Infinite Horizon Spatial Duopoly with Collusive Pricing and Noncollusive Location Choice

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January 1991

ABSTRACT

We investigate a spatial duopoly in which the firms simultaneously select locations at the beginning of time and, once chosen, the locations are fixed forever; however, the firms will choose prices in each of a countably infinite succession of time periods. We are interested in equilibrium behavior when the firms will collusively arrange a trigger strategy equilibrium in prices, and will select their locations knowing that a particular such trigger strategy price equilibrium will ensue. Specifically, we restrict our attention to prices that support an outcome on the profit possibility frontier at which the ratio of firm 1's profits to those of firm 2 is positively related to the similar ratio of profits at the single shot noncooperative equilibrium. The unique resulting location equilibrium involves the pairing of firms at the market center.

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ISSN-0771 3894

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1. Introduction

Ever since the seminal work of Harold Hotelling (1929), location models have proven to be very useful for describing differentiated product markets. The distinctive feature of these models, compared to other models of product differentiation, is that they allow an explicit representation of product choice by oligopolistic firms. Within this framework, the location space is considered as the array of potential variants of a product; a consumer's location corresponds to his ideal product; and the transportation cost is interpreted as the decrement of utility from not consuming the ideal product.

Unfortunately, as shown by d'Aspremont, Gabszewicz and Thisse (1979), Hotelling's original model was plagued by the non-existence of a price equilibrium in pure strategies when firms are located sufficiently close to each other. Recent work by Osborne and Pitchik (1987) casts some doubts on the possibility of using mixed strategies for solving this existence problem if the model is to be embedded within a more extended framework.

However, some reformulations of the Hotelling model have been developed that retain its main aspects while restoring existence of a unique price equilibrium in pure strategies for all location pairs. In particular, quadratic transportation costs have been used successfully to model spatial

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* We want to thank workshop participants at the Virginia Polytechnic Institute and the University of Rochester.

price competition and product positioning.[1]

In this paper we investigate a spatial duopoly in which the firms simultaneously select locations at the beginning of time and, once chosen, the locations are fixed forever; however, the firms will choose prices in each of a countably infinite succession of time periods. We are particularly interested in equilibrium behavior when the firms will collusively arrange a trigger strategy equilibrium in prices, and will select their locations knowing that a particular such trigger strategy price equilibrium will ensue. The market begins with the two firms choosing locations simultaneously. Following that, for times $t = 0, 1, 2, \ldots$ the firms simultaneously select prices in each period; however, we focus most of our attention on price policies for the firms that support an outcome on the profit possibility frontier at which the firms' profits are proportional to profits at the single shot noncooperative equilibrium. They choose locations presuming that these price policies will be followed subsequently. We have selected this sharing rule in the belief that the profits earned in colluding should somehow reflect the relative power of the firms. In general we think this power might be measured by capital stocks or noncollusive equilibrium profits or profit rates. In the model presented below there are no capital stocks; therefore noncollusive profits seem to us the appropriate measure.

However, strict proportionality of final profits to single shot noncooperative equilibrium profits is not required for our results and this restriction is dropped after most of the analysis is completed. It is sufficient that final profits is on the profit possibility frontier, that they are better for both firms than single shot noncooperative profits, and that the ratio of profits of firm 1 to those of firm 2 increase with an increase in the ratio of single shot noncooperative equilibrium profits.

One important characteristic of the present model, compared to other oligopolistic models using trigger strategies (see e.g., Friedman (1971)), is that the single-shot noncooperative equilibrium is not fixed but depends upon the locational choices made by firms at the beginning of the game. Similarly, the profit possibility frontier is a function of the two firms' locations. These two fundamental characteristics of our model make the analysis of the game substantially more involved than is typical in most existing models dealing with tacit collusion. Since we want to compare different location pairs to determine the location equilibrium, we need, therefore, tractable expressions for the equilibrium prices and payoffs of the single-shot game. The quadratic transportation cost reformulation of the Hotelling model yields such expressions.

Figure 1 measures profits for the two firms on the axes and illustrates the relationship between the single shot noncooperative equilibrium profits, marked N, and the trigger strategy outcome, marked T, on the profit possibility frontier (ATB). If firm 1, for example, would choose a different location, the profit possibility frontier would shift and the single shot noncooperative equilibrium would move; therefore, an alternative location would be better for firm 1 if the ray through the *new* single shot noncooperative equilibrium would intersect the profit possibility frontier at point to the right of the vertical line through T.

(place Figure 1 about here)

The remainder of the paper is organized as follows. In the next section, we introduce the setting and our notation. The single-shot equilibrium is described in Section 3. Section 4 is devoted to a detailed analysis of the profit possibility frontier as a function of firms locations. Our main results are discussed in Section 5 where it is shown that the central agglomeration of firms is the unique noncooperative equilibrium of the location game. We conclude the paper with a few remarks and make some suggestions for future research.

2. The Model

The basic model is a variant of Hotelling's (1929) spatial duopoly with quadratic transportation cost, a reservation price, and zero production costs.[2] Consumers are uniformly distributed along the unit interval. Demand is one per unit as long as the full price to the consumer does not exceed the reservation price. We use the following notation:

p_i = price of firm i,

D_i = demand and sales of firm i,

$R_i = p_i D_i$ = profit of firm i,

t = transport cost parameter,

π = reservation price, and

x_i = location of firm i.

