

$$\partial[-1,1]^3 = \bigcup_{i=0}^{2}\bigcup_{j=1}^{k_i} E_{i,j},\tag{3.4}$$

where $\{E_{0,j},\, j \in \{1,\ldots,k_0\}\} := ((1/n)\mathbb{Z}^3) \cap \partial[-1,1]^3$ and each set $E_{i,j}$ with $i \in \{1, 2\}$ is bilipschitz homeomorphic to $B^i_{1/n}$. It is worth noting that for each integer $l \in \mathbb{N}\backslash\{0\}$, the cube $(0,1)^l$ is bilipschitz homeomorphic to B_1^l (see, for instance, [32]).

Let $\delta \in (0,1]$ and $n = \lfloor 1/\delta \rfloor$, where $\lfloor\cdot\rfloor$ denotes the integer part. In view of (3.4), we obtain the following decomposition of $\mathbb{S}^2$ into mutually disjoint sets

$$\mathbb{S}^2 = \bigcup_{i=0}^{2}\bigcup_{j=1}^{k_i} F_{i,j} = \bigcup_{i=0}^{2}\bigcup_{j=1}^{k_i} \omega \circ \mathrm{P}_{\mathbb{S}^2}(E_{i,j}), \tag{3.5}$$

where $\mathrm{P}_{\mathbb{S}^2} : \mathbb{R}^3\backslash\{0\} \to \mathbb{S}^2$ is the radial projection ($\mathrm{P}_{\mathbb{S}^2}(x) = x/|x|$, $x \in \mathbb{R}^3\backslash\{0\}$), ω is an element of $SO(3)$, $\{F_{0,1},\dots,F_{0,k_0}\}$ is the collection of points and for each $F_{i,j}$ with $i \in \{1,2\}$, we have a bilipschitz homeomorphism

$$\Phi_{i,j} : F_{i,j} \to B_\delta^i \text{ with } \|D_\top \Phi_{i,j}\|_{L^\infty(F_{i,j})} + \|D(\Phi_{i,j})^{-1}\|_{L^\infty(B_\delta^i)} \leqq A_1, \tag{3.6}$$

where $A_1 > 0$ is an absolute constant. The map $\Phi_{i,j}$ in (3.6) extends by continuity to a bilipschitz map between $\overline{F}_{i,j}$ and $\overline{B}_\delta^i$ with the same bilipschitz constant A_1 as in (3.6).

If (3.5) and (3.6) hold for some $\omega \in SO(3)$, we shall call the decomposition (3.5) a uniform Lipschitz *triangulation* (or simply a Lipschitz triangulation) of step δ of $\mathbb{S}^2$. The points $F_{0,1},\dots,F_{0,k_0}$ will be called the vertices and the set $\bigcup_{i=0}^{1}\bigcup_{j=1}^{k_i} F_{i,j}$ the 1-skeleton.

To lighten the notation, we denote the collection of 1-skeletons of all Lipschitz triangulations of step δ of $\mathbb{S}^2$ by RT_δ, namely

$$\mathrm{RT}_\delta := \left\{ \bigcup_{i=0}^{1}\bigcup_{j=1}^{k_i} \omega \circ \mathrm{P}_{\mathbb{S}^2}(E_{i,j}) : \omega \in SO(3) \right\},$$

where the $E_{i,j}$'s come from (3.4) for $n = \lfloor 1/\delta \rfloor$. The importance of the rotation ω in the definition of RT_δ will be illustrated a little later, for example in Lemma 3.7.

Let us now define a Sobolev space on a finite union of bilipschitz images of d-cubes (or d-dimensional intervals), in particular, on the 1-skeleton of a uniform Lipschitz triangulation of the sphere $\mathbb{S}^2$.

Definition 3.5. Let $d, N \in \mathbb{N}\backslash\{0\}$. For each $p \in [1,+\infty)$ and for each set $\Sigma \subset \mathbb{R}^d$ being the closure of $\bigcup_{i=1}^N \Phi_i((a_{i1},b_{i1})\times\dots\times(a_{id},b_{id})) =: \bigcup_{i=1}^N \Delta_i^d$, where for each $i \in \{1,\dots,N\}$, $\Phi_i : (a_{i1},b_{i1})\times\dots\times(a_{id},b_{id}) \to \Delta_i^d$ is a bilipschitz homeomorphism and, in addition, $\Delta_i^d \cap \Delta_j^d = \emptyset$ if $i \neq j$, $j \in \{1,\dots,N\}$, we define $W^{1,p}(\Sigma,\mathbb{R}^k)$ by

$$\begin{aligned} W^{1,p}(\Sigma,\mathbb{R}^k) = \{u : \Sigma \to \mathbb{R}^k \text{ is Borel-measurable} : u|_{\Delta_i^d} \in W^{1,p}(\Delta_i^d,\mathbb{R}^k),\\ \mathrm{tr}_{\partial\Delta_i^d}(u|_{\Delta_i^d}) = u|_{\partial\Delta_i^d} \text{ for each } i \in \{1,\dots,N\}\}, \end{aligned}$$

where $\partial\Delta_i^d$ denotes the relative boundary of Δ_i^d. For $u \in W^{1,p}(\Sigma,\mathbb{R}^k)$, we set

$$\|u\|_{W^{1,p}(\Sigma,\mathbb{R}^k)} = \left(\sum_{i=1}^N \|u|_{\Delta_i^d}\|_{W^{1,p}(\Delta_i^d,\mathbb{R}^k)}^p\right)^{\frac{1}{p}}.$$

If $\mathcal{Y}$ is a compact Riemannian manifold which is isometrically embedded into $\mathbb{R}^k$, then $W^{1,p}(\Sigma, \mathcal{Y})$ is defined by

$$W^{1,p}(\Sigma, \mathcal{Y}) = \{u \in W^{1,p}(\Sigma, \mathbb{R}^k) : u \in \mathcal{Y} \text{ a.e. in } \Sigma\}.$$

Proposition 3.6. *Let $p \in [1, 2]$, $\delta \in (0, 1]$, $\Sigma \in \mathrm{RT}_\delta$ and $g \in W^{1,p}(\Sigma, \mathcal{N})$ (see Definition 3.5). Assume that for each 2-cell F from the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ, the map $g|_{\partial F}$ is homotopic to a constant, where ∂F is the relative boundary of F. Then there exists $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$, where $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the universal covering of $\mathcal{N}$, such that $\mathrm{tr}_\Sigma(\pi \circ \widetilde{v}) = g$ and*

$$\int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \, \mathrm{d}\mathcal{H}^2 \leqq C\delta \int_\Sigma |D_\top g|^p \, \mathrm{d}\mathcal{H}^1,$$

where $C = C(\mathcal{N}) > 0$.

Proof. We have $\mathbb{S}^2 = \bigcup F$ and $\Sigma = \bigcup \partial F$, where the unions are taken over all 2-cells from the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ, see (3.5).
Step 1. Fix an arbitrary 2-cell F from our Lipschitz triangulation of $\mathbb{S}^2$. We know that $g|_{\partial F}$ is homotopic to a constant. Let $\Phi : \overline{F} \to \overline{B^2_\delta}$ be the extension of the corresponding map coming from (3.6). Then, by the classical theory of continuous liftings, $g|_{\partial F} \circ \Phi^{-1}|_{\partial B^2_\delta}$ has a continuous lifting, which is also Sobolev; this, together with Lemma 2.13 and the estimate (3.6), implies that there exists $v \in W^{1,2}(F, \mathcal{N})$ such that $\mathrm{tr}_{\partial F}(v) = g|_{\partial F}$ and

$$\int_F |D_\top v|^p \, \mathrm{d}\mathcal{H}^2 \leqq C\delta \int_{\partial F} |D_\top g|^p \, \mathrm{d}\mathcal{H}^1, \tag{3.7}$$

where $C = C(\mathcal{N}) > 0$.

Step 2. Let $v \in L^\infty(\mathbb{S}^2, \mathcal{N})$ satisfy $v|_F = v_F$ for each 2-cell F from our Lipschitz triangulation of $\mathbb{S}^2$, where v_F is the mapping from *Step 1* corresponding to the cell F. Since $v_F \in W^{1,2}(F, \mathcal{N})$ and $\mathrm{tr}_{\partial F}(v_F) = g|_{\partial F}$, we have $v \in W^{1,2}(\mathbb{S}^2, \mathcal{N})$. In particular, there exists a map $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$ such that $v = \pi \circ \widetilde{v}$ (see, for instance, [11, Theorem 1]). Next, using (3.7) and the fact that each 1-cell of our Lipschitz triangulation of $\mathbb{S}^2$ lies in the relative boundary of exactly two 2-cells, since π is a local isometry, we get

$$\int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \, \mathrm{d}\mathcal{H}^2 = \int_{\mathbb{S}^2} |D_\top v|^p \, \mathrm{d}\mathcal{H}^2 = \sum \int_F |D_\top v|^p \, \mathrm{d}\mathcal{H}^2 \leqq 2C\delta \int_\Sigma |D_\top g|^p \, \mathrm{d}\mathcal{H}^1,$$

where the sum is taken over all 2-cells F. This completes our proof of Proposition 3.6. □

Lemma 3.7. *Let $p \in [1, +\infty)$ and $g \in W^{1,p}(\mathbb{S}^2, \mathcal{Y})$, where $\mathcal{Y}$ is either a Euclidean space or a compact Riemannian manifold. Let $\delta \in (0, 1]$ and $\Sigma \in \mathrm{RT}_\delta$. Then for $\mathcal{H}^3$-a.e. $\omega \in SO(3)$, $\mathrm{tr}_{\omega(\Sigma)}(g) = g|_{\omega(\Sigma)} \in W^{1,p}(\omega(\Sigma), \mathcal{Y})$. Furthermore, there exists a rotation $\omega \in SO(3)$ such that we have $\mathrm{tr}_{\omega(\Sigma)}(g) = g|_{\omega(\Sigma)} \in W^{1,p}(\omega(\Sigma), \mathcal{Y})$ and*

$$\int_{\omega(\Sigma)} |D_\top g|^p \, \mathrm{d}\mathcal{H}^1 \leqq \frac{A_2}{\delta} \int_{\mathbb{S}^2} |D_\top g|^p \, \mathrm{d}\mathcal{H}^2,$$

where $A_2 > 0$ is an absolute constant.

Proof. Since $\Sigma \in \mathrm{RT}_\delta$, there exists $\omega \in SO(3)$ such that $\Sigma = \bigcup_{i=0}^{1}\bigcup_{j=1}^{k_i}\omega(E_{i,j}) = \bigcup_{i=0}^{1}\bigcup_{j=1}^{k_i}F_{i,j}$ and hence $\Sigma = \bigcup_{j=1}^{k_1}\overline{F}_{1,j}$, where the $E_{i,j}$'s come from (3.4) for $n = \lfloor 1/\delta \rfloor$. Fix an arbitrary $j \in \{1,\ldots,k_0\}$. Let $\overline{F}_{1,j_1}$ and $\overline{F}_{1,j_2}$ be two edges emanating from $F_{0,j}$. Denote $V = \overline{F}_{1,j_1} \cup \overline{F}_{1,j_2}$. Then, in view of (3.6), there is a bilipschitz mapping $\Psi : \overline{B}^1_\delta \to V$. Denote $U = \Psi(B^1_\delta)$. Applying Lemma 2.17 (ii) with $v = g$ and $E = U$, we have $\mathrm{tr}_{\omega(U)}(g) = g|_{\omega(U)} \in W^{1,p}(\omega(U),\mathcal{Y})$ for $\mathcal{H}^3$-a.e. $\omega \in SO(3)$. Furthermore, for each $\omega \in SO(3)$ for which $g|_{\omega(U)} \in W^{1,p}(\omega(U),\mathcal{Y})$, there exists a unique mapping $h \in C(\omega(V),\mathcal{Y})$ such that $g|_{\omega(U)} = h$ $\mathcal{H}^1$-a.e. on $\omega(U)$, since $\omega(U)$ is bilipschitz homeomorphic to B^1_δ. Thus, successively applying Lemma 2.17 (ii), altogether we obtain that there exists $E \subset SO(3)$ such that $\mathcal{H}^3(E) = 0$ and the following holds. For each $\omega \in SO(3)\backslash E$ and $j \in \{1,\ldots,k_1\}$ we have $\mathrm{tr}_{\omega(F_{1,j})}(g) = g|_{\omega(F_{1,j})} \in W^{1,p}(\omega(F_{1,j}),\mathcal{Y})$ and there exists a unique mapping $h \in C(\omega(\Sigma),\mathcal{Y})$ such that $\mathrm{tr}_{\omega(\Sigma)}(g) = g|_{\omega(\Sigma)} = h$ $\mathcal{H}^1$-a.e. on $\omega(\Sigma)$, namely $g|_{\omega(\Sigma)} \in W^{1,p}(\omega(\Sigma),\mathcal{Y})$ (see Definition 3.5). Next, according to [30, Theorem 3.2.48],

$$\int_{O(3)} \mathrm{d}\theta_3(\omega) \int_{\omega(\Sigma)} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1 = \frac{\mathcal{H}^1(\Sigma)}{\mathcal{H}^2(\mathbb{S}^2)} \int_{\mathbb{S}^2} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2, \tag{3.8}$$

where $\theta_3 = (\mathcal{H}^3(O(3)))^{-1}\mathcal{H}^3 \llcorner O(3)$ is the Haar measure of the orthogonal group $O(3)$ such that $\theta_3(O(3)) = 1$. Denoting

$$A = \left\{\omega \in O(3) : \int_{\omega(\Sigma)} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1 < \frac{3\mathcal{H}^1(\Sigma)}{\mathcal{H}^2(\mathbb{S}^2)} \int_{\mathbb{S}^2} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2\right\},$$

we observe that $\theta_3(A) \geqq 2/3$, otherwise (3.8) would not be true. Thus, $\theta_3(A \cap SO(3)) \geqq 1/6$, since $\theta_3(O(3)\backslash SO(3)) = 1/2$. On the other hand, using (3.6), we get $\mathcal{H}^1(\Sigma) \leqq C/\delta$, where $C > 0$ is an absolute constant. This completes our proof of Lemma 3.7. □

Lemma 3.8. *Let $p_0 \in (1,2)$, $p \in [p_0,2)$, $\delta \in (0,1]$ and $\Sigma \in \mathrm{RT}_\delta$. Let F be a 2-cell of the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ. Then there exists a constant $\alpha_0 = \alpha_0(p_0,\mathcal{N}) > 0$ such that the following holds. If $u \in W^{1,p}(F,\mathcal{N})$, $\mathrm{tr}_{\partial F}(u) \in W^{1,p}(\partial F,\mathcal{N})$ and*

$$\int_F \frac{|D_\top u|^p}{p} \,\mathrm{d}\mathcal{H}^2 + \delta \int_{\partial F} \frac{|D_\top \,\mathrm{tr}_{\partial F}(u)|^p}{p} \,\mathrm{d}\mathcal{H}^1 \leqq \frac{\alpha_0 \delta^{2-p}}{2-p}, \tag{3.9}$$

then $\mathrm{tr}_{\partial F}(u)$ is homotopic to a constant, where ∂F denotes the relative boundary of F.

Proof. Let $\Phi : F \to B^2_\delta$ be a bilipschitz mapping coming from (3.6). To lighten the notation, denote $v := u \circ \Phi^{-1}$. According to Definition 2.7 and (2.4), if a map $\gamma \in W^{1,p}(\partial B^2_\delta,\mathcal{N})$ is not homotopic to a constant, then

$$\frac{(\mathrm{sys}(\mathcal{N}))^{\frac{p}{p-1}}(p-1)}{(2\pi)^{\frac{1}{p-1}}\,p} \leqq \mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma). \tag{3.10}$$

Let $\alpha_0 > 0$ be a constant, which will be defined a little later for the proof to work. Next, using Proposition 3.3, the chain rule for Sobolev mappings, [30, Theorem 3.2.5] and (3.9), we get

$$\begin{aligned} \frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\mathrm{tr}_{\partial B^2_\delta}(v))\delta^{2-p}}{2-p} &\leqq \int_{B^2_\delta} \frac{|Dv|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \delta \int_{\partial B^2_\delta} \frac{|D_\top \mathrm{tr}_{\partial B^2_\delta}(v)|^p}{p}\,\mathrm{d}\mathcal{H}^1 \\ &\leqq \frac{C\alpha_0\delta^{2-p}}{2-p}, \end{aligned} \tag{3.11}$$

where $C > 0$ is an absolute constant. Now we define

$$\alpha_0 = \min\left\{ \frac{(\mathrm{sys}(\mathcal{N}))^{\frac{p}{p-1}}(p-1)}{2C(2\pi)^{\frac{1}{p-1}}p} : p \in [p_0, 2] \right\}.$$

In view of the estimates (3.10) and (3.11), $\mathrm{tr}_{\partial B^2_\delta}(v)$ is homotopic to a constant, and hence $\mathrm{tr}_{\partial F}(u)$ is homotopic to a constant. This completes our proof of Lemma 3.8. □

The next lemma provides us with an extension to a 3-cell of given mappings on a 2-cell with appropriate energy control.

Lemma 3.9. *Let* $p \in [1, 2]$, $\delta \in (0, 1]$ *and for each* $i \in \{0, 1\}$, $g_i \in W^{1,p}((0,\delta)^2, \mathcal{N})$. *If* $\mathrm{tr}_{\partial(0,\delta)^2}(g_i) \in W^{1,p}(\partial(0,\delta)^2, \mathcal{N})$ *and if*

$$\mathrm{tr}_{\partial(0,\delta)^2}(g_0) = \mathrm{tr}_{\partial(0,\delta)^2}(g_1), \tag{3.12}$$

then there exists $w \in W^{1,p}((0,\delta)^3, \mathcal{N})$ *such that* $\mathrm{tr}_{(0,\delta)^2\times\{i\delta\}}(w) = g_i$ *for each* $i \in \{0, 1\}$ *and*

$$\begin{aligned} \int_{(0,\delta)^3} |Dw|^p\,\mathrm{d}x &\leqq A_3\delta \sum_{i=0}^{1} \int_{(0,\delta)^2} |Dg_i|^p\,\mathrm{d}\mathcal{H}^2 \\ &\quad + A_3\delta^2 \int_{\partial(0,\delta)^2} |D_\top \mathrm{tr}_{\partial(0,\delta)^2}(g_1)|^p\,\mathrm{d}\mathcal{H}^1, \end{aligned} \tag{3.13}$$

where $A_3 > 0$ *is an absolute constant.*

Proof. Denote $E = \partial(0,\delta)^2 \times (0,\delta)$. We define w on E by $w|_E(x) = \mathrm{tr}_{\partial(0,\delta)^2}(g_1)(x')$, where $x = (x', x_3) \in E$. Since $\mathrm{tr}_{\partial(0,\delta)^2}(g_1) \in W^{1,p}(\partial(0,\delta)^2, \mathcal{N})$, we have $w|_E \in W^{1,p}(E, \mathcal{N})$. Using [30, Theorem 3.2.22 (3)] and changing the variables, one has

$$\int_E |Dw|^p\,\mathrm{d}\mathcal{H}^2 = \delta \int_{\partial(0,\delta)^2} |D_\top \mathrm{tr}_{\partial(0,\delta)^2}(g_1)|^p\,\mathrm{d}\mathcal{H}^1. \tag{3.14}$$

Next, we define the map $w|_{\partial(0,\delta)^3} : \partial(0,\delta)^3 \to \mathcal{N}$ by

$$w|_{\partial(0,\delta)^3}(x) = \begin{cases} g_1(x') & \text{if } x = (x', x_3) \in (0,\delta)^2 \times \{\delta\}, \\ \mathrm{tr}_{\partial(0,\delta)^2}(g_1)(x') & \text{if } x = (x', x_3) \in \partial(0,\delta)^2 \times \{\delta\}, \\ w|_E(x) & \text{if } x \in E, \\ \mathrm{tr}_{\partial(0,\delta)^2}(g_0)(x') & \text{if } x = (x', x_3) \in \partial(0,\delta)^2 \times \{0\}, \\ g_0(x') & \text{if } x = (x', x_3) \in (0,\delta)^2 \times \{0\}. \end{cases}$$

Then, in view of (3.12), we deduce that $w|_{\partial(0,\delta)^3} \in W^{1,p}(\partial(0,\delta)^3, \mathcal{N})$ and $\mathrm{tr}_{(0,\delta)^2\times\{i\delta\}}(w) = g_i$ for each $i \in \{0,1\}$. Let $\Psi : \overline{B_\delta^3} \to [0,\delta]^3$ be a bilipschitz homeomorphism with an absolute bilipschitz constant (see [32, Corollary 3]). Notice that $w|_{\partial(0,\delta)^3} \circ \Psi|_{\partial B_\delta^3} \in W^{1,p}(\partial B_\delta^3, \mathcal{N})$. Thus, defining $v(x) = w|_{\partial(0,\delta)^3} \circ \Psi(\delta x/|x|)$ for each $x \in B_\delta^3 \backslash \{0\}$ and using that $p \in [1,2]$, we have $v \in W^{1,p}(B_\delta^3, \mathcal{N})$. Next, setting $\zeta(x) = x/|x|$ for $x \in \mathbb{R}^3 \backslash \{0\}$, using the special case of the coarea formula, [30, Theorem 3.2.5], the fact that $p \in [1,2]$ and the chain rule for Sobolev mappings, we obtain the following chain of estimates

$$\begin{aligned} \int_{B_\delta^3} |Dv|^p \, \mathrm{d}x &= \int_0^\delta \mathrm{d}r \int_{\partial B_r^3} \delta^p |D(w \circ \Psi)(\delta\zeta(\sigma)) \circ D\zeta(\sigma)|^p \, \mathrm{d}\mathcal{H}^2(\sigma) \\ &= \int_0^\delta \left(\frac{r}{\delta}\right)^{2-p} \mathrm{d}r \int_{\partial B_\delta^3} |D_\top(w \circ \Psi)|^p \, \mathrm{d}\mathcal{H}^2 \\ &\leqq \delta \int_{\partial B_\delta^3} |D_\top(w \circ \Psi)|^p \, \mathrm{d}\mathcal{H}^2 \\ &\leqq C\delta \int_{\partial(0,\delta)^3} |D_\top w|^p \, \mathrm{d}\mathcal{H}^2, \end{aligned} \tag{3.15}$$

where $C > 0$ is a constant depending only on the bilipschitz constant of Ψ. Then, defining for each $x \in (0,\delta)^3 \backslash \{\Psi(0)\}$, $w(x) = v(\Psi^{-1}(x))$, we observe that $w \in W^{1,p}((0,\delta)^3, \mathcal{N})$, since $v \in W^{1,p}(B_\delta^3, \mathcal{N})$ and Ψ is bilipschitz. Using [30, Theorem 3.2.5], the fact that Ψ is bilipschitz and (3.15), we have

$$\int_{(0,\delta)^3} |Dw|^p \, \mathrm{d}x \leqq C' \int_{B_\delta^3} |Dv|^p \, \mathrm{d}x \leqq C''\delta \int_{\partial(0,\delta)^3} |D_\top w|^p \, \mathrm{d}\mathcal{H}^2,$$

where $C'' > 0$ depends only on the bilipschitz constant of Ψ, which is an absolute constant. This, together with (3.14), yields (3.13) and completes our proof of Lemma 3.9. □

Using Lemma 3.9, we obtain the next Luckhaus-type lemma (see [42, Lemma 1]).

Lemma 3.10. *Let* $p_0 \in (1,2)$*,* $p \in [p_0, 2)$*,* $\delta \in (0,1)$ *and* $g \in W^{1,p}(\mathbb{S}^2, \mathcal{N})$*. Then there exists a constant* $\alpha = \alpha(p_0, \mathcal{N}) > 0$ *such that the following holds. Assume that*

$$\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, \mathrm{d}\mathcal{H}^2 \leqq \frac{\alpha\delta^{2-p}}{2-p}. \tag{3.16}$$

Then there exist mappings $\varphi_\delta \in W^{1,p}(B_1^3 \backslash \overline{B}_{1-\delta}^3, \mathcal{N})$ *and* $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$, *where* $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ *is the universal covering of* $\mathcal{N}$, *such that*

$$\mathrm{tr}_{\mathbb{S}^2}(\varphi_\delta) = g, \qquad \mathrm{tr}_{\mathbb{S}^2}(\varphi_\delta((1-\delta)\cdot)) = \pi \circ \widetilde{v}, \tag{3.17}$$

and the following estimates are satisfied

$$\int_{B_1^3 \backslash \overline{B}_{1-\delta}^3} |D\varphi_\delta|^p \, \mathrm{d}x \leqq C\delta \int_{\mathbb{S}^2} |D_\top g|^p \, \mathrm{d}\mathcal{H}^2, \tag{3.18}$$

$$\int_{\mathbb{S}^2} |D_\top(\pi \circ \widetilde{v})|^p \, \mathrm{d}\mathcal{H}^2 = \int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \, \mathrm{d}\mathcal{H}^2 \leqq C \int_{\mathbb{S}^2} |D_\top g|^p \, \mathrm{d}\mathcal{H}^2, \tag{3.19}$$

where $C = C(\mathcal{N}) > 0$.

Proof. Let $\alpha_0 = \alpha_0(p_0, \mathcal{N}) > 0$ be the constant given by Lemma 3.8. Define $\alpha = \alpha_0/2A_2$, where A_2 is the absolute constant of Lemma 3.7. Without loss of generality, we assume that $A_2 \geqq 1$. Then α depends only on p_0 and $\mathcal{N}$. In view of Lemma 3.7, the estimate (3.16) and the definition of α, there exists $\Sigma \in \mathrm{RT}_\delta$ such that $\mathrm{tr}_\Sigma(g) = g|_\Sigma \in W^{1,p}(\Sigma, \mathcal{N})$,

$$\int_\Sigma |D_\top g|^p \, \mathrm{d}\mathcal{H}^1 \leqq \frac{A_2}{\delta} \int_{\mathbb{S}^2} |D_\top g|^p \, \mathrm{d}\mathcal{H}^2 \tag{3.20}$$

and for each 2-cell F from the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ,

$$\int_F \frac{|D_\top g|^p}{p} \, \mathrm{d}\mathcal{H}^2 + \delta \int_{\partial F} \frac{|D_\top g|^p}{p} \, \mathrm{d}\mathcal{H}^1 \leqq 2A_2 \int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, \mathrm{d}\mathcal{H}^2 \leqq \frac{\alpha_0 \delta^{2-p}}{2-p},$$

where ∂F denotes the relative boundary of F. This, according to Lemma 3.8, implies that $g|_{\partial F}$ is homotopically trivial for each 2-cell F from the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ. By Proposition 3.6, there exists $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$ such that $\mathrm{tr}_\Sigma(\pi \circ \widetilde{v}) = \mathrm{tr}_\Sigma(g)$ and

$$\int_{\mathbb{S}^2} |D_\top(\pi \circ \widetilde{v})|^p \, \mathrm{d}\mathcal{H}^2 = \int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \, \mathrm{d}\mathcal{H}^2 \leqq C\delta \int_\Sigma |D_\top g|^p \, \mathrm{d}\mathcal{H}^1, \tag{3.21}$$

where $C = C(\mathcal{N}) > 0$. In (3.21) we used that π is a local isometry. Using (3.20) and (3.21), we get (3.19).

The Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ yields the decomposition of $B_1^3 \backslash \overline{B}_{1-\delta}^3$ into the cells

$$D_{i,j} = \left\{ x \in \mathbb{R}^3 : (1-\delta) < |x| < 1, \ \frac{x}{|x|} \in F_{i,j} \right\}, \tag{3.22}$$

where $i \in \{0, 1, 2\}$ and $j \in \{1, \ldots, k_i\}$ (see (3.5)). Notice that $D_{i,j}$ is of Hausdorff dimension $(i+1)$. Fix an arbitrary 3-cell $D := D_{2,j}$ from the above decomposition. Let F be the 2-cell of the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ such that $D = \{x \in \mathbb{R}^3 : (1-\delta) < |x| < 1, \ \frac{x}{|x|} \in F\}$. Fix a bilipschitz homeomorphism $\Phi : [0, \delta]^3 \to \overline{D}$ with an absolute bilipschitz constant (see (3.6)) such that $\Phi((0, \delta)^2 \times \{0\}) = (1-\delta)F$ and $\Phi((0, \delta)^2 \times \{\delta\}) = F$. Applying Lemma 3.9 with g_0 and g_1

taking the values of $\pi \circ \widetilde{v} \circ (\frac{\Phi}{1-\delta})|_{(0,\delta)^2\times\{0\}}$ and $g \circ \Phi|_{(0,\delta)^2\times\{1\}}$, respectively, we obtain a mapping $\varphi_j \in W^{1,p}(D, \mathcal{N})$ satisfying the estimate

$$\begin{aligned}\int_D |D\varphi_j|^p \,\mathrm{d}x \ &\leqq C'\delta \left(\int_F |D_\top \widetilde{v}|^p \,\mathrm{d}\mathcal{H}^2 + \int_F |D_\top g|^p \,\mathrm{d}\mathcal{H}^2 \right) \\ &\quad + C'\delta^2 \int_{\partial F} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1, \end{aligned} \tag{3.23}$$

where $C' > 0$ is an absolute constant.

Let $\varphi_\delta \in L^\infty(B_1^3 \backslash \overline{B}_{1-\delta}^3, \mathcal{N})$ satisfy $\varphi_\delta|_{D_{2,j}} = \varphi_j \in W^{1,p}(D_{2,j}, \mathcal{N})$ for each 3-cell $D_{2,j}$ from the decomposition (3.22). In view of our construction (see Lemma 3.9), we can ensure that the traces of the maps φ_j coincide on the mantles of the cells $D_{2,j}$ and hence $\varphi_\delta \in W^{1,p}(B_1^3 \setminus \overline{B}_{1-\delta}^3, \mathcal{N})$. Summing (3.23), where $D = D_{2,j}$ and $F = F_{2,j}$, over $j \in \{1, \ldots, k_2\}$, taking into account that each 1-cell of the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ lies in the relative boundary of exactly two 2-cells from this triangulation, and also using (3.19) together with (3.20), we deduce the following chain of estimates

$$\begin{aligned}\int_{B_1^3 \backslash \overline{B}_{1-\delta}^3} |D\varphi_\delta|^p \,\mathrm{d}x &\leqq C'\delta \sum_{j=1}^{k_2} \left(\int_{F_{2,j}} |D_\top \widetilde{v}|^p \,\mathrm{d}\mathcal{H}^2 + \int_{F_{2,j}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2 \right) \\ &\quad + 2C'\delta^2 \int_\Sigma |D_\top g|^p \,\mathrm{d}\mathcal{H}^1 \\ &\leqq C''\delta \int_{\mathbb{S}^2} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2, \end{aligned}$$

where $C'' = C''(\mathcal{N}) > 0$. This completes our proof of Lemma 3.10. □

We state now an η-extension with sublinear growth, which will turn out to be the key tool in the proof of our η-*compactness* lemma (see Lemma 3.12).

Lemma 3.11. *Let $p_0 \in (1, 2)$ and $p \in [p_0, 2)$. There exist constants $\eta_0, C > 0$, depending only on p_0 and $\mathcal{N}$, such that if $g \in W^{1,p}(\partial B_r^3, \mathcal{N})$ satisfies*

$$\int_{\partial B_r^3} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2 \leqq \frac{\eta_0 r^{2-p}}{2-p}, \tag{3.24}$$

then there exists $u \in W^{1,p}(B_r^3, \mathcal{N})$ such that $\mathrm{tr}_{\partial B_r^3}(u) = g$ and

$$\int_{B_r^3} \frac{|Du|^p}{p} \,\mathrm{d}x \leqq C r^{\frac{2}{p}} \left(\int_{\partial B_r^3} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2 \right)^{1-\frac{1}{p}}. \tag{3.25}$$

Proof. By scaling, we can assume that $r = 1$. Letting $\delta \in (0, 1/2]$, which will be fixed later for the proof to work, we define

$$\eta_0 = \alpha\delta^{2-p}, \tag{3.26}$$

where $\alpha = \alpha(p_0, \mathcal{N}) > 0$ is the constant of Lemma 3.10. Then, by (3.24) and (3.26), (3.16) is satisfied in Lemma 3.10 applied with δ and g, which yields $\varphi_\delta \in$

$W^{1,p}(B_1^3\setminus\overline{B}_{1-\delta}^3, \mathcal{N})$ and $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$, satisfying the condition (3.17) together with the estimates (3.18) and (3.19). Using Lemma 2.15 with the boundary datum $\pi \circ \widetilde{v}$, we obtain a map $w \in W^{1,p}(B_1^3, \mathcal{N})$ such that $\operatorname{tr}_{\mathbb{S}^2}(w) = \pi \circ \widetilde{v}$ and

$$\int_{B_1^3} |Dw|^p \,\mathrm{d}x \leqq C \left(\int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \,\mathrm{d}\mathcal{H}^2\right)^{1-\frac{1}{p}} \leqq C' \left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2\right)^{1-\frac{1}{p}}, \quad (3.27)$$

where $C = C(p_0, \mathcal{N}) > 0$ and $C' = C'(p_0, \mathcal{N}) > 0$. In the last estimate we used (3.19) under the condition that $p \in [p_0, 2)$. Now we define the mapping $u : B_1^3 \to \mathcal{N}$ by

$$u(x) = \begin{cases} \varphi_\delta(x) & \text{if} \quad x \in B_1^3 \setminus \overline{B}_{1-\delta}^3, \\ \pi \circ \widetilde{v}\left(\dfrac{x}{1-\delta}\right) & \text{if} \quad x \in \partial B_{1-\delta}^3, \\ w\left(\dfrac{x}{1-\delta}\right) & \text{if} \quad x \in B_{1-\delta}^3. \end{cases}$$

Observe that $u \in W^{1,p}(B_1^3, \mathcal{N})$ and $\operatorname{tr}_{\mathbb{S}^2}(u) = g$ (see (3.17)). Using this, (3.18) and (3.27), we have

$$\int_{B_1^3} \frac{|Du|^p}{p} \,\mathrm{d}x \leqq C' \left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2\right)^{1-\frac{1}{p}} + C'\delta \int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2, \quad (3.28)$$

where we assume that C' is large enough so that (3.18) is valid with $C = C'$. Now we need to distinguish between two further cases.

Case 1: If $\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2 \leqq 1$, then $\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2 \leqq (\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2)^{1-\frac{1}{p}}$, which, in view of (3.28), yields

$$\int_{B_1^3} \frac{|Du|^p}{p} \,\mathrm{d}x \leqq 2C' \left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2\right)^{1-\frac{1}{p}}. \quad (3.29)$$

Case 2: When $\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2 > 1$, we need to fix δ so that $\delta^{2-p} = \text{const}$ and $\delta \leqq 2 - p$. This holds for

$$\text{const} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \quad \text{and} \quad \delta = \left(\frac{1}{e}\right)^{\frac{1}{e(2-p)}}, \quad (3.30)$$

since $\inf_{x\in(0,+\infty)} x^x = (1/e)^{1/e}$. This defines our δ and hence η_0 (see (3.26)). Using (3.24), (3.26), (3.28), (3.30) and the fact that $\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2 > 1$, we obtain

$$\begin{aligned} \int_{B_1^3} \frac{|Du|^p}{p} \,\mathrm{d}x &\leqq C' \left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2\right)^{1-\frac{1}{p}} + \frac{C'\alpha\delta^{3-p}}{2-p} \\ &\leqq C'' \left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2\right)^{1-\frac{1}{p}} + C'' \quad (3.31) \\ &\leqq 2C'' \left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2\right)^{1-\frac{1}{p}}, \end{aligned}$$

where $C'' = C''(p_0, \mathcal{N}) > 0$. The estimates (3.29) and (3.31) imply then, up to a scaling, (3.25). This completes our proof of Lemma 3.11. $\square$

Now we prove our η-compactness lemma.

Lemma 3.12. *Let $\kappa \in (0,1)$, $p_0 \in (1,2)$, $p \in [p_0, 2)$ and let $E = \Psi(B^3_r) \subset \mathbb{R}^3$ be an open set, where $\Psi : B^3_r \to \mathbb{R}^3$ is a bilipschitz homeomorphism onto its image with a bilipschitz constant $L \geqq 1$. There exist $\eta, C > 0$, depending only on κ, p_0, L and $\mathcal{N}$, such that if $u_p \in W^{1,p}(E, \mathcal{N})$ is a p-minimizer and satisfies*

$$\int_E \frac{|Du_p|^p}{p}\,\mathrm{d}x \leqq \frac{\eta r^{3-p}}{2-p}, \tag{3.32}$$

then

$$\int_{\Psi(B^3_{\kappa r})} \frac{|Du_p|^p}{p}\,\mathrm{d}x \leqq C r^{3-p}. \tag{3.33}$$

Remark 3.13. Let us point out that the dependence on p in Lemma 3.12 is essential, since it is used in Lemma 5.2.

Proof of Lemma 3.12. We define $v_p = u_p \circ \Psi$, so that, by the chain rule, we have $v_p \in W^{1,p}(B^3_r, \mathcal{N})$ and

$$\int_{B^3_r} \frac{|Dv_p|^p}{p}\,\mathrm{d}x \leqq L^{3+p} \int_E \frac{|Du_p|^p}{p}\,\mathrm{d}x. \tag{3.34}$$

Let $\eta_0 > 0$ be the constant of Lemma 3.11. We define

$$G_p = \Bigg\{ \varrho \in (\kappa r, r) : \mathrm{tr}_{\partial B^3_\varrho}(v_p) = v_p|_{\partial B^3_\varrho} \in W^{1,p}(\partial B^3_\varrho, \mathcal{N}) \\ \text{and } \int_{\partial B^3_\varrho} \frac{|Dv_p|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{\eta_0 (\kappa r)^{2-p}}{2-p} \Bigg\}.$$

In view of (3.34) and (3.32),

$$\mathcal{H}^1((\kappa r, r) \setminus G_p) \leqq \frac{2-p}{\eta_0 (\kappa r)^{2-p}} \int_{B^3_r} \frac{|Dv_p|^p}{p}\,\mathrm{d}x \leqq \frac{L^{3+p}\eta r}{\eta_0 \kappa^{2-p}} \leqq \frac{L^5 \eta r}{\eta_0 \kappa}. \tag{3.35}$$

Choosing

$$\eta := \frac{\eta_0 \kappa (1-\kappa)}{2L^5}$$

and using (3.35), we obtain that

$$\mathcal{H}^1(G_p) \geqq \frac{(1-\kappa) r}{2}. \tag{3.36}$$

If $\varrho \in G_p$, then

$$\int_{\partial B^3_\varrho} \frac{|Dv_p|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{\eta_0(\kappa r)^{2-p}}{2-p} \leqq \frac{\eta_0 \varrho^{2-p}}{2-p}.$$

Thus, letting $v_p^\varrho : B^3_\varrho \to \mathcal{N}$ be the extension of $v_p|_{\partial B^3_\varrho}$ given by Lemma 3.11, we have by the p-minimizing property and the bilipschitz invariance,

$$\begin{aligned} L^{-3-p}\int_{B^3_\varrho} \frac{|Dv_p|^p}{p}\,\mathrm{d}x &\leqq \int_{\Psi(B^3_\varrho)} \frac{|Du_p|^p}{p}\,\mathrm{d}x \leqq \int_{\Psi(B^3_\varrho)} \frac{|D(v_p^\varrho\circ\Psi^{-1})|^p}{p}\,\mathrm{d}x \\ &\leqq L^{3+p}\int_{B^3_\varrho} \frac{|Dv_p^\varrho|^p}{p}\,\mathrm{d}x \leqq CL^{3+p}\varrho^{\frac{2}{p}}\left(\int_{\partial B^3_\varrho} \frac{|Dv_p|^p}{p}\,\mathrm{d}\mathcal{H}^2\right)^{1-\frac{1}{p}} \end{aligned} \tag{3.37}$$

for each $\varrho \in G_p$, where $C = C(p_0, \mathcal{N}) > 0$. We define now the function $F : [\kappa r, r] \to [0, +\infty)$ by

$$F(\varrho) = \int_{B^3_\varrho} \frac{|Dv_p|^p}{p}\,\mathrm{d}x.$$

Without loss of generality, we assume that $F(\kappa r) > 0$, because otherwise (3.33) follows. The function F is absolutely continuous and for every $\varrho \in G_p$, we have by (3.37),

$$F'(\varrho) = \int_{\partial B^3_\varrho} \frac{|Dv_p|^p}{p}\,\mathrm{d}\mathcal{H}^2 \geqq c\frac{F(\varrho)^{\frac{p}{p-1}}}{\varrho^{\frac{2}{p-1}}},$$

and thus

$$\begin{aligned} \frac{p-1}{F(\kappa r)^{\frac{1}{p-1}}} \geqq \frac{p-1}{F(\kappa r)^{\frac{1}{p-1}}} - \frac{p-1}{F(r)^{\frac{1}{p-1}}} &= \int_{\kappa r}^r \frac{F'(\varrho)}{F(\varrho)^{\frac{p}{p-1}}}\,\mathrm{d}\varrho \geqq c\int_{G_\varrho} \frac{\mathrm{d}\varrho}{\varrho^{\frac{2}{p-1}}} \\ &\geqq c\frac{\mathcal{H}^1(G_\varrho)}{r^{\frac{2}{p-1}}} \geqq \frac{c(1-\kappa)}{2r^{\frac{3-p}{p-1}}}, \end{aligned}$$

where $c = c(p_0, L, \mathcal{N}) > 0$ and the last estimate comes from (3.36). Then the estimate (3.33) follows through a bilipschitz change of variable, completing the proof of Lemma 3.12.