The full price for a consumer located at z who buys from firm i is $p_i + t(x_i - z)^2$. The consumer buys from the firm with the lower full price if that full price does not exceed π; otherwise the consumer buys nothing. We assume that $\pi \geq 3t$ which (as Lemmas 1 to 3 establish) assures that all consumers face full prices at or below π in any equilibrium. Without loss of generality we take $x_1 \leq x_2$ throughout the paper. The amount demanded of firm i when the firms are located at $x = (x_1, x_2)$ and the prices of the firms are $p = (p_1, p_2)$ is denoted $D_i(p, x)$

3. Characterizing the Single Shot Equilibrium

The trigger strategy equilibrium that we use is characterized by two conditions. First, profits are on the profit possibility frontier (for their given locations) and second profits are in the same proportion as at the single shot noncooperative equilibrium (for their given locations). Thus, in profit space, the trigger profit outcome lies on the payoff possibility frontier on a ray from the origin through the single shot equilibrium. Prices, outputs, and profits at the single shot equilibrium as a function of locations are:[3]

$$p_1^c(x) = \frac{t(x_2-x_1)(2+x_1+x_2)}{3}, \qquad D_1^c(x) = \frac{2+x_1+x_2}{6}, \quad R_1^c(x) = \frac{t(x_2-x_1)(2+x_1+x_2)^2}{18}$$

$$p_2^c(x) = \frac{t(x_2-x_1)(4-x_1-x_2)}{3}, \qquad D_2^c(x) = \frac{4-x_1-x_2}{6}, \quad R_2^c(x) = \frac{t(x_2-x_1)(4-x_1-x_2)^2}{18}.$$

It is easily seen that both $R_1^c(x)$ and $R_2^c(x)$ fall as x_1 rises; similarly, both profits fall as x_2 falls. In general, single shot equilibrium profits are higher as the firms move farther apart. But for firms that will be using single shot equilibrium profits to select a payoff frontier outcome, it is the ratio $\Lambda = R_1^c(x)/R_2^c(x)$ that matters. For $x_1 = x_2 < 1$, $\Lambda(x)$ is defined as the limit as x_2 converges to x_1 from above and for $x_1 = x_2 = 1$, $\Lambda(x)$ is defined as the limit as x_1 converges to x_2 from below. Holding the profit possibility frontier constant, raising Λ helps firm 1 and hurts firm 2. When $x_1 < x_2$

$$\frac{\partial \Lambda}{\partial x_1} = \frac{\partial \Lambda}{\partial x_2} = \frac{12(2+x_1+x_2)}{(4-x_1-x_2)^3}. \tag{1}$$

Eq. (1) is positive and, for symmetric locations, $\partial\Lambda/\partial x_1 = 4/3$. Thus if the profit possibility frontier did not depend upon the firms' locations, firm 1 would benefit by moving its location to the right and firm 2 would benefit by moving to the left. However, the profit possibility frontier changes with firms' locations so that the analysis of the location equilibrium is more involved.

4. The Profit Possibility Frontier as a Function of Firms' Locations

In general, for fixed locations, $x = (x_1, x_2)$, a profit frontier point is found by maximizing, say, R_1 subject to a constraint on R_2. There is sufficient structure in the present model to characterize the profit possibility frontier more fully. In Lemma 1 it is shown that $\pi \geq 3t$ is a sufficient condition for all consumers to face a full price of π or less when there is only one firm in the market, irrespective of the firm's location. This condition is shown in Lemma 3 to be sufficient to ensure that all consumers are served at any Pareto optimum when there are two firms in the market.[4] Lemma 3 provides an exact characterization of the profit possibility frontier price vector as a function of the firms' locations and the location of the consumer who faces the same full price from the two firms (called the *y-consumer*). This characterization permits the section to culminate in Theorem 1 in which the relationship between the production possibility frontier and the firms' locations is described.

LEMMA 1 If $\pi \geq 3t$ then a monopolist located at $x_1 \in [0, 1]$ will maximize profits by serving the whole market at the price $p_1 = \pi - t(1 - x_1)^2$ if $x_1 \leq \frac{1}{2}$ or at $p_1 = \pi - tx_1^2$ if $x_1 \geq \frac{1}{2}$.

PROOF It suffices to establish the result for the case of $x_1 \in [0, \frac{1}{2}]$, because the argument is parallel for $x_1 \in]\frac{1}{2}, 1]$. Clearly the price $p_1 = \pi - t(1 - x_1)^2$ is more profitable than any smaller price as sales will be one at this or lower prices while total revenue (equal profit) will be proportional to price. Profit is given by $R_1 = (\pi - t(y - x_1)^2)y$ when the consumer at y faces a full price of π and the consumer at zero faces a full price no greater than π. Profit is concave in y (for $y \geq x_1$) so that we seek a condition assuring that $y < 1$ is not optimal for the monopolist. Differentiating profit with respect to y yields

$$\pi - t(3y^2 - 4x_1y + x_1^2) \tag{2}$$

which we require to be nonnegative at $y = 1$. The latter is equivalent to

$$\frac{\pi}{t} \geq (3 - x_1)(1 - x_1) \tag{3}$$

which is clearly satisfied at all location if $\pi \geq 3t$. □

In Lemmas 2 and 3 it is shown that, given a pair of locations x, points on the profit possibility frontier can be found by following these steps: (1) Select y, the location of the consumer (called the *y-consumer*) who is indifferent about which firm he patronizes. (2) Find a price pair that maintains this indifference at y and is the largest price pair such that all consumers face no more than their reservation prices. This price pair satisfies the following conditions:

$$p_2 = p_1 + t(y - x_1)^2 - t(y - x_2)^2 \tag{4}$$

$$p_1 = \max \{\pi - t[(y - x_1)^2 - (y - x_2)^2 + (1 - x_2)^2],\ \pi - t(y - x_1)^2,\ \pi - tx_1^2\}. \quad (5)$$

Eq. (4) is the condition that the full price at y is the same for the two firms. Eq. (5) is the condition that the whole market is served at the highest feasible prices that satisfy eq. (4). That is, $p_1 + t[(y - x_1)^2 - (y - x_2)^2 + (1 - x_2)^2]$ is the full price from firm 2 at location 1, $p_1 + t(y - x_1)^2$ is the full price of either firm at y, and $p_1 + tx_1^2$ is the full price from firm 1 at location 0. To serve the whole market at the highest possible prices, given eq. (4), requires that the largest of these three full prices equals π. The consumer facing a full price of π is called the *marginal-to-the-market consumer*. Hence the marginal-to-the-market consumer is located at 0 or y or 1.

Lemma 2 characterizes the location of the marginal-to-the-market consumer as a function of the firms' locations and the location of the y-consumer subject to two conditions: (a) All consumers face full prices of no more than π and (b) the price vector is Pareto optimal given that all consumers are served. In Lemma 3 condition (a) is lifted and it is proved that Pareto optimal pricing requires that all consumers be served, no matter where the firms locate or where the y-consumer is located.

LEMMA 2 Given locations for the firms such that $0 \le x_1 < x_2 \le 1$ and given that all consumers face a full price of π or less, the marginal-to-the-market consumer at profit possibility frontier price vectors is characterized thus as a function of y:

(I) If $2x_1 \le 2x_2 - 1$ then

for $y \in [0, 2x_1]$ the marginal consumer is at zero,

for $y \in (2x_1, 2x_2 - 1)$ the marginal consumer is at y, and for $y \in [2x_2 - 1, 1]$ the marginal consumer is at one.

(II) If $2x_2 - 1 < 2x_1$ then define y^* by

$$y^* = 0 \text{ if } (2x_2 - 1) < 0,$$

$$y^* = \frac{2x_2 - 1}{2(x_2 - x_1)} \text{ if } 0 \leq (2x_2 - 1) \leq 2(x_2 - x_1)$$

$$y^* = 1 \text{ if } (2x_2 - 1) > 2(x_2 - x_1)$$

For $y \in [0, y^*]$ the marginal consumer is at zero and

for $y \in [y^*, 1]$ the marginal consumer is at one.

PROOF Take first case (I). If $y \in [0, 2x_1]$ and $p_1 = \pi - tx_1^2$ then all of the market is being served and the consumer at one faces a full price below π. If prices are raised with y unchanged, then the consumers near zero face prices above π. A parallel argument holds for $y \in [2x_2 - 1, 1]$. If $y \in]2x_1, 2x_2 - 1[$, then with $p_1 = \pi - t(y - x_1)^2$ and $p_2 = \pi - t(x_2 - y)^2$, all consumers are being served, the consumers at both zero and one face full prices below π, and any higher prices will cause some consumers near y to face full prices above π.

For case (II), it is immediate that the marginal-to-the-market consumer is at zero for $y \leq 2x_2 - 1$ (when $2x_2 - 1 \geq 0$) and at one for $y \geq 2x_1$ (when $2x_1 \leq 1$). Let $v_1(y) = \pi - tx_1^2 + t(y - x_1)^2$ and $v_2(y) = \pi - t(1 - x_2)^2 + t(y - x_2)^2$. $v_1(y)$ is the full price from firm 1 to a consumer at y given that the full price at zero is π, while $v_2(y)$ is the full price at y from firm 2 given that the full price at one is π. Thus v_1 characterizes full prices from firm 1 when the marginal-to-the-market consumer is at zero and v_2 has an analogous interpretation. Note that $v_1(2x_2 - 1) < \pi$ and $v_2(2x_2 - 1) = \pi$ while $v_1(2x_1) =$

π and $v_2(2x_1) < \pi$. As both functions are continuous, there must be a value, y', in the interval $(2x_2 - 1, 2x_1)$ where they are equal. Setting the expressions equal to one another, y' is unique and equals $(2x_2 - 1)/(2x_2 - 2x_1)$. As there is no assurance $y' \in [0, 1]$, $y^* = y'$ on for $y' \in [0, 1]$ and is otherwise defined as above in the lemma. For y values less than or equal to y^* Pareto optimal profits require that firm 1 present a full price of π to the consumer at zero; for y values greater than or equal to y^* a full price of π from firm 2 to the consumer at one is required. □

LEMMA 3 If $\pi \geq 3t$ and there are two firms located at $x = (x_1, x_2)$ then, at any profit possibility frontier prices $p = (p_1, p_2)$, no consumer faces a full price greater than π.

PROOF Let (x, y, p) satisfy (a) $0 \leq x_1 \leq x_2 \leq 1$, (b) $0 \leq y \leq 1$, and (c) p is Pareto optimal subject to all consumers being served. It will be shown that any price change that raises some consumer's full price above π will cause the profit of at least one firm to fall. Take first the case of $2x_1 \leq 2x_2 - 1$ and $y \in]2x_1, 2x_2 - 1[$. The marginal-to-the-market consumer is at y and, by Lemma 1, any price increase by a firm will lower the profit with no change in the profit of its rival.