4. Convergence to a Harmonic Map

The following variant (Lemma 4.1) of the Luckhaus lemma (see [42, Lemma 1], [43, Lemma 3]) is our key tool in the proof of Proposition 4.2 of the fact that a weakly convergent sequence of minimizing p-harmonic maps converges to a minimizing 2-harmonic map when $p \nearrow 2$. It says that if two $W^{1,p}$ mappings from a compact 2-dimensional Riemannian manifold $\mathcal{M}$ without boundary into $\mathcal{N}$ are close in L^p, then we can find a map w in $\mathcal{M} \times (0, T)$, having these two mappings as boundary values, such that w does not leave a tubular neighborhood of $\mathcal{N}$ and such that its $W^{1,p}$ norm is of order T.

Lemma 4.1. *Let $\mathcal{M}$ be a compact Riemannian manifold of dimension 2 without boundary and $p \in (1, 2]$. Then there exist positive constants C, T_0, depending only on $\mathcal{M}$, such that for every $u, v \in W^{1,p}(\mathcal{M}, \mathcal{N})$ and $T \in (0, T_0)$ there exists a map $w \in W^{1,p}(\mathcal{M} \times (0, T), \mathbb{R}^\nu)$ satisfying*

(i) $\mathrm{tr}_{\mathcal{M}\times\{0\}}(w) = u, \ \ \mathrm{tr}_{\mathcal{M}\times\{T\}}(w) = v$;

(ii) $\displaystyle\int_{\mathcal{M}\times(0,T)} |Dw|^p \,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t \leqq CT \int_{\mathcal{M}} \left(|D_\top u|^p + |D_\top v|^p + \frac{|u-v|^p}{T^p}\right) \mathrm{d}\mathcal{H}^2$;

(iii)
$$\begin{aligned}\underset{\mathcal{M}\times(0,T)}{\mathrm{ess1,\,sup1,}}\,\mathrm{dist}(\omega, \mathcal{N}) &\leqq C\,||u-v||_{L^\infty(\mathcal{M})}\\ &\leqq CT^{1-\frac{2}{p}}\left(\int_{\mathcal{M}}\left(|D_\top u|^p + |D_\top v|^p + \frac{|u-v|^p}{T^p}\right)\mathrm{d}\mathcal{H}^2\right)^{\frac{1}{p}\left(1-\frac{1}{2p}\right)}\\ &\quad\times\left(\int_{\mathcal{M}}\frac{|u-v|^p}{T^p}\,\mathrm{d}\mathcal{H}^2\right)^{\frac{1}{p}\frac{1}{2p}}.\end{aligned}$$

Proof. For a proof, we refer the reader to [43, Lemma 3] (see also [42, Lemma 1] for the original Luckhaus lemma, where $\mathcal{M} = \mathbb{S}^2$). Notice that the inductive assumption in [43, Lemma 3] may not be satisfied. However, if one slightly modifies the Luckhaus proof, namely, one defines $w(x,t) = \frac{t}{T}v(x) + \left(1 - \frac{t}{T}\right)u(x)$ for x belonging to the closure of the l-skeleton of $\mathcal{M}$ in the case when either $[p] < p$ and $l = [p]$ or $[p] = p$ and $l = [p-1]$, then the inductive assumption in [43, Lemma 3] is satisfied, and the proof works as written (observe that Luckhaus defines w as above for x belonging to the closure of the l-skeleton of $\mathcal{M}$ in the case when $l = [p-1]$). It is also possible to reduce the dependence of the constant C in [43, Lemma 3] on $\dim(\mathcal{N})$. □

Proposition 4.2. *Let $U \subset \mathbb{R}^3$ be an open set. Let $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$, $u_n \in W^{1,p_n}(U, \mathcal{N})$ and $p_n \nearrow 2$ as $n \to +\infty$. If for each open set $K \Subset U$ and each $n \in \mathbb{N}$, u_n is a p_n-minimizer in K and if*

$$\sup_{n\in\mathbb{N}} \int_K \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \leqq C, \tag{4.1}$$

where $C = C(K) > 0$, then there exists $u_* \in W^{1,2}_{\mathrm{loc}}(U, \mathcal{N})$ such that, up to a subsequence (not relabeled), the following assertions hold.

(i) *u_n converges to u_* weakly in $W^{1,p}_{\mathrm{loc}}(U, \mathbb{R}^\nu)$ for all $p \in (1, 2)$ and a.e. in U.*

(ii) *For each open set $K \Subset U$,*

$$\int_K \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \to \int_K \frac{|Du_*|^2}{2}\,\mathrm{d}x$$

as $n \to +\infty$. Furthermore, $\int_K |Du_n - Du_*|^{p_n}\,\mathrm{d}x \to 0$ and $Du_n \to Du_*$ a.e. in U as $n \to +\infty$.

(iii) *If $K \Subset U$ is open, then u_* is a 2-minimizer in K. Furthermore, there exists at most a countable and locally finite subset S_0 of U such that $u_* \in C^\infty(U \setminus S_0, \mathcal{N})$.*

Proof. Fix an open set $K \Subset U$. Since $\overline{K} \subset U$ is compact, for any cover by open balls, $\overline{K}$ admits a finite subcover. Then there exists an open set V such that $\overline{K} \subset V$, $\overline{V} \subset U$ and $\overline{V}$ is a compact smooth Riemannian manifold of dimension 3 with boundary ∂V (possibly disconnected, but with a finite number of connected components). Then ∂V is a 2-dimensional compact smooth Riemannian manifold without boundary. According to Lemma 2.1 with $\mathcal{N}$ replaced by ∂V, there exists $\lambda_{\partial V} > 0$ such that the nearest point retraction $\Pi_{\partial V} : \partial V_{\lambda_{\partial V}} \to \partial V$ is a well-defined and smooth mapping, where $\partial V_{\lambda_{\partial V}} = \{x \in \overline{V} : \operatorname{dist}(x, \partial V) \leqq \lambda_{\partial V}\}$. Choose $\lambda \in (0, \lambda_{\partial V})$ so that $\overline{K} \subset V \backslash \overline{A}_\lambda$, where $A_\lambda = \{x \in V : \operatorname{dist}(x, \partial V) < \lambda\}$.

If $p_n \in (p, 2)$ is large enough, then, applying Young's inequality, we have

$$\int_V \frac{|Du_n|^p}{p}\,\mathrm{d}x \leqq \int_V \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x + \Big(\frac{1}{p} - \frac{1}{p_n}\Big)|V| \tag{4.2}$$

so that

$$\limsup_{n\to+\infty} \int_V \frac{|Du_n|^p}{p}\,\mathrm{d}x < +\infty \tag{4.3}$$

(see (4.1)). Since $\mathcal{N}$ is compact, $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(U, \mathbb{R}^\nu)$, which, together with (4.3), implies that for each $p \in (1, 2)$, $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(V, \mathbb{R}^\nu)$. Thus, there exists a map $u_* : V \to \mathbb{R}^\nu$ such that, up to a subsequence (still denoted by n), for each $p \in (1, 2)$,

$$u_n \rightharpoonup u_* \text{ weakly in } W^{1,p}(V, \mathbb{R}^\nu) \qquad \text{and} \qquad u_n \to u_* \text{ a.e. in } V. \tag{4.4}$$

In particular, $u_* \in \mathcal{N}$ a.e. in V.

By the weak convergence, (4.2) and the fact that $p_n \nearrow 2$ as $n \to +\infty$, for each $p \in (1, 2)$ we have

$$\begin{aligned}\int_V \frac{|Du_*|^p}{p}\,\mathrm{d}x &\leqq \liminf_{n\to+\infty} \int_V \frac{|Du_n|^p}{p}\,\mathrm{d}x \\ &\leqq \liminf_{n\to+\infty} \int_V \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x + \Big(\frac{1}{p} - \frac{1}{2}\Big)|V|.\end{aligned} \tag{4.5}$$

Letting $p \nearrow 2$ in (4.5) and using Fatou's lemma, we obtain the estimate

$$\int_V \frac{|Du_*|^2}{2}\,\mathrm{d}x \leqq \liminf_{n\to+\infty} \int_V \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x, \tag{4.6}$$

which, together with (4.1), implies that $u_* \in W^{1,2}(V, \mathcal{N})$. We prove that u_* is a 2-minimizer in K and that, up to a subsequence (not relabeled),

$$\int_K \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \to \int_K \frac{|Du_*|^2}{2}\,\mathrm{d}x \tag{4.7}$$

and $\|Du_n - Du_*\|^{p_n}_{L^{p_n}(K, \mathbb{R}^\nu \otimes \mathbb{R}^3)} \to 0$ as $n \to +\infty$.

Applying [30, Theorem 3.2.22 (3)] with $W = A_\lambda$, $Z = [0, \lambda]$ and $f : W \to Z$ defined by $f(x) = |x - \Pi_{\partial V}(x)| = \operatorname{dist}(x, \partial V)$, and using also the fact that

$|Df(x)| = 1$ for each $x \in A_\lambda$ (see Lemma 2.1), Fatou's lemma and (4.1), we obtain

$$\int_0^\lambda \mathrm{d}\varrho \liminf_{n\to+\infty} \int_{f^{-1}(\{\varrho\})} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \leqq \liminf_{n\to+\infty} \int_{A_\lambda} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \leqq C.$$

This implies that

$$\mathcal{H}^1\left(\left\{\varrho \in (0,\lambda) : \liminf_{n\to+\infty} \int_{f^{-1}(\{\varrho\})} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \geqq \frac{4C}{\lambda}\right\}\right) \leqq \frac{\lambda}{4}.$$

So there exists $\varrho \in (0, \lambda/2)$ such that, up to a subsequence (not relabeled), we know that for each sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} &\mathrm{tr}_{f^{-1}(\{\varrho\})}(u_n) = u_n|_{f^{-1}(\{\varrho\})} \in W^{1,p_n}(f^{-1}(\{\varrho\}), \mathcal{N}), \\ &\int_{f^{-1}(\{\varrho\})} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 < \frac{4C}{\lambda} \end{aligned} \tag{4.8}$$

and $\mathrm{tr}_{f^{-1}(\{\varrho\})}(u_*) = u_*|_{f^{-1}(\{\varrho\})}$. Fix such $\varrho \in (0, \lambda/2)$ and an arbitrary $p \in (1, 2)$. Now we define $\mathcal{M} = f^{-1}(\{\varrho\})$. Notice that $\mathcal{M}$ is a 2-dimensional compact smooth Riemannian manifold without boundary. In view of (4.4) and the trace theorem, we have $u_n|_\mathcal{M} \rightharpoonup u_*|_\mathcal{M}$ weakly in $W^{1-1/p,p}(\mathcal{M}, \mathbb{R}^\nu)$ (we use that a bounded linear operator maps a weakly convergent sequence to a weakly convergent one). Then, by the compact embedding, up to a subsequence (not relabeled),

$$u_n|_\mathcal{M} \to u_*|_\mathcal{M} \text{ strongly in } L^q(\mathcal{M}, \mathbb{R}^\nu) \tag{4.9}$$

for each $q \in \left[1, \frac{2p}{2-p}\right)$. Since $u_n|_\mathcal{M} \in W^{1,p_n}(\mathcal{M}, \mathcal{N})$, (4.9) implies that $u_*(x) \in \mathcal{N}$ for $\mathcal{H}^2$-a.e. $x \in \mathcal{M}$. Using (4.8) and (4.9), we deduce that, up to a subsequence (not relabeled),

$$u_n|_\mathcal{M} \rightharpoonup u_*|_\mathcal{M} \text{ weakly in } W^{1,p}(\mathcal{M}, \mathbb{R}^\nu).$$

Then, by the weak convergence, Young's inequality and (4.8), we have

$$\begin{aligned} \int_\mathcal{M} \frac{|D_\top u_*|^p}{p}\,\mathrm{d}\mathcal{H}^2 &\leqq \liminf_{n\to+\infty} \int_\mathcal{M} \frac{|D_\top u_n|^p}{p}\,\mathrm{d}\mathcal{H}^2 \\ &\leqq \liminf_{n\to+\infty} \left(\int_\mathcal{M} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 + \left(\frac{1}{p} - \frac{1}{p_n}\right)\mathcal{H}^2(\mathcal{M})\right) \\ &\leqq \frac{4C}{\lambda} + \left(\frac{1}{p} - \frac{1}{2}\right)\mathcal{H}^2(\mathcal{M}), \end{aligned} \tag{4.10}$$

where we have also used that $|D_\top u_n| \leqq |Du_n|$. This implies that $u_*|_\mathcal{M} \in W^{1,2}(\mathcal{M}, \mathcal{N})$, because $p \in (1, 2)$ was arbitrarily chosen and we already know that $u_*(x) \in \mathcal{N}$ for $\mathcal{H}^2$-a.e. $x \in \mathcal{M}$. Moreover, $u_* \in L^\infty(\mathcal{M}, \mathbb{R}^\nu)$. Also, (4.10) yields

$$\int_\mathcal{M} \frac{|D_\top u_*|^2}{2}\,\mathrm{d}\mathcal{H}^2 \leqq \lim_{p\nearrow 2} \int_\mathcal{M} \frac{|D_\top u_*|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{4C}{\lambda}. \tag{4.11}$$

Next, we want to apply Lemma 4.1. Let $T_0 = T_0(\mathcal{M}) > 0$ be the constant of Lemma 4.1. Choose $T \in (0, T_0)$ sufficiently small and let $\Phi : A_\lambda \backslash A_\varrho \to \mathcal{M} \times [0, \lambda - \varrho)$ be defined by

$$\Phi(x) = (\Pi_{\mathcal{M}}(x), \operatorname{dist}(x, \mathcal{M})).$$

By Lemma 2.1, there exists $\lambda_{\mathcal{N}} > 0$ such that the nearest point retraction $\Pi_{\mathcal{N}} : \mathcal{N}_{\lambda_{\mathcal{N}}} \to \mathcal{N}$, where $\mathcal{N}_{\lambda_{\mathcal{N}}} = \{x \in \mathbb{R}^\nu : \operatorname{dist}(x, \mathcal{N}) < \lambda_{\mathcal{N}}\}$, is well defined and smooth. Setting $s_n = \|u_n - u_*\|_{L^{p_n}(\mathcal{M},\mathbb{R}^\nu)}$ (without loss of generality, we assume that for each $n \in \mathbb{N}$ large enough, $s_n > 0$) and using (4.8), (4.9) and (4.11), we observe that $s_n \to 0$ and for each sufficiently large $n \in \mathbb{N}$,

$$\int_{\mathcal{M}} \left(|D_\top u_n|^{p_n} + |D_\top u_*|^{p_n} + \frac{|u_n - u_*|^{p_n}}{s_n^{p_n/2}} \right) \mathrm{d}\mathcal{H}^2 \leqq C', \tag{4.12}$$

where C' is a positive constant independent of n. Applying Lemma 4.1 with $u = u_n|_{\mathcal{M}}$, $v = u_*|_{\mathcal{M}}$ and $T = s_n^{1/2}$, we obtain a map $w_n \in W^{1,p_n}(\mathcal{M} \times (0, s_n^{1/2}), \mathbb{R}^\nu)$ interpolating between $u_n|_{\mathcal{M}}$ and $u_*|_{\mathcal{M}}$. According to Lemma 4.1 (iii), (4.12) and the definition of s_n, if $n \in \mathbb{N}$ is large enough, we have $\operatorname{dist}(w_n(x, t), \mathcal{N}) < \lambda_{\mathcal{N}}$ for a.e. $(x, t) \in \mathcal{M} \times (0, s_n^{1/2})$. Thus, for each $n \in \mathbb{N}$ large enough, we can define the map

$$\varphi_n(y) = \Pi_{\mathcal{N}}(w_n(\Phi(y)))$$

for each $y \in \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}))$. Then we have the following: $\varphi_n \in W^{1,p_n}(\Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2})), \mathcal{N})$; $\operatorname{tr}_{\mathcal{M}}(\varphi_n) = u_n|_{\mathcal{M}}$, $\operatorname{tr}_{\Phi^{-1}(\mathcal{M}\times\{s_n^{1/2}\})}(\varphi_n) = u_*|_{\mathcal{M}} \circ \Phi_n \circ \Phi|_{\Phi^{-1}(\mathcal{M}\times\{s_n^{1/2}\})}$, where $\Phi_n : \mathcal{M} \times [s_n^{1/2}, T]$ is defined by $\Phi_n(x, t) = \Phi^{-1}(x, t - s_n^{1/2})$ and we have used that $\Phi^{-1}(x, 0) = x$; in view of Lemma 4.1 (ii) and (4.12),

$$\int_{\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}))} \frac{|D\varphi_n|^{p_n}}{p_n} \, \mathrm{d}y \leqq C'' s_n^{1/2}, \tag{4.13}$$

where $C'' > 0$ is a constant independent of n, but depending on the bilipschitz constant of Φ.

Now let $E = V \setminus \overline{A_\varrho}$. According to our construction, E is a finite union of bounded Lipschitz domains, and E contains K. Fix an arbitrary $w_* \in W^{1,2}(E, \mathcal{N})$ with $\operatorname{tr}_{\mathcal{M}}(w_*) = \operatorname{tr}_{\mathcal{M}}(u_*)$. Let $\zeta \in C_c([0, +\infty))$ be the Lipschitz map such that $\zeta = 1$ on $[0, 1]$, $\zeta(s) = 2 - s$ for $s \in [1, 2]$, $\zeta = 0$ on $[2, +\infty]$. Hence $\|\zeta'\|_{L^\infty([0,+\infty))} \leqq 1$. Next, for each sufficiently large $n \in \mathbb{N}$, define $w_n : E \to \mathcal{N}$ by

$$w_n(y) = \begin{cases} \varphi_n(y) & \text{if} \quad y \in \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}]), \\ w_*(\Psi_n(y)) & \text{if} \quad y \in E \setminus \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}]), \end{cases} \tag{4.14}$$

where $\Psi_n : E \backslash \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}]) \to E$ is the bilipschitz map defined by

$$\Psi_n(y) = \left(1 - \zeta\left(\frac{\operatorname{dist}(y, \mathcal{M})}{s_n^{1/3}}\right)\right) y + \zeta\left(\frac{\operatorname{dist}(y, \mathcal{M})}{s_n^{1/3}}\right) \Phi_n(\Phi(y)).$$

Notice that: $\mathrm{tr}_{\Phi^{-1}(\mathcal{M}\times\{s_n^{1/2}\})}(w_*\circ\Psi_n) = u_*|_{\mathcal{M}}\circ\Phi_n\circ\Phi|_{\Phi^{-1}(\mathcal{M}\times\{s_n^{1/2}\})}$; if $y\in E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])$ and $\mathrm{dist}(y,\mathcal{M})\leqq s_n^{1/3}$, then we have $\Psi_n(y)=\Phi_n(\Phi(y))$; if $y\in E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])$ and $\mathrm{dist}(y,\mathcal{M})\geqq 2s_n^{1/3}$, then $\Psi_n(y)=y$; for each $y\in E$, if $n\in\mathbb{N}$ is large enough, then $\Psi_n(y)$ is defined and $\Psi_n(y)\to y$ as $n\to+\infty$; for a.e. $y\in E\setminus\Phi(\mathcal{M}\times(0,s_n^{1/2}])$ and for each $n\in\mathbb{N}$ large enough,

$$\begin{aligned}|D\Psi_n(y)-\mathrm{Id}\,| &= \left|\zeta'\left(\frac{\mathrm{dist}(y,\mathcal{M})}{s_n^{1/3}}\right)\frac{D\mathrm{dist}(y,\mathcal{M})}{s_n^{1/3}}\otimes(\Phi_n(\Phi(y))-y)\right.\\ &\qquad\left.+\zeta\left(\frac{\mathrm{dist}(y,\mathcal{M})}{s_n^{1/3}}\right)(D\Phi_n(\Phi(y))\circ D\Phi(y)-\mathrm{Id})\right|\\ &\leqq Ls_n^{1/6},\end{aligned}\tag{4.15}$$

where $L>0$ is a constant independent of n, but depending on the bilipschitz constant of Φ. In the above computations we have used that $|\zeta'|\leqq 1$, $|D\mathrm{dist}(y,\mathcal{M})|=1$ and $|a\otimes b|\leqq|a||b|$ for $a,b\in\mathbb{R}^3$. In particular, when $n\in\mathbb{N}$ is large enough, $|D\Psi_n(y)-\mathrm{Id}\,|<1/2$. It is worth noting that, according to the definition of Ψ_n and [29, 4.8 (5)], Ψ_n is continuously differentiable on the three regions of $E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])$, where $\mathrm{dist}(\cdot,\mathcal{M})<s_n^{1/3}$, $\mathrm{dist}(\cdot,\mathcal{M})\in(s_n^{1/3},2\,s_n^{1/3})$, and $\mathrm{dist}(\cdot,\mathcal{M})>2\,s_n^{1/3}$, respectively. Then, using the inverse function theorem for Ψ_n on each of these three regions (namely, on compact subsets of these regions), and taking into account the definition of Ψ_n (namely, the injectivity of Ψ_n), one observes that Ψ_n admits the inverse Ψ_n^{-1} such that Ψ_n^{-1} is a.e. differentiable on E and $D\Psi_n^{-1}$ is sufficiently close to the identity a.e. on E. Furthermore, Ψ_n^{-1} is absolutely continuous on each segment lying in the closure of one of the images of Ψ_n of the three regions of $E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])$ considered above. Thus, Ψ_n is bilipschitz on $E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])$, as we mentioned earlier. It is also worth noting that for a.e. $y\in E$, if n is large enough, then $D\Psi_n(y)$ is defined, $D(w_*\circ\Psi_n)(y)=Dw_*(\Psi_n(y))\circ D\Psi_n(y)$ and $D\Psi_n(y)\to\mathrm{Id}$, $D(w_*\circ\Psi_n)(y)\to Dw_*(y)$ as $n\to+\infty$. In addition, if n is large enough, then for a.e. $y\in E$,

$$\begin{aligned}&\frac{|D(w_*\circ\Psi_n)(y)1_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])}(y)|^{p_n}}{p_n}\\ &\quad\leqq 2^{p_n}\frac{|Dw_*(\Psi_n(y))1_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])}(y)|^{p_n}}{p_n}\\ &\quad\leqq 2^{p_n}\left(\frac{|Dw_*(\Psi_n(y))1_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])}(y)|^{2}}{2}+\frac{1}{p_n}-\frac{1}{2}\right)\\ &\quad\leqq h(y),\end{aligned}$$

where we have used (4.15), the fact that $|\mathrm{Id}|=\sqrt{3}$, Young's inequality and since $|Dw_*|^2\in L^1(E)$, there exists $h\in L^1(E)$ such that the last estimate holds.

Taking into account the above observations and using the Lebesgue dominated convergence theorem, we deduce that

$$\int_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])} \frac{|Dw_n|^{p_n}}{p_n}\,\mathrm{d}y \to \int_E \frac{|Dw_*|^2}{2}\,\mathrm{d}y \tag{4.16}$$

as $n \to +\infty$. The mapping w_n is a competitor for u_n in E, and hence, in view of (4.14),

$$\int_E \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}y \leqq \int_{\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}))} \frac{|D\varphi_n|^{p_n}}{p_n}\,\mathrm{d}y + \int_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])} \frac{|Dw_n|^{p_n}}{p_n}\,\mathrm{d}y. \tag{4.17}$$

Using (4.6) with V replaced by E (which is possible thanks to (4.4) and proceeding as in (4.5)), together with (4.13), (4.16) and (4.17), we obtain

$$\begin{aligned}\int_E \frac{|Du_*|^2}{2}\,\mathrm{d}y = \lim_{p\nearrow 2}\int_E \frac{|Du_*|^p}{p}\,\mathrm{d}y &\leqq \liminf_{n\to+\infty}\int_E \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}y\\ &\leqq \limsup_{n\to+\infty}\int_E \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}y\\ &\leqq \int_E \frac{|Dw_*|^2}{2}\,\mathrm{d}y.\end{aligned}$$

This implies that $\int_E \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}y \to \int_E \frac{|Du_*|^2}{2}\,\mathrm{d}y$ and u_* is a 2-minimizer in E. In particular, according to Lemma 4.3 below, it holds

$$\|Du_n - Du_*\|^{p_n}_{L^{p_n}(E,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \to 0,$$

which proves (4.7).

Since u_* is a 2-minimizer in E and $K \Subset E$ is open, u_* is a 2-minimizer in K and, according to [52, Theorem II], there exists a finite set $S_K \subset K$ such that $u_* \in C^\infty(K \setminus S_K, \mathcal{N})$.

In order to reach the conclusion, we use a diagonal argument and rely on the fact that one can write $U = \bigcup_{m\in\mathbb{N}} K_m$, with $K_m \subset K_{m+1}$ so that if $K \Subset U$, then $K \subset K_m$ for some $m \in \mathbb{N}$. This completes our proof of Proposition 4.2. □

Lemma 4.3. *Let $U \subset \mathbb{R}^3$ be open and bounded, $u_* \in W^{1,2}(U,\mathbb{R}^\nu)$, $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_n}(U,\mathbb{R}^\nu)$. If $u_n \rightharpoonup u_*$ weakly in $W^{1,p}(U,\mathbb{R}^\nu)$ for all $p \in (1,2)$ and $\int_U |Du_n|^{p_n}\,\mathrm{d}x \to \int_U |Du_*|^2\,\mathrm{d}x$ as $n \to +\infty$, then $\int_U |Du_n - Du_*|^{p_n}\,\mathrm{d}x \to 0$ as $n \to +\infty$.*

Proof. The desired convergence comes by using Hanner's inequality, see [56, Lemma 6.3]. To apply [56, Lemma 6.3], we only need to prove that

$$4\|Du_*\|^2_{L^2(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \leqq \liminf_{n\to+\infty}\|Du_n + Du_*\|^{p_n}_{L^{p_n}(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)}. \tag{4.18}$$

Let $p \in (1,2)$ be arbitrary. Notice that $Du_n + Du_* \rightharpoonup 2Du_*$ weakly in $L^p(U, \mathbb{R}^\nu \otimes \mathbb{R}^3)$ and hence

$$\liminf_{n\to+\infty} \|Du_n + Du_*\|_{L^p(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \geqq 2\|Du_*\|_{L^p(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)}. \tag{4.19}$$

Using Hölder's inequality and (4.19), we obtain

$$\begin{aligned}\liminf_{n\to+\infty} \|Du_n + Du_*\|^{p_n}_{L^{p_n}(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} &\geqq \liminf_{n\to+\infty} |U|^{1-\frac{p_n}{p}} \|Du_n + Du_*\|^{p_n}_{L^p(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \\ &\geqq 4|U|^{1-\frac{2}{p}} \|Du_*\|^2_{L^p(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)}. \end{aligned}\tag{4.20}$$

Letting $p \nearrow 2$ and using the continuity of the L^p-norm, in view of (4.20), we obtain (4.18), which completes our proof of Lemma 4.3. □

5. The Singular Set

Let $p \in (1,2)$ and $u_p \in W^{1,p}(\Omega, \mathcal{N})$ be a p-minimizer in Ω. For each Borel-measurable set $E \in \mathcal{B}(\overline{\Omega})$, we define the positive Radon measure μ_p at E by

$$\mu_p(E) := (2-p)\int_E \frac{|Du_p|^p}{p}\,\mathrm{d}x.$$

Let $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of p_n-minimizers in Ω. We assume that for each $n \in \mathbb{N}$ large enough, setting $\mu_n = \mu_{p_n}$, it holds

$$\mu_n(\overline{\Omega}) \leqq C_0, \tag{5.1}$$

where $C_0 > 0$ is a constant independent of n. According to Proposition 2.21, the condition (5.1) is satisfied whenever for each $n \in \mathbb{N}$ large enough, $\mathrm{tr}_{\partial\Omega}(u_n) = g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$. Next, in view of (5.1), using the Banach-Alaoglu theorem, we obtain a positive Radon measure $\mu_* \in (C(\overline{\Omega}))'$ such that, up to a subsequence (not relabeled),

$$\mu_n \overset{*}{\rightharpoonup} \mu_* \text{ weakly* in } (C(\overline{\Omega}))'. \tag{5.2}$$

We denote the support of μ_* by $S_* \subset \overline{\Omega}$.

Remark 5.1. If the measure μ_* is independent of the subsequence, the entire sequence $(\mu_n)_{n\in\mathbb{N}}$ weakly* converges to μ_* as $n \to +\infty$, since in the latter case each subsequence of $(\mu_n)_{n\in\mathbb{N}}$ contains a subsequence weakly* converging to μ_*. In general, the limit measure μ_* does depend on the subsequence. To see this, we consider the projective plane $\mathcal{N} = \mathbb{RP}^2$, whose universal covering $\pi : \mathbb{S}^2 \to \mathbb{RP}^2$ identifies antipodal points (see e.g. [37, Example 0.4]), so that, in particular, $\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}/2\mathbb{Z}$. Defining first $\hat{g} : \mathbb{S}^2 \to \mathbb{S}^1$ by

$$\hat{g}(x) = \frac{(x_1x_2, x_3)}{(x_1^2 + x_3^2)^{1/2}(x_2^2 + x_3^2)^{1/2}},$$

(this is well defined since $x_1^2x_2^2+x_3^2 = x_1^2x_2^2+1-x_1^2-x_2^2 = (1-x_1^2)(1-x_2^2) = (x_2^2+x_3^2)(x_1^2+x_3^2)$) one checks that $\hat{g}(x_1, x_2, x_3) = \hat{g}(x_2, x_1, x_3) = -\hat{g}(x_1, -x_2, -x_3)$, that $\hat{g}$ has degree 1 around small circles around $(1, 0, 0)$ and $(-1, 0, 0)$ and degree -1 around small circles around $(0, 1, 0)$ and $(0, -1, 0)$, and that $\hat{g} \in W^{1/2,2}(\mathbb{S}^2, \mathbb{S}^1)$. We define $g = \tau \circ \hat{g}$, where the map $\tau \colon \mathbb{S}^1 \to \mathbb{RP}^2$ defined by the condition that $\tau(\cos(\theta), \sin(\theta)) = \pi(0, \cos(\frac{\theta}{2}), \sin(\frac{\theta}{2}))$. We have $g \in W^{1/2,2}(\mathbb{S}^2, \mathbb{RP}^2)$; since τ is not homotopic to a constant, g is topologically nontrivial in small circles around the same points as $\hat{g}$; we have $g(x_1, x_2, x_3) = g(x_2, x_1, x_3) = \sigma(g(x_1, -x_2, -x_3))$, where $\sigma \colon \mathbb{RP}^2 \to \mathbb{RP}^2$ is the isometry characterized by $\sigma(\pi(y_1, y_2, y_3)) = \pi(y_1, y_3, -y_2)$, so that $\sigma(\tau(z)) = \tau(-z)$. In particular, if $\rho(x_1, x_2, x_3) = (x_1, -x_2, -x_3)$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence of p_n-minimizers in B_1^3 having g as a trace on $\mathbb{S}^2$, $(\sigma \circ u_n \circ \rho)$ is also. Define a sequence $(v_k)_{k\in\mathbb{N}}$ of $p_{\lfloor \frac{k}{2} \rfloor}$-minimizers in B_1^3 having g as a trace on $\mathbb{S}^2$ by $v_{2k} = u_k$ and $v_{2k+1} = \sigma \circ u_k \circ \rho$ for each $k \in \mathbb{N}$. Letting $k \to +\infty$, by Proposition 7.25, the corresponding limit measure μ_* to the subsequence $(v_{2k})_{k\in\mathbb{N}}$ has as support either $[(1, 0, 0), (0, 1, 0)] \cup [(-1, 0, 0), (0, -1, 0)]$ or $[(1, 0, 0), (0, -1, 0)] \cup [(-1, 0, 0), (0, 1, 0)]$. Assume that the support of μ_* is $([(1, 0, 0), (0, 1, 0)] \cup [(-1, 0, 0), (0, -1, 0)])$. Then the corresponding limit measure to the subsequence $(v_{2k+1})_{k\in\mathbb{N}}$ has as support $([(1, 0, 0), (0, -1, 0)] \cup [(-1, 0, 0), (0, 1, 0)])$. Altogether, $(v_k)_{k\in\mathbb{N}}$ has two subsequences yielding two different limit measures.

Lemma 5.2. *Let $\eta > 0$ be the constant of Lemma 3.12, where $\kappa = 1/2$, $p_0 = 3/2$, $\Psi(x) = x + x_0$. Assume that $B_r^3(x_0) \subset \Omega$ and $\mu_*(\overline{B}_r^3(x_0)) < \eta r$. Then $\mu_*(B_{r/2}^3(x_0)) = 0$.*

Proof. In this proof, every ball is centered at point x_0. It is a well-known fact (see, for instance, [28, Section 1.9]) that the weak* convergence of the Borel measures μ_n is equivalent to the following two inequalities

$$\mu_*(F) \geqq \limsup_{n\to+\infty} \mu_n(F), \quad \mu_*(G) \leqq \liminf_{n\to+\infty} \mu_n(G) \tag{5.3}$$

whenever $F \subset \overline{\Omega}$ is closed and $G \subset \overline{\Omega}$ is open. Notice that this is where we take advantage of working with $(C(\overline{\Omega}))'$ instead of $(C_0(\Omega))'$. Using (5.3) and the facts that $\mu_*(\overline{B}_r^3) < \eta r$ and $\mu_n(\overline{B}_r^3) = \mu_n(B_r^3)$, we obtain the following estimate

$$\eta r > \limsup_{n\to+\infty} \mu_n(B_r^3).$$

This estimate, together with the fact that $\eta r^{3-p_n} \to \eta r$ as $n \to +\infty$, implies that for all sufficiently large $n \in \mathbb{N}$, it holds

$$\mu_n(B_r^3) < \eta r^{3-p_n}.$$

But then Lemma 3.12 says that for all sufficiently large $n \in \mathbb{N}$,

$$\mu_n(B_{r/2}^3) \leqq (2 - p_n)Cr^{3-p_n}, \tag{5.4}$$

where $C > 0$ is a constant independent of n. Letting n tend to $+\infty$ in (5.4) and using (5.3), we complete our proof of Lemma 5.2. □

The following monotonicity of the p-energy holds, which will be used later.

Lemma 5.3. *Let $p \in [1, 3)$, $x_0 \in \Omega$ and $0 < \varrho < r < \operatorname{dist}(x_0, \partial\Omega)$. Let $u_p \in W^{1,p}(\Omega, \mathcal{N})$ be a p-minimizer. Then*

$$\varrho^{p-3} \int_{B^3_\varrho(x_0)} \frac{|Du_p|^p}{p} \,\mathrm{d}x \leqq r^{p-3} \int_{B^3_r(x_0)} \frac{|Du_p|^p}{p} \,\mathrm{d}x.$$

Proof. For a proof, the reader may consult Sect. 4 in [34]. □

Corollary 5.4. *Let $x_0 \in \Omega$. Then the function*

$$r \in (0, \operatorname{dist}(x_0, \partial\Omega)) \mapsto \frac{\mu_*(\overline{B}^3_r(x_0))}{r}$$

is nondecreasing.

Proof of Corollary 5.4. Let $0 < \varrho < r < \operatorname{dist}(x_0, \partial\Omega)$. In view of Lemma 5.3, for each $n \in \mathbb{N}$ and for an arbitrary $\varepsilon \in (0, r - \varrho)$,

$$(\varrho + \varepsilon)^{p_n - 3} \mu_n(B^3_{\varrho+\varepsilon}(x_0)) \leqq r^{p_n - 3} \mu_n(\overline{B}^3_r(x_0)).$$

Letting n tend to $+\infty$ and using (5.3), we obtain

$$(\varrho + \varepsilon)^{-1} \mu_*(B^3_{\varrho+\varepsilon}(x_0)) \leqq r^{-1} \mu_*(\overline{B}^3_r(x_0)).$$

Then, letting $\varepsilon \searrow 0$ and using the fact that $\mu_*(\overline{\Omega}) < +\infty$, we conclude our proof of Corollary 5.4. □

Thanks to Corollary 5.4, for each $x_0 \in \Omega$, the 1-dimensional density

$$\Theta_1(\mu_*, x_0) := \lim_{r \to 0+} \frac{\mu_*(\overline{B}^3_r(x_0))}{2r} = \lim_{r \to 0+} \frac{\mu_*(B^3_r(x_0))}{2r} \tag{5.5}$$

exists and finite. In order to lighten the notation, we shall simply write $\Theta_1(x_0)$ instead of $\Theta_1(\mu_*, x_0)$ when appropriate.

Proposition 5.5. *Let $\eta > 0$ be the constant of Lemma 3.12, where $\kappa = 1/2$, $p_0 = 3/2$ and $\Psi(x) = x + x_0$ whenever $E = B^3_r(x_0)$. Then it holds*

$$\Omega \cap S_* = \{x \in \Omega : \Theta_1(x) > 0\} = \left\{x \in \Omega : \Theta_1(x) \geqq \frac{\eta}{2}\right\}.$$

Proof. First, assume by contradiction that for some $x \in \Omega$, $\Theta_1(x) = a \in (0, \eta/2)$. Fix

$$\varepsilon \in \left(0, \min\left\{\frac{a}{2}, \frac{\eta}{2} - a\right\}\right).$$

Then there exists $\delta > 0$ such that for all $r \in (0, \delta)$,

$$0 < a - \varepsilon < \frac{\mu_*(\overline{B}^3_r(x))}{2r} < \frac{\eta}{2}.$$

However, Lemma 5.2 says that $\mu_*(B^3_{r/2}(x)) = 0$, which leads to a contradiction and proves that

$$\{x \in \Omega : \Theta_1(x) > 0\} = \{x \in \Omega : \Theta_1(x) \geqq \eta/2\}.$$

On the other hand, by the definition of the support of a measure, $x \in \Omega \cap S_*$ if and only if for all sufficiently small $r > 0$, $\mu_*(\overline{B}^3_r(x)) > 0$, which holds, according to Lemma 5.2, if and only if for all sufficiently small $r > 0$, $\mu_*(\overline{B}^3_r(x)) \geqq \eta r$. The last holds if and only if $\Theta_1(x) \geqq \eta/2$. This proves that $\Omega \cap S_* = \{x \in \Omega : \Theta_1(x) \geqq \eta/2\}$ and completes our proof of Proposition 5.5. □

Proposition 5.6. *Let $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of p_n-minimizers in Ω satisfying (5.1). Then there exists a mapping $u_* \in W^{1,2}_{\mathrm{loc}}(\Omega \backslash S_*, \mathcal{N})$ such that, up to a subsequence (not relabeled), $u_n \to u_*$ a.e. in Ω and for which the conclusions of Proposition 4.2 apply with $U = \Omega \setminus S_*$.*

Remark 5.7. For interior quantitative estimates of the weak gradient of u_*, the reader may consult Proposition 5.23.

Proof of Proposition 5.6. Let $K \Subset \Omega \backslash S_*$ be open. Using (5.2), a standard covering argument and Lemma 3.12, we conclude that there exists a positive constant C, possibly depending only on K, Ω and $\mathcal{N}$, such that, up to a subsequence (not relabeled), for each $n \in \mathbb{N}$,

$$\int_K \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \leqq C.$$

Applying a diagonal argument, we observe that, up to a subsequence (not relabeled), the condition (4.1) of Proposition 4.2 is satisfied, where $U = \Omega\backslash S_*$, for our sequence of p_n-minimizers. Thus, the conclusions of Proposition 4.2 apply, yielding the desired mapping u_*. This completes our proof of Proposition 5.6. □

5.1. The Limit Measure is the Weight Measure of the Stationary Varifold

Recall that a set $E \subset \mathbb{R}^3$ is said to be countably $\mathcal{H}^1$-rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^1 \to \mathbb{R}^3$ such that

$$\mathcal{H}^1\Big(E \setminus \bigcup_{i=0}^{+\infty} f_i(\mathbb{R}^1)\Big) = 0.$$

Let $\mathrm{G}(3, 1)$ be the Grassmann manifold, which is the space of 1-dimensional subspaces of $\mathbb{R}^3$. Following Allard (see [2]), for each $A \in \mathrm{G}(3, 1)$, we shall also use "A" to denote the orthogonal projection of $\mathbb{R}^3$ onto A. That is, $\mathrm{G}(3, 1) = \{A \in M_3(\mathbb{R}) : A \circ A = A,\ A^{\mathrm{T}} = A \text{ and } \mathrm{Im}(A) = A\}$, where $M_3(\mathbb{R})$ denotes the space of real 3×3 matrices. We also define $\mathrm{G}_3(\Omega) = \Omega \times \mathrm{G}(3, 1)$. Recall that V is said to be a 1-dimensional varifold in Ω if V is a positive Radon measure on $\mathrm{G}_3(\Omega)$.

Let $S \subset \Omega$ be countably $\mathcal{H}^1$-rectifiable and $\mu \in (C(\overline{\Omega}))'$ be a positive Radon measure satisfying $\mu \llcorner \Omega = \theta \mathcal{H}^1 \llcorner S$ for some nonnegative $\theta \in L^1(\Omega)$. Rectifiability gives that for $\mathcal{H}^1$-a.e. $x \in S$ (or, equivalently, for μ-a.e. $x \in \Omega$), there exists a unique approximate tangent plane $T_x S \in \mathrm{G}(3,1)$ to S at x (see, for instance, [4, Theorem 2.83 (i)]). For μ-a.e. $x \in \Omega$, we denote by $A(x) \in \mathrm{G}(3,1)$ the orthogonal projection onto $T_x S$. We define the varifold V, which is naturally associated to μ, as the pushforward measure $V = (\mathrm{Id}, A)_{\#}\mu \llcorner \Omega$. Thus, for each Borel-measurable set $E \in \mathcal{B}(\mathrm{G}_3(\Omega))$, $V(E) = \mu(\{x \in \Omega : (x, A(x)) \in E\})$. The varifold V is said to be stationary (see Section 4.2 in [2]) if its first variation vanishes, namely

$$\int_\Omega A(x) : D\xi(x)\,\mathrm{d}\mu(x) = 0 \quad \forall \xi \in C_c^1(\Omega, \mathbb{R}^3).$$

Proposition 5.8. *The set $S_* \cap \Omega$ is countably $\mathcal{H}^1$-rectifiable and $\mathcal{H}^1(S_* \cap \Omega) < +\infty$. The measure $V_* = (\mathrm{Id}, A_*)_{\#}\mu_* \llcorner \Omega$ is a stationary 1-dimensional rectifiable varifold in Ω, where for μ_*-a.e. $x \in \Omega$, $A_*(x)$ represents the orthogonal projection onto the approximate tangent plane $T_x S_*$ to S_* at x. Furthermore, it holds $\mu_* \llcorner \Omega(dx) = \Theta_1(x)\mathcal{H}^1 \llcorner (S_* \cap \Omega)(dx)$, where the density Θ_1 is defined in (5.5).*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_n}(\Omega, \mathcal{N})$, where $(p_n)_{n\in\mathbb{N}} \subset (3/2, 2)$, be a sequence of p_n-minimizers satisfying (5.2). For each $n \in \mathbb{N}$, define the map $A^{p_n} : \Omega \to M_3(\mathbb{R})$ by

$$A_{i,j}^{p_n} = \frac{2 - p_n}{p_n}(|Du_n|^{p_n}\delta_{i,j} - p_n|Du_n|^{p_n-2}\langle D_i u_n, D_j u_n\rangle).$$

Then observe that

$$(A^{p_n})^{\mathrm{T}} = A^{p_n}, \quad \mathrm{tr}(A^{p_n}) = (3 - p_n)\frac{\mathrm{d}\mu_n}{\mathrm{d}x} \geqq \frac{\mathrm{d}\mu_n}{\mathrm{d}x} \tag{5.6}$$

and

$$|A^{p_n}| = \sqrt{A^{p_n} : A^{p_n}} = \sqrt{3 - 2p_n + p_n^2}\,\frac{\mathrm{d}\mu_n}{\mathrm{d}x} \leqq \sqrt{3}\,\frac{\mathrm{d}\mu_n}{\mathrm{d}x}, \tag{5.7}$$

where $\frac{\mathrm{d}\mu_n}{\mathrm{d}x} = (2 - p_n)\frac{|Du_n|^{p_n}}{p_n}$. For each $v \in \mathbb{S}^2$,

$$\sum_{i,j=1}^3 \langle A_{i,j}^{p_n} v_i, v_j\rangle = \frac{2 - p_n}{p_n}\left(|Du_n|^{p_n} - p_n|Du_n|^{p_n-2}\left|\sum_{i=1}^3 v_i D_i u_n\right|^2\right) \leqq \frac{\mathrm{d}\mu_n}{\mathrm{d}x}, \tag{5.8}$$

which implies that all eigenvalues of A^{p_n} are less than or equal to $\frac{\mathrm{d}\mu_n}{\mathrm{d}x}$. Furthermore, by (2.28),

$$\int_\Omega A^{p_n}(x) : D\xi(x)\,\mathrm{d}x = 0 \quad \forall \xi \in C_c^1(\Omega, \mathbb{R}^3). \tag{5.9}$$

In view of (5.1) and (5.7), there exists $\widetilde{A} \in (C_0(\Omega, M_3(\mathbb{R})))'$ such that, up to a subsequence (not relabeled), $A^{p_n}\,\mathrm{d}x \overset{*}{\rightharpoonup} \widetilde{A}$ weakly* in $(C_0(\Omega, M_3(\mathbb{R})))'$. By (5.2) and (5.7),

$$|\widetilde{A}| \leqq \sqrt{3}\mu_* \llcorner \Omega, \tag{5.10}$$

which implies that $\widetilde{A}$ is absolutely continuous with respect to $\mu_* \llcorner \Omega$. Then, by the Radon-Nikodým theorem, there exists $F \in L^1(\Omega, M_3(\mathbb{R}), \mu_*)$ such that $\widetilde{A} = F\mu_* \llcorner \Omega$ in $(C_0(\Omega, M_3(\mathbb{R})))'$. Letting n tend to $+\infty$ and taking into account (5.6), (5.8) and (5.9), we observe that for μ_*-a.e. $x \in \Omega$, $(F(x))^{\mathrm{T}} = F(x)$, $\mathrm{tr}(F(x)) \geqq 1$, the eigenvalues of $F(x)$ are less than or equal to 1 and

$$\int_\Omega F(x) : D\xi(x)\, \mathrm{d}\mu_*(x) = 0 \quad \forall \xi \in C_c^1(\Omega, \mathbb{R}^3). \tag{5.11}$$

According to [6, Proposition 3.1] and [6, Lemma 2.3], the identity (5.11) implies that there exists an $\mathcal{H}^1$-rectifiable set $R \subset \Omega$ and a map $\lambda \in L^\infty(R, M_3(\mathbb{R}))$ satisfying the following conditions. For $\mathcal{H}^1$-a.e. $x_0 \in R$, $|\lambda(x_0)| = 1$, $\lambda(x_0) \in \mathrm{G}(3, 1)$ is the orthogonal projection matrix onto $T_{x_0}R$ such that $\widetilde{A} \llcorner \{\Theta_1^*(|\widetilde{A}|) > 0\}(dx) = \Theta_1^*(|\widetilde{A}|, x)\lambda(x)\mathcal{H}^1 \llcorner R(dx)$, where $T_{x_0}R$ is the approximate tangent plane to R at x_0 and $\Theta_1^*(|\widetilde{A}|, x_0) = \limsup_{r\to 0+} \frac{|\widetilde{A}|(B_r^3(x_0))}{2r}$. Then, for μ_*-a.e. $x \in \Omega$, two eigenvalues of $F(x)$ are zeros. Since for μ_*-a.e. $x \in \Omega$, $\mathrm{tr}(F(x)) \geqq 1$ and the eigenvalues of $F(x)$ are less than or equal to 1, we deduce that the eigenvalues of $F(x)$ (up to a permutation) are $(1, 0, 0)$ and $|F(x)| = 1$ for μ_*-a.e. $x \in \Omega$. Using this, together with (5.5) and Proposition 5.5, we observe that $|\widetilde{A}| = \mu_* \llcorner \Omega$, $\Theta_1^*(|\widetilde{A}|) = \Theta_1(\mu_*)$ and $R = S_* \cap \Omega$. In particular, for μ_*-a.e. $x \in \Omega$, $F(x) = A_*(x)$ and $V_*(dA, dx) = \delta_{A_*(x)}(dA) \otimes \mu_*(dx) = \delta_{A_*(x)}(dA) \otimes \Theta_1(\mu_*, x)\mathcal{H}^1 \llcorner (S_* \cap \Omega)(dx)$, which, together with (5.11), implies that V_* is a 1-dimensional stationary rectifiable varifold (see [2,26]). This completes our proof of Proposition 5.8. □

Remark 5.9. Our proof of Proposition 5.8, based on [6, Proposition 3.1], differs from Canevari's proof for the three-dimensional Landau-de Gennes model [21, Proposition 55]. His strategy consists in first showing that the set $S_* \cap \Omega$ is countably $\mathcal{H}^1$-rectifiable using the Moore-Preiss rectifiability theorem (see [47, Theorem 3.5], [50, Theorem 5.3]); next, using the Ambrosio-Soner construction (see [5, Lemma 3.9]), he proves that the Radon-Nikodým derivative of the limit tensor has the eigenvalues (up to a permutation) $(1, 0, 0)$, and concludes that the naturally associated one-dimensional varifold is stationary; the rectifiability of the latter then follows from Rectifiability Theorem in [2, Section 5.5]. Our more direct strategy of proof could be adapted to work in Canevari's setting, and his strategy of proof would provide another pathway to Proposition 5.8.

5.2. Energy Bounds on a Cylinder

Let us define the notion of a uniform Lipschitz triangulation on the lateral surface of a (finite) cylinder $\Lambda_{1,L} := B_1^2 \times (-L, L)$. Let $L \geqq 1$ and $h \in (0, 1)$. Set $n := \lfloor 1/h \rfloor \geqq 1$. For each $k \in [0, n-1]$, we define the line segment $S_k = [(e^{\frac{2\pi i k}{n}}, -L), (e^{\frac{2\pi i k}{n}}, L)] \subset \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$ and, for $j \in [0, 2\lfloor L \rfloor n]$ the points

$$a_j^k := \left(e^{\frac{2\pi i k}{n}}, -L + \frac{Lj}{\lfloor L \rfloor n}\right) \in \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3.$$

For each $k \in [0, n-1]$ and $j \in [0, 2\lfloor L \rfloor n - 1]$, let γ_j^k be the union of two geodesics (*a geodesic cross*) on $\overline{\Gamma}_{1,L}$, where one of which connecting a_j^k with a_{j+1}^{k+1} and the

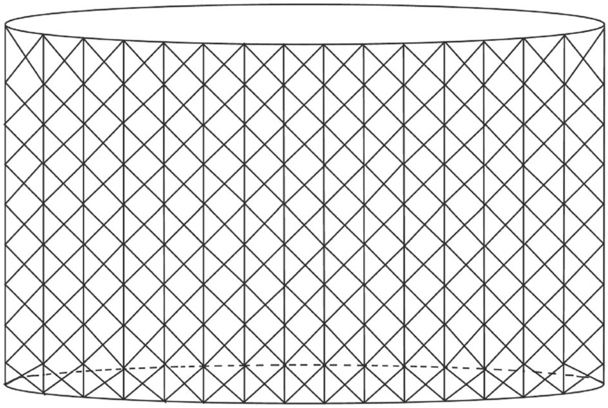

Fig. 1. A uniform Lipschitz triangulation of the lateral surface of a cylinder

other connecting a^k_{j+1} with a^{k+1}_j. By $c_{k,z}$, where $z \in \{-L, L\}$, we denote the arc on $\mathbb{S}^1 \times \{z\}$ connecting $a^k_{2\lfloor L\rfloor n}$ with $a^{k+1}_{2\lfloor L\rfloor n}$ if $z = L$ and a^k_0 with a^{k+1}_0 if $z = -L$. Observe that the union of all the line segments S_k, all the geodesic crosses γ^k_j and all the arcs $c_{k,z}$ provides us with the decomposition of $\overline{\Gamma}_{1,L}$ into mutually disjoint 0, 1 and 2-dimensional cells. Hereinafter, we denote by $SO(2)$ the subgroup of $SO(3)$ of rotations with respect to the axis $\{0\}\times\mathbb{R}$, and whose action on the plane $\mathbb{R}^2\times\{0\}$ coincides with the action of the orthogonal group $SO(2)$ on the plane. Thus, for each $\omega \in SO(2)$, we obtain a decomposition of $\overline{\Gamma}_{1,L}$ into mutually disjoint 0, 1 and 2-dimensional cells, namely

$$\overline{\Gamma}_{1,L} = \bigcup_{i=0}^{2}\bigcup_{j=1}^{k_i} \omega(E_{i,j}) = \bigcup_{i=0}^{2}\bigcup_{j=1}^{k_i} F_{i,j}, \tag{5.12}$$

where $\{F_{0,1}, \dots, F_{0,k_0}\}$ is the set of points and for each $i \in \{1, 2\}$, there exists a bilipschitz homeomorphism

$$\Phi_{i,j} : F_{i,j} \to B^i_h \text{ with } \|D_\top \Phi_{i,j}\|_{L^\infty(F_{i,j})} + \|D_\top(\Phi_{i,j})^{-1}\|_{L^\infty(B^i_h)} \leqq A_4, \tag{5.13}$$

where $A_4 > 0$ is an absolute constant; see Fig. 1. The map $\Phi_{i,j}$ in (5.13) extends by continuity to a bilipschitz map between $\overline{F}_{i,j}$ and $\overline{B}^i_h$ with the same bilipschitz constant A_4 as in (5.13).

The decomposition (5.12) we shall call a uniform Lipschitz (or simply a Lipschitz) triangulation of $\overline{\Gamma}_{1,L}$ of step h with the 1-skeleton $\Sigma = \bigcup_{i=0}^{1}\bigcup_{j=0}^{k_i} F_{i,j}$.

To lighten the notation, we denote the collection of 1-skeletons of all Lipschitz triangulations of $\overline{\Gamma}_{1,L}$ of step h by $\mathrm{RC}_{L,h}$, namely

$$\mathrm{RC}_{L,h} := \left\{ \bigcup_{i=0}^{1}\bigcup_{j=1}^{k_i} \omega(E_{i,j}) : \omega \in SO(2) \right\},$$

where $\bigcup_{i=0}^{1}\bigcup_{j=1}^{k_i} E_{i,j}$ is the union of all the line segments S_k, all the geodesic crosses γ_j^k and all the arcs $c_{k,z}$.

It is worth noting that Lemmas 5.10, 5.11, 5.12 are the counterparts of Lemmas 3.7, 3.8, 3.10 on the lateral surface of a cylinder. A consequence of Lemma 5.12 is Corollary 5.13, which provides us with the key arguments that will be used in the proof of Proposition 5.14. The latter gives us the estimates of the p-energy on a cylinder, which will be used to prove the fact that the limit measure has a finite set of values.

Lemma 5.10. *Let $p \in [1, +\infty)$, $L \geqq 1$, $h \in (0, 1)$ and $\mathcal{Y}$ be a compact Riemannian manifold or a Euclidean space. Let $g \in W^{1,p}(\Gamma_{1,L}, \mathcal{Y})$ and $\Sigma \in \mathrm{RC}_{L,h}$. Then for $\mathcal{H}^1$-a.e. $\omega \in SO(2)$ we have $\mathrm{tr}_{\omega(\Sigma\cap\Gamma_{1,L})}(g) = g|_{\omega(\Sigma\cap\Gamma_{1,L})} \in W^{1,p}(\omega(\Sigma \cap \Gamma_{1,L}), \mathcal{Y})$. Furthermore, there exists $\omega \in SO(2)$ such that*

$$\mathrm{tr}_{\omega(\Sigma\cap\Gamma_{1,L})}(g) = g|_{\omega(\Sigma\cap\Gamma_{1,L})} \in W^{1,p}(\omega(\Sigma \cap \Gamma_{1,L}), \mathcal{Y})$$

and

$$\int_{\omega(\Sigma\cap\Gamma_{1,L})} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1 \leqq \frac{A_5}{h}\int_{\Gamma_{1,L}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2,$$

where $A_5 > 0$ is an absolute constant.

Proof. For a proof, we refer to [33, Lemma 3.4]. The main ingredient is the integralgeometric formula

$$\int_{SO(2)} \mathrm{d}\omega \int_{\omega(\Sigma\cap\Gamma_{1,L})} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1 = \frac{\mathcal{H}^1(\Sigma\cap\Gamma_{1,L})}{\mathcal{H}^2(\Gamma_{1,L})}\int_{\Gamma_{1,L}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2$$

(for the integralgeometric formulas, the reader may consult [20,30]; see also [33, Lemma 3.3]). □

Lemma 5.11. *Let $p_0 \in (1, 2)$, $p \in [p_0, 2)$, $L \geqq 1$ and $h \in (0, 1)$. Then there exists a constant $\kappa_0 = \kappa_0(p_0, \mathcal{N}) > 0$ such that the following holds. Let $\Sigma \in \mathrm{RC}_{L,h}$ and F be a 2-cell of the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ. Let $g \in W^{1,p}(F, \mathcal{N})$ and $\mathrm{tr}_{\partial F}(g) \in W^{1,p}(\partial F, \mathcal{N})$, where ∂F denotes the relative boundary of F. If*

$$\int_F \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 + h\int_{\partial F} \frac{|D_\top \,\mathrm{tr}_{\partial F}(g)|^p}{p}\,\mathrm{d}\mathcal{H}^1 \leqq \frac{\kappa_0 h^{2-p}}{2-p},$$

then $\mathrm{tr}_{\partial F}(g)$ is nullhomotopic.

Proof. The proof follows by reproducing the proof of Lemma 3.8 with minor modifications. Essentially, it is enough to replace the 2-dimensional cell in the proof of Lemma 3.8 by the 2-cell F of Lemma 5.11. □

Lemma 5.12. *Let* $p_0 \in (1,2)$, $p \in [p_0, 2)$, $L_0 \geqq 2$, $h \in (0, 1/2]$ *and* $g \in W^{1,p}(\Gamma_{1,L_0}, \mathcal{N})$, *where* Γ_{1,L_0} *is defined in* (2.18). *Then there exists* $\kappa = \kappa(p_0, \mathcal{N}) > 0$ *such that the following holds. Assume that*

$$\int_{\Gamma_{1,L_0}} \frac{|D_\top g|^p}{p} \, \mathrm{d}\mathcal{H}^2 \leqq \frac{\kappa h^{2-p}}{2-p}. \tag{5.14}$$

There exist $z_- \in (-L_0 + h, -L_0 + 2h)$, $z_+ \in (L_0 - 2h, L_0 - h)$, $\varphi_h \in W^{1,p}((B_1^2 \backslash \overline{B}_{1-h}^2) \times (z_-, z_+), \mathcal{N})$ *such that the following assertions hold.*

(i) $\mathrm{tr}_{\mathbb{S}^1 \times (z_-, z_+)}(\varphi_h) = g|_{\mathbb{S}^1 \times (z_-, z_+)}$, $v_h := \mathrm{tr}_{\partial B_{1-h}^2 \times (z_-, z_+)}(\varphi_h) \in W^{1,2}(\partial B_{1-h}^2 \times (z_-, z_+), \mathcal{N})$.

(ii) *For each* $z \in \{z_-, z_+\}$, $\mathrm{tr}_{(B_1^2 \backslash \overline{B}_{1-h}^2) \times \{z\}}(\varphi_h) \in W^{1,p}((B_1^2 \backslash \overline{B}_{1-h}^2) \times \{z\}, \mathcal{N})$ *and*

$$\int_{(B_1^2 \backslash \overline{B}_{1-h}^2) \times \{z\}} |D_\top \, \mathrm{tr}_{(B_1^2 \backslash \overline{B}_{1-h}^2) \times \{z\}}(\varphi_h)|^p \, \mathrm{d}\mathcal{H}^2 \leqq A_6 \int_{\Gamma_{1,L_0}} |D_\top g|^p \, \mathrm{d}\mathcal{H}^2,$$

where $A_6 > 0$ *is an absolute constant.*

(iii) *There exists a constant* $C = C(\mathcal{N}) > 0$ *such that*

$$\int_{\partial B_{1-h}^2 \times (z_-, z_+)} |D_\top v_h|^p \, \mathrm{d}\mathcal{H}^2 \leqq C \int_{\Gamma_{1,L_0}} |D_\top g|^p \, \mathrm{d}\mathcal{H}^2 \tag{5.15}$$

and

$$\int_{(B_1^2 \backslash \overline{B}_{1-h}^2) \times (z_-, z_+)} |D\varphi_h|^p \, \mathrm{d}x \leqq Ch \int_{\Gamma_{1,L_0}} |D_\top g|^p \, \mathrm{d}\mathcal{H}^2. \tag{5.16}$$

Proof. Using the coarea formula, we find $z_- \in (-L_0 + h, -L_0 + 2h)$ and $z_+ \in (L_0 - 2h, L_0 - h)$ such that for each $z \in \{z_-, z_+\}$, $\mathrm{tr}_{\mathbb{S}^1 \times \{z\}}(g) = g|_{\mathbb{S}^1 \times \{z\}} \in W^{1,p}(\mathbb{S}^1 \times \{z\}, \mathcal{N})$ and

$$\int_{\mathbb{S}^1 \times \{z\}} |D_\top g|^p \, \mathrm{d}\mathcal{H}^1 \leqq \frac{2}{h} \int_{\Gamma_{1,L_0}} |D_\top g|^p \, \mathrm{d}\mathcal{H}^2. \tag{5.17}$$

Define $L := (z_+ - z_-)/2$, $\mathrm{e} := (0, 0, (z_+ + z_-)/2)$ and $u(y) := g(\mathrm{e} + y)$ for each $y \in \Gamma_{1,L}$. Let $\kappa_0 = \kappa_0(p_0, \mathcal{N}) > 0$ be the constant given by Lemma 5.11. Define $\kappa = \kappa_0 / 2A_5$, where $A_5 > 0$ is the absolute constant of Lemma 5.10. Without loss of generality, we assume that $A_5 \geqq 2$. Then κ depends only on p_0 and $\mathcal{N}$. Using Lemma 5.10, the estimates (5.14), (5.17) and taking into account the definition of κ, we obtain $\Sigma \in \mathrm{RC}_{L,h}$ such that $\mathrm{tr}_\Sigma(u) \in W^{1,p}(\Sigma, \mathcal{N})$, $\mathrm{tr}_{\Sigma \cap \Gamma_{1,L}}(u) = u|_{\Sigma \cap \Gamma_{1,L}} \in W^{1,p}(\Sigma \cap \Gamma_{1,L}, \mathcal{N})$,

$$\int_{\Sigma \cap \Gamma_{1,L}} |D_\top u|^p \, \mathrm{d}\mathcal{H}^1 \leqq \frac{A_5}{h} \int_{\Gamma_{1,L}} |D_\top u|^p \, \mathrm{d}\mathcal{H}^2 \tag{5.18}$$

and for each 2-cell F from the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ,

$$\int_F \frac{|D_\top u|^p}{p} \, \mathrm{d}\mathcal{H}^2 + h \int_{\partial F} \frac{|D_\top \, \mathrm{tr}_{\partial F}(u)|^p}{p} \, \mathrm{d}\mathcal{H}^1 \leqq 2A_5 \int_{\Gamma_{1,L_0}} \frac{|D_\top g|^p}{p} \, \mathrm{d}\mathcal{H}^2 \leqq \frac{\kappa_0 h^{2-p}}{2-p},$$

where ∂F denotes the relative boundary of F. This, according to Lemma 5.11, implies that $\mathrm{tr}_{\partial F}(u)$ is nullhomotopic for each 2-cell F from the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ. Proceeding similarly as in Proposition 3.6 (changing $\mathbb{S}^2$ to $\overline{\Gamma}_{1,L}$ and the geometry of cells, namely changing "squares" to "squares or triangles" which are all bilipschitz homeomorphic to B_h^2), we obtain $w_h \in W^{1,2}(\Gamma_{1,L}, \mathcal{N})$ such that $\mathrm{tr}_\Sigma(w_h) = \mathrm{tr}_\Sigma(u)$, $\mathrm{tr}_{\Sigma\cap\Gamma_{1,L}}(w_h) = w_h|_{\Sigma\cap\Gamma_{1,L}} = u|_{\Sigma\cap\Gamma_{1,L}}$ and

$$\int_{\Gamma_{1,L}} |D_\top w_h|^p \,\mathrm{d}\mathcal{H}^2 \leqq Ch \int_\Sigma |D_\top u|^p \,\mathrm{d}\mathcal{H}^1, \tag{5.19}$$

where $C = C(\mathcal{N}) > 0$. Using (5.17), (5.18) and (5.19), we obtain the estimate

$$\int_{\Gamma_{1,L}} |D_\top w_h|^p \,\mathrm{d}\mathcal{H}^2 \leqq C' \int_{\Gamma_{1,L_0}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2,$$

where $C' = C'(\mathcal{N}) > 0$. Now we define $v_h(x) = w_h((x_1/(1-h), x_2/(1-h), x_3) - \mathrm{e})$ for $x \in \partial B_{1-h}^2 \times (z_-, z_+)$. In view of the last estimate, we get (5.15).

From the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ we obtain the decomposition of $(B_1^2 \backslash \overline{B}_{1-h}^2) \times (z_-, z_+)$ into the cells

$$D_{i,j} = \left\{ x = (x', x_3) \in \mathbb{R}^3 : (1-h) < |x'| < 1,\ \left(\frac{x'}{|x'|}, x_3\right) \in (F_{i,j} + \mathrm{e}) \right\}, \tag{5.20}$$

where $D_{i,j}$ is of Hausdorff dimension $(i+1)$. Fix an arbitrary 3-cell $D_{2,j}$ from the above decomposition. Let $F_{2,j}$ be the 2-cell of the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ such that $D_{2,j} = \{x \in \mathbb{R}^3 : (1-h) < |x'| < 1,\ (\frac{x'}{|x'|}, x_3) \in (F_{2,j} + \mathrm{e})\}$. Let $\Phi_j : [0,h]^3 \to \overline{D}_{2,j}$ be a bilipschitz homeomorphism with an absolute bilipschitz constant (see (5.13)) such that $\Phi_j((0,h)^2 \times \{0\}) = \{x \in \mathbb{R}^3 : |x'| = 1-h,\ (\frac{x'}{|x'|}, x_3) \in (F_{2,j} + \mathrm{e})\}$ and $\Phi_j((0,h)^2 \times \{1\}) = F_{2,j} + \mathrm{e}$. Applying Lemma 3.9 with g_0 and g_1 taking the values of $v_h \circ \Phi_j|_{(0,h)^2\times\{0\}}$ and $g \circ \Phi_j|_{(0,h)^2\times\{1\}}$, respectively, and changing the variables, we obtain a map $\varphi_j \in W^{1,p}(D_{2,j}, \mathcal{N})$ satisfying the estimate

$$\begin{aligned}\int_{D_{2,j}} |D\varphi_j|^p \,\mathrm{d}x \leqq C''h\left(\int_{\Phi_j((0,h)^2\times\{0\})} |D_\top v_h|^p \,\mathrm{d}\mathcal{H}^2 + \int_{F_{2,j}+\mathrm{e}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2\right)\\ +C''h^2 \int_{\partial F_{2,j}+\mathrm{e}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1,\end{aligned} \tag{5.21}$$

where $C'' > 0$ is an absolute constant.

Let $\varphi_h \in L^\infty((B_1^2 \backslash \overline{B}_{1-h}^2) \times (z_-, z_+), \mathcal{N})$ satisfy $\varphi_h|_{D_{2,j}} = \varphi_j \in W^{1,p}(D_{2,j}, \mathcal{N})$ for each 3-cell $D_{2,j}$ from the decomposition (5.20). In view of our construction (see Lemma 3.9), we can ensure that the traces of the maps φ_j coincide on the mantles of the 3-cells $D_{2,j}$. Thus, we deduce that $\varphi_h \in W^{1,p}((B_1^2 \backslash \overline{B}_{1-h}^2) \times$

$(z_-, z_+), \mathcal{N})$. Observe that (i) and (ii) are satisfied, where (ii) comes from the fact that

$$\int_{(B_1^2 \setminus \overline{B}_{1-h}^2)\times\{z\}} |D_\top \operatorname{tr}_{(B_1^2\setminus \overline{B}_{1-h}^2)\times\{z\}}(\varphi_h)|^p \,\mathrm{d}\mathcal{H}^2 = \int_{1-h}^1 r^{1-p}\,\mathrm{d}r \int_{\mathbb{S}^1\times\{z\}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1$$
$$\leqq 2h \int_{\mathbb{S}^1\times\{z\}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1$$

for each $z \in \{z_-, z_+\}$ and (5.17). Furthermore, summing (5.21) over $j \in \{1, \ldots, k_2\}$, taking into account that each 1-cell of the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ lies in the relative boundary of at most two 2-cells from this triangulation, and also using (5.15) together with (5.17) and (5.18), we deduce the following chain of estimates

$$\begin{aligned}
&\int_{(B_1^2\setminus \overline{B}_{1-h}^2)\times(z_-,z_+)} |D\varphi_h|^p \,\mathrm{d}x \\
&\leqq C''h\left(\int_{\partial B_{1-h}^2\times(z_-,z_+)} |D_\top v_h|^p \,\mathrm{d}\mathcal{H}^2 + \int_{\mathbb{S}^1\times(z_-,z_+)} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2\right) \\
&\qquad + 2C''h^2 \int_{\Sigma+\mathrm{e}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^1 \\
&\leqq C'''h \int_{\Gamma_{1,L_0}} |D_\top g|^p \,\mathrm{d}\mathcal{H}^2,
\end{aligned}$$

where $C''' = C'''(\mathcal{N}) > 0$. This completes our proof of (iii) and hence of Lemma 5.12. □

Corollary 5.13. *Let $p_0 \in (1, 2)$, $p \in [p_0, 2)$, $\delta \in (0, 1/2]$ and $g \in W^{1,p}(\Gamma_{\delta r,r}, \mathcal{N})$, where $\Gamma_{\delta r,r}$ is defined in (2.18). Then there exists $\tau = \tau(p_0, \mathcal{N}) > 0$ such that the following holds. Assume that*

$$\int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2 \leqq \frac{\tau(\delta r)^{2-p}}{2-p}. \tag{5.22}$$

Then there exist points $z_- \in (-r + \delta r/2, -r + \delta r)$, $z_+ \in (r - \delta r, r - \delta r/2)$ and a mapping $\varphi_\delta \in W^{1,p}((B_{\delta r}^2 \setminus \overline{B}_{\delta r/2}^2) \times (z_-, z_+), \mathcal{N})$ such that the following assertions hold.

(i) $\operatorname{tr}_{\partial B_{\delta r}^2\times I}(\varphi_\delta) = g|_{\partial B_{\delta r}^2\times I}$, $v_\delta := \operatorname{tr}_{\partial B_{\delta r/2}^2\times I}(\varphi_\delta) \in W^{1,2}(\partial B_{\delta r/2}^2 \times I, \mathcal{N})$, *where $I = (z_-, z_+)$.*

(ii) *For each $z \in \{z_-, z_+\}$, $\operatorname{tr}_{(B_{\delta r}^2\setminus \overline{B}_{\delta r/2}^2)\times\{z\}}(\varphi_\delta) \in W^{1,p}((B_{\delta r}^2 \setminus \overline{B}_{\delta r/2}^2) \times \{z\}, \mathcal{N})$ and*

$$\int_{(B_{\delta r}^2\setminus \overline{B}_{\delta r/2}^2)\times\{z\}} \frac{|D_\top \operatorname{tr}_{(B_{\delta r}^2\setminus \overline{B}_{\delta r/2}^2)\times\{z\}}(\varphi_\delta)|^p}{p} \,\mathrm{d}\mathcal{H}^2 \leqq A_6 \int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2.$$

(iii) *There exists a constant $C = C(\mathcal{N}) > 0$ such that*

$$\int_{\partial B^2_{\delta r/2}\times(z_-,z_+)} \frac{|D_\top v_\delta|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq C\int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2$$

and

$$\int_{(B^2_{\delta r}\setminus\overline{B}^2_{\delta r/2})\times(z_-,z_+)} \frac{|D\varphi_\delta|^p}{p}\,\mathrm{d}x \leqq C\delta r\int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2.$$

Proof of Corollary 5.13. Define $f(x) := g(\delta r x)$ for each $x \in \Gamma_{1,1/\delta}$. Let $\kappa = \kappa(p_0,\mathcal{N}) > 0$ be the constant of Lemma 5.12. Fix $\tau \in (0,\kappa/2]$. Then, in view of (5.22), it holds

$$\int_{\Gamma_{1,1/\delta}} \frac{|D_\top f|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{\tau}{2-p} \leqq \frac{\kappa}{2(2-p)} < \frac{\kappa(1/2)^{2-p}}{2-p}.$$

This allows us to apply Lemma 5.12 with $L_0 = 1/\delta$, $h = 1/2$ for the map $f \in W^{1,p}(\Gamma_{1,1/\delta},\mathcal{N})$. Thus, we obtain points $\xi_- \in (-1/\delta+1/2, -1/\delta+1)$, $\xi_+ \in (1/\delta-1, 1/\delta-1/2)$ and maps $w \in W^{1,2}(\partial B^2_{1/2}\times(\xi_-,\xi_+),\mathcal{N})$, $\psi \in W^{1,p}((B^2_1\setminus\overline{B}^2_{1/2})\times(\xi_-,\xi_+),\mathcal{N})$ such that the assertions (i)-(iii) of Lemma 5.12 hold for z_-, z_+, v_h, φ_h replaced by ξ_-, ξ_+, w, ψ, respectively. Scaling back to $\Gamma_{\delta r,r}$, namely, defining $z_- = \delta r\xi_-$, $z_+ = \delta r\xi_+$, $v_\delta(\cdot) = w(\cdot/\delta r)$ and $\varphi_\delta(\cdot) = \psi(\cdot/\delta r)$, we complete our proof of Corollary 5.13. □

Proposition 5.14. *Let $p_0 \in (1,2)$, $p \in [p_0,2)$, $\delta \in (0,1/2]$ and $g \in W^{1,p}(\partial\Lambda_{\delta r,r},\mathcal{N})$, where $\Lambda_{\delta r,r}$ and $\Gamma_{\delta r,r}$ are defined in (2.18). Let $\tau = \tau(p_0,\mathcal{N}) > 0$ be the constant of Corollary 5.13. Assume that*

$$\int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{\tau(\delta r)^{2-p}}{2-p}.$$

Let u be a p-minimizer in $\Lambda_{\delta r,r}$ with $\mathrm{tr}_{\partial\Lambda_{\delta r,r}}(u) = g$. Then there exists $\gamma \in [\mathbb{S}^1,\mathcal{N}]$ such that the following estimates hold

$$\begin{aligned}\int_{\Lambda_{\delta r,r}} \frac{|Du|^p}{p}\,\mathrm{d}x \leqq{}& \frac{2\mathcal{E}^{\mathrm{sg}}_p(\gamma)r^{3-p}}{2-p} + C\delta^{1-p}r\int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2\\ &+C\delta r\left(\int_{B^2_{\delta r}\times\{-r\}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \int_{B^2_{\delta r}\times\{r\}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2\right)\\ &+\frac{C(\delta r)^{3-p}}{2-p}\end{aligned} \tag{5.23}$$

and

$$(2-2\delta)\frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma)\delta^{2-p}r^{3-p}}{2-p} - C\delta r\int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \int_{\Lambda_{\delta r,r}} \frac{|Du|^p}{p}\,\mathrm{d}x, \tag{5.24}$$

where $C = C(\mathcal{N}) > 0$.

Remark 5.15. The singular p-energies $\mathcal{E}_p^{\mathrm{sg}}(\gamma)$ and $\mathcal{E}_{p/(p-1)}^{\mathrm{sg}}(\gamma)$ appearing in (5.23) and (5.24), respectively, are not the same in general (see Definition 2.7).

Before proving Proposition 5.14, let us consider the case where $r = 1$, $\gamma \in C^1(\mathbb{S}^1, \mathcal{N})$ and the map $g \in W^{1,p}(\partial\Lambda_{\delta,1}, \mathcal{N})$ is defined on the closure of the lateral surface $\Gamma_{\delta,1}$ (see (2.18)) as $g(\sigma, z) = \gamma(\sigma/\delta)$ for each point $(\sigma, z) \in \partial B_\delta^2 \times [-1, 1]$ and on the ball $B_\delta^2 \times \{z\}$ with $z \in \{-1, 1\}$, as the extension of the mapping $g(\cdot, z) \in C^1(\partial B_\delta^2 \times \{z\}, \mathcal{N})$ coming from Lemma 2.18, so that g is of class C^1 on $\Gamma_{\delta,1}$ and its homotopy class on $\Gamma_{\delta,1}$ (see Remark 2.19) is given by $[\gamma] \in [\mathbb{S}^1, \mathcal{N}]$. The upper bound (5.23) for the p-energy on $\Lambda_{\delta,1}$ of a minimizer $u \in W^{1,p}(\Lambda_{\delta,1}, \mathcal{N})$ with $\operatorname{tr}_{\partial\Lambda_{\delta,1}}(u) = g$ comes from Lemma 2.20 (i), while the lower bound (5.24) for the p-energy of u on $\Lambda_{\delta,1}$ is a consequence of the application of the coarea formula and Proposition 3.3.