Now suppose the marginal-to-the-market consumer is at zero. Thus $R_1 = p_1y$. Consider a price increase by firm 1 accompanied by a change in p_2 that keeps R_1 constant. Then

$$dR_1 = \left[y - \frac{p_1}{2t(x_2 - x_1)} - \frac{p_1}{2tx_1}\right]dp_1 + \frac{p_1}{2t(x_2 - x_1)}dp_2 = 0. \tag{6}$$

The bracketed term in eq. (6), $\partial R_1/\partial p_1$, is composed of two effects. The first two terms are $y + p_1 \partial y/\partial p_1$, which would be $\partial R_1/\partial p_1$ if the firm lost no consumers near zero, and the third term expresses the profit decrease due to the loss of some consumers in the neighborhood of zero. The remaining term in eq. (6) is $\partial R_1/\partial p_2$ (times dp_2). Eq. (6) implies

$$dp_2 = \frac{p_1 x_2 - 2ty(x_2 - x_1)x_1}{p_1 x_1} dp_1. \tag{7}$$

Therefore, using eq. (7) we have

$$\frac{dR_2}{dp_1} = \frac{-4t^2(x_2 - x_1)y(1 - y)x_1 - p_1 p_2 + 2t[p_2 y x_1 + p_1(1 - y)x_2]}{2tp_1 x_1}. \tag{8}$$

The term $-4t^2(x_2 - x_1)y(1 - y)x_1$ is non-positive; hence, if $-p_1p_2 + 2t[p_2yx_1 + p_1(1 - y)x_2] < 0$ the lemma is proved for this case. Using $p_2 + t(y - x_2)^2 = p_1 + t(y - x_1)^2$ and $p_1 = \pi - tx_1^2$,

$$\begin{aligned} -p_1p_2 + 2t[p_2yx_1 + p_1(1 - y)x_2] &= -\pi^2 - \pi t[4y(x_2 - x_1) - 2x_2 - (x_2^2 + x_1^2)] \\ &\quad - t^2[x_1^2x_2^2 + 2yx_1(x_2 - x_1)^2 + 2x_1^2x_2 - 4y^2(x_2 - x_1)x_1]. \end{aligned} \tag{9}$$

Eq. (9) is strictly decreasing in y so that the maximum value occurs at $y = 0$. Using $y = 0$ in eq. (9) yields

$$-\pi^2 + \pi t(2x_2 + x_2^2 + x_1^2) - t^2(x_1^2x_2^2 + 2x_1^2x_2) = -[\pi - tx_1^2][\pi - tx_2(2 + x_2)] \tag{10}$$

which is strictly negative except at $x_2 = 1$. A parallel argument holds when the marginal-to-the-market consumer is at one, which completes the proof. □

Lemmas 4 and 5 establish, respectively, that a positively sloped ray from the origin always intersects the profit possibility frontier and that the frontier is parameterized by y with R_1 increasing and R_2 decreasing as y rises.

LEMMA 4 For $\pi \geq 3t$ and given x, the boundary of the attainable payoff space included in the positive orthant is monotone decreasing.

PROOF If the upper boundary is not monotone decreasing, then some firm's payoff reaches a local maximum somewhere on the boundary other than at one of the endpoints. Without loss of generality, suppose R_1 reaches a local maximum. There are three possibilities to consider according to the location of the marginal-to-the-market consumer. If the marginal consumer is at zero, then y is in a region where p_1 is constant as y increases. Thus p_2 can be increased which increases both y and $R_1 = p_1 y$, the profit of firm 1. This contradicts the existence of a local maximum in this region.

If y is in a region where the marginal consumer is at one; hence $p_2 = \pi - t(1 - x_2)^2$. That R_1 is at a local maximum in a region of the upper boundary where p_2 cannot be changed means that $\partial R_1/\partial p_1 = 0$. In addition, concavity of R_1 in p_1 means that any change in p_1 from its maximizing value will lower R_1. However, if p_1 is lowered to $\pi - t(1 - x_1)^2$, then firm 1 will receive the monopolist's maximum profit, given the location x_1. This is the largest profit firm 1 could receive, no matter what firm 2 does, given the location x_1; therefore this profit is no lower that profit at the supposed maximum. This contradicts the existence of a local maximum in this region.

If y is in a region where the marginal consumer is at y, then y must exceed $2x_1$ and firm 1 is receiving the profit that a monopolist would have if he were at x_1 and decided to serve consumers in the [0, y] interval. If p_1 is lowered and p_2 is raised so as to increase y, then profit must rise for firm 1, because a monopolist would gain by serving more of the market. Thus there can be no local maximum on the upper boundary for firm 1, which completes the proof. □

LEMMA 5 Let $\hat{p}(x, y)$ denote the largest price vector at which the whole market is served given y, the location of the y-consumer, and x, the firms' locations. Then for any x such that $0 \leq x_1 < x_2 \leq 1$, $R_1 = \hat{p}_1(x, y)y$ is increasing in y and $R_2 = \hat{p}_2(x, y)(1 - y)$ is decreasing in y.

PROOF First suppose the marginal-to-the-market consumer is at y. By Lemma 2 this implies that $2x_1 \leq 2x_2 - 1$ and $y \in (2x_1, 2x_2 - 1)$. Increasing y requires that p_1 fall and p_2 rise. That R_1 rises follows from the same argument used in Lemma 1 to show that a monopolist maximizes profit by serving the whole market. That R_2 falls follows from the same consideration.

Now suppose the marginal-to-the-market consumer is at 1 and consider separately the case of $y \geq x_2$ versus $y < x_2$. In the former, increasing y requires that p_1 fall, but having the marginal consumer at 1 means that p_2 cannot change. Clearly R_2 must fall because price is constant while sales diminish. But R_1 must rise because the amount of price decrease that is required to gain another ϵ length of consumers is less than would be the case for a monopolist at x_1 because the consumer's reservation prices for switching from firm 2 to firm 1 are beneath π. Such a decrease would benefit a monopolist, so à fortiori the duopolist benefits. If $y < x_2$ solve for y in terms of p_1 and p_2 from $p_1 + (y - x_1)^2 = p_2 + (y - x_2)^2$. Then use $p_2 = \pi - t(1 - x_2)^2$ to allow the profit of firm 1 to be expressed as a function of

p_1 alone:

$$p_1 y = \frac{[\pi - t(1 - x_2)^2 - p_1 + t(x_2^2 - x_1^2)]p_1}{2t(x_2 - x_1)}. \tag{11}$$

Again, as y increases p_1 must be reduced and R_2 necessarily falls; therefore, if the derivative of eq. (11) with respect to p_1 is negative, R_1 rises. That derivative is

$$\frac{\pi - t(1 - 2x_2 + x_1^2) - 2p_1}{2t(x_2 - x_1)} = \frac{t[1 - 2x_2 + x_1^2 + 4y(x_2 - x_1)] - \pi}{2t(x_2 - x_1)}. \tag{12}$$

Eq. (12) is negative if $1 - 2x_2 + x_1^2 + 4y(x_2 - x_1) < \pi/t$. The left hand side of this inequality is maximized when $y = 1$, $x_1 = 0$, and $x_2 = 1$. Using these values yields $3 < \pi/t$ which is the critical relation found for the monopolist in Lemma 1. Of course a parallel argument can be made when the marginal-to-the-market consumer is located at zero, which completes the proof. □

Thus the boundary of the profit space is downward sloping in the positive quadrant (i.e., coincides with the profit possibility frontier), all consumers buy at a Pareto optimum, and the points along the frontier are parameterized by y, the location of the consumer indifferent between the two firms. As x_1 or x_2 changes the profit possibility frontier shifts. Imagine locations $x = (x_1, x_2)$ for the firms. Given x there is a profit possibility frontier and it will be useful to know how that frontier shifts as a function of x_1. In Theorem 1 it is shown that when the consumer at zero is the marginal-to-the-market consumer, then an increase in x_1 results in lower Pareto optimal profits for both firms. The converse holds if the marginal-to-the-market consumer is at y or at one and parallel statements can be made for changes in x_2.

THEOREM 1 Let x be the firms' locations. For any given y, let p be the profit possibility frontier price vector relative to x such that the consumer indifferent between the two firms is at y. If (a) $p_1 = \pi - tx_1^2$ and either x_1 rises or x_2 rises, (b) $p_2 = \pi - t(1 - x_2^2)$ and either x_1 falls or x_2 falls, or (c) $p_1 < \pi - tx_1^2$, $p_2 < \pi - t(1 - x_2^2)$ and either x_1 falls or x_2 rises, then the profit possibility frontier profits $[p_1y, p_2(1 - y)]$ will lie outside the new profit possibility frontier .

PROOF Let $x_1' > x_1$, let p' be a profit possibility frontier price vector relative to (x_1', x_2) such that the y-consumer is at $y' = y$. If $\pi = p_1 + tx_1^2$, then $p' \ll p$ so that $p_1'y < p_1y$ and $p_2'(1 - y) < p_2(1 - y)$. To see this, note that as $\pi = p_1 + tx_1^2$, p_1 must be reduced (to keep full price at π for the consumer at zero) and p_2 must be reduced (to keep full price at y the same for both firms). This results in lower frontier profit for both firms, so that $[p_1y, p_2(1 - y)]$ is no longer attainable. This covers the movement of x_1 in (a) and obvious parallel arguments cover the converse of (a) for a movement of x_1 as well as the movement of x_2 in (b) and the converse of (b).

Let x_1' and p' be as above. If $\pi > p_1 + tx_1^2$, then $p_2' = p_2$ and $p_1' > p_1$ so that $p_1'y > p_1y$ and $p_2'(1 - y) = p_2(1 - y)$. To see this note that the consumer facing a full price of π remains where he was (i.e., at y or at 1). Therefore, p_2 is unchanged, because the distances from x_2 to both y and 1 are unchanged. Meanwhile, p_1 must be raised (to keep the full price at y the same from both firms). This raises the profit of firm 1 and keeps the profit of firm 2 unchanged; therefore, $[p_1y, p_2(1 - y)]$ is interior to the attainable profit space. This covers the movement of x_1 for the converse of (b) and the converse of (c) an obvious parallel arguments cover the remaining cases. □

Theorem 1 is illustrated in Figure 2. Suppose x is such that the profi

(place Figure 2 about here)

possibility frontier is ABDE and let B correspond to $\bar{y} = \min\{2x_1, (2x_2 - 1)/(2x_2 - 2x_1)\}$. Then along the segment AB the points correspond to values of $y \in [0, \bar{y}]$. If x_1 is increased, the frontier changes to something like ACDF. The content of the theorem is also summarized in Table 1. An entry in the cell marked S indicates that the profit possibility frontier shrinks inward and a cell marked E indicates that the frontier expands outward.