Proof of Proposition 5.14. By scaling, it is enough to prove Proposition 5.14 in the case when $r = 1$. So we assume that $r = 1$. In view of the conditions of Proposition 5.14, we can apply Corollary 5.13 to g in $\Gamma_{\delta,1}$. Let points $z_- \in (-1 + \delta/2, -1 + \delta)$, $z_+ \in (1 - \delta, 1 - \delta/2)$ and maps $v_\delta \in W^{1,2}(\partial B_{\delta/2}^2 \times (z_-, z_+), \mathcal{N})$, $\varphi_\delta \in W^{1,p}((B_\delta^2 \setminus \overline{B}_{\delta/2}^2) \times (z_-, z_+), \mathcal{N})$ be given by Corollary 5.13 applied to our $g \in W^{1,p}(\Gamma_{\delta,1}, \mathcal{N})$. According to Remark 2.19, v_δ has a well-defined "homotopy class" on $\partial B_{\delta/2}^2 \times (z_-, z_+)$. Thus, there exists $\gamma \in [\mathbb{S}^1, \mathcal{N}]$ such that for $\mathcal{H}^1$-a.e. $z \in (z_-, z_+)$, $[v_\delta|_{\partial B_{\delta/2}^2 \times \{z\}}] = [\gamma]$. This defines our γ. Hereinafter in this proof, C will denote a positive constant that can depend only on $\mathcal{N}$ and can be different from line to line.

Step 1. We prove the upper bound (5.23). For convenience, in order to construct a competitor for u, we define the following subdomains of $\Lambda_{\delta,1}$:

$$\begin{aligned} \Lambda_\delta^- &:= B_\delta^2 \times (-1, z_-), & \Lambda_\delta^0 &:= B_{\delta/2}^2 \times (z_-, z_+), \\ E_\delta &:= (B_\delta^2 \setminus \overline{B}_{\delta/2}^2) \times (z_-, z_+), & \Lambda_\delta^+ &:= B_\delta^2 \times (z_+, 1). \end{aligned}$$

Let us construct a competitor w for u, namely $w \in W^{1,p}(\Lambda_{\delta,1}, \mathcal{N})$ with $\operatorname{tr}_{\partial\Lambda_{\delta,1}}(w) = g$. We define w in Λ_δ^0 by using Lemma 2.20 (i), namely we obtain a map $w|_{\Lambda_\delta^0} \in W^{1,p}(\Lambda_\delta^0, \mathcal{N})$ such that

$$\begin{aligned} \int_{\Lambda_\delta^0} \frac{|Dw|^p}{p} \,\mathrm{d}x &\leqq \frac{(z_+ - z_-)\mathcal{E}_p^{\mathrm{sg}}(\gamma)\delta^{2-p}}{(2-p)2^{2-p}} \\ &\quad + C\left(\frac{(z_+ - z_-)^p}{2\delta^{p-1}} + \frac{\delta}{2}\right) \int_{\partial B_{\delta/2}^2 \times (z_-, z_+)} \frac{|D_\top v_\delta|^p}{p} \,\mathrm{d}\mathcal{H}^2 \\ &\leqq \frac{2\mathcal{E}_p^{\mathrm{sg}}(\gamma)}{2-p} + \frac{C}{\delta^{p-1}} \int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p} \,\mathrm{d}\mathcal{H}^2, \end{aligned} \tag{5.25}$$

where we have also used that $(z_+ - z_-) \leqq 2$, $\delta \in (0, 1/2]$, $p \in (1, 2)$ and Corollary 5.13 (iii). Next, we define the mapping $g^+ : \partial\Lambda_\delta^+ \to \mathcal{N}$ by

$$g^+(x) := \begin{cases} g(x) & \text{if } x \in \partial\Lambda_\delta^+ \cap \partial\Lambda_{\delta,1}, \\ \mathrm{tr}_{(B_\delta^2 \setminus \overline{B}_{\delta/2}^2)\times\{z_+\}}(\varphi_\delta)(x) & \text{if } x \in (B_\delta^2 \setminus \overline{B}_{\delta/2}^2) \times \{z_+\}, \\ \mathrm{tr}_{\overline{B}_{\delta/2}^2\times\{z_+\}}(w)(x) & \text{if } x \in \overline{B}_{\delta/2}^2 \times \{z_+\}. \end{cases}$$

By Lemma 2.20 (ii), $g^+|_{B_{\delta/2}^2\times\{z_+\}} \in W^{1,p}(B_{\delta/2}^2 \times \{z_+\}, \mathcal{N})$. On the other hand, according to Corollary 5.13 (ii), $g^+|_{(B_\delta^2\setminus\overline{B}_{\delta/2}^2)\times\{z_+\}} \in W^{1,p}((B_\delta^2\setminus\overline{B}_{\delta/2}^2) \times \{z_+\}, \mathcal{N})$. Thus, taking into account that the traces of g and φ_δ coincide on $\partial B_\delta^2 \times \{z_+\}$, and the traces of φ_δ and w coincide on $\partial B_{\delta/2}^2 \times \{z_+\}$, we conclude that $g^+ \in W^{1,p}(\partial\Lambda_\delta^+, \mathcal{N})$. In particular, using Lemma 2.20 (ii), we obtain

$$\begin{aligned} \int_{\partial\Lambda_\delta^+} \frac{|D_\top g^+|^p}{p}\,\mathrm{d}\mathcal{H}^2 &\leqq \int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \int_{B_\delta^2\times\{1\}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \frac{\mathcal{E}_p^{\mathrm{sg}}(\gamma)\delta^{2-p}}{(2-p)2^{2-p}} \\ &\quad + C\left(\left(\frac{z_+ - z_-}{\delta}\right)^{p-1} + \frac{\delta}{z_+ - z_-}\right)\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 \\ &\leqq \frac{C}{\delta^{p-1}}\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \int_{B_\delta^2\times\{1\}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \frac{C\delta^{2-p}}{2-p}, \end{aligned} \tag{5.26}$$

where the last estimate comes since $(z_+ - z_-) \leqq 2$, $\delta \in (0, 1/2]$, $p \in (1, 2)$ and $\#[\mathbb{S}^1, \mathcal{N}] < +\infty$ (hence $\mathcal{E}_p^{\mathrm{sg}}(\gamma) \leqq C$). Notice that $\overline{\Lambda}_\delta^+$ is bilipschitz homeomorphic to $\overline{B}_\delta^3$, namely, there exists a bilipschitz mapping $\Phi : \overline{\Lambda}_\delta^+ \to \overline{B}_\delta^3$ with an absolute bilipschitz constant. We define

$$w|_{\Lambda_\delta^+}(x) := g^+\left(\Phi^{-1}\left(\frac{\delta\Phi(x)}{|\Phi(x)|}\right)\right) \qquad \text{for } x \in \Lambda_\delta^+ \setminus \{\Phi^{-1}(0)\}.$$

Then, using (5.26), we obtain

$$\begin{aligned} \int_{\Lambda_\delta^+} \frac{|Dw|^p}{p}\,\mathrm{d}x &\leqq C\delta\int_{\partial\Lambda_\delta^+} \frac{|D_\top g^+|^p}{p}\,\mathrm{d}\mathcal{H}^2 \\ &\leqq C\delta^{2-p}\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 + C\delta\int_{B_\delta^2\times\{1\}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \frac{C\delta^{3-p}}{2-p}. \end{aligned} \tag{5.27}$$

We carry out a similar construction on the set Λ_δ^- and obtain a map $w|_{\Lambda_\delta^-} \in W^{1,p}(\Lambda_\delta^-, \mathcal{N})$ satisfying the same type of energy control as in (5.27). In E_δ, we define $w|_{E_\delta} = \varphi_\delta$. By Corollary 5.13 (iii),

$$\int_{E_\delta} \frac{|Dw|^p}{p}\,\mathrm{d}x \leqq C\delta\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2. \tag{5.28}$$

Altogether, we have constructed the map $w \in W^{1,p}(\Lambda_{\delta,1}, \mathcal{N})$ with $\mathrm{tr}_{\partial\Lambda_{\delta,1}}(w) = g$ (we refer to Corollary 5.13 (i)). Since u is a p-minimizer in $\Lambda_{\delta,1}$ with $\mathrm{tr}_{\partial\Lambda_{\delta,1}}(u) = g$

and, in view of (5.25), (5.27), (5.28), the facts that $\delta \in (0, 1/2]$ and $p \in (1, 2)$, we have

$$\int_{\Lambda_{\delta,1}} \frac{|Du|^p}{p}\,\mathrm{d}x \leqq \int_{\Lambda_{\delta,1}} \frac{|Dw|^p}{p}\,\mathrm{d}x \leqq \frac{2\mathcal{E}_p^{\mathrm{sg}}(\gamma)}{2-p} + \frac{C}{\delta^{p-1}} \int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2$$
$$+ C\delta \left(\int_{B_\delta^2\times\{-1\}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \int_{B_\delta^2\times\{1\}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 \right) + \frac{C\delta^{3-p}}{2-p},$$

which yields (5.23).

Step 2. We prove the lower bound (5.24). For each $(\varrho, \sigma, z) \in [0, \delta) \times \mathbb{S}^1 \times (z_-, z_+)$, define the map $\psi : B_\delta^2 \times (z_-, z_+)$ by

$$\psi(\varrho\sigma, z) := \begin{cases} u(2\varrho\sigma, z) & \text{if} \quad \varrho \in [0, \delta/2) \text{ and } z \in (z_-, z_+), \\ \varphi_\delta((3\delta/2 - \varrho)\sigma, z) & \text{if} \quad \varrho \in [\delta/2, \delta) \text{ and } z \in (z_-, z_+). \end{cases}$$

Then $\psi \in W^{1,p}(B_\delta^2 \times (z_-, z_+), \mathcal{N})$ and $\mathrm{tr}_{\partial B_\delta^2\times(z_-,z_+)}(\psi) = v_\delta(\cdot/2, \cdot) \in W^{1,2}(\partial B_\delta^2 \times (z_-, z_+), \mathcal{N})$. By the definition of ψ and Corollary 5.13 (iii),

$$\begin{aligned}\int_{B_\delta^2\times(z_-,z_+)} \frac{|D\psi|^p}{p}\,\mathrm{d}x &\leqq \int_{B_\delta^2\times(z_-,z_+)} \frac{|Du|^p}{p}\,\mathrm{d}x + \int_{E_\delta} \frac{|D\varphi_\delta|^p}{p}\,\mathrm{d}x \\ &\leqq \int_{B_\delta^2\times(z_-,z_+)} \frac{|Du|^p}{p}\,\mathrm{d}x + C\delta \int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2.\end{aligned} \tag{5.29}$$

Using the coarea formula and that $\mathrm{tr}_{\partial B_\delta^2\times(z_-,z_+)}(\psi) = v_\delta(\cdot/2, \cdot) \in W^{1,2}(\partial B_\delta^2 \times (z_-, z_+), \mathcal{N})$, we observe that $\mathrm{tr}_{\partial B_\delta^2\times\{z\}}(\psi) = \psi|_{\partial B_\delta^2\times\{z\}} \in W^{1,2}(\partial B_\delta^2 \times \{z\}, \mathcal{N})$ and $[\psi|_{\partial B_\delta^2\times\{z\}}] = [\gamma]$ for $\mathcal{H}^1$-a.e. $z \in (z_-, z_+)$. Then, applying Proposition 3.3, for $\mathcal{H}^1$-a.e. $z \in (z_-, z_+)$, we obtain

$$\int_{B_\delta^2\times\{z\}} \frac{|D\psi|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \delta \int_{\partial B_\delta^2\times\{z\}} \frac{|D_\top\psi|^p}{p}\,\mathrm{d}\mathcal{H}^1 \geqq \frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma)\delta^{2-p}}{2-p}.$$

Integrating both sides of the above inequality over (z_-, z_+) with respect to d z, we get

$$\begin{aligned}\int_{B_\delta^2\times(z_-,z_+)} \frac{|D\psi|^p}{p}\,\mathrm{d}x &\geqq \frac{(z_+ - z_-)\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma)\delta^{2-p}}{2-p} \\ &\quad - \frac{\delta}{2^{p-1}} \int_{\partial B_{\delta/2}^2\times(z_-,z_+)} \frac{|Dv_\delta|^p}{p}\,\mathrm{d}\mathcal{H}^2 \\ &\geqq (2-2\delta)\frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma)\delta^{2-p}}{2-p} - C\delta \int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2,\end{aligned} \tag{5.30}$$

where we have also used Corollary 5.13 (iii). Combining (5.29) and (5.30), we obtain (5.24). This completes our proof of Proposition 5.14. ◻

5.3. The Limit Measure has the Density with a Finite Set of Values

Recall that μ_* is the positive Radon measure on $\overline{\Omega}$ defined at the beginning of Sect. 5 and its support (i.e., $\operatorname{supp}(\mu_*)$) is the set S_*. We prove the following

Proposition 5.16. *For $\mathcal{H}^1$-a.e. $x \in S_* \cap \Omega$, we have*

$$\Theta_1(\mu_*, x) \in \{\mathcal{E}_2^{\mathrm{sg}}(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\}\backslash\{0\}.$$

Proof. According to Proposition 5.8 and [4, Theorem 2.83 (i)], μ_* admits an approximate tangent space with multiplicity $\Theta_1(x_0)$ for $\mathcal{H}^1$-a.e. $x_0 \in S_* \cap \Omega$. Namely, for $\mathcal{H}^1$-a.e. $x_0 \in S_* \cap \Omega$ (or, equivalently, for μ_*-a.e. $x_0 \in \Omega$), there exists a unique approximate tangent plane $T_{x_0} S_* \in \mathrm{G}(3, 1)$ to S_* at x_0 and

$$\mu_{x_0,r} = T_{\#}^{x_0,r} \mu_* \llcorner \Omega \overset{*}{\rightharpoonup} \mu_0 = \Theta_1(x_0)\mathcal{H}^1 \llcorner T_{x_0} S_* \quad \text{weakly* in } (C_0(\mathbb{R}^3))' \text{ as } r \searrow 0, \tag{5.31}$$

where $T^{x_0,r}(x) := (x - x_0)/r$. Up to rotation and translation, to lighten the notation, we shall assume that $x_0 = 0$ and $T_{x_0} S_* = \{(0, 0)\} \times \mathbb{R}$. Recall that $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$, $p_n \nearrow 2$ as $n \to +\infty$ and for each $n \in \mathbb{N}$ large enough,

$$\int_{\Omega} \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x \leqq \frac{C_0}{2 - p_n}, \tag{5.32}$$

where $C_0 > 0$ is a constant independent of n (see (5.1)).

Let $\delta \in (0, 1/2)$ be fairly small, which we shall carefully choose later for the proof to work. Let $\tau > 0$ be the constant of Corollary 5.13, where $p_0 = 3/2$. Let $\eta > 0$ be the constant of Lemma 3.12, where $\kappa = 1/2$, $p_0 = 3/2$ and Ψ is a bilipschitz homeomorphism between a ball and a cube with an absolute bilipschitz constant (see, for instance, [32, Corollary 3]). We fix some $\varepsilon \in (0, \min\{\eta, \tau\})$. Next, since $\mu_0((\overline{B}^2_{3\delta/2} \backslash B^2_{\delta/2}) \times [-2, 2]) = 0$, in view of the weak* convergence (see (5.31)), there exists $r_0 = r_0(\varepsilon, \delta) \in (0, \min\{1, \operatorname{dist}(x_0, \partial\Omega)/4\})$ such that

$$\mu_*((\overline{B}^2_{3\delta r/2} \setminus B^2_{\delta r/2}) \times [-2r, 2r]) < \frac{\varepsilon\delta r}{100} \quad \text{for each } r \in (0, r_0]. \tag{5.33}$$

Taking into account (5.2) and (5.3), the estimate (5.33), together with the Fatou lemma, implies that for a fixed $r \in (0, r_0]$,

$$\int_{\delta r/2}^{3\delta r/2} \mathrm{d}\varrho \liminf_{n\to+\infty} (2 - p_n) \int_{\Gamma_{\varrho,r}} \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}\mathcal{H}^2 < \frac{\varepsilon\delta r}{100}. \tag{5.34}$$

Due to (5.34), there exist $\varrho \in (3\delta r/4, 5\delta r/4)$ and $n_0 = n_0(\delta, r, \varepsilon) \in \mathbb{N}$ such that, up to a subsequence (not relabeled), for each $n \geqq n_0$, $\operatorname{tr}_{\Gamma_{\varrho,r}}(u_n) = u_n|_{\Gamma_{\varrho,r}} \in W^{1,p_n}(\Gamma_{\varrho,r}, \mathcal{N})$ (see (2.18)) and

$$\int_{\Gamma_{\varrho,r}} \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}\mathcal{H}^2 < \frac{\varepsilon\varrho^{2-p_n}}{2 - p_n}, \tag{5.35}$$

where we have used that $\varrho^{2-p_n} \to 1$ as $n \to +\infty$. To simplify the notation, without loss of generality, we assume that $\varrho = \delta r$. Notice that $\Lambda_{\delta r, 5r/4} \subset B^3_{\sqrt{2}r}$, where $\sqrt{2}r < d_0 := \mathrm{dist}(x_0, \partial\Omega)/2$ (since it holds $r \leqq r_0 < d_0/2$). Thus, using the monotonicity of the p-energy (see Lemma 5.3) and (5.32), for each $n \in \mathbb{N}$ large enough, we get

$$\int_{\Lambda_{\delta r,5r/4}} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \leqq \left(\frac{\sqrt{2}r}{d_0}\right)^{3-p_n} \int_{B^3_{d_0}} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \leqq \left(\frac{\sqrt{2}r}{d_0}\right)^{3-p_n} \frac{C_0}{2-p_n}.$$

Using again the Fatou lemma, it holds

$$\int_{-5r/4}^{5r/4} \mathrm{d}z \liminf_{n\to+\infty}(2-p_n)\int_{B^2_{\delta r}\times\{z\}} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{C_0\sqrt{2}r}{d_0}.$$

This implies that there exist $z^r_- \in (-5r/4, -3r/4)$ and $z^r_+ \in (3r/4, 5r/4)$ such that, up to a subsequence (still denoted by the same index), for each $z \in \{z^r_-, z^r_+\}$ we have the following $\mathrm{tr}_{B^2_{\delta r}\times\{z\}}(u_n) = u_n|_{B^2_{\delta r}\times\{z\}} \in W^{1,p_n}(B^2_{\delta r}\times\{z\}, \mathcal{N})$ and for each $n \in \mathbb{N}$ large enough,

$$\int_{B^2_{\delta r}\times\{z\}} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{C_1}{2-p_n}, \tag{5.36}$$

where $C_1 = C_1(C_0, d_0) > 0$. Without loss of generality, to lighten the notation, we shall assume that $z^r_- = -r$ and $z^r_+ = r$. Thus, $\mathrm{tr}_{\partial\Lambda_{\delta r,r}}(u_n) = u_n|_{\partial\Lambda_{\delta r,r}} \in W^{1,p_n}(\partial\Lambda_{\delta r,r}, \mathcal{N})$. From (5.35) and (5.36) we obtain

$$\begin{aligned} &\int_{\Gamma_{\delta r,r}} \frac{|D_\top \mathrm{tr}_{\Gamma_{\delta r,r}}(u_n)|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{\varepsilon(\delta r)^{2-p_n}}{2-p_n} \quad \text{and} \\ &\int_{B^2_{\delta r}\times\{z\}} \frac{|D_\top \mathrm{tr}_{B^2_{\delta r}\times\{z\}}(u_n)|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{C_1}{2-p_n} \end{aligned} \tag{5.37}$$

for each $z \in \{-r, r\}$ and $n \in \mathbb{N}$ large enough. Thus, for each sufficiently large $n \in \mathbb{N}$, u_n fulfills the conditions of Proposition 5.14 (recall that $0 < \varepsilon < \tau$). Using Proposition 5.14 and (5.37), we obtain $\gamma_n \in [\mathbb{S}^1, \mathcal{N}]$ such that the following holds. Firstly,

$$\int_{\Lambda_{\delta r,r}} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \leqq \frac{2\mathcal{E}^{\mathrm{sg}}_{p_n}(\gamma_n)r^{3-p_n}}{2-p_n} + \frac{C\varepsilon\delta^{3-2p_n}r^{3-p_n}}{2-p_n} + \frac{C\delta r}{2-p_n} + \frac{C(\delta r)^{3-p_n}}{2-p_n} \tag{5.38}$$

for each sufficiently large $n \in \mathbb{N}$, where $C = C(C_1, \mathcal{N}) > 0$. Secondly,

$$(2-2\delta)\frac{\mathcal{E}^{\mathrm{sg}}_{p_n/(p_n-1)}(\gamma_n)\delta^{2-p_n}r^{3-p_n}}{2-p_n} - \frac{C'\varepsilon(\delta r)^{3-p_n}}{2-p_n} \leqq \int_{\Lambda_{\delta r,r}} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \tag{5.39}$$

for each sufficiently large $n \in \mathbb{N}$, where $C' = C'(\mathcal{N}) > 0$.

Now, applying Lemma 5.2, due to (5.33), we can cover the compact set $\partial B^2_{\delta r} \times [-r, r]$ by a finite number of small cubes (which are bilipschitz homeomorphic to balls) being subsets of $\Omega\backslash S_*$ and obtain that the set

$$E_{\delta,r} = (\overline{B}^2_{101\delta r/100} \setminus B^2_{99\delta r/100}) \times [-r - \delta r/100, r + \delta r/100]$$

is a subset of $\Omega \setminus S_*$. Denote by $E^0_{\delta,r}$ the interior of $E_{\delta,r}$. Then, by Proposition 5.6, $u_n \rightharpoonup u_*$ weakly in $W^{1,p}(E^0_{\delta,r}, \mathbb{R}^\nu)$ for all $p \in (1, 2)$. Furthermore, there exists a finite set $\{x_1, \dots, x_l\} \subset E^0_{\delta,r}$ such that $u_* \in C^\infty(E^0_{\delta,r}\backslash\{x_1, \dots, x_l\}, \mathcal{N})$. By the weak convergence and the Sobolev embedding, without loss of generality, $u_n|_{\partial B^2_{\delta r}\times\{z\}}$ converges uniformly to $u_*|_{\partial B^2_{\delta r}\times\{z\}} \in C(\partial B^2_{\delta r} \times \{z\}, \mathcal{N})$ for each $z \in \{-r, r\}$.

We claim that for each $n \in \mathbb{N}$ large enough, $[\gamma_n] = [u_*|_{\partial B^2_{\delta r}\times\{-r\}}] = [u_*|_{\partial B^2_{\delta r}\times\{r\}}]$. Indeed, a close inspection of the proofs of Lemma 5.12 and Corollary 5.13 shows that the mapping $v_{\delta,n} \in W^{1,2}(\partial B^2_{\delta r/2}\times (z^n_-, z^n_+), \mathcal{N})$, coming from Corollary 5.13, has a well-defined homotopy class on the surface $\partial B^2_{\delta r/2}\times (z^n_-, z^n_+)$ and for each $z \in \{z^n_-, z^n_+\}$, $[v_{\delta,n}|_{\partial B^2_{\delta r/2}\times\{z\}}] = [\gamma_n]$, where $z^n_- \in (-r + \delta r/2, -r + \delta r)$ and $z^n_+ \in (r - \delta r, r - \delta r/2)$ (the reader may consult the arguments given immediately above the estimate (5.19), take into account Remark 2.19, and observe that the trace on $\mathbb{S}^1 \times \{L\}$ of the map $w_h \in W^{1,2}(\Gamma_{1,L}, \mathcal{N})$ constructed in the proof of Proposition 5.14 admits a continuous representative). We can actually extend $\varphi_{\delta,n}$ on $(B^2_{\delta r}\backslash\overline{B}^2_{\delta r/2}) \times (-r, r)$ and $v_{\delta,n}$ on $\partial B^2_{\delta r/2} \times (-r, r)$, where $\varphi_{\delta,n}$ comes from Corollary 5.13, so that these extensions have the following properties: $\varphi_{\delta,n} \in W^{1,p}((B^2_{\delta r}\backslash\overline{B}^2_{\delta r/2}) \times (-r, r), \mathcal{N})$, $v_{\delta,n} \in W^{1,2}(\partial B^2_{\delta r/2} \times (-r, r), \mathcal{N})$, $\mathrm{tr}_{\partial B^2_{\delta r/2}\times(-r,r)}(\varphi_{\delta,n}) = v_{\delta,n}$ (the proof consists in constructing the map on the union of surfaces $(\partial B^2_{\delta r/2} \times (-r, z^n_-)) \cup (\partial B^2_{\delta r/2} \times (z^n_+, r))$ by similar way as was constructed $v_{\delta,n}$ on $\partial B^2_{\delta r/2} \times (z^n_-, z^n_+)$, namely, by choosing appropriate triangulations on the sets $\partial B^2_{\delta r} \times (-r, z^n_-)$, $\partial B^2_{\delta r} \times (z^n_+, r)$ and projecting them onto $\partial B^2_{\delta r/2} \times (-r, z^n_-)$, $\partial B^2_{\delta r/2} \times (z^n_+, r)$, respectively, and then proceed as in the construction of the maps $v_{\delta,n}$, $\varphi_{\delta,n}$; the traces of the obtained extensions of $\varphi_{\delta,n}$ and $v_{\delta,n}$ will coincide with the traces of $\varphi_{\delta,n}$ and $v_{\delta,n}$ on $\partial B^2_{\delta r} \times \{z^n_-\}$, $\partial B^2_{\delta r} \times \{z^n_+\}$ and $\partial B^2_{\delta r/2} \times \{z^n_-\}$, $\partial B^2_{\delta r/2} \times \{z^n_+\}$, respectively, and hence we obtain a $W^{1,p}$ extension of $\varphi_{\delta,n}$ on $(B^2_{\delta r}\backslash\overline{B}^2_{\delta r/2}) \times (-r, r)$ and a $W^{1,2}$ extension of $v_{\delta,n}$ on $\partial B^2_{\delta r/2} \times (-r, r)$). Since for each $z \in \{-r, r\}$, $\mathrm{tr}_{\partial B^2_{\delta r/2}\times\{z\}}(v_{\delta,n})$ is homotopic to $u_n|_{\partial B^2_{\delta r}\times\{z\}}$, we deduce that for each $n \in \mathbb{N}$ large enough, $[\gamma_n] = [u_*|_{\partial B^2_{\delta r}\times\{z\}}]$. This proves our claim.

Thus, there exists $\gamma \in [\mathbb{S}^1, \mathcal{N}]$ such that for each $n \in \mathbb{N}$ large enough, $[\gamma_n] = [\gamma]$. Next, multiplying both sides of (5.38) and (5.39) by $(2 - p_n)$ and then passing to the limit when n tend to $+\infty$, we obtain

$$r(2 - 2\delta)\mathcal{E}^{\mathrm{sg}}_2(\gamma) - C'\varepsilon\delta r \leqq \mu_*(\Lambda_{\delta r,r}) \leqq 2r\mathcal{E}^{\mathrm{sg}}_2(\gamma) + \frac{C\varepsilon r}{\delta} + 2C\delta r, \quad (5.40)$$

where we have used Lemma 2.8. Now we choose $\delta = \varepsilon^{1/2}$ and hence $\delta \searrow 0$ as $\varepsilon \searrow 0$. Dividing both sides of (5.40) by r and letting $r \searrow 0$ and then $\varepsilon \searrow 0$, we obtain

$$2\mathcal{E}_2^{\rm sg}(\gamma) \leqq \Theta_1(x_0)\mathcal{H}^1(\{(0,0)\}\times[-1,1]) \leqq 2\mathcal{E}_2^{\rm sg}(\gamma).$$

This implies that $\Theta_1(x_0) = \mathcal{E}_2^{\rm sg}(\gamma)$. Since $x_0 \in S_* \cap \Omega$, by Proposition 5.5, $\Theta_1(x_0) > 0$ and hence $\mathcal{E}_2^{\rm sg}(\gamma) > 0$. This completes our proof of Proposition 5.16. □

According to Lemma 2.8 and Proposition 5.16, the density of the one-dimensional varifold V_* coming from Proposition 5.8, which was *naturally* associated with $\mu_* \llcorner \Omega$, is bounded from below $\|V_*\|$-a.e. in Ω (i.e., μ_*-a.e. in Ω or, equivalently, $\mathcal{H}^1$-a.e. in $S_* \cap \Omega$) and has a finite set of values. Thus, due to Allard and Almgren's theorem [3, Theorem, p. 89], we obtain the following structure for $S_* \cap \Omega$.

Proposition 5.17. *Any compact set $K \subset \Omega$ such that $S_* \cap K \neq \emptyset$ has an open neighborhood $U \Subset \Omega$ with a Lipschitz boundary and a finite number of connected components such that $S_* \cap \overline{U}$ is a union of a finite number of closed straight line segments, each of which has a positive length, and the interiors of these segments (i.e., the segments without their endpoints) are pairwise disjoint. If K is connected, then U is connected; otherwise, U may have several connected components, but their closures are pairwise disjoint. Furthermore, $S_* \cap \partial U$ is a finite set of points such that for each point $x \in S_* \cap \partial U$ there is exactly one segment lying in $S_* \cap \overline{U}$ and emanating from x.*

Proof. If $\pi_1(\mathcal{N}) = \{0\}$, then by Proposition 5.16 and Proposition 5.5, it holds $S_* \cap \Omega = \emptyset$ and the proof is trivial.

Assume that $\pi_1(\mathcal{N}) \neq \{0\}$. According to the definition of the varifold V_* (see Proposition 5.8), the weight of V_* (or, also the projection of V_* on Ω), defined by $\|V_*\|(E) = V_*(E \times \mathrm{G}(3,1))$ for each $E \in \mathcal{B}(\Omega)$, is equal to $\mu_* \llcorner \Omega = \Theta_1\mathcal{H}^1 \llcorner (S_* \cap \Omega)$. Then, by Proposition 5.5, there exists $c = c(\mathcal{N}) > 0$ such that for each $x \in S_* \cap \Omega$,

$$\Theta_1(\|V_*\|, x) := \lim_{r\to 0+} \frac{\|V_*\|(B_r^3(x))}{2r} = \lim_{r\to 0+} \frac{\mu_*(B_r^3(x))}{2r} = \Theta_1(x) \geqq c. \tag{5.41}$$

In view of Proposition 5.8 and (5.41), we can apply [3, Theorem, p. 89] and obtain that there exists a countable family $\mathcal{P}$ of bounded open intervals (i.e., segments without endpoints) contained in $S_* \cap \Omega$ such that $V_* = \sum\{\Theta_1(I)|I| : I \in \mathcal{P}\}$, where $\Theta_1(I)$ is the unique member of the range of Θ_1 restricted to the interval I and $|I| \in (C_0(\mathrm{G}_3(\Omega)))'$ is a finite positive Radon measure defined by $|I| = \delta_{\rm e\otimes e}(dT) \otimes \mathcal{H}^1 \llcorner I(dx)$, where e is the unit direction vector of I. Taking into account Lemma 2.8 and Proposition 5.16, we have that $\{\Theta_1(I) : I \in \mathcal{P}\} \subset \{\mathcal{E}_2^{\rm sg}(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\}$ is a finite set of values. Furthermore, the proof of [3, Theorem, p. 89] says that for each $x \in S_* \cap \Omega$, there exists a radius $r > 0$ such that $S_* \cap \Omega \cap \overline{B}_r^3(x)$ is a finite union of radii of $\overline{B}_r^3(x)$, that is, closed straight line segments joining x with a point on $\partial B_r^3(x)$. Let $K \subset \Omega$ be compact and satisfy $K \cap S_* \neq \emptyset$. Then there exists a finite collection of open balls $B_{2r_1}^3(x_1), \dots, B_{2r_l}^3(x_l)$ whose closures are contained in Ω, centered at S_* and the following holds: $K \cap S_* \subset \bigcup_{i=1}^{l} B^3_{r_i}(x_i)$; $S_* \cap \overline{B}^3_{2r_i}(x_i)$ is a finite union of radii of $\overline{B}^3_{2r_i}(x_i)$ for each $i \in \{1, \dots, l\}$ (at least two radii are contained in each of these covering balls, because otherwise we would obtain a contradiction with the stationarity of V_*). Now fix an arbitrary $i \in \{1, \dots, l\}$. If for some $j \in \{1, \dots, l\}$, $j \neq i$ a radius L_j of $\overline{B}^3_{r_j}(x_j)$ (i.e., a closed segment joining x_j with a point on $\partial B^3_{r_j}(x_j)$) lying in $S_* \cap \overline{B}^3_{r_j}(x_j)$ intersects $\overline{B}^3_{r_i}(x_i)$, then, by construction of our covering, there exists a radius L_i of $\overline{B}^3_{r_i}(x_i)$ and a straight line containing both L_i and L_j (the union of these radii is a straight line segment). Covering the rest of K with a finite number of chains of sufficiently small balls, one can appropriately choose a neighborhood U of K satisfying the conditions of Proposition 5.17 by passing from ball to ball in each chain from the resulting cover and successively constructing ∂U. This completes our proof of Proposition 5.17. □

As a byproduct of the proof of Proposition 5.16, taking into account Proposition 5.17, we obtain the following.

Corollary 5.18. *Let $L \subset S_*$ be a closed straight line segment. Let x be a point lying in the relative interior of L and $D \subset \Omega$ be a closed 2-disk (which is an embedded submanifold of $\mathbb{R}^3$) with center x such that $D \cap S_* = \{x\}$ and $\partial D \cap S_0 = \emptyset$, where S_0 is at most a countable and locally finite subset of $\Omega \backslash S_*$ coming from Proposition 4.2 (iii), which applies to u_* in $\Omega \backslash S_*$ (see Proposition 5.6). Then the homotopy class of $u_*|_{\partial D}$ is nontrivial and for each point y in the relative interior of L, $\Theta_1(\mu_*, y) = \mathcal{E}^{\mathrm{sg}}_2(u_*|_{\partial D})$.*

Proof of Corollary 5.18. Fix an arbitrary point y lying in the relative interior of L. According to Proposition 5.6 and Proposition 5.17, we can find an open set $U \Subset \Omega$ such that $x, y \in U$, $D \subset U$ and $u_* \in W^{1,2}_{\mathrm{loc}}(U \backslash L, \mathcal{N}) \cap C^\infty(U \backslash (L \cup S_0), \mathcal{N})$. Since S_0 is locally finite, there exists a closed 2-disk $D_y \subset U$ such that $D_y \cap S_* = \{y\}$, $\partial D_y \cap S_0 = \emptyset$ and $u_*|_{\partial D}$ is continuously homotopic to $u_*|_{\partial D_y}$. Then $\mathcal{E}^{\mathrm{sg}}_2(u_*|_{\partial D}) = \mathcal{E}^{\mathrm{sg}}_2(u_*|_{\partial D_y})$. On the other hand, as a byproduct of the proof of Proposition 5.16, $\Theta_1(y) = \mathcal{E}^{\mathrm{sg}}_2(u_*|_{\partial D})$, which completes our proof of Corollary 5.18. □

The stationarity of V_* and Proposition 5.17 yield the following observation.

Corollary 5.19. *Let $x_0 \in S_* \cap \Omega$. Then, according to Proposition 5.17, there exists a radius $r > 0$ such that $\overline{B}^3_r(x_0) \subset \Omega$ and $S_* \cap \overline{B}^3_r(x_0) = \bigcup_{i=1}^{l} L_i$, where $l \in \mathbb{N} \backslash \{0\}$ and L_i is a closed straight line segment emanating from x_0. Let $v_i \in \mathbb{S}^2$ be the direction vector of L_i pointing outward from x_0. Then the stationarity of V_* yields that the weighted sum of the v_i's is zero, namely*

$$\sum_{i=1}^{l} \lambda_i v_i = 0, \tag{5.42}$$

where $\lambda_i = \mathcal{E}^{\mathrm{sg}}_2(u_*|_{\partial D_i}) > 0$ with D_i being a closed 2-disk (which is an embedded submanifold of $\mathbb{R}^3$) centered inside L_i and lying in $B^3_r(x_0)$ on the 2-plane orthogonal to L_i such that $D_i \cap S_*$ is the singleton, $\partial D_i \cap S_0 = \emptyset$, where S_0 is at most a countable and locally finite subset of $\Omega \backslash S_*$ coming from Proposition 4.2 (iii), which applies to u_* in $\Omega \backslash S_*$ (see Proposition 5.6).

Proof of Corollary 5.19. Fix an arbitrary $\chi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$. Since the first variation of V_* vanishes, where $V_*(d\,T,\, \mathrm{d}x) = \delta_{A_*(x)}(dT) \otimes \Theta_1(x)\mathcal{H}^1 \llcorner (S_* \cap \Omega)(dx)$ with $A_*(x)$ being the orthogonal projection matrix onto the approximate tangent plane $T_x S_*$ to S_* at x, we have

$$0 = \int_\Omega A_*(x) : D\chi(x)\,\mathrm{d}\mu_*(x) = \int_{S_* \cap B_r^3(x_0)} \Theta_1(x) A_*(x) : D\chi(x)\,\mathrm{d}\mathcal{H}^1(x). \tag{5.43}$$

Next, using that $S_* \cap \overline{B}_r^3(x_0) = \bigcup_{i=1}^l L_i$, $\mu_* \llcorner B_r^3(x_0) = \sum_{i=1}^l \lambda_i \mathcal{H}^1 \llcorner L_i$ and $A_*|_{L_i} = v_i \otimes v_i$, in view of (5.43), we get

$$\begin{aligned} 0 &= \sum_{i=1}^l \lambda_i \int_{L_i} v_i \otimes v_i : D\chi(x)\,\mathrm{d}\mathcal{H}^1(x) \\ &= \sum_{i=1}^l \lambda_i \int_0^{\mathcal{H}^1(L_i)} \frac{\mathrm{d}}{\mathrm{d}t} \langle \chi(x_0 + t v_i), v_i \rangle\,\mathrm{d}t \\ &= -\sum_{i=1}^l \langle \chi(x_0), \lambda_i v_i \rangle. \end{aligned} \tag{5.44}$$

Since $\chi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$ was arbitrarily chosen, (5.44) yields (5.42), which completes our proof of Corollary 5.19. □

Remark 5.20. We obtain several direct consequences of Corollary 5.19.

(i) For each $x_0 \in S_* \cap \Omega$, at least two segments emanate from x_0, because otherwise x_0 would be an endpoint for some segment lying in $S_* \cap \Omega$, but this would contradict (5.42).

(ii) If locally around x_0 the set $S_* \cap \Omega$ is a union of exactly two segments emanating from x_0, then the angle between them is π, since a smaller angle would contradict (5.42).

(iii) If exactly three segments emanate from x_0, then their direction vectors lie in the same two-dimensional plane.

(iv) If exactly four segments emanate from x_0, then $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = 0$, which yields a system of equations. Firstly, $|\lambda_1 v_1 + \lambda_2 v_2|^2 = |\lambda_3 v_3 + \lambda_4 v_4|^2$ and hence

$$\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 \langle v_1, v_2 \rangle = \lambda_3^2 + \lambda_4^2 + 2\lambda_3\lambda_4 \langle v_3, v_4 \rangle.$$

Secondly, for each $j \in \{1, \dots, 4\}$,

$$\lambda_j + \sum_{i=1,\, i \neq j}^4 \lambda_i \langle v_i, v_j \rangle = 0,$$

since $|v_j| = 1$. Thirdly, for each $j \in \{1, \dots, 4\}$, $|\lambda_j v_j|^2 = |\sum_{i=1,\, i \neq j}^4 \lambda_i v_i|^2$ and hence

$$\lambda_j^2 = \sum_{i=1,\, i \neq j}^4 \lambda_i^2 + 2 \sum_{i,\, k \in (\{1,\dots,4\} \setminus \{j\}) \cap \{i<k\}} \lambda_i \lambda_k \langle v_i, v_k \rangle.$$

If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ in (5.42), then, up to relabeling, $v_1 + v_2 = -v_3 - v_4$, and there exists a double cone with apex 0 such that v_1, v_2 lie on the upper nappe and v_3, v_4 lie on the lower nappe of the cone, respectively. Furthermore, the angle between v_1 and v_2 is the same as between v_3 and v_4, which is twice the angle of the double cone.

Moreover, in [21, Proposition 2] it was proved that in the case when $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$ (for instance, $\mathcal{N} = SO(n)$ or $\mathcal{N} = \mathbb{RP}^{n-1}$ for $n \in \mathbb{N}$, $n \geqq 3$), then only an even number of segments can emanate from $x \in S_* \cap \Omega$. We prove that in the case when $\pi_1(\mathcal{N}) = \mathbb{Z}/2\mathbb{Z}$, there cannot be branching points inside Ω (see Corollary 6.8).

5.4. *Interior Estimates and $W^{1,q}_{\mathrm{loc}}(\Omega)$ compactness for p-minimizers when $q < 2$*

We prove that the q-energy of p_n-minimizers with $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$ and $p_n \nearrow 2$ as $n \to +\infty$ is locally bounded inside Ω for each $q \in [1, 2)$ under the condition (5.1).

For the reader's convenience, let us recall the construction of dyadic cubes in $\mathbb{R}^3$ (see, for instance, [7]).

Definition 5.21. Let $[0, 1)^3$ be the reference cube, $j \in \mathbb{Z}$ and $x \in \mathbb{Z}^3$. Then define the dyadic cube of generation j with lower left corner $2^{-j}x$ by

$$Q_{j,x} := 2^{-j}(x + [0, 1)^3) = \{y \in \mathbb{R}^3 : 2^j y - x \in [0, 1)^3\},$$

the set of generation j dyadic cubes by $\mathcal{D}_j = \{Q_{j,x} : x \in \mathbb{Z}^3\}$ and the set of all dyadic cubes by

$$\mathcal{D} := \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j = \{Q_{j,x} : j \in \mathbb{Z} \text{ and } x \in \mathbb{Z}^3\}.$$

For each $k \in \mathbb{N} \setminus \{0\}$, we also define the enlarged cubes

$$kQ_{j,x} := \left\{ y \in \mathbb{R}^3 : 2^j y - x \in \left[-\frac{k-1}{2}, \frac{k+1}{2} \right)^3 \right\}.$$

According to [7, Theorem 2.2.3 (ii)], for each $j \in \mathbb{Z}$ and $x \in \mathbb{Z}^3$, there exists a unique $y \in \mathbb{Z}^3$ such that $Q_{j,x} \subset Q_{j-1,y}$. The cube $Q_{j-1,y}$ will be called the *parent* of $Q_{j,x}$. For each $j \in \mathbb{Z}$ and $y \in \mathbb{Z}^3$, the cubes $Q_{j+1,x}$ for which $Q_{j,y}$ is the parent are called the *children* of $Q_{j,y}$.

The distance from $U \subset \mathbb{R}^3$ to $V \subset \mathbb{R}^3$ is given by

$$\mathrm{dist}(U, V) := \inf\{|x - y| : x \in U, \ \ y \in V\}.$$

Proposition 5.22. *Let $V \subset \mathbb{R}^3$ be a bounded open set. Let C_1, C_2 be positive constants, $p \in (1, 2]$ and $f \in L^p(V)$. Assume that for each dyadic cube Q_{j,x_0} such that $2Q_{j,x_0} \subset V$ and*

$$C_1 \int_{2Q_{j,x_0}} |f|^p \,\mathrm{d}x \leqq 2^{-j(3-p)} \tag{5.45}$$

it holds

$$\int_{Q_{j,x_0}} |f|^p \,\mathrm{d}x \leqq C_2 2^{-j(3-p)}. \tag{5.46}$$

Then for each $q \in [1, p)$, $f \in L^q(V)$ *and*

$$\int_V (\mathrm{dist}(x, \partial V)|f(x)|)^q \,\mathrm{d}x \leqq 10^4 C_2^{\frac{q}{p}} |V| + 10^4 \frac{C_1 C_2^{\frac{q}{p}}}{p-q} \int_V (\mathrm{dist}(x, \partial V)|f(x)|)^p \,\mathrm{d}x. \tag{5.47}$$

Proof. According to the Whitney covering theorem for dyadic cubes, namely [7, Theorem 2.3.1], there exists a collection of mutually disjoint dyadic cubes $\mathcal{F} = \{Q_i\}_{i \in I}$, where I is countable, such that

$$\frac{1}{30}\mathrm{dist}(Q_i, \partial V) \leqq \mathrm{diam}(Q_i) \leqq \frac{1}{10}\mathrm{dist}(Q_i, \partial V) \tag{5.48}$$

and

$$V = \bigcup_{i \in I} Q_i. \tag{5.49}$$

Since V is bounded, there exists (the minimal) $j_0 \in \mathbb{Z}$ such that for some $x_0 \in \mathbb{Z}^3$, $Q_{j_0,x_0} \in \mathcal{F}$ and for each $x \in \mathbb{Z}^3$ and $j \in (-\infty, j_0) \cap \mathbb{Z}$, $Q_{j,x} \notin \mathcal{F}$. For each $j \in [j_0, +\infty) \cap \mathbb{Z}$, define $\mathcal{F}_j := \mathcal{D}_j \cap \mathcal{F}$. Since V is bounded, $\mathcal{F}_j$ is finite. Notice also that for each $Q \in \mathcal{F}$, $3Q \subset V$ (this comes from (5.48), see [7, Proposition 2.3.3 (i)]). We say that a dyadic cube $Q \in \mathcal{F}_j$ is *good* if (5.45) holds with Q_{j,x_0} replaced by Q, and it is *bad* otherwise. Let us now define the collection of all good cubes of $\mathcal{F}_j$ by $\mathcal{F}_{j,0}$ and their union by $G_{j,0}$. For each bad cube $Q \in \mathcal{F}_j$, we consider the children of Q. Similarly, we say that a child Q' of Q is good if (5.45) holds with Q_{j,x_0} replaced by Q', and it is bad otherwise. We define the collection of all good children of bad cubes of $\mathcal{F}_j$ by $\mathcal{F}_{j,1}$ and their union by $G_{j,1}$. Next, we select good children of bad children of bad cubes in $\mathcal{F}_j$ and denote this collection by $\mathcal{F}_{j,2}$ and their union by $G_{j,2}$. Denote the number of bad cubes in $\mathcal{F}_j$ by $N_{j,0}$. The number of bad children of bad cubes of $\mathcal{F}_j$ denote by $N_{j,1}$ and the number of bad children of bad children of bad cubes of $\mathcal{F}_j$ denote by $N_{j,2}$. Repeating this procedure, we obtain sequences $(\mathcal{F}_{j,n})_{n \in \mathbb{N}}$, $(G_{j,n})_{n \in \mathbb{N}}$ and $(N_{j,n})_{n \in \mathbb{N}}$. Define $G_j := \bigcup_{n \in \mathbb{N}} G_{j,n}$. We claim that G_j has full measure in $\bigcup_{Q \in \mathcal{F}_j} Q$. Indeed, by [7, Theorem 2.2.3], for each $y \in \mathbb{R}^3$, there exists a sequence of dyadic cubes $(Q_{k,y_k})_{k \in \mathbb{N}}$ such that $\{y\} = \bigcap_{k \in \mathbb{N}} Q_{k,y_k}$. If $y \in \bigcup_{Q \in \mathcal{F}_j} Q \setminus G_j$, then, by construction, for each $k \in \mathbb{N}$ large enough, $2Q_{k,y_k} \subset B^3_{2^{-k+1}}(y) \subset V$ and

$$C_1 \int_{B^3_{2^{-k+1}}(y)} |f|^p \,\mathrm{d}x \geqq 2^{-k(3-p)},$$

which implies that

$$\liminf_{k \to +\infty} \frac{1}{|B^3_{2^{-k+1}}|} \int_{B^3_{2^{-k+1}}(y)} |f|^p \,\mathrm{d}x = +\infty.$$

Thus, in view of the Lebesgue differentiation theorem (see [28, Theorem 1.32]) and the fact that $f \in L^p(V)$, $|\bigcup_{Q\in\mathcal{F}_j} Q \setminus G_j| = 0$. This proves our claim. On the other hand, observe that for each $n \in \mathbb{N}$,

$$\#\mathcal{F}_{j,n+1} \leqq 2^3 N_{j,n} \tag{5.50}$$

and, in view of (5.45),

$$N_{j,n} \leqq 3^3 2^{(j+n)(3-p)} C_1 \int_{G_j} |f|^p \, \mathrm{d}x, \tag{5.51}$$

where we have also used that if $Q \in \mathscr{D}_j$, then there exists exactly $3^3 - 1$ other dyadic cubes $Q' \in \mathscr{D}_j$ such that $2Q \cap 2Q' \neq \emptyset$, and that G_j has full measure in $\bigcup_{Q\in\mathcal{F}_j} Q$. Combining (5.50) with (5.51), for each $n \in \mathbb{N}$, we get

$$\#\mathcal{F}_{j,n+1} \leqq 6^3 2^{(j+n)(3-p)} C_1 \int_{G_j} |f|^p \, \mathrm{d}x. \tag{5.52}$$

It is also worth noting that

$$\#\mathcal{F}_{j,0} \leqq 2^{3j} |G_j|. \tag{5.53}$$

Applying Hölder's inequality and (5.46), for each $Q \in \mathcal{F}_{j,n}$, we obtain

$$\begin{aligned} \int_Q |f|^q \, \mathrm{d}x &\leqq |Q|^{1-\frac{q}{p}} \left(\int_Q |f|^p \, \mathrm{d}x \right)^{\frac{q}{p}} \leqq C_2^{\frac{q}{p}} 2^{-3(j+n)\left(1-\frac{q}{p}\right)} 2^{-(j+n)\left(\frac{3q}{p}-q\right)} \\ &\leqq C_2^{\frac{q}{p}} 2^{-(j+n)(3-q)}. \end{aligned}$$

Using this, (5.52) and (5.53), we deduce that

$$\begin{aligned} &\int_{G_j} |f|^q \, \mathrm{d}x \\ &= \sum_{n=0}^{+\infty} \int_{G_{j,n}} |f|^q \, \mathrm{d}x = \sum_{Q\in\mathcal{F}_{j,0}} \int_Q |f|^q \, \mathrm{d}x + \sum_{n=0}^{+\infty} \sum_{Q\in\mathcal{F}_{j,n+1}} \int_Q |f|^q \, \mathrm{d}x \\ &\leqq \#\mathcal{F}_{j,0} C_2^{\frac{q}{p}} 2^{-j(3-q)} + \sum_{n=0}^{+\infty} \#\mathcal{F}_{j,n+1} C_2^{\frac{q}{p}} 2^{-(j+n+1)(3-q)} \\ &\leqq C_2^{\frac{q}{p}} 2^{jq} |G_j| + 3^3 2^q C_1 C_2^{\frac{q}{p}} \sum_{n=0}^{+\infty} 2^{-(j+n)(p-q)} \int_{G_j} |f|^p \, \mathrm{d}x \\ &\leqq C_2^{\frac{q}{p}} 2^{jq} |G_j| + 3^3 2^{jq+p} \frac{C_1 C_2^{\frac{q}{p}}}{\ln(2)(p-q)} \int_{G_j} 2^{-jp} |f|^p \, \mathrm{d}x, \end{aligned}$$

where, in the last estimate, we have used that $2^{p-q} - 1 \geqq \ln(2)(p-q)$. Hence

$$\int_{G_j} 2^{-jq}|f|^q \,\mathrm{d}x \leqq C_2^{\frac{q}{p}}|G_j| + 3^3 2^p \frac{C_1 C_2^{\frac{q}{p}}}{\ln(2)(p-q)} \int_{G_j} 2^{-jp}|f|^p \,\mathrm{d}x. \quad (5.54)$$

Since $1 \leqq q < p \leqq 2$, from (5.48), (5.54) and the fact that G_j has full measure in $\bigcup_{Q \in \mathcal{F}_j} Q$, we obtain

$$\int_{G_j} (\mathrm{dist}(x, \partial V)|f(x)|)^q \,\mathrm{d}x \leqq 10^4 C_2^{\frac{q}{p}}|G_j| + 10^4 \frac{C_1 C_2^{\frac{q}{p}}}{p-q} \int_{G_j} (\mathrm{dist}(x, \partial V)|f(x)|)^p \,\mathrm{d}x.$$

Summing both sides of the above inequality over $j \in [j_0, +\infty) \cap \mathbb{Z}$ and taking into account that $\bigcup_{j=j_0}^{+\infty} G_j$ has full measure in V (this comes from (5.49) and the fact that G_j has full measure in $\bigcup_{Q \in \mathcal{F}_j} Q$), yields

$$\int_V (\mathrm{dist}(x, \partial V)|f(x)|)^q \,\mathrm{d}x \leqq 10^4 C_2^{\frac{q}{p}}|V| + 10^4 \frac{C_1 C_2^{\frac{q}{p}}}{p-q} \int_V (\mathrm{dist}(x, \partial V)|f(x)|)^p \,\mathrm{d}x.$$

This completes our proof of Proposition 5.22. □

As a consequence of Proposition 5.22, we obtain an improvement of Proposition 5.6.

Proposition 5.23. *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of p_n-minimizers in Ω. Then the following assertions hold.*

(i) *There exists $C = C(\mathrm{diam}(\Omega), \mathcal{N}) > 0$ such that for each $q \in [1, 2)$,*

$$\limsup_{n \to +\infty} \int_\Omega (\mathrm{dist}(x, \partial\Omega)|Du_n(x)|)^q \,\mathrm{d}x \leqq C|\Omega| + \frac{C}{2-q} \limsup_{n \to +\infty} (2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x.$$

(ii) *Let $(u_n)_{n \in \mathbb{N}}$ satisfy (5.1) and let u_* be given by Proposition 5.6. Then for each $q \in [1, 2)$, $u_* \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathcal{N})$ and, up to a subsequence (not relabeled), $u_n \to u_*$ in $W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^\nu)$ and*

$$\int_\Omega (\mathrm{dist}(x, \partial\Omega)|Du_*(x)|)^q \,\mathrm{d}x \leqq C|\Omega| + \frac{C}{2-q} \limsup_{n \to +\infty} (2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x, \quad (5.55)$$

where $C = C(\mathrm{diam}(\Omega), \mathcal{N}) > 0$.

Proof. Let $\eta, C > 0$ be the constants of Lemma 3.12, where $\kappa = 1/2$, $p_0 = 3/2$ and Ψ comes from [32, Corollary 3]. Observe that if Q is a dyadic cube with barycenter x_Q and sidelength $r > 0$, then, denoting its interior by Q°, we have $Q^\circ = \Psi(B^3_{r/2}) + x_Q$, $2Q^\circ = \Psi(B^3_r) + x_Q$ and the bilipschitz constant of the map $\Psi + x_Q$ is the bilipschitz constant of Ψ. Thus, applying Proposition 5.22 with $V = \Omega$, $p = p_n$, $f = |Du_n|$, $C_2 = C$, $C_1 = \frac{2-p_n}{\eta}$ and taking into account that $\operatorname{dist}(\cdot, \partial\Omega) \leqq \operatorname{diam}(\Omega)$ everywhere in Ω, for each $q \in [1, 2)$ we get

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_\Omega (\operatorname{dist}(x, \partial\Omega)|Du_n(x)|)^q \,\mathrm{d}x \\
&\leqq C'|\Omega| + \frac{C'}{2-q} \limsup_{n\to+\infty} (2-p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x,
\end{aligned}$$

where $C' = C'(\operatorname{diam}(\Omega), \mathcal{N}) > 0$. This yields (i).

Now we prove (ii). Let $q \in [1, 2)$ and $\mathcal{F} = \{Q_i\}_{i\in I}$ be a Whitney covering of $\Omega\backslash S_*$ by dyadic cubes such that $\Omega\backslash S_* = \bigcup_{i\in I} Q_i$ and $2Q_i \subset \Omega\backslash S_*$ for each $i \in I$ (see, for example, [7, Theorem 2.3.1]). Notice that I is countable. Using the fact that $|S_*| = 0$ (see Proposition 5.8), the weak convergence (see Proposition 4.2 (i)), together with the convexity, and also using Fatou's lemma, up to a subsequence (not relabeled), one has

$$\begin{aligned}
\int_\Omega (\operatorname{dist}(x, \partial\Omega)|Du_*(x)|)^q \,\mathrm{d}x &= \sum_{i\in I} \int_{Q_i} (\operatorname{dist}(x, \partial\Omega)|Du_*(x)|)^q \,\mathrm{d}x \\
&\leqq \sum_{i\in I} \liminf_{n\to+\infty} \int_{Q_i} (\operatorname{dist}(x, \partial\Omega)|Du_n(x)|)^q \,\mathrm{d}x \\
&\leqq \liminf_{n\to+\infty} \sum_{i\in I} \int_{Q_i} (\operatorname{dist}(x, \partial\Omega)|Du_n(x)|)^q \,\mathrm{d}x \\
&= \liminf_{n\to+\infty} \int_\Omega (\operatorname{dist}(x, \partial\Omega)|Du_n(x)|)^q \,\mathrm{d}x.
\end{aligned}$$

This, in view of (i) and (5.1), implies (5.55) and the fact that $u_* \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathcal{N})$ for all $q \in [1, 2)$. Next, let $q_0 \in (q, 2)$ and $U \Subset \Omega$ be an open set. Fix a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets such that $K_n \Subset U\backslash S_*$, $K_n \subset K_{n+1}$ and $\bigcup_{n\in\mathbb{N}} K_n = U \setminus S_*$. Applying Hölder's inequality and using the fact that $|Du_n - Du_*|^{q_0} \leqq 2^{q_0-1}(|Du_n|^{q_0} + |Du_*|^{q_0})$ (by convexity), for each $n \in \mathbb{N}$ large enough we obtain

$$\begin{aligned}
\int_U |Du_n - Du_*|^q \,\mathrm{d}x &\leqq |U \setminus K_n|^{1-\frac{q}{q_0}} 2^{q-\frac{q}{q_0}} \left(\int_{U\setminus K_n} (|Du_n|^{q_0} + |Du_*|^{q_0}) \,\mathrm{d}x \right)^{\frac{q}{q_0}} \\
&\quad + |K_n|^{1-\frac{q}{p_n}} \left(\int_{K_n} |Du_n - Du_*|^{p_n} \,\mathrm{d}x \right)^{\frac{q}{p_n}}.
\end{aligned}$$

By carefully choosing K_n and letting n tend to $+\infty$, in view of Proposition 4.2 (ii), up to a subsequence (not relabeled), we can ensure that the last two terms tend to 0 as $n \to +\infty$, and hence $u_n \to u_*$ in $W^{1,q}(U, \mathbb{R}^\nu)$ as $n \to +\infty$. Using a diagonal argument, we deduce (ii), which completes our proof of Proposition 5.23. $\square$

6. Interior Minimality of the Singular Set

In this section, we prove some properties of local minimality inside Ω of the positive Radon measure $\mu_* \in (C(\overline{\Omega}))'$ and its support S_*, defined at the beginning of Section 5.

First, we define the notion of an *admissible chain* for a bounded Lipschitz domain $U \subset \mathbb{R}^3$ and a boundary datum $g \in W^{1/2,2}(\partial U, \mathcal{N})$.

Definition 6.1. Given a bounded Lipschitz domain $U \subset \mathbb{R}^3$ and a map $g \in W^{1/2,2}(\partial U, \mathcal{N})$, we say that a closed set $S \subset \overline{U}$ is an *admissible chain* for U and g if the following properties are satisfied.

(P.1) S is the union of a finite number of closed straight line segments, each of which has a positive length, and the interiors of these segments (i.e., the segments without their endpoints) are pairwise disjoint.
(P.2) $S \cap \partial U$ is a finite set of points.
(P.3) S has no endpoints in U, namely, if $L \subset S$ is a segment having an endpoint $x \in U$, then there exists another segment $L' \subset S$ emanating from x.
(P.4) There exists a map $u \in W^{1,2}_{\mathrm{loc}}(U \backslash S, \mathcal{N})$ such that $\mathrm{tr}_{\partial U}(u) = g$.

The set of all admissible chains for U and g will be denoted by $\mathscr{C}(U, g)$.

Remark 6.2. Assume that there exists $u \in W^{1,2}_{\mathrm{loc}}(U \backslash S, \mathcal{N})$ satisfying (P.4). Then there exists $v \in W^{1,2}_{\mathrm{loc}}(U \backslash S, \mathcal{N}) \cap C(U \backslash (S \cup S_v), \mathcal{N})$ such that $\mathrm{tr}_{\partial U}(v) = g$, where S_v is at most a countable and locally finite subset of $U \setminus S$ (see [52, Theorem II]). Let L be a segment in the chain S and x_1, x_2 be interior points of L such that $x_1 \neq x_2$. Since $v \in W^{1,2}_{\mathrm{loc}}(U \backslash S, \mathcal{N})$, using the coarea formula, we can find a sufficiently small radius $r > 0$ such that the following holds. There exists $\gamma \in [\mathbb{S}^1, \mathcal{N}]$ such that for $\mathcal{H}^1$-a.e. $x \in [x_1, x_2]$, letting ∂D to be the circle with center x and radius r lying in the 2-plane orthogonal to L, we have $\mathrm{tr}_{\partial D}(v) = v|_{\partial D} \in W^{1,2}(\partial D, \mathcal{N})$, $v|_{\partial D}$ is homotopic to γ and $\mathcal{E}_2^{\mathrm{sg}}(v|_{\partial D}) = \mathcal{E}_2^{\mathrm{sg}}(\gamma)$. On the other hand, since $v \in C(U \backslash (S \cup S_v), \mathcal{N})$ and S_v is locally finite, we can choose x_1 and x_2 to be arbitrarily close to the endpoints of L and repeat the previous procedure for a smaller radius $\varrho \in (0, r)$, but without changing $\gamma \in [\mathbb{S}^1, \mathcal{N}]$. Thus, we can define the *mass* of S with respect to the extension v of g as the sum $\sum_{i=1}^{l} \lambda_i \mathcal{H}^1(L_i)$, where $\bigcup_{i=1}^{l} L_i = S$ is the union of the segments lying in S and $\lambda_i = \mathcal{E}_2^{\mathrm{sg}}(\gamma_i)$, where $\gamma_i \in [\mathbb{S}^1, \mathcal{N}]$ is defined for each L_i with respect to v. Denote this mass by $\mathbb{M}(v, g, S)$. It is worth noting that for different extensions $u_1 \in W^{1,2}_{\mathrm{loc}}(U \backslash S, \mathcal{N}) \cap C(U \backslash (S \cup S_{u_1}), \mathcal{N})$ and $u_2 \in W^{1,2}_{\mathrm{loc}}(U \backslash S, \mathcal{N}) \cap C(U \backslash (S \cup S_{u_2}), \mathcal{N})$ of g satisfying the property (P.4) of Definition 6.1, we can obtain different masses of S with respect to u_1 and u_2, however, the following

$$\inf\{\mathbb{M}(u, g, S) : u \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N}) \cap C(U \setminus (S \cup S_u), \mathcal{N}),\ \ \mathrm{tr}_{\partial U}(u) = g\} \tag{6.1}$$

is achieved. Indeed, if $u_n \in W^{1,2}_{\mathrm{loc}}(U \backslash S, \mathcal{N})$ is a minimizing sequence for (6.1), then, since $\#[\mathbb{S}^1, \mathcal{N}] < +\infty$, the set $\{\mathcal{E}_2^{\mathrm{sg}}(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\}$ is finite, and hence $\{\mathbb{M}(u_n, g, S) : n \in \mathbb{N}\}$ is finite. Thus, for each sufficiently large $n \in \mathbb{N}$, $\mathbb{M}(u_n, g, S)$

is equal to the infimum in (6.1), and each u_n, for sufficiently large $n \in \mathbb{N}$, realizes the infimum in (6.1).

Next, using Remark 6.2, we define the notion of the *mass* of an admissible chain.

Definition 6.3. Given a bounded Lipschitz domain $U \subset \mathbb{R}^3$ and a map $g \in W^{1/2,2}(\partial U, \mathcal{N})$, if $S \in \mathscr{C}(U, g)$, we define its mass by

$$\mathbb{M}(g, S) = \mathbb{M}(u, g, S),$$

where $u \in W^{1,2}_{\mathrm{loc}}(U \backslash S, \mathcal{N}) \cap C(U \backslash (S \cup S_u), \mathcal{N})$ realizes the infimum in (6.1) for g and S.

When appropriate, we shall simply write $\mathbb{M}(S)$ instead of $\mathbb{M}(g, S)$. Notice that the definition of $\mathbb{M}(g, S)$ also depends on $\mathcal{N}$, but we decided not to mention it explicitly to simplify the notation.

Now we prove the interior minimality of S_*.

Proposition 6.4. *Let $u_* \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus S_*, \mathcal{N})$ and $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$ be given by Proposition 5.6. Then for any domain $K \Subset \Omega$ such that $\overline{K} \cap S_* \neq \emptyset$, there exists a Lipschitz domain $U \Subset \Omega$ containing K such that $\mathrm{tr}_{\partial U}(u_*) = u_*|_{\partial U} \in W^{1/2,2}(\partial U, \mathcal{N})$, $S_* \cap \overline{U} \in \mathscr{C}(U, u_*|_{\partial U})$ and for each $S \in \mathscr{C}(U, u_*|_{\partial U})$,*

$$\mu_*(S_* \cap U) = \mathbb{M}(u_*|_{\partial U}, S_* \cap \overline{U}) \leqq \mathbb{M}(u_*|_{\partial U}, S). \tag{6.2}$$

Remark 6.5. Proposition 6.4 allows the choice of the domain U to be arbitrarily close to K.

Proof of Proposition 6.4. According to Proposition 5.17, since $\overline{K}$ is connected, there exists a Lipschitz domain $V \Subset \Omega$ containing $\overline{K}$ such that $S_* \cap \overline{V}$ is a union of a finite number of closed straight line segments, each of which has a positive length, and the interiors of these segments are pairwise disjoint. Also $S_* \cap \partial V$ is a finite set of points such that for each point $x \in S_* \cap \partial V$ there is exactly one segment lying in $S_* \cap \overline{V}$ and emanating from x. Changing, if necessary, ∂V locally around each point of $S_* \cap \partial V$, we can assume that for each $x \in S_* \cap \partial V$, ∂V coincides with its tangent plane at x in a small neighborhood of x, and the segment lying in $S_* \cap \overline{V}$ and emanating from x is orthogonal to this tangent plane.

Step 1. We define U. Fix $\lambda > 0$ small enough such that for each $t \in (0, \lambda]$, all the above properties prescribed for V concerning S_* and K hold also for $V_t = \{y \in V : \mathrm{dist}(y, \partial V) > t\}$, which is a Lipschitz domain. Applying [30, Theorem 3.2.22 (3)] with $W = V \backslash \overline{V}_\lambda$, $Z = (0, \lambda)$ and $f : W \to Z$ defined by $f(y) = \mathrm{dist}(y, \partial V)$, and using also the fact that $|Df(y)| = 1$ for a.e. $y \in W$, Fatou's lemma, (5.1) and Proposition 5.23 (ii), we obtain

$$\begin{aligned} \int_0^\lambda \mathrm{d}\varrho \ \liminf_{n\to+\infty} \int_{f^{-1}(\{\varrho\})} (|u_n - u_*|^q + |Du_n - Du_*|^q)\, \mathrm{d}\mathcal{H}^2 \\ \leqq \liminf_{n\to+\infty} \int_W (|u_n - u_*|^q + |Du_n - Du_*|^q)\, \mathrm{d}y = 0 \end{aligned} \tag{6.3}$$

for each $q \in (1, 2)$ and

$$\int_0^\lambda \mathrm{d}\varrho \liminf_{n\to+\infty}(2-p_n)\int_{f^{-1}(\{\varrho\})}(|Du_n|^{p_n}+|Du_*|^{p_n})\,\mathrm{d}\mathcal{H}^2 \leqq \liminf_{n\to+\infty}(2-p_n)\int_W(|Du_n|^{p_n}+|Du_*|^{p_n})\,\mathrm{d}y \leqq C',$$

where $C' = C'(\mathrm{dist}(V, \Omega), \mathrm{diam}(\Omega), C_0, \mathcal{N}) > 0$ and $C_0 > 0$ is the constant coming from (5.1). On the other hand, using Proposition 5.6 (thanks to which we know that, up to a subsequence (not relabeled), $u_n \to u_*$ in $L^p(\Omega, \mathbb{R}^\nu)$ for all $p \in [1, +\infty)$ (see Proposition 4.2 (i))) and the continuity of the L^p-norm, we deduce that $\|u_n - u_*\|^{p_n}_{L^{p_n}(W,\mathbb{R}^\nu)} \to 0$ as $n \to +\infty$ and hence

$$\lim_{n\to+\infty}\int_0^\lambda \mathrm{d}\varrho\int_{f^{-1}(\{\varrho\})}|u_n-u_*|^{p_n}\,\mathrm{d}\mathcal{H}^2 = 0.$$

Then there exists $t \in (0, \lambda)$ such that for $U = V_t$, up to a subsequence (not relabeled), for each $n \in \mathbb{N}$ large enough: $\mathrm{tr}_{\partial U}(u_n) = u_n|_{\partial U}$, $\mathrm{tr}_{\partial U}(u_*) = u_*|_{\partial U} \in W^{1,p_n}(\partial U, \mathcal{N}) \subset W^{1/2,2}(\partial U, \mathcal{N})$ and

$$\int_{\partial U}(|Du_n|^{p_n}+|Du_*|^{p_n})\,\mathrm{d}\mathcal{H}^2 \leqq \frac{C''}{2-p_n}, \tag{6.4}$$

where $C'' = C''(t, \mathrm{dist}(V, \Omega), \mathrm{diam}(\Omega), C_0, \mathcal{N}) > 0$;

$$\|u_n|_{\partial U} - u_*|_{\partial U}\|^{p_n}_{L^{p_n}(\partial U,\mathbb{R}^\nu)} \to 0 \tag{6.5}$$

as $n \to +\infty$;

$$\|u_n|_{\partial U} - u_*|_{\partial U}\|_{L^\infty(\partial U;\mathbb{R}^\nu)} \to 0 \tag{6.6}$$

as $n \to +\infty$, in view of (6.3) and Morrey's embedding for Sobolev mappings. Since t and V uniquely determine U, we can assume that C'' actually depends on U, $\mathrm{diam}(\Omega)$, C_0 and $\mathcal{N}$. Thus, we have defined U. According to our construction and Proposition 5.6, the properties (P.1)-(P.4) of Definition 6.1 are fulfilled for U, $u_*|_{\partial U}$ and $S_* \cap \overline{U}$. Therefore, $S_* \cap \overline{U} \in \mathscr{C}(U, u_*|_{\partial U})$.
Step 2. Given $S \in \mathscr{C}(U, u_*|_{\partial U})$, we construct a sequence of mappings $w_n \in W^{1,p_n}(U, \mathcal{N})$ such that $\mathrm{tr}_{\partial U}(w_n) = u_*|_{\partial U}$ and

$$\limsup_{n\to+\infty}(2-p_n)\int_U\frac{|Dw_n|^{p_n}}{p_n}\,\mathrm{d}y \leqq \mathbb{M}(S). \tag{6.7}$$

Consider a map $u \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N})$ minimizing the expression in (6.1), where $g = u_*|_{\partial U}$. Let $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$ be small enough and δ be fairly small with respect to ε. Then, using the coarea formula and taking into account that $u \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N})$, we obtain $\delta' \in (3\delta/4, 5\delta/4)$ and $\varepsilon' \in (3\varepsilon/4, 5\varepsilon/4)$ such that, defining $S_{\delta'} = \{y \in U : \mathrm{dist}(y, S) \leqq \delta'\}$, we have the following. Firstly, $\mathrm{tr}_{\partial S_{\delta'}\cap U}(u) = u|_{\partial S_{\delta'}\cap U} \in W^{1,2}(\partial S_{\delta'} \cap U, \mathcal{N})$. Secondly, for each branching point $x \in S \cap \overline{U}$ of S (if such a point exists), $\mathrm{tr}_{(\partial B^3_{\varepsilon'}(x)\setminus S_{\delta'})\cap U}(u) = u|_{(\partial B^3_{\varepsilon'}(x)\setminus S_{\delta'})\cap U} \in$

$W^{1,2}((\partial B^3_{\varepsilon'}(x)\backslash S_{\delta'})\cap U,\mathcal{N})$, where $(\partial B^3_{\varepsilon'}(x)\backslash S_{\delta'})\cap U = \partial B^3_{\varepsilon'}(x)\backslash S_{\delta'}$ if $x \in U$ (we recall that δ is fairly small with respect to ε and the structure of S is given by Definition 6.1). To lighten the notation, we assume that $\delta' = \delta$ and $\varepsilon' = \varepsilon$. Let L be a segment of S. Up to rotation and translation, we can assume that $L = \{(0,0)\} \times [-h,h]$, where $h = \mathcal{H}^1(L)/2 > 0$. Let $\Lambda := B^2_\delta \times (z(0,0,-h), z(0,0,h))$, where $z(0,0,-h) := -h + \sqrt{\varepsilon^2-\delta^2}$ if $(0,0,-h)$ is a branching point of S (i.e., several segments of S emanate from this point), $z(0,0,h) := h - \sqrt{\varepsilon^2-\delta^2}$ if $(0,0,h)$ is a branching point of S, $z(0,0,-h) := -h + \kappa\delta$ if $(0,0,-h) \in \partial U$ is an endpoint of S (i.e., only one segment of S emanates from this point, namely L), $z(0,0,h) = h - \kappa\delta$ if $(0,0,h) \in \partial U$ is an endpoint of S, where $\kappa = \kappa(S) > 0$ will be defined later for the proof to work. Hereinafter in this proof, C will denote a positive constant that is independent of n, ε and δ and can be different from line to line. We define w_n in Λ by using Lemma 2.20 (i), namely we obtain a map $w_n|_\Lambda \in W^{1,p_n}(\Lambda,\mathcal{N})$ such that

$$\int_\Lambda \frac{|Dw_n|^{p_n}}{p_n}\,\mathrm{d}y \leqq \frac{\mathcal{H}^1(L)\mathcal{E}^{\mathrm{sg}}_{p_n}(\gamma)\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}}\int_\Gamma \frac{|D_\top u|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2, \tag{6.8}$$

where Γ is the lateral surface of Λ and $\gamma \in [\mathbb{S}^1,\mathcal{N}]$ is homotopic to $u|_{\partial D} \in C(\partial D,\mathcal{N})$, where ∂D is the circle being the boundary of a 2-disk orthogonal to L with fairly small radius and whose center belongs to the interior of L. We have used that $\Gamma \subset (\partial S_\delta \cap U)$ and $u|_{\partial S_\delta\cap U} \in W^{1,2}(\partial S_\delta \cap U,\mathcal{N})$.

We need to distinguish between two further cases.