		x_1		x_2	
		rise	fall	rise	fall
marginal to the market consumer	0	S	E	S	E
	y	E	S	S	E
	1	E	S	E	S

Table 1

5. Existence of a Unique Equilibrium

Eq. (1) and Theorem 1 yield sufficient information to show that there i a unique equilibrium with the firms located at $x^* = (\frac{1}{2}, \frac{1}{2})$. This is shown in three steps, the first of which is Lemma 6 establishing that equilibrium requires the firms to be at the same location and the second is Theorem 2, showing that $x = (\frac{1}{2}, \frac{1}{2})$ is the only candidate location. To carry out the third step establishing that $(\frac{1}{2}, \frac{1}{2})$ is indeed an equilibrium location choice, the repeated game and equilibrium trigger strategy combinations must be specified and examined.

LEMMA 6 Given two firms and $\pi \geq 3t$, equilibrium cannot occur at x if $x_1 \neq x_2$.

PROOF The key to the proof is that, from any location x, at least one firm can move toward the other so as to expand the profit possibility frontier and raise the ratio of his Cournot profit to that of his rival. Note that eq. (1) is always positive which means that when $x_1 < x_2$, $R_1^C(x)/R_2^C(x)$ will rise if x_1 rises and $R_1^C(x)/R_2^C(x)$ will fall if x_2 falls. Putting this together with Table 1, at the candidate equilibrium (1) if the marginal-to-the-market consumer is at zero, then firm 2 will increase his equilibrium profit by reducing x_2, (2) if the marginal-to-the-market consumer is at one, then firm 1 will increase his equilibrium profit by increasing x_1, and (3) if the marginal-to-the-market consumer is at y, then either firm will increase his equilibrium profit by moving toward the other firm. □

THEOREM 2 Given two firms and $\pi \geq 3t$, if equilibrium exists, it is unique and occurs at $x = (\frac{1}{2}, \frac{1}{2})$.

PROOF Two things must be shown: (1) (x_1, x_2) is not an equilibrium when $x_1 = x_2 \neq \frac{1}{2}$ and (2) (x_1, x_2) is an equilibrium when $x_1 = x_2 = \frac{1}{2}$. Taking the former first, note that $R_1^C(x) = R_2^C(x) = 0$ if $x_1 = x_2$, but, for $x_1 < x_2$ with x_1 almost equal to x_2, $R_1^C(x)/R_2^C(x)$ is approximately $\nu(x_1) = (1 + x_1)^2/(2 - x_1)^2$. Consider now $x_1 = x_2 < \frac{1}{2}$ where $\nu(x_1) < 1$. If firm 2 were to locate just to the right of firm 1, then the profit possibility frontier is virtually unchanged, while $R_1^c(x)/R_2^c(x)$ drops to a value that is bounded strictly below 1 (by $\nu(x_1)$ in the limit as the rightward movement goes to zero). Thus the equilibrium is moved to a point on the frontier to the benefit of firm 2. Conversely, if $x_1 = x_2 > \frac{1}{2}$ where $\nu(x_1) > 1$, a similar argument can be made to show that firm 1 will gain by making a small movement to the left. Thus $x_1 = x_2 \neq \frac{1}{2}$ cannot be an equilibrium.

Consider now $x_1 = x_2 = \frac{1}{2}$ and compare this for firm 1 with some alternate x' where $x_1' = \frac{1}{2} - \epsilon$ $(\epsilon > 0)$ and $x_2' = \frac{1}{2}$. At such an x' the ratio $R_1^c(x)/R_2^c(x) = (3 - \epsilon)^2/(3 + \epsilon)^2 < 1$. As x_1 is increased toward $\frac{1}{2}$, $R_1^c(x)/R_2^c(x)$ increases and the frontier moves outward, so that firm 1 cannot gain by x_1 less than $\frac{1}{2}$. Moving to the right runs into an equal disadvantage. □

Turning now to describing this locational duopoly as a repeated game, the set of players is, of course, $N = \{1, 2\}$, the firms share a common discount parameter, $\alpha \in]0, 1[$, and, letting $p_t = (p_{1t}, p_{2t})$ denote the price vector chosen at time t, the discounted profit of firm i, given locations x and a sequence of price vectors $p_0, p_1, p_2, \ldots$, is

$$\sum_{t=0}^{\infty}\alpha^t p_{it} D_i(p_t, x)$$

Relative to the locations x, let $p^*(x)$ denote the profit possibility frontier price vector for which profits are $R_1^*(x)/R_2^*(x) = p_1^*(x)D_1(p^*(x), x)/p_2^*(x)D_2(p^*(x), x) = R_1^c(x)/R_2^c(x)$. The strategy of firm i is $\sigma_i = (x_i, \phi_{i0}, \phi_{i1}, \phi_{i2}, \ldots)$ where x_i is the location selected by the firm, ϕ_{i0} is a decision rule that selects p_{i0} as a function of x, and each ϕ_{it} $(t > 0)$ is a decision rule that selects p_{it} as a function of the locations and all past price choices, $(x, p_0, \ldots, p_{t-1})$. Our interest lies in determining whether the following trigger strategies are a subgame perfect noncooperative equilibrium in the infinitely repeated game: $\sigma^* = (\sigma_1^*, \sigma_2^*)$ and $\sigma_i^* = (x_i^*, \phi_{i0}^*, \phi_{i1}^*, \phi_{i2}^*, \ldots)$ $(i = 1, 2)$ where