Case 1. Let $x \in \partial U$ be an endpoint of S (if it exists). Then, taking into account the properties (P.1) and (P.2) of Definition 6.1 and the fact that for each $x \in S \cap \partial U$, ∂U coincides with its tangent plane at x in a small neighborhood of x, we can find $\kappa = \kappa(S) > 0$ such that for each endpoint $y \in \partial U$ of S, if L is a segment of S emanating from y, then the 2-disk orthogonal to L with center $\xi \in L$ such that $|\xi - y| = \kappa\delta$ lies entirely in U. Now let L denote the segment of S emanating from x. Up to rotation and translation, $L = \{(0,0)\} \times [-h,h]$ and $x = (0,0,h)$, where $h := \mathcal{H}^1(L)/2 > 0$. Let $\Lambda_\delta := \{y = (y_1,y_2,t) \in U : \operatorname{dist}(y,L) < \delta \text{ and } t \in (h-\kappa\delta,h)\}$. Then $\overline{\Lambda}_\delta$ is bilipschitz homeomorphic to $\overline{B}^3_\delta$ with a bilipschitz constant depending on κ. Thus, there exists a bilipschitz map $\Psi : \overline{\Lambda}_\delta \to \overline{B}^3_\delta$. We have constructed w_n on $\Lambda = B^2_\delta \times (z(0,0,-h), h-\kappa\delta)$. By Lemma 2.20 (ii), $\mathrm{tr}_{B^2_\delta\times\{h-\kappa\delta\}}(w_n) \in W^{1,p_n}(B^2_\delta \times \{h-\kappa\delta\},\mathcal{N})$ and

$$\begin{aligned}\int_{B^2_\delta\times\{h-\kappa\delta\}} \frac{|D\,\mathrm{tr}_{B^2_\delta\times\{h-\kappa\delta\}}(w_n)|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 &\leqq \frac{\mathcal{E}^{\mathrm{sg}}_{p_n}(\gamma)\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}}\int_\Gamma \frac{|D_\top u|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2\\ &\leqq \frac{C\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}}\int_\Gamma \frac{|D_\top u|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2,\end{aligned} \tag{6.9}$$

where $\gamma \in [\mathbb{S}^1, \mathcal{N}]$ comes from (6.8), Γ is the lateral surface of Λ, and we have used that $\#[\mathbb{S}^1, \mathcal{N}] < +\infty$. Now we define the mapping $g_n : \partial\Lambda_\delta \to \mathcal{N}$ by

$$g_n(y) = \begin{cases} u_*|_{\partial U}(y) & \text{if } \; y \in \partial\Lambda_\delta \cap \partial U, \\ u|_{\partial S_\delta \cap U}(y) & \text{if } \; y \in \partial S_\delta \cap U, \\ \mathrm{tr}_{B^2_\delta \times \{h-\kappa\delta\}}(w_n) & \text{if } \; y \in B^2_\delta \times \{h-\kappa\delta\}. \end{cases}$$

Then, since the traces of $u_*|_{\partial U}$ and $u|_{\partial S_\delta \cap U}$ coincide on $\partial S_\delta \cap \partial U$, and the traces of $u|_{\partial S_\delta \cap U}$ and $\mathrm{tr}_{B^2_\delta \times \{h-\kappa\delta\}}(w_n)$ coincide on $\partial B^2_\delta \times \{h-\kappa\delta\}$, we deduce that $g_n \in W^{1,p_n}(\partial\Lambda_\delta, \mathcal{N})$. Using (6.4), (6.9) and the fact that $\delta \in (0,1)$ is fairly small, we obtain

$$\int_{\partial\Lambda_\delta} \frac{|D_\top g_n|^{p_n}}{p_n} \,\mathrm{d}\mathcal{H}^2 \leqq \frac{C}{2-p_n} + \frac{C\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}} \int_{\partial S_\delta \cap U} \frac{|D_\top u|^{p_n}}{p_n} \,\mathrm{d}\mathcal{H}^2. \tag{6.10}$$

Next, we define $w_n|_{\Lambda_\delta}(y) = g_n(\Psi^{-1}(\delta\Psi(y)/|\Psi(y)|))$ for $y \in \Lambda_\delta \backslash \{\Psi^{-1}(0)\}$. Then, using (6.10), we deduce the following

$$\begin{aligned} \int_{\Lambda_\delta} \frac{|Dw_n|^{p_n}}{p_n} \,\mathrm{d}y &\leqq C\delta \int_{\partial\Lambda_\delta} \frac{|D_\top g_n|^{p_n}}{p_n} \,\mathrm{d}\mathcal{H}^2 \\ &\leqq \frac{C\delta}{2-p_n} + \frac{C\delta^{3-p_n}}{2-p_n} + C\delta^{2-p_n} \int_{\partial S_\delta \cap U} \frac{|D_\top u|^{p_n}}{p_n} \,\mathrm{d}\mathcal{H}^2. \end{aligned} \tag{6.11}$$

Case 2. Let x be a branching point of S (if it exists). Then, according to our construction, $u|_{(\partial B^3_\varepsilon(x)\backslash S_\delta)\cap U} \in W^{1,2}((\partial B^3_\varepsilon(x)\backslash S_\delta) \cap U, \mathcal{N})$. Furthermore, $(\partial B^3_\varepsilon(x)\backslash S_\delta) \cap U = \partial B^3_\varepsilon(x)\backslash S_\delta$ if $x \in U$. Let L_i, $i \in \{1, \ldots, s\}$ be the segments of S emanating from x. We fill the holes in $(\partial B^3_\varepsilon(x)\backslash S_\delta) \cap U$ with the closures of the open 2-disks D_i such that D_i is orthogonal to L_i, has radius δ, and its center lies on L_i at a distance of $\sqrt{\varepsilon^2 - \delta^2}$ from the point x. Thus, $\mathcal{M} = (\partial U \cap \overline{B}^3_\varepsilon(x)) \cup ((\partial B^3_\varepsilon(x)\backslash S_\delta) \cap U) \cup \bigcup_{i=1}^{l} \overline{D}_i$ is a compact 2-manifold without boundary, which is bilipschitz homeomorphic to ∂B^3_ε. Furthermore, according to our construction, for each $i \in \{1, \ldots, l\}$, $\mathrm{tr}_{D_i}(w_n) \in W^{1,p_n}(D_i, \mathcal{N})$. Since the traces of $u_*|_{\partial U}$ and $u|_{(\partial B^3_\varepsilon(x)\backslash S_\delta)\cap U}$ coincide on $\partial U \cap \partial B^3_\varepsilon(x)$, and the traces of $u|_{(\partial B^3_\varepsilon(x)\backslash S_\delta)\cap U}$ and $\mathrm{tr}_{D_i}(w_n)$ coincide on ∂D_i, defining $v_n : \mathcal{M} \to \mathcal{N}$ by

$$v_n(y) = \begin{cases} u_*|_{\partial U}(y) & \text{if } \; y \in \partial U \cap \overline{B}^3_\varepsilon(x), \\ u|_{(\partial B^3_\varepsilon(x)\backslash S_\delta)\cap U}(y) & \text{if } \; y \in (\partial B^3_\varepsilon(x) \setminus S_\delta) \cap U, \\ \mathrm{tr}_{\overline{D}_i}(w_n)(y) & \text{if } \; y \in \overline{D}_i, \end{cases}$$

we observe that $v_n \in W^{1,p_n}(\mathcal{M}, \mathcal{N})$. Next, let E be an open bounded set whose boundary is $\mathcal{M}$. Let $\Phi : \overline{E} \to \overline{B}^3_\varepsilon$ be a bilipschitz homeomorphism. We define

$w_n|_E(y) = v_n(\Phi^{-1}(\varepsilon\Phi(y)/|\Phi(y)|))$ for $y \in E\setminus\{\Phi^{-1}(0)\}$. Then, using (6.4), (6.9) and Definition 6.1, we obtain

$$\begin{aligned}\int_E \frac{|Dw_n|^{p_n}}{p_n}\,\mathrm{d}y &\leqq C\varepsilon\int_{\mathcal{M}} \frac{|D_\top v_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2\\ &\leqq C\varepsilon\int_{\partial U\cap B^3_\varepsilon(x)} \frac{|D_\top u_*|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 + C\varepsilon\int_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U}\frac{|D_\top u|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2\\ &\quad + C\varepsilon\sum_{i=1}^{l}\int_{D_i}\frac{|D_\top \operatorname{tr}_{D_i}(w_n)|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \qquad (6.12)\\ &\leqq \frac{C\varepsilon}{2-p_n} + C\varepsilon\int_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U}\frac{|D_\top u|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 + \frac{C\varepsilon\delta^{2-p_n}}{2-p_n}\\ &\quad + \frac{C\varepsilon}{\delta^{p_n-1}}\int_{\partial S_\delta\cap U}\frac{|D_\top u|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

Finally, we define w_n for each y in U lying outside the set $E_{\varepsilon,\delta}$, which is the union of S_δ with the union of $\overline{B}^3_\varepsilon(x)\cap U$ over all branching points x of S, by $w_n(y) = u(y)$. Since the traces of the piecewise definitions of w_n coincide, $w_n \in W^{1,p_n}(U,\mathcal{N})$. Next, gathering together (6.8), (6.11), (6.12) and using the facts that $\varepsilon, \delta \in (0, 1)$ are sufficiently small, δ is fairly small with respect to ε, $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$ and $p_n \nearrow 2$ as $n \to +\infty$, S consists of a finite union of segments (and hence it has a finite number of branching points and endpoints), for all sufficiently large $n \in \mathbb{N}$, we deduce the following

$$\begin{aligned}\int_U \frac{|Dw_n|^{p_n}}{p_n}\,\mathrm{d}y \leqq \frac{\mathbb{M}(S)\delta^{2-p_n}}{2-p_n} + \frac{C\varepsilon}{2-p_n} + C\varepsilon\sum_{x\in P}\int_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U}\frac{|D_\top u|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2\\ + \frac{C\varepsilon\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}}\int_{\partial S_\delta\cap U}\frac{|D_\top u|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 + \int_{U\setminus E_{\varepsilon,\delta}}\frac{|Du|^{p_n}}{p_n}\,\mathrm{d}y,\end{aligned}$$

where P is the set of branching points of S. Multiplying both sides of the above inequality by $(2 - p_n)$, then letting n tend to $+\infty$ and using the continuity of the L^p-norm and the facts that $u \in W^{1,2}(\partial S_\delta\cap U,\mathcal{N})$, $u \in W^{1,2}((\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U,\mathcal{N})$ for each branching point x of S, $u \in W^{1,2}(U\setminus E_{\varepsilon,\delta},\mathcal{N})$, we deduce that

$$\limsup_{n\to+\infty}(2-p_n)\int_U \frac{|Dw_n|^{p_n}}{p_n}\,\mathrm{d}y \leqq \mathbb{M}(S) + C\varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily small and C is independent of ε, the last estimate implies (6.7).

Step 3. We prove (6.2). Let the sequence of maps $w_n \in W^{1,p_n}(U,\mathcal{N})$ be as in *Step 2*. Using Lemma 4.1, we construct a competitor for the minimizer u_n by interpolating between u_n and w_n. In view of (6.5) and the fact that $\operatorname{tr}_{\partial U}(w_n) = u_*|_{\partial U}$,

$$\|u_n|_{\partial U} - \operatorname{tr}_{\partial U}(w_n)\|^{p_n}_{L^{p_n}(\partial U,\mathbb{R}^\nu)} \to 0 \qquad (6.13)$$

as $n \to +\infty$. Let $T_0 = T_0(\partial U) > 0$ be the constant of Lemma 4.1. Choose $T' \in (0, T_0)$ sufficiently small and $\Phi : \partial U \times [0, T'] \to \overline{E}_{r_0}$ such that Φ is bilipschitz onto its image, $\Phi(\sigma, 0) = \sigma$, $D_\top \Phi(\sigma, 0)$ is an isometry for $\sigma \in \partial U$, where $E_{r_0} = \{x \in U : \operatorname{dist}(x, \partial U) < r_0\}$ for some fairly small constant $r_0 > 0$ depending on T'. According to Lemma 2.1, there exists a constant $\lambda_{\mathcal{N}} > 0$ such that the nearest point retraction $\Pi_{\mathcal{N}} : \mathcal{N}_{\lambda_{\mathcal{N}}} \to \mathcal{N}$ is well defined and smooth, where $\mathcal{N}_{\lambda_{\mathcal{N}}} = \{x \in \mathbb{R}^\nu : \operatorname{dist}(x, \mathcal{N}) < \lambda_{\mathcal{N}}\}$. Setting $s_n = \|u_n|_{\partial U} - \operatorname{tr}_{\partial U}(w_n)\|_{L^{p_n}(\partial U, \mathbb{R}^\nu)}$ and using (6.4) and (6.13), we observe that $s_n \to 0$ and for each sufficiently large $n \in \mathbb{N}$,

$$\int_{\partial U} \left(|D_\top u_n|^{p_n} + |D_\top \operatorname{tr}_{\partial U}(w_n)|^{p_n} + \frac{|u_n - u_*|^{p_n}}{s_n^{p_n/2}} \right) \mathrm{d}\mathcal{H}^2 \leqq \frac{C}{2 - p_n}. \tag{6.14}$$

Then, by Lemma 4.1, applied with $u = u_n|_{\partial U}$, $v = \operatorname{tr}_{\partial U}(w_n) = u_*|_{\partial U}$ and $T = s_n^{1/2}$, there exists a mapping $\xi_n \in W^{1,p_n}(\partial U \times (0, s_n^{1/2}), \mathbb{R}^\nu)$ interpolating between $u_n|_{\partial U}$ and $\operatorname{tr}_{\partial U}(w_n)$. By (6.6), we deduce that

$$\operatorname*{ess1, sup}_{(\sigma a, t) \in \partial U \times (0, S_n^{1/2})} 1, \operatorname{dist}(1xi_n(\sigma, t), \mathcal{N}) \leqq \|u_n|_{\partial U} - u_*|_{\partial U}\|_{L^\infty(\partial U, \mathbb{R}^\nu)} \to 0$$

as $n \to +\infty$. This implies that for each $n \in \mathbb{N}$ large enough, it holds

$$\operatorname{dist}(\xi_n(\sigma, t), \mathcal{N}) < \frac{\lambda_{\mathcal{N}}}{10} \tag{6.15}$$

for a.e. $(\sigma, t) \in \partial U \times (0, s_n^{1/2})$.

In view of (6.15) and Lemma 2.1, for each $n \in \mathbb{N}$ large enough we can define the mapping $\varphi_n(y) := \Pi_{\mathcal{N}}(\xi_n(\Phi^{-1}(y)))$ for each $y \in \Phi(\partial U \times (0, s_n^{1/2}))$. Then the following assertions hold. First $\varphi_n \in W^{1,p_n}(\Phi(\partial U \times (0, s_n^{1/2})), \mathcal{N})$. Second,

$$\operatorname{tr}_{\partial U}(\varphi_n) = u_n|_{\partial U}, \quad \operatorname{tr}_{\Phi(\partial U \times \{s_n^{1/2}\})}(\varphi_n) = u_*|_{\partial U} \circ \Phi_n \circ \Phi^{-1}|_{\Phi(\partial U \times \{s_n^{1/2}\})},$$

where $\Phi_n : \partial U \times [s_n^{1/2}, T']$ is defined by $\Phi_n(\sigma, t) := \Phi(\sigma, t - s_n^{1/2})$ and we have used the fact that $\Phi(\sigma, 0) = \sigma$. Thirdly, using [30, Theorem 3.2.5], the fact that Ψ is bilipschitz, Lemma 4.1 (ii), where $T = s_n^{1/2}$, and (6.14), we have the following

$$\begin{aligned}
(2 - p_n) \int_{\Phi(\partial U \times (0, s_n^{1/2}))} \frac{|D\varphi_n|^{p_n}}{p_n} \,\mathrm{d}y &\leqq C(2 - p_n) \int_{\partial U \times (0, s_n^{1/2})} \frac{|D\xi_n|^{p_n}}{p_n} \,\mathrm{d}\mathcal{H}^2 \,\mathrm{d}t \\
&\leqq C(2 - p_n) s_n^{1/2} \int_{\partial U} \left(|D_\top u_n|^{p_n} + |D_\top \operatorname{tr}_{\partial U}(w_n)|^{p_n} + \frac{|u_n - u_*|^{p_n}}{s_n^{p_n/2}} \right) \mathrm{d}\mathcal{H}^2 \\
&\leqq C s_n^{1/2}.
\end{aligned} \tag{6.16}$$

Let $\zeta \in C_c([0, +\infty))$ be the Lipschitz map such that $\zeta = 1$ on $[0, 1]$, $\zeta(s) = 2 - s$ for $s \in [1, 2]$, $\zeta = 0$ on $[2, +\infty)$. Hence $\|\zeta'\|_{L^\infty([0,+\infty))} \leqq 1$. Next, for each sufficiently large $n \in \mathbb{N}$, define $\psi_n : U \to \mathcal{N}$ by

$$\psi_n(y) := \begin{cases} \varphi_n(y) & \text{if } \ y \in \Phi(\partial U \times (0, s_n^{1/2})), \\ u_*|_{\partial U} \circ \Phi_n \circ \Phi^{-1}|_{\Phi(\partial U \times \{s_n^{1/2}\})}(y) & \text{if } \ y \in \Phi(\partial U \times \{s_n^{1/2}\}), \\ w_n(\Psi_n(y)) & \text{if } \ y \in U \setminus \Phi(\partial U \times (0, s_n^{1/2}]), \end{cases} \tag{6.17}$$

where $\Psi_n : U \setminus \Phi(\partial U \times (0, s_n^{1/2}]) \to U$ is the map defined by

$$\Psi_n(y) := \left(1 - \zeta\left(\frac{\operatorname{dist}(y, \partial U)}{s_n^{1/3}}\right)\right) y + \zeta\left(\frac{\operatorname{dist}(y, \partial U)}{s_n^{1/3}}\right) \Phi_n(\Phi^{-1}(y)).$$

We note that $\psi_n \in W^{1,p_n}(U, \mathcal{N})$, $\operatorname{tr}_{\partial U}(\psi_n) = u_n|_{\partial U}$, Ψ_n is bilipschitz, $|D\Psi_n - \mathrm{Id}\,| = O(s_n^{1/6})$ a.e. in $U \setminus \Phi(\partial U \times (0, s_n^{1/2}])$ (for more details, see the proof of Proposition 4.2). Thus, ψ_n is a competitor for u_n. Then, using (6.16), (6.17), [30, Theorem 3.2.5] and (6.7), we obtain

$$\begin{aligned} &\liminf_{n\to+\infty}(2 - p_n) \int_U \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}y \\ &\quad \leqq \liminf_{n\to+\infty}(2 - p_n) \int_U \frac{|D\psi_n|^{p_n}}{p_n}\,\mathrm{d}y \\ &\quad \leqq \limsup_{n\to+\infty}\left(Cs_n^{1/2} + (2 - p_n)(1 + o(s_n^{1/10})) \int_U \frac{|Dw_n|^{p_n}}{p_n}\,\mathrm{d}y\right) \\ &\quad \leqq \mathbb{M}(S). \end{aligned} \tag{6.18}$$

On the other hand, using (5.3), the fact that U is open, Corollary 5.18 and the geometry of $S_* \cap \overline{U}$ (see the beginning of the proof and *Step 1*), we deduce that

$$\mu_*(S_* \cap U) = \mathbb{M}(S_* \cap \overline{U}) \leqq \liminf_{n\to+\infty}(2 - p_n) \int_U \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}y. \tag{6.19}$$

Combining (6.18) with (6.19), yields (6.2). This completes our proof of Proposition 6.4. □

We shall apply Proposition 6.4 to prove that in the case when $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$ the set S_* has no branching points inside Ω. To this end, we need the next extension result. The following statement is similar to [22, Lemma 3].

Proposition 6.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded strictly convex domain, $\pi_1(\mathcal{N})$ be Abelian (and endowed with the law $+$) and $a_1, \dots, a_k \subset \overline{\Omega}$ be distinct points. Let $g \in C(\partial\Omega\setminus\{a_1, \dots, a_k\}, \mathcal{N})$ and $\alpha : \{a_1, \dots, a_k\}^2 \to \pi_1(\mathcal{N})$ satisfy the following conditions:*

(i) $\alpha(a_i, a_j) = -\alpha(a_j, a_i)$*, so that, in particular,* $\alpha(a_i, a_i) = 0$*;*
(ii) *if $a_i \in \partial\Omega$, then $\sum_{j=1}^k \alpha(a_i, a_j)$ is the homotopy class of g around a_i;*
(iii) *if $a_i \in \Omega$, then* $\sum_{j=1}^k \alpha(a_i, a_j) = 0$.

Then there exists $u \in C(\Omega\setminus(\Sigma \cup F), \mathcal{N})$, where F is finite and $\Sigma = \bigcup\{[a_i, a_j] : \alpha(a_i, a_j) \neq 0\}$, and for every $i, j \in \{1, \dots, k\}$, u is homotopic to $\alpha(a_i, a_j)$ on small circles transversal to $[a_i, a_j]$.

The proof of Proposition 6.6 will use the following two-dimensional construction.

Proposition 6.7. *Let $U \subset \mathbb{R}^2$ be a bounded strictly convex domain, $\pi_1(\mathcal{N})$ be Abelian and $a_1, \ldots, a_k \subset U$ be distinct points. Assume that $h \in C(\partial U, \mathcal{N})$ and $\gamma : \{a_1, \ldots, a_k\} \to \pi_1(\mathcal{N})$ satisfy the condition that $\sum_{i=1}^{k} \gamma(a_i)$ is the homotopy class of h on ∂U. Then there exists $u \in C(\overline{U} \backslash \{a_1, \ldots, a_k\}, \mathcal{N})$, such that $u|_{\partial U} = h$ and u is homotopic to $\gamma(a_i)$ on small circles transversal around a_i.*

Proof. We consider a set L of segments in $\mathbb{R}^2$ such that $\{a_1, \ldots, a_k\} \subset L$ and $U \backslash L$ is simply connected. For $r > 0$ small enough, the disks $B^2_{2r}(a_1), \ldots, B^2_{2r}(a_k)$ are pairwise disjoint. We fix u on $\partial B^2_r(a_i)$ so that $u|_{\partial B^2_r(a_i)}$ has the prescribed homotopy class $\gamma(a_i)$. We extend then this map u to $(U \cap L) \backslash \bigcup_{i=1}^{k} \overline{B}^2_r(a_i)$ by connectedness of $\mathcal{N}$. Since $V = U \backslash (L \cup \bigcup_{i=1}^{k} \overline{B}^2_r(a_i))$ is simply connected and by our assumption on the homotopy classes, u has a trivial homotopy on ∂V. Thus, we can extend u to $\overline{V}$ in such a way that $u = h$ on ∂U. Finally, u is extended homogeneously on the set $B^2_r(a_i) \backslash \{a_i\}$ for each $i \in \{1, \ldots, k\}$. □

Proof of Proposition 6.6. As a first step, given a plane $P \subset \mathbb{R}^3$ that is transversal to Σ and does not contain F, the map g can be extended to $(\partial\Omega \cup (P \cap \Omega)) \backslash \Sigma$. Indeed, the homotopy class of g on $\partial\Omega$ is equal to the sum of the homotopy classes on small circles around $\{a_1, \ldots, a_k\} \cap \partial\Omega$ on one side of P. By the conditions (ii) and (iii), the homotopy class of g on $\partial\Omega$ is equal to the sum of the homotopy classes on small circles around $\Sigma \cap P$. By Proposition 6.7, g can be extended to $P \cap \Omega \backslash \Sigma$ with the prescribed homotopy on small circles around points of $\Sigma \cap P$. Reiterating this construction finitely many times, the problem is reduced to the case where either $k = 0$ and one can perform a homogeneous extension with respect to any interior point, or the set Σ is starshaped with respect to some point in $\overline{\Omega}$ and we perform then a homogeneous extension with respect to that point. □

Corollary 6.8. *Let $\pi_1(\mathcal{N}) = \mathbb{Z}/2\mathbb{Z}$. Then S_* has no branching points inside Ω.*

Proof of Corollary 6.8. Assume by contradiction that S_* has a branching point $x_0 \in \Omega$. Then, according to Remark 5.20, locally around x_0, S_* is a finite union of an even number of segments emanating from x_0. Fix two segments from this union that form the smallest angle among all possible pairs. Denote these segments by L_1 and L_2. Then there exists a truncated cone $K \subset \Omega$ such that $\overline{K} \cap S_* = \overline{K} \cap (L_1 \cup L_2)$ and $\partial K \cap (L_1 \cup L_2) = \{x_1, x_2, y_1, y_2\}$, where x_1, y_1 and x_2, y_2 are the endpoints of $L_1 \cap \overline{K}$, $L_2 \cap \overline{K}$, respectively. Furthermore, x_1, x_2 belong to the smaller circular base of K and y_1, y_2 belong to the bigger circular base of K. Choosing K so that its smaller circular base is fairly close to x_0 and using the triangle inequality, we can ensure that $|x_1 - x_2| + |y_1 - y_2| < |x_1 - y_1| + |x_2 - y_2|$. This, together with Propositions 6.4, 6.6, yields a contradiction, since the set U of Proposition 6.4, which contains K, can be chosen to be convex and arbitrarily close to K (for instance, we can choose $U = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, K) < \varepsilon\}$ for some sufficiently small $\varepsilon \in (0, 1)$). Therefore, S_* has no branching points inside Ω, which completes our proof of Corollary 6.8.

7. Analysis Up to the Boundary

7.1. Preliminary Results

If the fractional seminorm of the boundary datum is fairly small, we may have a concentration of the p-energy as $p \nearrow 2$ only near the boundary of the domain. Namely, the smaller the fractional seminorm of the boundary datum, the closer to $\partial\Omega$ the singular set is located.

Proposition 7.1. *Let $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, $r_0 \in (0, 1)$, $\Omega_{r_0} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r_0\} \neq \emptyset$, $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_n}(\Omega, \mathcal{N})$ be a sequence of p_n-minimizers with $\operatorname{tr}_{\partial\Omega}(u_n) = g$. Then there exists $\varepsilon = \varepsilon(r_0, \partial\Omega, \mathcal{N}) > 0$ such that the following holds. Assume that $|g|_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)} \leqq \varepsilon$. Then $S_* \cap \Omega_{r_0} = \emptyset$.*

Proof. Fix arbitrary $x_0 \in \Omega_{r_0}$ and $r \in (0, r_0)$. Let $\eta > 0$ be the constant of Lemma 3.12, where $\kappa = 1/2$, $p_0 = 3/2$ and $\Psi(x) = x+x_0$ whenever $E = B^3_r(x_0)$. Let $\varepsilon > 0$ be small enough so that for each $n \in \mathbb{N}$,

$$(2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x \leqq \eta r_0^{3/2}, \tag{7.1}$$

which is possible in view of Proposition 2.21 and the assumption $|g|_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)} \leqq \varepsilon$. Then, applying Lemma 5.3 and using (7.1), for each fairly small $\delta > 0$ we have

$$\begin{aligned}(2 - p_n) \int_{B^3_{r+\delta}(x_0)} \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x &\leqq (2 - p_n) \left(\frac{r + \delta}{r_0}\right)^{3-p_n} \int_{B^3_{r_0}(x_0)} \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x \\ &\leqq (2 - p_n) \left(\frac{r + \delta}{r_0}\right)^{3-p_n} \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x \\ &\leqq \eta r_0^{3/2-3+p_n} (r + \delta)^{3-p_n}.\end{aligned}$$

Letting n tend to $+\infty$, we have $\mu_*(\overline{B}^3_r(x_0)) \leqq \mu_*(B^3_{r+\delta}(x_0)) \leqq \eta r_0^{1/2}(r + \delta)$ (see (5.3)). Letting δ tend to $0+$, yields $\mu_*(\overline{B}^3_r(x_0)) < \eta r$, since $r_0 \in (0, 1)$. Then, according to Lemma 5.2, $\mu_*(B^3_{r/2}(x_0)) = 0$ and hence $B^3_{r/2}(x_0) \subset \Omega\backslash S_*$. Since $x_0 \in \Omega_{r_0}$ was arbitrary, $S_* \cap \Omega_{r_0} = \emptyset$, which completes our proof of Proposition 7.1. □

The next proposition says that if the boundary of the domain is homeomorphic to $\mathbb{S}^2$ and the boundary datum is regular enough, then no singular set arises in the limit.

Proposition 7.2. *Let $\partial\Omega$ be homeomorphic to $\mathbb{S}^2$, $g \in W^{1,2}(\partial\Omega, \mathcal{N})$, $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_n}(\Omega, \mathcal{N})$ be a sequence of p_n-minimizers such that for each $n \in \mathbb{N}$, $\operatorname{tr}_{\partial\Omega}(u_n) = g$. Then $S_* = \emptyset$ and the map u_* coming from Proposition 5.6 is a minimizing harmonic extension of g to Ω. Furthermore, $u_* \in C^\infty(\Omega\backslash S_0, \mathcal{N})$, where $S_0 \subset \Omega$ is a discrete set.*

Remark 7.3. In Proposition 7.2, the topology of Ω is simple, namely $\Omega \simeq B_1^3$. However, the singular set S_* does not occur in the limit in some cases even if the topology of Ω is not simple, for example when g is the restriction of a smooth map on $\overline{\Omega}$.

Proof of Proposition 7.2. Since $\Omega \subset \mathbb{R}^3$ is a bounded (locally) Lipschitz domain and $\partial\Omega$ is homeomorphic to $\mathbb{S}^2$, there exists a bilipschitz homeomorphism $\Phi : \overline{\Omega} \to \overline{B}_1^3$. Let $h = g \circ \Phi^{-1}|_{\mathbb{S}^2}$. Then $h \in W^{1,2}(\mathbb{S}^2, \mathcal{N})$. Next, defining, as it was done in [52, Proposition 2.4], $u(x) = h(x/|x|)$ for $x \in B_1^3 \backslash \{0\}$, we have $u \in W^{1,2}(B_1^3, \mathcal{N})$ and $\mathrm{tr}_{\mathbb{S}^2}(u) = h$. Since Φ is bilipschitz and $u \circ \Phi$ is a competitor for u_n, we deduce that

$$\mu_n(\overline{\Omega}) \leqq C(2-p_n)\int_{B_1^3} \frac{|Du|^{p_n}}{p_n}\,\mathrm{d}x = \frac{C(2-p_n)}{3-p_n}\int_{\mathbb{S}^2} \frac{|D_\top h|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \to 0 \tag{7.2}$$

as $n \to +\infty$, where $C > 0$ is a constant depending only on the bilipschitz constant of Φ, and we have used that $\mu_n(\Omega) = \mu_n(\overline{\Omega})$. Thus, $\mu_*(\overline{\Omega}) = 0$ (here we again take advantage of considering the μ_n's as elements of $(C(\overline{\Omega}))'$ instead of $(C_0(\Omega))'$, because any constant belongs to $C(\overline{\Omega})$, and hence the weak* convergence preserves mass, namely, since $\mu_n \overset{*}{\rightharpoonup} \mu_*$ weakly* in $(C(\overline{\Omega}))'$, it holds $\mu_n(\overline{\Omega}) \to \mu_*(\overline{\Omega})$). Therefore, the support of μ_* is the empty set, that is $S_* = \emptyset$. On the other hand, in view of (7.2), the condition (4.1) of Proposition 4.2 is valid for the sequence $(u_n)_{n\in\mathbb{N}}$ and the set $K = \Omega$. Thus, the conclusions of Proposition 4.2 apply for some mapping $u_* \in W^{1,2}(\Omega, \mathcal{N})$, which comes from Proposition 5.6. In particular, u_* is a minimizing harmonic extension of g to Ω. Then, according to [52, Theorem II], $u_* \in C^\infty(\Omega\backslash S_0, \mathcal{N})$, where $S_0 \subset \Omega$ is a discrete set, which completes our proof of Proposition 7.2. □

As an immediate consequence of Lemma 3.11, we have its counterpart on a half-ball. Recall that $\mathbb{R}^3_+ = \{(x', x_3) \in \mathbb{R}^3 : x_3 \in (0, +\infty)\}$.

Lemma 7.4. *Let $p_0 \in (1,2)$, $p \in [p_0, 2)$. There exist constants $\eta_0, C > 0$, depending only on p_0 and $\mathcal{N}$, such that if $g \in W^{1,p}(\partial(B_r^3 \cap \mathbb{R}^3_+), \mathcal{N})$ satisfies*

$$\int_{\partial(B_r^3\cap\mathbb{R}^3_+)} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{\eta_0 r^{2-p}}{2-p},$$

then there exists $u \in W^{1,p}(B_r^3 \cap \mathbb{R}^3_+, \mathcal{N})$ such that $\mathrm{tr}_{\partial(B_r^3\cap\mathbb{R}^3_+)}(u) = g$ *and*

$$\int_{B_r^3\cap\mathbb{R}^3_+} \frac{|Du|^p}{p}\,\mathrm{d}x \leqq Cr^{\frac{2}{p}}\left(\int_{\partial(B_r^3\cap\mathbb{R}^3_+)} \frac{|D_\top g|^p}{p}\,\mathrm{d}\mathcal{H}^2\right)^{1-\frac{1}{p}}.$$

Next, we prove the η-compactness lemma up to the boundary of Ω, which is the counterpart of Lemma 3.12 on the boundary.

Proposition 7.5. *Let $\kappa \in (0,1)$, $p_0 \in (1,2)$, $p \in [p_0, 2)$ and let $U = \Psi(B_r^3 \cap \mathbb{R}_+^3, \mathcal{N})$ be an open set, where $\Psi : \overline{B}_r^3 \cap \overline{\mathbb{R}}_+^3 \to \mathbb{R}^3$ is a bilipschitz homeomorphism onto its image $\overline{U}$ with a bilipschitz constant $L \geqq 1$. There exist $\eta, C > 0$, depending only on κ, p_0, L and $\mathcal{N}$, such that if $u_p \in W^{1,p}(U, \mathcal{N})$ is a p-minimizer and if*

$$\int_U \frac{|Du_p|^p}{p}\,\mathrm{d}x + r\int_{\Psi(B_r^3\cap\mathbb{R}^2\times\{0\})} \frac{|D_\top \operatorname{tr}_{\Psi(B_r^3\cap\mathbb{R}^2\times\{0\})}(u_p)|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{\eta r^{3-p}}{2-p}, \tag{7.3}$$

then

$$\int_{\Psi(B_{\kappa r}^3\cap\mathbb{R}_+^3)} \frac{|Du_p|^p}{p}\,\mathrm{d}x \leqq C r^{3-p}. \tag{7.4}$$

Proof. Given the mapping $v_p := u_p \circ \Psi \in W^{1,p}(B_r^3 \cap \mathbb{R}_+^3, \mathcal{N})$ satisfying

$$\int_{B_r^3\cap\mathbb{R}_+^3} \frac{|Dv_p|^p}{p}\,\mathrm{d}x \leqq L^{3+p}\int_U \frac{|Du_p|^p}{p}\,\mathrm{d}x \tag{7.5}$$

and the constant $\eta_0 = \eta_0(p_0, \mathcal{N}) > 0$ of Lemma 7.4, we define the set

$$G_p = \left\{\varrho \in (\kappa r, r) : \operatorname{tr}_{\partial(B_\varrho^3\cap\mathbb{R}_+^3)}(v_p) = v_p|_{\partial(B_\varrho^3\cap\mathbb{R}_+^3)} \in W^{1,p}(\partial(B_\varrho^3\cap\mathbb{R}_+^3), \mathcal{N}) \right.$$
$$\left. \text{and} \int_{\partial B_\varrho^3\cap\mathbb{R}_+^3} \frac{|Dv_p|^p}{p}\,\mathrm{d}\mathcal{H}^2 \leqq \frac{\eta_0(\kappa r)^{2-p}}{2(2-p)} \right\}. \tag{7.6}$$

As a consequence of (7.3), (7.5) and (7.6),

$$\begin{aligned}\mathcal{H}^1((\kappa r, r)\setminus G_p) &\leqq \frac{2(2-p)}{\eta_0(\kappa r)^{2-p}}\int_{B_r^3\cap\mathbb{R}_+^3}\frac{|Dv_p|^p}{p}\,\mathrm{d}x \leqq \frac{2(2-p)L^{3+p}}{\eta_0(\kappa r)^{2-p}}\int_U\frac{|Du_p|^p}{p}\,\mathrm{d}x\\ &\leqq \frac{(1-\kappa)r}{2},\end{aligned}$$

provided we choose η satisfying

$$\eta \leqq \frac{\eta_0\kappa(1-\kappa)}{4L^5}; \tag{7.7}$$

equivalently we have then

$$\mathcal{H}^1(G_p) \geqq \frac{(1-\kappa)r}{2}. \tag{7.8}$$

Fix η satisfying (7.7). For every $\varrho \in G_p$, the following holds

$$\begin{aligned}\int_{\partial(B_\varrho^3\cap\mathbb{R}_+^3)}\frac{|D_\top v_p|^p}{p}\,\mathrm{d}\mathcal{H}^2 &= \int_{\partial B_\varrho^3\cap\mathbb{R}_+^3}\frac{|D_\top v_p|^p}{p}\,\mathrm{d}\mathcal{H}^2 + \int_{B_\varrho^3\cap\mathbb{R}^2\times\{0\}}\frac{|D_\top v_p|^p}{p}\,\mathrm{d}\mathcal{H}^2\\ &\leqq \frac{\eta_0(\kappa r)^{2-p}}{2(2-p)} + \frac{\eta r^{2-p}L^{2+p}}{2-p} \leqq \frac{\eta_0\varrho^{2-p}}{2-p},\end{aligned}$$

where we have used (7.3), the fact that $|D_\top v_p| \leqq |Dv_p|$ and (7.7). Under this condition, according to Lemma 7.4,

$$\int_{B^3_\varrho \cap \mathbb{R}^3_+} \frac{|Dv_p|^p}{p}\,\mathrm{d}x \leqq C\varrho^{\frac{2}{p}} \left(\int_{\partial(B^3_\varrho \cap \mathbb{R}^3_+)} \frac{|D_\top v_p|^p}{p}\,\mathrm{d}\mathcal{H}^2 \right)^{1-\frac{1}{p}}. \tag{7.9}$$

To obtain (7.4), we conclude as in the proof of Lemma 3.12. □

The next corollary is a direct consequence of Proposition 7.5 and Definition 2.4, so we omit its proof.

Corollary 7.6. *Let $\kappa \in (0,1)$ and $g \in W^{1,2}(E,\mathcal{N})$, where E is a relatively open subset of $\partial\Omega$. Let $r_{\partial\Omega} > 0$ be the number coming from Definition 2.4, which is applied with $U = \Omega$. Then there exist constants $p_1 = p_1(\partial\Omega, \mathcal{N}, \|D_\top g\|_{L^2(E,\mathbb{R}^\nu\otimes\mathbb{R}^3)}) \in [3/2, 2)$, $\eta = \eta(\kappa, \partial\Omega, \mathcal{N}) > 0$ and $C = C(\kappa, \partial\Omega, \mathcal{N}) > 0$ such that the following holds. Let $p \in [p_1, 2)$ and $u_p \in W^{1,p}(\Omega, \mathcal{N})$ be a p-minimizer such that $\mathrm{tr}_E(u_p) = g$. Then, for each $r \in (0, r_{\partial\Omega})$ and $x_0 \in E$ such that $B^3_r(x_0) \cap \partial\Omega \subset E$ and*

$$\int_{B^3_r(x_0)\cap\Omega} \frac{|Du_p|^p}{p}\,\mathrm{d}x \leqq \frac{\eta r^{3-p}}{2-p},$$

it holds

$$\int_{B^3_{\kappa r}(x_0)\cap\Omega} \frac{|Du_p|^p}{p}\,\mathrm{d}x \leqq C r^{3-p}.$$

Corollary 7.7. *Let E be a relatively open subset of $\partial\Omega$. Assume that (5.1) holds and that for all $n \in \mathbb{N}$ large enough, $\mathrm{tr}_E(u_n) = g \in W^{1,2}(E, \mathcal{N})$. Then there exists a closed set $S_* \subset \overline{\Omega}$ and a mapping $u_* \in W^{1,2}_{\mathrm{loc}}((\Omega \cup E)\backslash S_*, \mathcal{N})$ satisfying the assertions of Proposition 4.2 and the assertion (ii) of Proposition 5.23 such that $\mathrm{tr}_E(u_*) = g$ and, up to a subsequence (not relabeled),*

$$u_n \rightharpoonup u_* \ \text{weakly in} \ W^{1,p}_{\mathrm{loc}}((\Omega \cup E) \setminus S_*, \mathbb{R}^\nu) \ \text{for all} \ p \in (1,2).$$

Proof of Corollary 7.7. The proof of the assertion (ii) of Proposition 5.23 remains unchanged. One only needs to take Corollary 7.6 into account in the proof of Proposition 4.2. □

7.2. Global Estimates and $W^{1,q}$ Compactness

It is well known that the set $C^\infty(\partial\Omega, \mathcal{N})$ is dense in $W^{1/2,2}(\partial\Omega, \mathcal{N})$ if and only if $\pi_1(\mathcal{N}) \simeq \{0\}$ (we refer the reader to [17, Theorem 4]). Although $C^\infty(\partial\Omega, \mathcal{N})$ is not dense in $W^{1/2,2}(\partial\Omega, \mathcal{N})$ if $\pi_1(\mathcal{N})$ is nontrivial, the set of smooth maps outside a finite set of points whose Euclidean norm of the gradient behaves like the distance to the power of -1 close to these points is dense in $W^{1/2,2}(\partial\Omega, \mathcal{N})$ (see [48, Theorem 1] and [17, Theorem 3]). Namely, letting $\mathcal{R}^1_0(\partial\Omega, \mathcal{N})$ to be the class of maps $\varphi \in C^1(\partial\Omega\backslash\{a_1, \ldots, a_l\}, \mathcal{N})$ for some $a_1, \ldots, a_l \in \partial\Omega$ such that

$$|D_\top \varphi(x)| \leqq \frac{C}{\mathrm{dist}(x, \bigcup_{i=1}^l a_i)}, \tag{7.10}$$

where C is a positive constant independent of x, we have

$$\overline{\mathcal{R}_0^1(\partial\Omega,\mathcal{N})}^{W^{1/2,2}} = W^{1/2,2}(\partial\Omega,\mathcal{N}).$$

Hereinafter, for each $g \in \mathcal{R}_0^1(\partial\Omega,\mathcal{N})$, we shall denote by $S(g)$ the singular set of g, namely the set of points $a \in \partial\Omega$ such that for each sufficiently small radius $\varrho > 0$, $g|_{\partial B_\varrho^{\partial\Omega}(a)}$ is not nullhomotopic, where $B_\varrho^{\partial\Omega}(a)$ is the geodesic ball in $\partial\Omega$ with center a and radius ϱ such that $B_\varrho^{\partial\Omega}(a)$ is far enough from $S(g)\backslash\{a\}$.

It is worth noting that for each $g \in \mathcal{R}_0^1(\partial\Omega,\mathcal{N})$, there exists the *minimal* set of points $\{a_1,\dots,a_l\} \subset \partial\Omega$ such that $g \in C^1(\partial\Omega\backslash\{a_1,\dots,a_l\},\mathcal{N})$. Moreover, $S(g) \subset \{a_1,\dots,a_l\}$ and the inclusion can be strict.

Lemma 7.8. *Let $g \in \mathcal{R}_0^1(\partial\Omega,\mathcal{N})$ and $\{a_1,\dots,a_l\} \subset \partial\Omega$ be the minimal set of different points such that $g \in C^1(\partial\Omega\backslash\{a_1,\dots,a_l\},\mathcal{N})$. Then for each $q \in [1,2)$, $g \in W^{1,q}(\partial\Omega,\mathcal{N})$ and*

$$\int_{\partial\Omega} |D_\top g|^q \,\mathrm{d}\mathcal{H}^2 \leqq \frac{C}{2-q},$$

where $C > 0$ is a constant depending only on $\partial\Omega$, the set $\{a_1,\dots,a_l\}$ and the constant of the estimate (7.10) applied with $\varphi = g$.

Proof. Fix an arbitrary $q \in [1,2)$. Define

$$\bar{\varrho}_{\partial\Omega}(a_1,\dots,a_l) = \min\left\{\frac{|a_i-a_j|}{2} : i,j \in \{1,\dots,l\} \text{ and } i \neq j\right\}$$

so that if $\varrho \in (0,\bar{\varrho}_{\partial\Omega}(a_1,\dots,a_l))$, then $\overline{B}_\varrho^3(a_i) \cap \overline{B}_\varrho^3(a_j) = \emptyset$ for each $i,j \in \{1,\dots,l\}$ such that $i \neq j$. Then for each $i \in \{1,\dots,l\}$ and for each sufficiently small $\varrho \in (0,\bar{\varrho}_{\partial\Omega}(a_1,\dots,a_l))$,

$$\int_{B_\varrho^3(a_i)\cap\partial\Omega} |D_\top g|^q \,\mathrm{d}\mathcal{H}^2 \leqq C\int_0^\varrho \frac{\mathrm{d}t}{t^{1-q}} \leqq \frac{C}{2-q}, \tag{7.11}$$

where $C > 0$ is a constant depending only on $\partial\Omega$, $\bar{\varrho}_{\partial\Omega}(a_1,\dots,a_l)$ and the constant of the estimate (7.10) applied with $\varphi = g$. On the other hand, in view of (7.10), if $\varrho > 0$ and $x \in \partial\Omega\backslash\bigcup_{i=1}^l \overline{B}_\varrho^3(a_i)$, then $|D_\top g(x)| \leqq C/\varrho$. Taking this and (7.11) into account, we complete our proof of Lemma 7.8. □

Assuming that $g \in \mathcal{R}_0^1(\partial\Omega,\mathcal{N})$, we obtain the next improvement of Proposition 5.23.

Proposition 7.9. *Let $g \in \mathcal{R}_0^1(\partial\Omega,\mathcal{N})$ be such that $g \in C^1(\partial\Omega\backslash S(g),\mathcal{N})$. Let $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of p_n-minimizers such that $\mathrm{tr}_{\partial\Omega}(u_n) = g$ for each $n \in \mathbb{N}$. Let u_* be a map given by Proposition 5.6. Assume that there exists $r_0 > 0$ such that if $x \in \mathbb{R}^3\backslash\Omega$ and $\mathrm{dist}(x,\partial\Omega) \leqq r_0$, there exists a unique point $P(x) \in \partial\Omega$ such that $\mathrm{dist}(x,\partial\Omega) = |x-P(x)|$. Then the following assertions hold.*

(i) *There exists* $C = C(g, \Omega, \mathcal{N}) > 0$ *such that for each* $q \in [1, 2)$,

$$\limsup_{n\to+\infty} \int_\Omega |Du_n|^q \,\mathrm{d}x \leqq C + \frac{C}{2-q} \limsup_{n\to+\infty} (2-p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x.$$

(ii) *For each* $q \in [1, 2)$, $u_* \in W^{1,q}(\Omega, \mathcal{N})$ *and, up to a subsequence (not relabeled),* $u_n \to u_*$ *in* $W^{1,q}(\Omega, \mathbb{R}^\nu)$ *and for some* $C = C(g, \Omega, \mathcal{N}) > 0$,

$$\int_\Omega |Du_*|^q \,\mathrm{d}x \leqq C + \frac{C}{2-q} \limsup_{n\to+\infty} (2-p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x.$$

Remark 7.10. If Ω is of class C^2 or is convex, then it has the *unique nearest point property*, namely, the assumption of Proposition 7.9 applies (for more details, we refer to [29], [39]). It is also worth noting that the constants in the assertions (i), (ii) depend on $S(g)$ and the constant coming from (7.10), where $\varphi = g$. Since the latter constant depends only on g, we can assume that the constants in (i), (ii) depend only on g, Ω and $\mathcal{N}$.

Proof of Proposition 7.9. Fix $R > 0$ large enough so that $\Omega \Subset B_R^3$ and for each $x \in \Omega$ the distance from x to ∂B_R^3 is bounded from below by 1. Without loss of generality, we assume that r_0 is small enough with respect to R and that P is a 2-Lipschitz map (the reader may consult [29, Theorem 4.8 (8)]). To lighten the notation, we denote $W = \{x \in B_R^3 : \mathrm{dist}(x, \Omega) \leqq r_0\}$ and $V = \{x \in B_R^3 \backslash \Omega : \mathrm{dist}(x, \partial\Omega) \leqq r_0\}$. For each $n \in \mathbb{N}$, define the map $f_n : B_R^3 \to [0, +\infty)$ by $f_n = |Du_n|$ in Ω, $f_n = |D(g \circ P)|$ in V and $f_n = 0$ in $B_R^3 \backslash W$. Since P is 2-Lipschitz and $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N})$, using Lemma 7.8, we observe that

$$(2-p_n) \int_{B_R^3 \backslash \Omega} f_n^{p_n} \,\mathrm{d}x = (2-p_n) \int_V f_n^{p_n} \,\mathrm{d}x \leqq C, \tag{7.12}$$

where $C = C(g, \Omega, \mathcal{N}) > 0$ (here C depends on r_0, $S(g)$ and the constant coming from (7.10), where $\varphi = g$, but r_0 depends only on Ω and the latter constant depends only on g). Then the proof of the assertion (i) follows by reproducing the proof of the assertion (i) of Proposition 5.23 with minor modifications, namely, applying Proposition 5.22 with $f = f_n$, $p = p_n$, $V = B_R^3$, $C_1 = \frac{2-p_n}{\eta}$ and $C_2 = C$, where $n \in \mathbb{N}$ is large enough and $\eta, C > 0$ depend only on g, Ω, $\mathcal{N}$ and will be fixed later for the proof to work. Let us fix an arbitrary dyadic cube Q such that $2Q \subset B_R^3$. Let $r > 0$ be the sidelength of Q. To apply Proposition 5.22 with the above parameters, we need to prove the following: if $n \in \mathbb{N}$ is large enough and

$$\int_{2Q} f_n^{p_n} \,\mathrm{d}x \leqq \frac{\eta r^{3-p_n}}{2-p_n}, \tag{7.13}$$

then

$$\int_Q f_n^{p_n} \,\mathrm{d}x \leqq C r^{3-p_n}. \tag{7.14}$$

First, notice the following. Fix an arbitrary $x_0 \in S(g)$. Let $B_\varrho^{\partial\Omega}(x_0)$ be the geodesic ball in $\partial\Omega$ with center x_0 and radius $\varrho > 0$ such that every point in $S(g)\backslash\{x_0\}$ is far enough from $B_\varrho^{\partial\Omega}(x_0)$. Then, using Proposition 3.3 and the facts that $p_n \in [3/2, 2)$ for each $n \in \mathbb{N}$ large enough and $B_\varrho^{\partial\Omega}(x_0)$ is bilipschitz homeomorphic to $B_\varrho^2(x_0)$, one has

$$\begin{aligned}\int_{B_\varrho^{\partial\Omega}(x_0)} |D_\top g|^{p_n}\, \mathrm{d}\mathcal{H}^2 + \varrho \int_{\partial B_\varrho^{\partial\Omega}(x_0)} |D_\top(g|_{\partial B_\varrho^{\partial\Omega}(x_0)})|^{p_n}\, \mathrm{d}\mathcal{H}^1 \\ \geqq \frac{\mathcal{E}^{\mathrm{sg}}_{p_n/(p_n-1)}(g|_{\partial B_\varrho^{\partial\Omega}(x_0)})\varrho^{2-p_n}}{C_0(2-p_n)},\end{aligned}$$

where $\partial B_\varrho^{\partial\Omega}(x_0)$ denotes the relative boundary of $B_\varrho^{\partial\Omega}(x_0)$, $C_0 = C_0(\Omega) > 0$ and we have used that $\mathrm{tr}_{\partial B_\varrho^{\partial\Omega}(x_0)}(g) = g|_{\partial B_\varrho^{\partial\Omega}(x_0)}$. Then, proceeding as in the proof of Lemma 3.8, we deduce that

$$\int_{B_\varrho^{\partial\Omega}(x_0)} |D_\top g|^{p_n}\, \mathrm{d}\mathcal{H}^2 + \varrho \int_{\partial B_\varrho^{\partial\Omega}(x_0)} |D_\top(g|_{\partial B_\varrho^{\partial\Omega}(x_0)})|^{p_n}\, \mathrm{d}\mathcal{H}^1 \geqq \frac{2\varepsilon_0\varrho^{2-p_n}}{2-p_n}, \tag{7.15}$$

for each $n \in \mathbb{N}$ large enough, where $\varepsilon_0 = \varepsilon_0(C_0, \mathcal{N}) > 0$. Since C_0 depends only on Ω, we can assume that ε_0 depends only on Ω and $\mathcal{N}$. Also, since $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N})$, we have the estimate

$$\varrho \int_{\partial B_\varrho^{\partial\Omega}(x_0)} |D_\top(g|_{\partial B_\varrho^{\partial\Omega}(x_0)})|^{p_n}\, \mathrm{d}\mathcal{H}^1 \leqq C'\varrho\mathcal{H}^1(\partial B_\varrho^{\partial\Omega}(x_0)),$$

where $C' > 0$ is a constant depending only on the constant coming from (7.10), which depends only on g. This, since $p_n \nearrow 2$ as $n \to +\infty$, implies that for each $n \in \mathbb{N}$ large enough (depending possibly only on g, Ω and $\mathcal{N}$),

$$\varrho \int_{\partial B_\varrho^{\partial\Omega}(x_0)} |D_\top(g|_{\partial B_\varrho^{\partial\Omega}(x_0)})|^{p_n}\, \mathrm{d}\mathcal{H}^1 \leqq \frac{\varepsilon_0\varrho^{2-p_n}}{2-p_n}. \tag{7.16}$$

Combining (7.15) and (7.16), we get the next estimate

$$\int_{B_\varrho^{\partial\Omega}(x_0)} |D_\top g|^{p_n}\, \mathrm{d}\mathcal{H}^2 \geqq \frac{\varepsilon_0\varrho^{2-p_n}}{2-p_n} \tag{7.17}$$

for each $n \in \mathbb{N}$ large enough. On the other hand, if $E \subset \partial\Omega$ is Borel and $\mathrm{dist}(E, S(g)) \geqq \varrho > 0$, then, using (7.10), we obtain

$$\int_E |D_\top g|^{p_n}\, \mathrm{d}\mathcal{H}^2 \leqq \frac{C'\mathcal{H}^2(E)}{\varrho^{p_n}}. \tag{7.18}$$

Next, observe that, since Ω is a bounded Lipschitz domain having the unique nearest point property in V, there exist fairly small constants $0 < a < \delta < 1/2$ and a number $N_0 \in \mathbb{N}\backslash\{0\}$, depending only on Ω, such that the following holds. We need to distinguish between two further cases.

Case 1: $Q \subset W$. If $\operatorname{dist}(P(Q \cap V), S(g)) \geqq \delta r$, then we proceed as follows. Applying (7.18) with $E = P(Q \cap V)$, we have

$$\int_{Q\cap V} f_n^{p_n} \,\mathrm{d}x \leqq Lr \int_{P(Q\cap V)} |D_\top g|^{p_n} \,\mathrm{d}\mathcal{H}^2 \leqq C'' r^{3-p_n}, \tag{7.19}$$

where $L = L(P) > 0$ and $C'' = C''(g, \Omega, \mathcal{N}) > 0$. Next, there exists at most N_0 balls $B^3_{ar}(x_1), \ldots, B^3_{ar}(x_k)$ such that $Q \cap \Omega \subset \bigcup_{i=1}^k B^3_{ar}(x_i) \subset 2Q$ and for each $i \in \{1, \ldots, k\}$, either $B^3_{9ar/8}(x_i) \subset \Omega$ or $x_i \in \partial\Omega$ and $\operatorname{dist}(B^3_{9ar/8}(x_i), S(g)) \geqq \delta r/2$. We shall apply the corresponding η-compactness argument for each ball $B^3_{9ar/8}(x_i)$, namely Lemma 3.12 in the case when $B^3_{9ar/8}(x_i) \subset \Omega$ and Proposition 7.5 otherwise. Let $\eta, C''' > 0$ be constants that are valid for both Lemma 3.12 and Proposition 7.5, where $\kappa = 8/9$, $p_0 = 3/2$ and the corresponding bilipschitz homeomorphism whose bilipschitz constant can depend only on Ω. Thus, we can assume that η, C''' depend only on Ω and $\mathcal{N}$. Assume that

$$\int_{2Q} f_n^{p_n} \,\mathrm{d}x \leqq \frac{\eta r^{3-p_n}}{2(2-p_n)}. \tag{7.20}$$

In the case when $B^3_{9ar/8}(x_i) \subset \Omega$, by Lemma 3.12 and (7.20), we immediately obtain the estimate

$$\int_{B^3_{ar}(x_i)} |Du_n|^{p_n} \,\mathrm{d}x \leqq C''' r^{3-p_n}. \tag{7.21}$$

Let now $x_i \in \partial\Omega$. Using (7.19) and the fact that $p_n \nearrow 2$, we observe that

$$\frac{9ar}{8} \int_{B^3_{\frac{9ar}{8}}(x_i)\cap\partial\Omega} |D_\top g|^{p_n} \,\mathrm{d}\mathcal{H}^2 \leqq \frac{\eta r^{3-p_n}}{2(2-p_n)} \tag{7.22}$$

for each $n \in \mathbb{N}$ large enough depending only on g, Ω and $\mathcal{N}$. Thus, in view of (7.20) and (7.22),

$$\int_{B^3_{\frac{9ar}{8}}(x_i)\cap\Omega} |Du_n|^{p_n} \,\mathrm{d}x + \frac{9ar}{8} \int_{B^3_{\frac{9ar}{8}}(x_i)\cap\partial\Omega} |D_\top g|^{p_n} \,\mathrm{d}\mathcal{H}^2 \leqq \frac{\eta r^{3-p_n}}{2-p_n},$$

for each $n \in \mathbb{N}$ large enough, which according to Proposition 7.5, implies that

$$\int_{B^3_{ar}(x_i)} |Du_n|^{p_n} \,\mathrm{d}x \leqq C''' r^{3-p_n}. \tag{7.23}$$

Combining (7.21) and (7.23), we get

$$\int_{Q\cap\Omega} f_n^{p_n} \,\mathrm{d}x \leqq N_0 C''' r^{3-p_n} \tag{7.24}$$

for each $n \in \mathbb{N}$ large enough. Taking into account the estimates (7.19) and (7.24), we observe that (7.13) implies (7.14), where $C = C'' + N_0 C'''$, which can depend only on g, Ω and $\mathcal{N}$.

If $\operatorname{dist}(P(Q\cap V), S(g)) < \delta r$, then there exists $x_0 \in S(g)$ such that $2Q\cap V$ contains a cylinder $U \simeq B^{\partial\Omega}_{ar}(x_0) \times (0, ar)$ such that, in view of (7.17) and the definition of f_n in V, it holds

$$\int_{2Q} f_n^{p_n}\,\mathrm{d}x \geqq \int_U f_n^{p_n}\,\mathrm{d}x \geqq \frac{\varepsilon_0(a^{10}r)^{3-p_n}}{2-p_n} \tag{7.25}$$

for each $n\in\mathbb{N}$ large enough. Thus, taking η small enough with respect to $\varepsilon_0 a^{15}$ so that Proposition 7.5 applies with 2η, under the condition that $\int_{2Q} f_n^{p_n}\,\mathrm{d}x \leqq \frac{\eta r^{3-p_n}}{2-p_n}$, we exclude the situation where $\operatorname{dist}(P(Q\cap V), S(g)) < \delta r$.

Case 2: $Q\cap(B^3_R\backslash W)\neq\emptyset$. If $Q\subset B^3_R\backslash W$, then, clearly, (7.14) holds, since $f_n = 0$ on Q by definition. Otherwise, we repeat the strategy of the *Case 1* for $Q\cap W$ to conclude that for each $n\in\mathbb{N}$ large enough, (7.13) implies (7.14) for some $\eta, C > 0$ depending only on g, Ω and $\mathcal{N}$.

After all, we have showed that there exist constants $\eta, C > 0$ depending only on g, Ω and $\mathcal{N}$ such that (7.13) implies (7.14) for each $n\in\mathbb{N}$ large enough depending on g, Ω and $\mathcal{N}$. Therefore, we can apply Proposition 5.22 with $f = f_n$, $V = B^3_R$, $C_1 = \frac{2-p_n}{\eta}$ and $C_2 = C$ for each $n\in\mathbb{N}$ large enough. Taking into account (7.12) and the fact that $f_n = 0$ in $B^3_R\backslash W$, we obtain (i).

The assertion (ii) then comes from a diagonal argument, proceeding by the same way as in the proof of Proposition 5.23 (ii). This completes our proof of Proposition 7.9. □

7.3. *Proof of Theorem 1.1*

Now we prove our first main theorem.

Proof of Theorem 1.1. The proof follows from Propositions 5.6, 5.8, 5.17, 5.23, 6.4, 7.2, 7.9 and Corollary 5.18. □

7.4. *Boundary Repulsion Property*

In this subsection, we derive the Pohozaev-type identity for p-minimizers and prove that the varifold associated to the limit of the stress-energy tensors of p-minimizers as $p \nearrow 2$ is *minimizing area to first order* in the Brian White sense (see [57]). As a consequence, we prove that if $\overline{\Omega}$ is strongly convex at every point of $\partial\Omega$, the boundary datum g is an element of $\mathcal{R}^1_0(\partial\Omega,\mathcal{N})$ and $g\in C^1(\partial\Omega\backslash S(g),\mathcal{N})$, then $S_*\cap\partial\Omega = S(g)$, where S_* is the energy concentration set supporting the varifold and $S(g)$ is the singular set of g.

We begin by establishing the boundary stress-energy tensor identity. For convenience, if $U\subset\mathbb{R}^3$ is open, we define for $u\in W^{1,p}(U,\mathbb{R}^\nu)$ the map $T^p_u : U\to\mathbb{R}^3\otimes\mathbb{R}^3$ by

$$T^p_u = \frac{|Du|^p}{p}\mathrm{Id} - \frac{Du\otimes Du}{|Du|^{2-p}}, \tag{7.26}$$

where $Du \otimes Du = (Du)^{\mathrm{T}} Du$. If U is a bounded Lipschitz domain and u is a minimizing p-harmonic map in U, we have $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(U, \mathbb{R}^3)$ (see (2.28)).

Proposition 7.11. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in [1, +\infty)$. If $u \in W^{1,p}(\Omega \cap B_r^3(x_0), \mathbb{R}^\nu)$ satisfies $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(\Omega \cap B_r^3(x_0), \mathbb{R}^3)$ and if Du is continuous at every point of $\partial\Omega \cap B_r^3(x_0)$, then for each $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$,*

$$\int_{\Omega \cap B_r^3(x_0)} T_u^p : D\xi \,\mathrm{d}x = \int_{\partial\Omega \cap B_r^3(x_0)} \langle \nu, T_u^p[\xi] \rangle \,\mathrm{d}\mathcal{H}^2, \tag{7.27}$$

where $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ stands for the outward pointing unit normal vector field to $\partial\Omega$.

Proof. To lighten the notation, denote $B = B_r^3(x_0)$. Observe that $\partial\Omega$ is a 2-dimensional compact Riemannian manifold of class C^2 without boundary. Hence there exists $r_0 = r_0(\partial\Omega) > 0$ such that the function $f(x) = \operatorname{dist}(x, \partial\Omega)$, $x \in \Omega$ is of class C^1 in $V = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < r_0\}$ (see [29, Theorem 4.12]) and, furthermore, $|Df(x)| = 1$ for $x \in V$ (see [29, 4.8 (5), 4.8 (3)]). Let us fix a sufficiently small $\varepsilon > 0$ and define

$$g_\varepsilon(t) = \begin{cases} 1 & \text{if} \quad t \in [\varepsilon, +\infty), \\ \frac{t}{\varepsilon} & \text{if} \quad t \in [0, \varepsilon]. \end{cases}$$

Since g_ε is a Lipschitz function on $[0, +\infty)$, for each $\xi \in W_0^{1,\infty}(B, \mathbb{R}^3)$, it is clear that the function $\varphi_\varepsilon(x) := g_\varepsilon(f(x))\xi(x)$ is an element of $W_0^{1,\infty}(\Omega \cap B, \mathbb{R}^3)$. Since $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(\Omega \cap B, \mathbb{R}^3)$, $u \in W^{1,p}(\Omega \cap B, \mathbb{R}^\nu)$ and Du is continuous at every point of $\partial\Omega \cap B$, by approximation, we have $\int_{\Omega \cap B} T_u^p : D\xi \,\mathrm{d}x = 0$ for each $\xi \in W_0^{1,\infty}(\Omega \cap B, \mathbb{R}^3)$. Thus, using the function φ_ε as a test function in the weak formulation of $\operatorname{div}(T_u^p) = 0$, we get

$$\begin{aligned} 0 &= \int_{\Omega \cap B} T_u^p : g_\varepsilon(f) D\xi \,\mathrm{d}x + \int_{\Omega \cap B} \langle D(g_\varepsilon(f)), T_u^p[\xi] \rangle \,\mathrm{d}x \\ &= \int_{\Omega \cap B} T_u^p : g_\varepsilon(f) D\xi \,\mathrm{d}x + \int_{\Omega \cap B} \langle g_\varepsilon'(f) Df, T_u^p[\xi] \rangle \,\mathrm{d}x. \end{aligned}$$

Letting $\varepsilon \searrow 0$ and using the Lebesgue dominated convergence theorem, we have

$$\int_{\Omega \cap B} T_u^p : g_\varepsilon(f) D\xi \,\mathrm{d}x \to \int_{\Omega \cap B} T_u^p : D\xi \,\mathrm{d}x. \tag{7.28}$$

On the other hand, using the coarea formula (see [30, Theorem 3.2.22 (3)]), the fact that $|Df| = 1$ in V and changing the variables, we get

$$\begin{aligned}\int_{\Omega\cap B}\langle g'_\varepsilon(f)Df, T_u^p[\xi]\rangle\,\mathrm{d}x &= \frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{d}t\int_{\{f=t\}\cap B}\langle Df(\sigma), T_u^p[\xi](\sigma)\rangle\,\mathrm{d}\mathcal{H}^2(\sigma)\\ &= \frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{d}t\int_{\partial\Omega\cap B}\langle Df(y-t\nu(y)), T_u^p[\xi](y)\rangle\,\mathrm{d}\mathcal{H}^2(y)\\ &\quad + o(1)_{\varepsilon\searrow 0}\\ &\to -\int_{\partial\Omega\cap B}\langle \nu, T_u^p[\xi]\rangle\,\mathrm{d}\mathcal{H}^2,\end{aligned} \tag{7.29}$$

as $\varepsilon \searrow 0$. In fact, since $u \in W^{1,p}(\Omega \cap B, \mathbb{R}^\nu)$, Du is continuous at every point of $\partial\Omega \cap B$, $\xi \in C_c^1(B, \mathbb{R}^3)$ and $Df(y - t\nu(y)) \to -\nu(y)$ as $t \searrow 0$ for each $y \in \partial\Omega$, the function

$$r \in (0, a] \mapsto \Psi(r) := \int_0^a \mathrm{d}t\int_{\partial\Omega\cap B}\langle Df(y-t\nu(y)), T_u^p[\xi](y)\rangle\,\mathrm{d}\mathcal{H}^2(y),$$

where $a > 0$ is fairly small, has the right derivative at 0 equal to $\Psi'(0+) = -\int_{\partial\Omega\cap B}\langle \nu, T_u^p[\xi]\rangle\,\mathrm{d}\mathcal{H}^2$. Combining (7.28) and (7.29), we deduce the desired stress-energy identity, which completes our proof of Proposition 7.11. ◻

Now we deduce the normal boundary estimate.

Proposition 7.12. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in (1, 2]$. Let $K \Subset \partial\Omega \cap B_r^3(x_0)$ be relatively open. If $u \in W^{1,p}(\Omega \cap B_r^3(x_0), \mathbb{R}^\nu)$ satisfies $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(\Omega \cap B_r^3(x_0), \mathbb{R}^3)$ and if Du is continuous at every point of $\partial\Omega \cap B_r^3(x_0)$, then*

$$\int_K |Du|^p\,\mathrm{d}\mathcal{H}^2 \leqq \frac{C}{p-1}\left(\int_{\Omega\cap B_r^3(x_0)}\frac{|Du|^p}{p}\,\mathrm{d}x + \int_{\partial\Omega\cap B_r^3(x_0)}\frac{|D_\top u|^p}{p}\,\mathrm{d}\mathcal{H}^2\right), \tag{7.30}$$

where $C = C(K) > 0$.

To prove Proposition 7.12, we need the next estimate.

Lemma 7.13. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in [1, 2]$. Let $u \in W^{1,p}(\Omega \cap B_r^3(x_0), \mathbb{R}^\nu)$ and Du be continuous at every point of $\partial\Omega \cap B_r^3(x_0)$. Then the following estimate holds*

$$(p-1)|D_\perp u|^p \leqq -p\langle \nu, T_u^p[\nu]\rangle + (3-p)|D_\top u|^p \tag{7.31}$$

on $\partial\Omega\cap B_r^3(x_0)$, where $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ stands for the outward pointing unit normal vector field to $\partial\Omega$.

Proof. Applying successively Young's inequality for products, the definition of the stress-energy tensor T_u^p and the subadditivity of the function $t \in (0, +\infty) \mapsto t^{p/2}$, we have

$$\begin{aligned} |D_\perp u|^p &= |D_\perp u|^p |Du|^{\frac{p(p-2)}{2}} |Du|^{\frac{p(2-p)}{2}} \\ &\leqq \frac{p}{2}|D_\perp u|^2 |Du|^{p-2} + \frac{2-p}{2}|Du|^p \\ &= -\frac{p}{2}\left(\frac{|Du|^p}{p} - \frac{|D_\perp u|^2}{|Du|^{2-p}}\right) + \frac{3-p}{2}|Du|^p \\ &= -\frac{p}{2}\langle \nu, T_u^p[\nu]\rangle + \frac{3-p}{2}(|D_\top u|^2 + |D_\perp u|^2)^{\frac{p}{2}} \\ &\leqq -\frac{p}{2}\langle \nu, T_u^p[\nu]\rangle + \frac{3-p}{2}(|D_\top u|^p + |D_\perp u|^p), \end{aligned}$$

which yields (7.31) by rearranging the terms. □

Proof of Proposition 7.12. Let $\theta \in C^1(\partial\Omega, \mathbb{R})$ be such that $\theta = 1$ on K, $0 \leqq \theta \leqq 1$ and $\theta = 0$ in $\partial\Omega \backslash B_r^3(x_0)$. Fix $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$ such that $\xi = \theta\nu$ on $\partial\Omega$. Applying Proposition 7.11 and using the Cauchy-Schwarz inequality, and also the facts that $|\mathrm{Id}| = \sqrt{3}$, $|Du \otimes Du| = |Du|^2$, we get

$$\begin{aligned} \left|\int_{\partial\Omega\cap B_r^3(x_0)} \theta\langle \nu, T_u^p[\nu]\rangle \,\mathrm{d}\mathcal{H}^2\right| &= \left|\int_{\Omega\cap B_r^3(x_0)} T_u^p : D\xi \,\mathrm{d}x\right| \\ &= \left|\int_{\Omega\cap B_r^3(x_0)} \left(\frac{|Du|^p}{p}\mathrm{Id} - \frac{Du\otimes Du}{|Du|^{2-p}}\right) : D\xi \,\mathrm{d}x\right| \\ &\leqq \int_{\Omega\cap B_r^3(x_0)} \left(\frac{\sqrt{3}}{p} + 1\right)|Du|^p |D\xi| \,\mathrm{d}x \\ &\leqq (2+\sqrt{3})\|D\xi\|_{L^\infty(B_r^3(x_0),\mathbb{R}^3\otimes\mathbb{R}^3)} \int_{\Omega\cap B_r^3(x_0)} \frac{|Du|^p}{p}\,\mathrm{d}x. \end{aligned} \tag{7.32}$$

Setting $E = \{x \in \partial\Omega \cap B_r^3(x_0) : (p-1)|D_\perp u(x)|^2 \geqq |D_\top u(x)|^2\}$, observe that $-\langle \nu, T_u^p[\nu]\rangle \geqq 0$ on E and hence

$$-\frac{p}{p-1}\int_{K\cap E} \langle \nu, T_u^p[\nu]\rangle \,\mathrm{d}\mathcal{H}^2 \leqq -\frac{p}{p-1}\int_E \theta\langle \nu, T_u^p[\nu]\rangle \,\mathrm{d}\mathcal{H}^2, \tag{7.33}$$

where we have used that $K \Subset \partial\Omega \cap B_r^3(x_0)$, $\theta = 1$ on K and $0 \leqq \theta \leqq 1$. On the other hand, on $(\partial\Omega \cap B_r^3(x_0))\backslash E$ it holds

$$\begin{aligned} 0 \leqq \langle \nu, T_u^p[\nu]\rangle &= |Du|^{p-2}\left(\frac{|D_\top u|^2 + |D_\perp u|^2}{p} - |D_\perp u|^2\right) \\ &\leqq |D_\top u|^{p-2}\left(\frac{|D_\top u|^2 + (p-1)|D_\perp u|^2}{p}\right) \leqq \frac{2}{p}|D_\top u|^p. \end{aligned} \tag{7.34}$$

Hereinafter in this proof, C denotes an absolute positive constant that can be different from line to line. Using Lemma 7.13, the estimates (7.32), (7.33) and (7.34), we deduce the following chain of estimates

$$\begin{aligned}\int_K |Du|^p \,\mathrm{d}\mathcal{H}^2 &\leqq \int_K (|D_\top u|^p + |D_\perp u|^p)\,\mathrm{d}\mathcal{H}^2\\ &\leqq -\frac{p}{p-1}\int_K \langle \nu, T_u^p[\nu]\rangle\,\mathrm{d}\mathcal{H}^2 + \frac{C}{p-1}\int_K \frac{|D_\top u|^p}{p}\,\mathrm{d}\mathcal{H}^2\\ &\leqq -\frac{p}{p-1}\int_{\partial\Omega\cap B_r^3(x_0)} \theta\langle \nu, T_u^p[\nu]\rangle\,\mathrm{d}\mathcal{H}^2 + \frac{C}{p-1}\int_{\partial\Omega\cap B_r^3(x_0)} \frac{|D_\top u|^p}{p}\,\mathrm{d}\mathcal{H}^2\\ &\leqq \frac{C}{p-1}\left(\|D\xi\|_\infty \int_{\Omega\cap B_r^3(x_0)} \frac{|Du|^p}{p}\,\mathrm{d}x + \int_{\partial\Omega\cap B_r^3(x_0)} \frac{|D_\top u|^p}{p}\,\mathrm{d}\mathcal{H}^2\right),\end{aligned}$$

which completes our proof of Proposition 7.12. □

Next, we analyze the boundary and inward perturbations for the limit stress-energy tensor, and we prove that the set supporting the associated varifold to this tensor is *minimizing area to first order* in the Brian White sense. First, we analyze the boundary perturbations.

Proposition 7.14. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in [1, +\infty)$. If $u \in W^{1,p}(\Omega\cap B_r^3(x_0), \mathbb{R}^\nu)$ satisfies $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(\Omega \cap B_r^3(x_0), \mathbb{R}^3)$ and if Du is continuous at every point of $\partial\Omega \cap B_r^3(x_0)$, then, for each $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$,*

$$\left|\int_{\Omega\cap B_r^3(x_0)} T_u^p : D\xi\,\mathrm{d}x\right| \leqq \left(1+\frac{1}{p}\right)\|\xi\|_{L^\infty(\partial\Omega\cap B_r^3(x_0),\mathbb{R}^3)} \int_{\partial\Omega\cap B_r^3(x_0)} |Du|^p\,\mathrm{d}\mathcal{H}^2. \tag{7.35}$$

Proof. Let $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ be the outward pointing unit normal vector field to $\partial\Omega$ and let $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$. Applying the triangle and the Cauchy-Schwarz inequalities, we have

$$|\langle \nu, T_u^p[\xi]\rangle| \leqq (1+1/p)|Du|^p|\xi|$$

on $\partial\Omega\cap B_r^3(x_0)$. Using this estimate and Proposition 7.11, we complete our proof of Proposition 7.14. □

Proposition 7.15. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small, $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$ and $p_n \nearrow 2$ as $n \to +\infty$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of mappings such that $u_n \in W^{1,p_n}(\Omega\cap B_r^3(x_0), \mathbb{R}^\nu)$, Du_n be continuous at every point of $\partial\Omega \cap B_r^3(x_0)$ and $\operatorname{div}(T_{u_n}^{p_n}) = 0$ in $\mathscr{D}'(\Omega \cap B_r^3(x_0), \mathbb{R}^3)$. Assume also that $(2-p_n)T_{u_n}^{p_n} \overset{*}{\rightharpoonup} T_*$ weakly* in $(C(\overline{\Omega\cap B_r^3(x_0)}, M_3(\mathbb{R})))'$. If*

$$\limsup_{n\to+\infty}(2-p_n)\int_{\Omega\cap B_r^3(x_0)} \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x < +\infty \tag{7.36}$$

and

$$\limsup_{n\to+\infty}(2-p_n)\int_{\partial\Omega\cap B_r^3(x_0)} \frac{|D_\top u_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 < +\infty, \tag{7.37}$$

then there exists a constant $C > 0$ such that for each $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$ we have

$$\left| \int_{\Omega \cap B_r^3(x_0)} T_* : D\xi \, \mathrm{d}x \right| \leqq C \|\xi\|_{L^\infty(\partial\Omega \cap B_r^3(x_0), \mathbb{R}^3)}. \tag{7.38}$$

Proof. By the weak* convergence and the convexity, and also using Proposition 7.12, Proposition 7.14 and the estimates (7.36), (7.37), we obtain the following chain of estimates

$$\begin{aligned} \left| \int_{\Omega \cap B_r^3(x_0)} T_* : D\xi \, \mathrm{d}x \right| &\leqq \liminf_{n \to +\infty} \left| (2 - p_n) \int_{\Omega \cap B_r^3(x_0)} T_{u_n}^{p_n} : D\xi \, \mathrm{d}x \right| \\ &\leqq \limsup_{n \to +\infty} 2\|\xi\|_{L^\infty(\partial\Omega \cap B_r^3(x_0), \mathbb{R}^3)} (2 - p_n) \int_{\partial\Omega \cap B_r^3(x_0)} |Du_n|^{p_n} \, \mathrm{d}\mathcal{H}^2 \\ &\leqq \limsup_{n \to +\infty} C' \|\xi\|_{L^\infty(\partial\Omega \cap B_r^3(x_0), \mathbb{R}^3)} (2 - p_n) \\ &\qquad \left(\int_{\Omega \cap B_r^3(x_0)} \frac{|Du_n|^{p_n}}{p_n} \, \mathrm{d}x + \int_{\partial\Omega \cap B_r^3(x_0)} \frac{|D_\top u_n|^{p_n}}{p_n} \, \mathrm{d}\mathcal{H}^2 \right) \\ &\leqq C \|\xi\|_{L^\infty(\partial\Omega \cap B_r^3(x_0), \mathbb{R}^3)}, \end{aligned}$$

where $C', C > 0$ are constants independent of ξ. This leads to (7.38) and thereby completes our proof of Proposition 7.15. □

Now we analyze the inward perturbations.

Proposition 7.16. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in [1, 2]$. If $u \in W^{1,p}(\Omega \cap B_r^3(x_0), \mathbb{R}^\nu)$ satisfies $\operatorname{div}(T_u^p) = 0$ in $\mathcal{D}'(\Omega \cap B_r^3(x_0), \mathbb{R}^3)$ and if Du is continuous at every point of $\partial\Omega \cap B_r^3(x_0)$, then for each $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$ such that $\langle \xi, \nu \rangle \leqq 0$ on $\partial\Omega \cap B_r^3(x_0)$, where $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ denotes the outward pointing unit normal vector field to $\partial\Omega$, we have*

$$\int_{\Omega \cap B_r^3(x_0)} T_u^p : D\xi \, \mathrm{d}x \geqq -\|\xi\|_{L^\infty(U, \mathbb{R}^3)} \int_U \left((3 - p) \frac{|D_\top u|^p}{p} + |D_\perp u|^{p-1} |D_\top u| \right) \mathrm{d}\mathcal{H}^2,$$

where $U = \partial\Omega \cap B_r^3(x_0)$.

Proof. Using Lemma 7.13 and the fact that $\langle \xi, \nu \rangle \leqq 0$ on U, we deduce that

$$\begin{aligned} \langle \nu, T_u^p[\xi] \rangle &= \langle \nu, \xi \rangle \langle \nu, T_u^p[\nu] \rangle - |Du|^{p-2} \langle Du[\xi - \langle \nu, \xi \rangle \nu], Du[\nu] \rangle \\ &\geqq \langle \nu, \xi \rangle \left(\frac{1 - p}{p} \right) |D_\perp u|^p + \langle \nu, \xi \rangle \left(\frac{3 - p}{p} \right) |D_\top u|^p - \|\xi\|_{L^\infty(U, \mathbb{R}^3)} |D_\perp u|^{p-1} |D_\top u| \\ &\geqq -\|\xi\|_{L^\infty(U, \mathbb{R}^3)} \left(\frac{3 - p}{p} \right) |D_\top u|^p - \|\xi\|_{L^\infty(U, \mathbb{R}^3)} |D_\perp u|^{p-1} |D_\top u|. \end{aligned}$$

The conclusion then follows from Proposition 7.11. □

The next proposition says that the set supporting the varifold associated to the tensor T_* of Proposition 7.15 is *minimizing area to first order* in the Brian White sense (see [57]).