$$x_i^* = \tfrac{1}{2},$$

$$\phi_{i0}^*(x) = p_i^*(x)$$

$$\phi_{it}^*(x, p_0, \ldots, p_{t-1}) = p_i^*(x) \text{ if } p_\tau = p^*(x)$$
$$\text{for } \tau = 0, \ldots, t-1, \text{ and}$$

$$\phi_{it}^*(x, p_0, \ldots, p_{t-1}) = p_i^C(x) \text{ otherwise.}$$

THEOREM 3 There exists $\alpha^0 \in]0, 1[$ such that if $\alpha \in]\alpha^0, 1[$ the strategy combination σ^* is a subgame perfect noncooperative equilibrium in the infinitely repeated game.

PROOF Because the strategies of the firms do not trigger from a profit possibility frontier price vector to a single-shot Nash equilibrium price vector in response to the location choices, but only trigger in response to deviations from the frontier prices, it is necessary that the strategy combination prescribe equilibrium behavior that supports frontier profits given any locations. Let $R_1^d(x) = \max_{p_i} p_1 D_1(p_1, p_2^*(x), x)$, define $R_2^d(x)$ analogously, and let

$$\alpha^0 = \max_{0 \le x_1 \le x_2 \le 1} \frac{R_1^d(x) - R_1^*(x)}{R_1^d(x) - R_1^c(x)} \tag{1$$

Clearly $\alpha^0 \in]0, 1[$ because $R_1^d(x) - R_1^*(x)$ and $R_1^*(x) - R_1^c(x)$ are both positive and bounded strictly away from zero. To see that σ^* is a subgame perfect equilibrium, suppose an arbitrary time t and arbitrary history of play throu time t - 1. A history falls into one of two categories: (a) no player i ha deviated from choosing $p_i^*(x)$ at any time in the past or (b) at least one player has deviated. If (a) holds, then, by deviating, player i receives a discounted profit for the remainder of the game that is no greater than $R_1^d($ + $\alpha R_1^c(x)/(1 - \alpha)$ and if he does not deviate, his discounted profit is $R_1^*(x)/$ - α). If $\alpha \in]\alpha^0, 1[$, then the latter profit is at least as large as the former. If (b) holds, then the firms choose single shot equilibrium prices, which constitutes equilibrium behavior. Thus the strategy combination σ^* induces an equilibrium on every subgame and Theorem 2 implies that the only equilibrium choice of locations is (½, ½), the choice specified by σ^*. □

On the equilibrium path the players thus choose $x^* = (½, ½)$ and $p^*(x^*)$ $(\pi - ½t, \pi - ½t)$. A subgame perfect trigger strategy equilibrium for the subgame following the choice of (½, ½) can be sustained as long as $\alpha > ½$.

The firms' trigger strategy profits have been taken to be proportional to single-shot equilibrium profits in all discussion to this point; however, large variety of payoff rules yield the same result as Theorem 3. Suppose, particular, that the firms are agreed that their prices will be chosen so tl their profits per period will be F(x) and suppose that F satisfies the following conditions: (a) F(x) is on the profit possibility frontier relat: to x, (b) $F_i(x) > R_i^c(x)$, i = 1, 2, and (c) $F_1(x)/F_2(x)$ increases as $R_1^c(x)/R_2^c($

increases. Our rule is a special case of conditions (a) to (c); however, the proofs of Theorems 2 and 3 rely only upon (a) to (c). The proofs do not make use of equality of $F_1(x)/F_2(x)$ and $R_1^c(x)/R_2^c(x)$, which means there is a corollary to Theorem 3. Let $p^F(x)$ denote the price pair that achieves the single-shot profits $F(x)$, let ϕ_{it}^F be defined precisely as ϕ_{it}^* above except that p^F replaces p^*, and let σ^F be defined as σ^* except that ϕ^F is substituted for ϕ^*. Then we have

COROLLARY Let F satisfy (a) to (c). Then there is some discount parameter, $\alpha^F \in]0, 1[$ such that, if $\alpha \in]\alpha^F, 1[$, then σ^F is a subgame perfect noncooperative equilibrium.

PROOF We omit details of proof because the proof is too similar to typical trigger strategy equilibrium existence proofs and to the proof of Theorem 3. The precise value of α^F will, of course, depend upon $F(x)$. □

6. Concluding Remarks

In this paper, we have taken a very different route from those explored up to now in spatial competition. As in Hotelling and successors, we have assumed that firms choose locations (or products) once-and-for-all at the noncooperative equilibrium of a game. However, unlike Hotelling and successors, we have also assumed that price choice takes place forever. Firms choose their prices at a subgame-perfect equilibrium of a repeated game. We have identified a trigger strategy supporting an outcome on the profit possibility frontier at which firms' profits are proportional to profits at the single-shot equilibrium.

Changing the time horizon of the price stage has a dramatic impact on firms' locational decisions. *While firms choose to locate at the market endpoints in order to relax price competition in the one-shot market, they choose to locate at the market center when due account is taken of the repetition of the price choices.* Hence minimum differentiation emerges as th only equilibrium outcome when collusion on price follows competition on location. Price competition was enough to destroy Hotelling's belief that spatial competition should result in "excessive sameness"; price collusion is enough to support it. Note, however, that firms do not exploit the market as fully as they could if they merged and were located optimally from the standpoint of a two-plant monopolist.