Proposition 7.17. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$ and $r > 0$ be fairly small. Let $(p_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ and T_* be as in Proposition 7.15. Let $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ be the outward pointing unit normal vector field to $\partial\Omega$. Assume that*

$$\lim_{n\to+\infty}(2-p_n)\int_{\partial\Omega\cap B_r^3(x_0)} \frac{|D_\top u_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 = 0. \tag{7.39}$$

Then for each $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$ such that $\langle \xi, \nu\rangle \leqq 0$ on $\partial\Omega \cap B_r^3(x_0)$,

$$\int_{\Omega\cap B_r^3(x_0)} T_* : D\xi\,\mathrm{d}x \geqq 0.$$

Proof. Set $U = \partial\Omega \cap B_r^3(x_0)$. Taking into account (7.36), (7.37) and Proposition 7.12, we get

$$A := \limsup_{n\to+\infty}(2-p_n)\int_U |Du_n|^{p_n}\,\mathrm{d}\mathcal{H}^2 < +\infty. \tag{7.40}$$

Next, using Proposition 7.16, Hölder's inequality, (7.40) and (7.39), we deduce that

$$\begin{aligned}
&\int_{\Omega\cap B_r^3(x_0)} T_* : D\xi\,\mathrm{d}x \\
&\geqq -\|\xi\|_{L^\infty(U,\mathbb{R}^3)}\limsup_{n\to+\infty}(2-p_n)\int_U \frac{|D_\top u_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \\
&\quad - \|\xi\|_{L^\infty(U,\mathbb{R}^3)}\limsup_{n\to+\infty}(2-p_n)\|Du_n\|^{1-\frac{1}{p_n}}_{L^{p_n}(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)}\|D_\top u_n\|_{L^{p_n}(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \\
&\geqq -\|\xi\|_{L^\infty(U,\mathbb{R}^3)}A^{\frac{1}{2}}\left(\limsup_{n\to+\infty}(2-p_n)\int_U \frac{|D_\top u_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2\right)^{\frac{1}{2}} = 0,
\end{aligned}$$

which completes our proof of Proposition 7.17. □

We shall say that $\overline{\Omega}$ is *strongly convex* at a point $x_0 \in \partial\Omega$ whenever the two principal curvatures of $\partial\Omega$ at x_0, with respect to the inward pointing unit normal vector to $\partial\Omega$ at x_0, are positive. If Ω is convex and $\partial\Omega$ has the second fundamental form that is positive semidefinite with respect to the inward pointing unit normal vector field to $\partial\Omega$, then $\overline{\Omega}$ is strongly convex at $x_0 \in \partial\Omega$ whenever the Gaussian curvature of $\partial\Omega$ is positive at x_0.

As a result of our analysis in this subsection, we obtain the next boundary repulsion property.

Theorem 7.18. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2 such that $\overline{\Omega}$ is strongly convex at every point of $\partial\Omega$. Let $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N})$ be such that $g \in C^1(\partial\Omega\backslash S(g), \mathcal{N})$. Let $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of p_n-minimizers in Ω such that for each $n \in \mathbb{N}$, $\operatorname{tr}_{\partial\Omega}(u_n) = g$. Assume that for each $n \in \mathbb{N}$, Du_n is continuous on $\partial\Omega\backslash S(g)$. Then there exists $\mu_* \in (C(\overline{\Omega}))'$ such that, up to a subsequence, (5.2) holds. Moreover, S_* lies in the convex hull of $S(g)$ and $S_* \cap \partial\Omega = S(g)$, where S_* is the support of μ_*.*

Remark 7.19. According to [34], every p-minimizer u in Ω is locally $C^{1,\alpha}$ regular in $\Omega\backslash S(u)$ for some relatively closed subset $S(u)$ of Ω which has Hausdorff dimension at most 1 when $p \in (1, 2]$. Furthermore, p-minimizers with smooth boundary values on smooth domains are Hölder continuous near the boundary and, in view of [53], it is reasonable to assume that they are C^1 regular near the boundary. In Theorem 7.18, we assume that the gradient of a p-minimizer is continuous on a portion of the boundary, where the boundary datum is smooth. This assumption is significantly less demanding than the assumption that the minimizer is C^1 regular outside the union of its singular set with the singular set of the boundary datum.

Proof of Theorem 7.18. Using Proposition 2.21, the fact that $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N}) \subset W^{1/2,2}(\partial\Omega, \mathcal{N})$ and the Banach-Alaoglu theorem, we obtain $\mu_* \in (C(\overline{\Omega}))'$ such that, up to a subsequence, (5.2) holds. Fix an arbitrary point $x_0 \in \partial\Omega \setminus S(g)$ and a fairly small $r > 0$ such that $\overline{B_r^3}(x_0) \cap S(g) = \emptyset$. In view of Proposition 2.21, the fact that $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ and Lemma 7.8, (7.36) and (7.37) hold. Then, taking into account (5.7), Proposition 2.21 and using the Banach-Alaoglu theorem, we deduce that there exists $T_* \in (C_0(\overline{\Omega \cap B_r^3(x_0)}, M_3(\mathbb{R})))'$ such that, up to a subsequence (still denoted by the same index), $(2-p_n)T_{u_n}^{p_n} \overset{*}{\rightharpoonup} T_*$ weakly* in $(C_0(\overline{\Omega \cap B_r^3(x_0)}, M_3(\mathbb{R})))'$. Keeping the same notation, let us define $T_*(\cdot) = T_*(\cdot \cap \overline{\Omega} \cap B_r^3(x_0)) \in (C_0(B_r^3(x_0), M_3(\mathbb{R})))'$. By Proposition 7.15, $\operatorname{div}(T_*) \in (C_0(B_r^3(x_0), \mathbb{R}^3))'$. Then, according to [6, Proposition 3.1], there exists an $\mathcal{H}^1$-rectifiable set $R \subset B_r^3(x_0) \cap \overline{\Omega}$ and a map $\lambda \in L^\infty(R, M_3(\mathbb{R}))$ satisfying the following conditions. For $\mathcal{H}^1$-a.e. $x_0 \in R$, $|\lambda(x_0)| = 1$, $\lambda(x_0) \in \mathrm{G}(3, 1)$ is the orthogonal projection matrix onto $T_{x_0}R$ such that $T_* \llcorner \{\Theta_1^*(|T_*|) > 0\}(dx) = \Theta_1^*(|T_*|, x)\lambda(x)\mathcal{H}^1 \llcorner R(dx)$, where $T_{x_0}R$ is the approximate tangent plane to R at x_0. One has that $|T_*| = \mu_*$, $\Theta_1^*(|T_*|) = \Theta_1(\mu_*)$ and $R = S_* \cap B_r^3(x_0)$ (we refer the reader to the proof of Proposition 5.8). In particular, we deduce that $V_*(dA, dx) = \delta_{\lambda_*(x)}(dA) \otimes |T_*|(dx)$ is a 1-dimensional rectifiable varifold (see [2,26]). Since $g \in C^1(\partial\Omega \cap B_r^3(x_0), \mathcal{N})$, $(u_n)_{n\in\mathbb{N}}$ fulfill the condition (7.39) of Proposition 7.17, which says that V_* is *minimizing area to first order* in $(\Omega \cap B_r^3(x_0)) \cup (\partial\Omega \cap B_r^3(x_0))$ in the Brian White sense (see [57]). Then, by [57, Theorem 1], $x_0 \notin S_*$. Thus, $S_* \cap \partial\Omega \subset S(g)$. Denote by E the convex hull of $S(g)$. By contradiction, assume that $S_* \cap (\Omega\backslash E) \neq \emptyset$. Then, since $S_* \cap \partial\Omega \subset S(g)$, the set M of points $x_0 \in S_*$ such that $\operatorname{dist}(x_0, E) = \max\{\operatorname{dist}(x, E) : x \in S_*\}$ is a subset of $\Omega \setminus E$. Fix a point $x_0 \in M$ such that the segments of S_* emanating from x_0 do not lie in the same plane. Then these segments lie in the same half-space (lying under a 2-plane parallel to some face of E) and form a vertex, which cannot happen in view of the stationarity of the varifold coming from Proposition 5.8 (see Corollary 5.19). Therefore, $S_* \subset E$.

Next, let us prove that $S(g) \subset S_* \cap \partial\Omega$. Using Corollary 7.7 and the fact that $S_* \cap \partial\Omega \subset S(g)$, for all fairly small $\delta > 0$, we have $\operatorname{tr}_{\partial\Omega\backslash S_g^\delta}(u_*) = g|_{S_g^\delta}$, where u_* is a map coming from Corollary 7.7 and $S_g^\delta = \{x \in \partial\Omega : \operatorname{dist}(x, S(g)) \geqq \delta\}$. For each $a \in S(g)$, denoting by $\partial B_\varrho^{\partial\Omega}(a)$ the boundary of the geodesic ball in $\partial\Omega$ with center a and radius $\varrho > 0$, we know that $u_*|_{\partial B_\varrho^{\partial\Omega}(a)} = g|_{\partial B_\varrho^{\partial\Omega}(a)}$ is not nullhomotopic for each sufficiently small $\varrho > 0$. This, together with the fact that

$S_* \subset E$ and Corollary 5.18, implies that $a \in S_* \cap \partial\Omega$ and hence $S(g) \subset S_* \cap \partial\Omega$. Therefore, $S_* \cap \partial\Omega = S(g)$, which completes our proof of Theorem 7.18. □

7.5. Structure of the Singular Set and the Limit Measure

Theorem 7.20. *Let Ω, g, $S(g)$, $(u_n)_{n\in\mathbb{N}}$, μ_* and S_* be as in Theorem 7.18. Let u_* be the map of Proposition 5.6 being an element of $W^{1,2}_{\mathrm{loc}}(\Omega \setminus S_*, \mathcal{N})$. Then S_* is a finite union of finite chains of segments connecting the points of $S(g)$ and lying in the convex hull of $S(g)$. More precisely, there exists a finite set of segments $L_1, \ldots, L_n \subset \overline{\Omega}$ such that $S_* = \bigcup_{i=1}^n L_i$ and $u_* \in C^\infty(\Omega \backslash (\bigcup_{i=1}^n L_i \cup S_0), \mathcal{N})$, where S_0 is a locally finite subset of $\Omega \backslash \bigcup_{i=1}^n L_i$. If x is a point inside L_i, then $\Theta_1(\mu_*, x) = \mathcal{E}_2^{\mathrm{sg}}([u_*|_{\partial D}]) =: \lambda_i$, where D is a closed 2-disk in Ω such that $D \cap S_* = \{x\}$, $\partial D \cap S_0 = \emptyset$ and x is an interior point of D. If x is a branching point of S_*, namely there exists $L_{i_1}, \ldots, L_{i_k}$ emanating from x, then $\sum_{j=1}^k \lambda_{i_j} v_{i_j} = 0$, where v_{i_j} is the unit direction vector of L_{i_j} pointing from x. Furthermore, $\mu_* = \sum_{i=1}^n \lambda_i \mathcal{H}^1 \llcorner L_i$. Finally, for each $i \in \{1, \ldots, n\}$, each endpoint of L_i is either a branching point of S_* lying in Ω or it belongs to $S(g)$. If x is an endpoint of some L_i lying in $S(g)$, then $[g|_{\partial D_x}] = [u_*|_{\partial D}]$, where D_x is the geodesic disk on $\partial\Omega$ with center x and lying far from $S(g)\backslash\{x\}$, and D is a closed 2-disk in Ω centered inside L_i with ∂D lying outside the convex hull of $S(g)$ and outside of S_0.*

Proof. Denote by E the convex hull of $S(g)$. The fact that $S_* \subset E$ comes from Theorem 7.18. Fix an arbitrary $x_0 \in S(g)$. Further in this proof, the center of each ball and sphere is at x_0. In view of (5.1), for each $r_0 > 0$,

$$\limsup_{n\to+\infty} (2 - p_n) \int_{B^3_{r_0} \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, \mathrm{d}x < +\infty.$$

Then, up to extracting a subsequence (not relabeled) and using the coarea formula, together with Fatou's lemma, we deduce that $\mathrm{tr}_{\partial B^3_r \cap \Omega}(u_n) = u_n|_{\partial B^3_r \cap \Omega} \in W^{1,p_n}(\partial B^3_r \cap \Omega, \mathcal{N})$ and

$$\limsup_{n\to+\infty} (2 - p_n) \int_{\partial B^3_r \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, \mathrm{d}\mathcal{H}^2 < +\infty$$

for $\mathcal{H}^1$-a.e. $r \in (0, r_0]$. Let $r_0 > 0$ be small enough so that $\overline{B}^3_{r_0}$ does not contain any point of $S(g)$ except of x_0. Furthermore, by virtue of Proposition 5.17 and the fact that $S_* \subset E$, we can also assume that $\partial B^3_{r_0}$ does not contain any branching point of S_*. Next, we recall that $\mu_* \llcorner \Omega(dx) = \Theta_1(x)\mathcal{H}^1 \llcorner (S_* \cap \Omega)(dx)$, where the density Θ_1 is defined in (5.5). For each $r \in (0, r_0]$, define $M(r) = \int_{S_* \cap \partial B^3_r} \Theta_1(x) \, \mathrm{d}\mathcal{H}^0(x)$, where $\mathcal{H}^0$ denotes the 0-dimensional Hausdorff measure in $\mathbb{R}^3$, namely, for a discrete set of points $X \subset \mathbb{R}^3$, $\mathcal{H}^0(X) = \#X$ (the cardinality of X). In view of Lemma 2.8, Proposition 5.16, Proposition 5.17 and the fact that $S_* \subset E$, M is a bounded from below piecewise constant function with a discrete set of values. Indeed, there exists an at most countable collection of intervals $\{(a_i, b_i] : i \in \mathbb{N} \cap [0, i_0]\}$, where $i_0 \in \mathbb{N} \cup \{+\infty\}$, such that $b_0 = r_0$, $(0, r_0] = \bigcup_{i=0}^{i_0} (a_i, b_i]$ and for each $i \in \mathbb{N} \cap [0, i_0]$: M is constant on (a_i, b_i); $a_i = b_{i+1}$; for each $r \in (a_i, b_i)$, ∂B_r^3 does not meet any branching point of S_*; $\partial B_{a_i}^3$ meets a branching point of S_*. By definition of r_0, M is constant on $(a_0, b_0]$. Notice that, by the upper semicontinuity of Θ_1 (see, for instance, [3, 2(8)]) and Corollary 5.18, for each $i \in \mathbb{N} \cap [0, i_0]$ and $x \in (a_i, b_i) \cup (a_{i+1}, b_{i+1})$, $M(x) \leqq M(b_i)$. Also, taking into account Proposition 5.17 and the facts that $S_* \subset E$, $E \cap \partial\Omega = S(g)$ (since $\overline{\Omega}$ is strongly convex at every point of $\partial\Omega$), we observe that $\mu_*(\partial(B_r^3 \cap \Omega)) = 0$ for each $r \in (0, r_0]$. This, together with the weak* convergence of μ_n to μ_* (see (5.2)), implies that for each $i \in \mathbb{N} \cap [0, i_0]$,

$$\begin{aligned}
&\mu_n((B_{b_i}^3 \setminus B_{a_i}^3) \cap \Omega) - \mu_*((B_{b_i}^3 \setminus B_{a_i}^3) \cap \Omega) \\
&\quad = \int_{a_i}^{b_i} \mathrm{d}r \left((2 - p_n) \int_{\partial B_r^3 \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, \mathrm{d}\mathcal{H}^2 - M(r) \right) \to 0
\end{aligned}$$

as $n \to +\infty$, where we have used the coarea formula. Applying Fatou's lemma, up to a subsequence (not relabeled), we get

$$\int_{a_i}^{b_i} \mathrm{d}r \left(\limsup_{n \to +\infty} (2 - p_n) \int_{\partial B_r^3 \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, \mathrm{d}\mathcal{H}^2 - M(r) \right) \leqq 0. \tag{7.41}$$

Fix an arbitrary $i \in \mathbb{N} \cap [0, i_0]$. Then, either for $\mathcal{H}^1$-a.e. $r \in [a_i, b_i]$,

$$\limsup_{n \to +\infty} (2 - p_n) \int_{\partial B_r^3 \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, \mathrm{d}\mathcal{H}^2 \geqq M(r), \tag{7.42}$$

or there exists a set $I \subset [a_i, b_i]$ of positive $\mathcal{H}^1$-measure such that for each $r \in I$, (7.42) does not hold. In the case when (7.42) holds for $\mathcal{H}^1$-a.e. $r \in [a_i, b_i]$, according to (7.41), we have

$$\limsup_{n \to +\infty} (2 - p_n) \int_{\partial B_r^3 \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, \mathrm{d}\mathcal{H}^2 = M(r)$$

for $\mathcal{H}^1$-a.e. $r \in [a_i, b_i]$.

Thus, there exists $r \in (a_i, b_i)$ such that $\mathrm{tr}_{\partial B_r^3 \cap \Omega}(u_n) = u_n|_{\partial B_r^3 \cap \Omega} \in W^{1,p_n}(\partial B_r^3 \cap \Omega, \mathcal{N})$ for each n and

$$\limsup_{n \to +\infty} (2 - p_n) \int_{\partial B_r^3 \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, \mathrm{d}\mathcal{H}^2 \leqq M(r). \tag{7.43}$$

Next, we construct a competitor v_n for u_n. Define $v_n = u_n$ in $\Omega \setminus B_r^3$. For each x in the cone $K = \{x \in B_r^3 \cap \Omega : (x_0 + r(x - x_0)/|x - x_0|) \in \partial B_r^3 \cap \overline{\Omega}\}$, set

$$v_n(x) = \mathrm{tr}_{\partial B_r^3 \cap \overline{\Omega}}(u_n) \left(x_0 + \frac{r(x - x_0)}{|x - x_0|} \right). \tag{7.44}$$

In the region $V = (B_r^3 \cap \Omega) \setminus K$, we define v_n by homotopy, using [12, Theorem 2]. More precisely, fix a point $x \in V$ and consider a 2-plane passing through x, which is orthogonal to the axis of the cone K. The intersection of this 2-plane with V is an annulus whose boundary consists of two curves $\gamma_1 \subset \partial\Omega$ and $\gamma_2 \subset \partial K$. By definition, $g|_{\gamma_1}$ is continuously homotopic to $v_n|_{\gamma_2}$. Then, by [12, Theorem 2], on this annulus there exists a $W^{1,2}$ map with values in $\mathcal{N}$ and with boundary values $g|_{\gamma_1}$ and $v_n|_{\gamma_2}$ on γ_1 and γ_2, respectively. Observe that the set of values of $v_n|_{\gamma_2}$ coincides with the set of values of $g|_{\partial B^3_r \cap \partial\Omega}$. Continuing this process, namely filling V, we obtain a map $v_n|_V$ such that $v_n|_V \in W^{1,2}(V \setminus \overline{B}^3_\delta, \mathcal{N})$ for each $\delta \in (0, r)$ and

$$\limsup_{n\to+\infty}(2 - p_n) \int_V \frac{|D(v_n|_V)|^{p_n}}{p_n} \,\mathrm{d}x = 0, \tag{7.45}$$

which is possible due to the fact that $g \in \mathcal{R}^1_0(\partial\Omega, \mathcal{N})$ (see Lemma 7.8), since the p_n-energy of $v_n|_V$ in $V \cap B^3_\delta$ is of order $\delta/(2 - p_n)$ for $\delta \in (0, r)$. Thus, we have constructed a competitor v_n for u_n.

Next, again by the weak* convergence of μ_n to μ_* and the fact that $\mu_*(\partial(B^3_r \cap \Omega)) = 0$, we have

$$\int_0^r M(\varrho)\,\mathrm{d}\varrho = \mu_*(B^3_r \cap \Omega) = \lim_{n\to+\infty}(2 - p_n) \int_{B^3_r\cap\Omega} \frac{|Du_n|^{p_n}}{p_n} \,\mathrm{d}x. \tag{7.46}$$

Since v_n is a competitor for u_n, $v_n = u_n$ in $\Omega\setminus B^3_r$, using (7.43), (7.44), (7.45) and (7.46), we deduce that

$$\begin{aligned} \int_0^r M(\varrho)\,\mathrm{d}\varrho &\leqq \limsup_{n\to+\infty}(2 - p_n) \int_K \frac{|Dv_n|^{p_n}}{p_n} \,\mathrm{d}x \\ &\leqq \limsup_{n\to+\infty} \int_0^r \varrho^{2-p_n}\,\mathrm{d}\varrho\, (2-p_n) \int_{\partial B^3_r\cap\Omega} \frac{|D_\top u_n|^{p_n}}{p_n} \,\mathrm{d}\mathcal{H}^2 \\ &\leqq rM(r), \end{aligned} \tag{7.47}$$

where we have also used that $p_n \nearrow 2$ and $|D_\top u_n| \leqq |Du_n|$. We can choose $i \in \mathbb{N} \cap [0, i_0]$ and $r > 0$ so that, apart from (7.47), $M(r)$ is the minimal value of M on $(0, r_0]$. In fact, since M is bounded from below and has a discrete set of values, it achieves its minimum at every point of the interval (a_i, b_i) for some $i \in \mathbb{N}\cap[0, i_0]$ (notice that it can achieve the minimum at a_i or b_i, but in this case it achieves its minimum at every point of (a_i, b_i), since $M(x) \leqq \min\{M(a_i), M(b_i)\}$ as was observed earlier). Thus, taking such $i \in \mathbb{N}\cap[0, i_0]$ and $r \in (a_i, b_i)$, it holds $M = M(r)$ on $(0, r]$, because otherwise (7.47) would not hold. Then, since M has a discrete set of values, there exists a bounded set $J \subset \mathbb{N}$ such that for each $\varrho \in (0, r]$, the number of branching points of $S_* \cap \partial B^3_\varrho$ belongs to J. This, together with the stationarity of the associated varifold V_* (see Corollary 5.19), implies that the only possible situation describing $S_* \cap \overline{B}^3_r$ is when $S_* \cap \overline{B}^3_r$ is the union of closed straight line segments connecting the points of $S_* \cap \partial B^3_r$ with x_0 and whose relative interiors are mutually disjoint.

We complete our proof of Theorem 7.20 using Propositions 5.6, 5.17, Corollaries 5.18, 5.19, 7.7 and Theorem 7.18. □

The next corollary is a direct consequence of Theorem 7.20, so we omit its proof.

Corollary 7.21. *If the set $S(g)$ of Theorem 7.20 lies in a 2-plane, then S_* is the segment connecting the points of $S(g)$ if $S(g)$ has only two points, is contained in the triangle with vertices from $S(g)$ if $S(g)$ has only three points and is contained in the polygon with vertices from $S(g)$ if $S(g)$ has more than four points.*

We provide a revised version of Corollary 7.21 for the case when $S(g)$ has only two points, also without proof.

Corollary 7.22. *Let Ω, g and $(u_n)_{n\in\mathbb{N}}$ be as in Theorem 7.18. Let $S(g) = \{a, b\}$. Then $S_* = [a, b]$ and for each $x \in (a, b)$, $\Theta_1(\mu_*, x) = \mathcal{E}_2^{\mathrm{sg}}([g|_{\partial D_a}]) = \mathcal{E}_2^{\mathrm{sg}}([g|_{\partial D_b}])$, where for $z \in \{a, b\}$, D_z is a geodesic (closed) disk in $\partial\Omega$ with center z, not containing b if $z = a$ and a if $z = b$. Furthermore, $\mu_* = \mathcal{E}_2^{\mathrm{sg}}([g|_{\partial D_a}])\mathcal{H}^1 \llcorner [a, b]$.*

7.6. Minimality of the Singular Set

Now we prove the global minimality of S_*.

Proposition 7.23. *Let Ω, g, $S(g)$, $(u_n)_{n\in\mathbb{N}}$, S_* and μ_* be as in Theorem 7.18. Let u_* be the map of Proposition 5.6 being an element of $W^{1,2}_{\mathrm{loc}}(\Omega \setminus S_*, \mathcal{N})$. Then $S_* \in \mathscr{C}(\Omega, g)$ and for each $S \in \mathscr{C}(\Omega, g)$,*

$$\mu_*(S_*) = \mathbb{M}(g, S_*) \leqq \mathbb{M}(g, S).$$

Proof. According to Theorem 7.20, $S_* \in \mathscr{C}(\Omega, g)$ (see Definition 6.1). Let $S \in \mathscr{C}(\Omega, g)$. By reproducing the *Step 2* of the proof of Proposition 6.4, we obtain a sequence of competitors $v_n \in W^{1,p_n}(\Omega, \mathcal{N})$ for u_n such that

$$\limsup_{n\to+\infty}(2 - p_n) \int_\Omega \frac{|Dv_n|^{p_n}}{p_n}\,\mathrm{d}x \leqq \mathbb{M}(g, S), \tag{7.48}$$

such reproduction is possible thanks to Lemma 7.8. By Theorem 7.20 and Definition 6.3, $\mu_*(S_*) = \mathbb{M}(g, S_*)$. Using this, (5.2), the fact that v_n is a competitor for u_n and (7.48), we obtain

$$\begin{aligned}
\mathbb{M}(g, S_*) &= \lim_{n\to+\infty}(2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n}\,\mathrm{d}x \\
&\leqq \limsup_{n\to+\infty}(2 - p_n) \int_\Omega \frac{|Dv_n|^{p_n}}{p_n}\,\mathrm{d}x \\
&\leqq \mathbb{M}(g, S),
\end{aligned}$$

which completes our proof of Proposition 7.23. □

For convenience, we provide the definition of a *minimal connection*.

Definition 7.24. A minimal connection of a set $\{a_1, \ldots, a_{2n}\} \subset \mathbb{R}^3$ is a union S of mutually disjoint closed segments, each of which connects two points from $\{a_1, \ldots, a_{2n}\}$ so that the sum of the lengths of these segments is minimal among all possible such connections.

Proposition 7.25. *Let Ω, g, $S(g)$, S_* be as in Theorem 7.18, $S(g) \neq \emptyset$ and $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$. Then $S(g)$ has an even number of elements and S_* is a minimal connection of $S(g)$ in the sense of Definition 7.24.*

Proof. First, notice that $\#S(g)$ is even, because otherwise, using the fact that $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$, we would obtain a map which is not nullhomotopic, but having zero degree, which is a contradiction. Next, in view of Corollary 6.8 and Theorem 7.20, S_* is a finite union of segments whose relative interiors are mutually disjoint, and each of these segments connects two points from $S(g)$. Let S be a union of mutually disjoint closed segments, each of which connects two points from $S(g)$. Then, according to Proposition 6.6 and Definition 6.1, $S \in \mathscr{C}(\Omega, g)$. By Proposition 7.23,

$$\mathbb{M}(g, S_*) \leqq \mathbb{M}(g, S).$$

Since $\pi_1(\mathcal{N})$ has only one nontrivial element, the above estimate implies that $S_* \in \mathscr{C}(\Omega, g)$ is a minimal connection of $S(g)$ in the sense of Definition 7.24. This completes our proof of Proposition 7.25. □

7.7. Proof of Theorem 1.2

Finally, we prove our second main theorem.

Proof of Theorem 1.2. The proof follows from Propositions 7.9, 7.23 and Theorem 7.20. □

Appendix A. Auxiliary Results

We provide a proof of the following lemma leading to Corollary A.2, which was used in the proofs of Lemmas 2.13, 2.15.

Lemma A.1. *Let $m \in \mathbb{N}\setminus\{0,1\}$, $p \in (1,+\infty)$ and $U \subset \mathbb{R}^{m-1}$ be open and convex. Let $\varphi : U \to \mathbb{R}^m$ be a bilipschitz mapping and $E = \varphi(U)$. Then there exists a constant $C = C(m) > 0$ such that for every $u \in W^{1,p}(E,\mathbb{R}^k)$,*

$$|u|^p_{W^{1-1/p,p}(E)} \leqq \frac{2^p}{p-1} L^{4m-5+2p} C \|u\|_{L^p(E)} \|D_\top u\|^{p-1}_{L^p(E)} \tag{A.1}$$

and

$$|u|^p_{W^{1-1/p,p}(E)} \leqq 2^p L^{2m-3+p} C\Big(\frac{2}{\operatorname{diam}(U)}\Big)^{p-1} \|u\|^p_{L^p(E)} + L^{4m-5+2p} C\Big(\frac{\operatorname{diam}(U)}{2}\Big) \|D_\top u\|^p_{L^p(E)}, \tag{A.2}$$

where $|u|^p_{W^{1-1/p,p}(E)}$ denotes the integral $\int_E \int_E \frac{|u(y)-u(x)|^p}{|y-x|^{m-2+p}} \,\mathrm{d}\mathcal{H}^{m-1}(y)\,\mathrm{d}\mathcal{H}^{m-1}(x)$ and $L > 0$ is a bilipschitz constant of φ.

Proof. Let $\|D_\top u\|_{L^p(E)} > 0$, because otherwise (A.1) and (A.2) are satisfied. According to [30, Theorem 3.2.5, p. 244], we have

$$|u|^p_{W^{1-1/p,p}(E)} = \int_U \int_U \frac{|u(\varphi(y)) - u(\varphi(x))|^p}{|\varphi(y)-\varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x)) \,\mathrm{d}y\,\mathrm{d}x, \tag{A.3}$$

where J_{m-1} denotes the $(m-1)$-dimensional Jacobian (see, for instance, Section 3.2.1, p. 241 in [30]). We define for each $\delta > 0$ the sets

$$W_\delta = \{(x,y) \in U^2 : |y-x| \geqq \delta\} \quad \text{and} \quad F_\delta = \{(x,y) \in U^2 : |y-x| < \delta\}.$$

Then $U^2 = W_\delta \cup F_\delta$. Hereinafter in this proof, C denotes a positive constant that can only depend on m and can be different from line to line. We first prove (A.1). For each $\delta > 0$,

$$\begin{aligned}
&\iint_{W_\delta} \frac{|u(\varphi(y)) - u(\varphi(x))|^p}{|\varphi(y)-\varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x)) \,\mathrm{d}y\,\mathrm{d}x \\
&\quad \leqq 2^{p-1} \iint_{W_\delta} \frac{|u(\varphi(y))|^p + |u(\varphi(x))|^p}{|\varphi(y)-\varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x)) \,\mathrm{d}y\,\mathrm{d}x \\
&\quad \leqq 2^{p-1} L^{m-2+p} \iint_{W_\delta} \frac{|u(\varphi(y))|^p + |u(\varphi(x))|^p}{|y-x|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x)) \,\mathrm{d}y\,\mathrm{d}x \\
&\quad \leqq 2^p L^{2m-3+p} \int_E |u|^p \,\mathrm{d}\mathcal{H}^{m-1} \int_{\mathbb{R}^{m-1}\setminus B^{m-1}_\delta} \frac{\mathrm{d}z}{|z|^{m-2+p}} \\
&\quad = \frac{2^p C L^{2m-3+p}}{(p-1)\delta^{p-1}} \int_E |u|^p \,\mathrm{d}\mathcal{H}^{m-1},
\end{aligned} \tag{A.4}$$

where we have used that φ is bilipschitz with the bilipschitz constant L and [30, Theorem 3.2.5]. Next, using Jensen's inequality, Fubini's theorem and the fact that φ is bilipschitz with the bilipschitz constant L, we have

$$\begin{aligned}
&\iint_{F_\delta} \frac{|u(\varphi(y)) - u(\varphi(x))|^p}{|\varphi(y) - \varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x))\, \mathrm{d}y\, \mathrm{d}x \\
&\quad = \iint_{F_\delta} \left| \int_0^1 \frac{\partial}{\partial t} u(\varphi((1-t)x + ty))\, \mathrm{d}t \right|^p \frac{J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x))}{|\varphi(y) - \varphi(x)|^{m-2+p}}\, \mathrm{d}y\, \mathrm{d}x \\
&\quad \leqq L^{4m-5+2p} \int_0^1 \left(\iint_{F_\delta} \frac{|D_\top u(\varphi((1-t)x + ty))|^p |y-x|^p}{|y-x|^{m-2+p}} J_{m-1}(\varphi((1-t)x + ty))\, \mathrm{d}y\, \mathrm{d}x \right) \mathrm{d}t \\
&\quad \leqq L^{4m-5+2p} \int_0^1 \left(\iint_{F_\delta} \frac{|D_\top u(\varphi((1-t)x + ty))|^p}{|y-x|^{m-2}} J_{m-1}(\varphi((1-t)x + ty))\, \mathrm{d}y\, \mathrm{d}x \right) \mathrm{d}t.
\end{aligned} \tag{A.5}$$

Performing the change of variable $z = (1-t)x + ty$ in the integral with respect to x, we have

$$\begin{aligned}
&\iint_{F_\delta} \frac{|D_\top u(\varphi((1-t)x + ty))|^p}{|y-x|^{m-2}} J_{m-1}(\varphi((1-t)x + ty))\, \mathrm{d}y\, \mathrm{d}x \\
&\qquad = \frac{1}{1-t} \iint_{F_{(1-t)\delta}} \frac{|D_\top u(\varphi(z))|^p}{|y-z|^{m-2}} J_{m-1}(\varphi(z))\, \mathrm{d}y\, \mathrm{d}z.
\end{aligned} \tag{A.6}$$

Therefore, integrating (A.6) with respect to $t \in [0, 1]$, we get

$$\begin{aligned}
&\int_0^1 \left(\iint_{F_\delta} \frac{|D_\top u(\varphi((1-t)x + ty))|^p}{|y-x|^{m-2}} J_{m-1}(\varphi((1-t)x + ty))\, \mathrm{d}y\, \mathrm{d}x \right) \mathrm{d}t \\
&\quad = \int_0^1 \frac{1}{1-t} \iint_{F_{(1-t)\delta}} \frac{|D_\top u(\varphi(z))|^p}{|y-z|^{m-2}} J_{m-1}(\varphi(z))\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}t \\
&\quad \leqq \int_E |D_\top u|^p\, \mathrm{d}\mathcal{H}^{m-1} \int_0^1 \frac{1}{1-t} \int_{B^{m-1}_{(1-t)\delta}} \frac{1}{|\xi|^{m-2}}\, \mathrm{d}\xi\, \mathrm{d}t \\
&\quad \leqq C\delta \int_E |D_\top u|^p\, \mathrm{d}\mathcal{H}^{m-1}.
\end{aligned} \tag{A.7}$$

Then, taking $\delta > 0$ such that

$$\delta = \frac{\|u\|_{L^p(E)}}{\|D_\top u\|_{L^p(E)}},$$

we obtain (A.1) from (A.3), (A.4), (A.5) and (A.7).

Next, we prove (A.2). Let us now choose $\delta = \mathrm{diam}(U)/2$. Then, changing $\int_{\mathbb{R}^{m-1} \setminus B^{m-1}_\delta} \frac{\mathrm{d}z}{|z|^{m-2+p}}$ to $\int_{B^{m-1}_{2\delta} \setminus B^{m-1}_\delta} \frac{\mathrm{d}z}{|z|^{m-2+p}}$ and using that $\frac{1-(1/2)^{p-1}}{p-1} \leqq \ln(2)$ in (A.4), we obtain

$$\begin{aligned}
&\iint_{W_\delta} \frac{|u(\varphi(y)) - u(\varphi(x))|^p}{|\varphi(y) - \varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x))\, \mathrm{d}y\, \mathrm{d}x \\
&\qquad \leqq 2^p \ln(2) L^{2m-3+p} C \left(\frac{1}{\delta}\right)^{p-1} \int_E |u|^p\, \mathrm{d}\mathcal{H}^{m-1},
\end{aligned}$$

which, together with (A.7), implies (A.2). This completes our proof of Lemma A.1. □

Corollary A.2. *Let* $p \in (1, 2]$. *Then there exists a constant* $C = C(m) > 0$ *such that for every* $u \in W^{1,p}(\partial B_r^m(x_0), \mathbb{R}^k)$,

$$|u|^p_{W^{1-1/p,p}(\partial B_r^m(x_0))} \leqq Cr \|D_\top u\|^p_{L^p(\partial B_r^m(x_0))}. \tag{A.8}$$

Furthermore, if $\mathcal{Y}$ *is a compact Riemannian manifold, which is isometrically embedded into* $\mathbb{R}^k$, $p \in [p_0, 2]$ *for some* $p_0 \in (1, 2]$ *and* $u \in W^{1,p}(\partial B_r^m(x_0), \mathcal{Y})$, *then there exists a constant* $C = C(p_0, m, \mathcal{Y}) > 0$ *such that*

$$|u|^p_{W^{1-1/p,p}(\partial B_r^m(x_0))} \leqq Cr^{\frac{m-1}{p}} \|D_\top u\|^{p-1}_{L^p(\partial B_r^m(x_0))}. \tag{A.9}$$

Proof of Corollary A.2. First, let us prove (A.8). By scaling and translation, it is enough to show that for each $u \in W^{1,p}(\mathbb{S}^{m-1}, \mathbb{R}^k)$,

$$|u|^p_{W^{1-1/p,p}(\mathbb{S}^{m-1})} \leqq C\|D_\top u\|^p_{L^p(\mathbb{S}^{m-1})}. \tag{A.10}$$

Recall that $m \geqq 2$ and for each $j \in \{1, \dots, m\}$, define $E_{2j-1} = \{x \in \mathbb{S}^{m-1} : x_j < 3/4\}$ and $E_{2j} = \{x \in \mathbb{S}^{m-1} : x_j > -3/4\}$. For every $x \in \mathbb{S}^{m-1}$, there is at most one $i \in \{1, \dots, 2m\}$ such that $x \notin E_i$. Thus, $\mathbb{S}^{m-1} \times \mathbb{S}^{m-1} = \bigcup_{i=1}^{2m} E_i \times E_i$. Moreover, the E_i's are bilipschitz images of the unit ball B_1^{m-1}.

Hereinafter in this proof, C will denote a positive constant that can depend only on m and can be different from line to line. Then, in view of (A.2) with $U = B_1^{m-1}$, we have

$$|u|^p_{W^{1-1/p,p}(E_i)} \leqq C(\|u\|^p_{L^p(E_i)} + \|D_\top u\|^p_{L^p(E_i)}). \tag{A.11}$$

Applying (A.11) with $v = u - \frac{1}{\mathcal{H}^{m-1}(E_i)} \int_{E_i} u \, \mathrm{d}\mathcal{H}^{m-1}$ and using the Poincaré inequality for v (observe that the constant in the Poincaré inequality can be chosen independently of $p \in (1, 2]$), we get the estimate $|u|^p_{W^{1-1/p,p}(E_i)} \leqq C\|D_\top u\|^p_{L^p(E_i)}$. Then, summing over i, we obtain (A.10), which implies (A.8).

Next, we prove (A.9). To this end, we show that if $p \in [p_0, 2]$ and $u \in W^{1,p}(\partial B_r^m(x_0), \mathcal{Y})$, then

$$|u|^p_{W^{1-1/p,p}(\partial B_r^m(x_0))} \leqq C' \|u\|_{L^p(\partial B_r^m(x_0))} \|D_\top u\|^{p-1}_{L^p(\partial B_r^m(x_0))}, \tag{A.12}$$

where $C' = C'(p_0, m) > 0$. Notice that if (A.12) would hold, then it would be "scale-invariant". Thus, it is enough to prove (A.12) in the case when $\partial B_r^m(x_0) = \mathbb{S}^{m-1}$. Applying Lemma A.1 with $E = E_i$ as above for each $i \in \{1, \dots, 2m\}$ and then summing over i, we get

$$|u|^p_{W^{1-1/p,p}(\mathbb{S}^{m-1})} \leqq \frac{C}{p_0 - 1} \|u\|_{L^p(\mathbb{S}^{m-1})} \|D_\top u\|^{p-1}_{L^p(\mathbb{S}^{m-1})}. \tag{A.13}$$

Setting $C' = C/(p_0 - 1) > 0$, (A.13) yields (A.12). The inequality (A.9) is a direct consequence of (A.12) and the fact that $\mathcal{Y}$ is a compact Riemannian manifold, which is isometrically embedded into $\mathbb{R}^k$. This completes our proof of Corollary A.2. □

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