Locating at the same point means that firms' ability to punish one another for defection is maximized once the equilibrium locations are selected. By minimizing the extent of their product differentiation, the firms can support their collusion using the maximal interval of discount parameters. In this case the single-shot equilibrium profits at the repeated game equilibrium are zero which coincides with the firms' minimax payoffs. Thus a byproduct of the equilibrium is that, once the locations ($\frac{1}{2}$, $\frac{1}{2}$) have been chosen, the punishment for defecting is naturally the most severe punishment possible within the model and the corresponding equilibrium can be sustained by any discount parameter above $\frac{1}{2}$.[5] (See Abreu (1988) and Fudenberg and Maskin (1986).)

Our reformulation of the spatial competition model suggests some natura extensions. Firstly, it would be worth knowing whether minimum differentiation still holds for more than two firms. Clearly, the equilibriu configuration of firms will be affected by collusive pricing, but it is not obvious that all firms will choose to locate together. Some might want to get away from the market center. Secondly, instead of assuming that firms select their location simultaneously, one could consider the case where they choose sequentially. This problem becomes especially relevant when firms are not confined to establish themselves inside the market. In this case, indeed, there is a first-mover advantage with the first entrant locating at the market center, the second outside the market, and both selecting prices simultaneously. If collusion on price takes place after entry, then it seems reasonable to expect the second firm to locate at the same place as the first one, thus destroying the first-mover advantage observed in the one-shot price game. Finally, alternative profit sharing rules should be analyzed. We believe, however, that the tendency toward central agglomeration uncovered in this paper will be robust to several alternative rules.[6]

The particular game we analyzed in this paper is representative of a class of games in which players have the ability, in a prior stage of such a game, to determine the set of outcomes that are admissible in the subsequent repeated subgames. Clearly, potential applications of such games are many in industrial organization, and the results derived here invite us to develop them.

Notes

1. Models of product differentiation involving a quadratic utility loss function include William Novshek and Hugo Sonnenschein (1979), Curtis Eaton and Myrna Wooders (1985), Damien Neven (1985), and Nicholas Economides (1989 It is also worth mentioning that many ideal-point models used in marketing assume that preferences are negatively related to the square of the Euclidea distance between the stimulus and the individual's ideal point (see, e.g. Pa Green and V. Srinivasan (1978)).

2. We need a reservation price in order to guarantee the existence of a collusive solution in price.

3. See d'Aspremont, Gabszewicz and Thisse (1979) for further details.

4. We use Pareto optimality with reference to only the two firms; therefore an outcome is Pareto optimal if and only if it is on the profit possibility frontier.

5. These most severe punishments are possible *given that the firms locate a the same place*; therefore, such punishments are not available as of the beginning of the game *before locations have been selected* nor are they available if $x_1 \neq x_2$.

6. For instance Jehiel (1990) also finds that firms locate at the market center when the Nash bargaining solution is used in the second stage of the game. It is not clear whether the Nash bargaining solution falls within the class of rules covered in our Corollary to Theorem 3 because the Nash soluti is sensitive to the slope of the payoff possibility frontier and that slope changes as the firms' locations vary. We were not obliged to deal explicitl with such slope changes.

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Figure 1

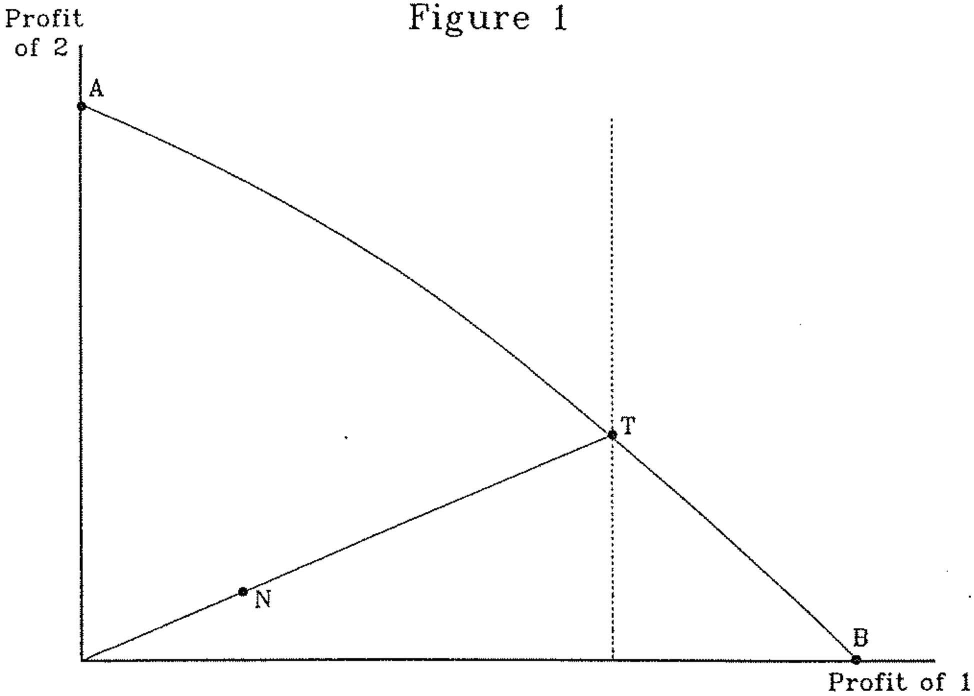

Figure 2